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IRMA Lectures in Mathematics and Theoretical Physics
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Handbook of Pseudo-Riemannian Geometry and Supersymmetry

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To the memory of Krzysztof Galicki

Preface

The project for this special volume on pseudo-Riemannian geometry and supersymmetry grew out of the 77th "Encounter between Mathematicians and Theoretical Physicists" at the Institut de Recherche Mathématique Avancée in Strasbourg in 2005.

A number of authors of this volume participated in that meeting, including our friend Kris Galicki, who died as a consequence of a tragic hiking accident in 2007. He was always interested in the differential geometric structures occurring in physical theories and was one of the rare scientists who mastered the language of mathematics as well as that of physics.

The aim of this handbook is to cover recent developments in the field in a language comprehensible to both, mathematicians and theoretical physicists. The intended audience consists of advanced students and researchers working in differential geometry, string theory and related areas. It includes a chapter about each of the following subjects: special geometry and supersymmetry, generalized geometry, geometries with torsion, para-geometries, holonomy theory, symmetric spaces and spaces of constant curvature, conformal geometry and other topics of recent interest. The contents of each chapter is briefly summarised in the introduction.

Acknowledgements. I am grateful to the authors for their beautiful expositions and results. I would like to thank Vladimir Turaev for inviting me to realise this book project and Manfred Karbe and Irene Zimmermann from the EMS Publishing House for the good cooperation. Finally, I would like to thank Eva Kuhlmann for assisting me in the editorial process.

Hamburg, January 2010 Vicente Cortés

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Introduction

The present handbook focuses on recent developments in pseudo-Riemannian geometry and supersymmetry. In this introduction we give a short overview of the material contained in the various parts of the volume.

Part A. Special geometry and supersymmetry

A classical field theory is usually specified by a Lagrangian L. The scalar fields of the theory are functions $\phi^1, \ldots, \phi^n$ on space-time $\mathbb{M}$. They can be interpreted as the components of a map $\phi\colon \mathbb{M} \to M$ from space-time into a target manifold M with respect to some system of local coordinates $x^1, \ldots, x^n$ on M. The kinetic term $-\frac{1}{2}\sum g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$ for the scalars in the Lagrangian L defines a pseudo-Riemannian or even Riemannian metric $g = \sum g_{ij}dx^i dx^j$ on M, provided that the symmetric matrix (g_{ij}) of the scalar couplings is nondegenerate or even positive definite. Since the discovery of the first supersymmetric field theories, physicists have found that supersymmetry is often reflected in geometric properties of the target metric g. The specific restrictions imposed by supersymmetry depend on the dimension and signature of space-time, as well as on the field content of the theory. When the number of supercharges increases, the allowed target geometry is more and more restricted and becomes finally locally symmetric. The most interesting case is that of eight (real) supercharges. The corresponding geometry is called *special geometry*. In four-dimensional supergravity theories the matter fields are assembled in vector and hyper multiplets, which are coupled to the supergravity multiplet. The special geometry of supergravity coupled to vector mutiplets is (projective) special Kähler geometry. That of supergravity coupled to hyper mutiplets is quaternionic Kähler geometry. Both geometries are related by the so-called c-map, which associates a quaternionic Kähler manifold N of real dimension $4n + 4$ (and negative scalar curvature) to any special Kähler manifold M of real dimension $2n$. Any quaternionic Kähler manifold N (of non-zero scalar curvature) is the base of an $\mathbb{H}^*/\{\pm 1\}$-bundle $U(N) \to N$ called the Swann bundle of N. The Swann bundle is a hyper-Kähler cone. In particular, it carries a hyper-Kähler metric, the signature of which depends on the sign of the scalar curvature of the quaternionic Kähler base N. The hyper-Kähler metric can be locally derived from a (real valued) hyper-Kähler potential f, which effectively encodes also the geometry of N. Similarly, the geometry of the special Kähler manifold M is locally encoded in a holomorphic function F (called the holomorphic prepotential).

In Chapter 1 by Martin Roček, Cumrum Vafa and Stefan Vandoren, the hyper-Kähler potential f is determined in terms of the holomorphic prepotential F. The c-map is thus reduced to the correspondence $F \mapsto f$.

In Gregor Weingart's contribution, the bundle of differential forms on a quaternionic Kähler (or hyper-Kähler) manifold is decomposed into parallel subbundles. In

particular, the multiplicities of the corresponding irreducible $\mathrm{Sp}(n)\mathrm{Sp}(1)$-representations (or $\mathrm{Sp}(n)$-representations in the hyper-Kähler case) are explicitly calculated.

Charles Boyer and Kris Galicki discuss Sasakian manifolds and their relation to special holonomy groups and supersymmetry in Chapter 3. Sasakian manifolds are intimately related to Kähler manifolds, which are fundamental objects in mathematics and theoretical physics. In fact, the metric cone over a Sasakian manifold is Kähler and the geometry transversal to the Sasakian vector field is also Kähler. Similarly, 3-Sasakian manifolds are intimately related to hyper-Kähler and quaternionic Kähler manifolds.

In the chapter by María A. Lledó, Oscar Maciá, Antoine Van Proeyen and Veeravalli S. Varadarajan the space-time signature remains Lorentzian but the signature of the special (pseudo-)Kähler target metric is arbitrary.

In Chapter 5, Thomas Mohaupt explains the role of special geometry in the theory of supersymmetric black holes. In particular, he shows how Euclidian supersymmetry in three dimensions can be used to study stationary black hole solutions in four dimensions. As one can see already from this example, the geometric structure of the target manifold of a supersymmetric theory can change significantly when the space-time signature changes from Lorentzian to Euclidian. Here not only the target metric changes from Riemannian to neutral, but from quaternionic Kähler to para-quaternionic Kähler. Such para-geometries are further discussed below.

Part B. Generalized geometry

Mirror symmetry relates deformations of complex structures to deformations of symplectic structures (on the mirror manifold). Nigel Hitchin's notion of a *generalized complex structure* provides a superordinate conceptual framework in which complex and symplectic structures can be treated symmetrically.

The chapter by Hitchin is an introduction to the rapidly developing subject of generalized geometry, which incorporates central concepts of supergravity and string theory. In particular, the B-field and the 3-form gauge field H occur naturally in the twisting of the generalized tangent bundle by a gerbe. Moreover, the three-form H plays also the role of the torsion of a metric connection on the base manifold.

Alexei Kotov and Thomas Strobl focus on the role of such 'generalized' geometries encoded in some algebroid structure as targets of supersymmetric sigma models. In particular, generalized Kähler manifolds occur as such targets.

Ulf Lindström, Martin Roček, Rikard von Unge and Maxim Zabzine show in Chapter 8 that generalized Kähler structures can be derived from a generalized Kähler potential.

Part C. Geometries with torsion

Let $G \subset \mathrm{O}(n)$ be a closed subgroup. A G-structure on an n-dimensional manifold admits a torsion-free connection only if the holonomy group of the Levi-Civita connection is a subgroup of G. Therefore, for a given G-structure, there may be no torsion-free connection at all. One is led to allow connections with non-zero torsion and to look for conditions on the torsion which ensure the uniqueness of the connection. It turns out that complete skew-symmetry of the torsion provides such a condition for certain G-structures. Moreover, connections with totally skew-symmetric torsion occur naturally in string theory and supergravity, as explained in the broad survey about geometries with torsion by Ilka Agricola.

A beautiful example of a G-structure admitting a unique connection with totally skew-symmetric torsion is provided by the class of nearly Kähler manifolds, which are almost Hermitian manifolds such that the Levi-Civita covariant derivative of the almost complex structure is completely skew-symmetric. General almost Hermitian structures admitting a unique connection with totally skew-symmetric torsion are discussed by Paul-Andi Nagy, whereas Jean-Baptiste Butruille explains the classification of homogeneous nearly Kähler manifolds.

Nearly Kähler structures with indefinite metric are considered by Lars Schäfer and Fabian Schulte-Hengesbach in Chapter 12. In particular, they prove that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ admits a unique left-invariant nearly pseudo-Kähler structure.

As mentioned in Part A of the introduction, the Swann bundle provides a fundamental correspondence $M \mapsto U(M)$, which associates a hyper-Kähler cone $U(M)$ with any quaternionic Kähler manifold M. The inverse construction, which associates (at least locally) a quaternionic Kähler manifold $M(U)$ with any hyper-Kähler cone U, is known as the superconformal quotient in the physics literature. It relates a superconformal field theory with scalar manifold U to a Poincaré supergravity theory with target $M(U)$. It turns out that geometric and field theoretic constructions are often much simpler when described in terms of hyper-Kähler geometry. In Chapter 13 Andrew Swann explains how these results extend to the framework of geometries with torsion. The underlying superconformal algebra is now the one-parameter family of simple Lie superalgebras $D(2,1;\alpha)$, which occurs, for instance, in the work of Michelson and Strominger on superconformal quantum mechanics.

Part D. Para-geometries

A complex structure on a (smooth) manifold M can be defined as an endomorphism field $J \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -\mathrm{Id}$ and such that the eigendistributions $T^{1,0}M, T^{0,1}M \subset TM \otimes \mathbb{C}$ are involutive. Similarly, a *para-complex* structure on a manifold is an endomorphism field J such that $J^2 = \mathrm{Id}$ with involutive eigendistributions $T^+M, T^-M \subset TM$ of the same dimension. In virtue of the Frobenius theorem, a para-complex structure is simply a local product structure with factors of equal dimension. Nevertheless, it is helpful to make use of the analogy between complex

and para-complex manifolds. There is a useful para-holomorphic calculus in which the role of the field of complex numbers $\mathbb{C} = \mathbb{R}[i]$, $i^2 = -1$, is played by the ring of para-complex numbers $\mathbb{R}[e]$, $e^2 = 1$. Many interesting structures in Riemannian geometry have natural "para-analogues". In particular, there is a notion of para-Kähler (or bi-Lagrangian), special para-Kähler, para-hyper-Kähler (or hypersymplectic) and para-quaternionic Kähler manifold. These manifolds carry pseudo-Riemannian metrics of split signature. Remarkably, these structures occur as special geometries of supersymmetric field theories, when the Lorentzian space-time metric is replaced by a positive definite metric, see the chapter by Thomas Mohaupt.

In Chapter 14, Stefan Ivanov, Ivan Minchev and Simeon Zamkovoy discuss twistor spaces of general almost para-quaternionic manifolds.

Matthias Krahe establishes a Darboux theorem for para-holomorphic symplectic and contact structures. This fundamental result can be applied, for instance, in the twistor theory of para-quaternionic Kähler manifolds. His contribution develops the twistor theory of para-pluriharmonic maps into symmetric spaces.

Dmitri V. Alekseevsky, Constantino Medori and Adriano Tomassini classify maximally homogeneous para-CR manifolds of semisimple type.

Part E. Holonomy theory

The *holonomy group* of a pseudo-Riemannian manifold M of signature (p, q) at a point $x \in M$ is the subgroup $\mathrm{Hol}_x \subset \mathrm{O}(T_xM) \cong \mathrm{O}(p, q)$ generated by parallel transports along loops based at x. For connected manifolds this yields a subgroup $\mathrm{Hol} \subset \mathrm{O}(p, q)$ well defined up to conjugation in the pseudo-orthogonal group $\mathrm{O}(p, q)$. Holonomy groups were introduced by Élie Cartan in the twenties for the study of Riemannian symmetric spaces and became a powerful tool in Riemannian geometry with Berger's classification of holonomy groups of complete simply connected Riemannian manifolds in the fifties.

Anton Galaev and Thomas Leistner review recent developments in the holonomy theory of pseudo-Riemannian manifolds. These include their classification of Lorentzian holonomy groups and Anton Galaev's classification of holonomy groups which are subgroups of $\mathrm{U}(1, n)$. The general classification problem for holonomy groups of pseudo-Riemannian manifolds of arbitrary signature remains unsolved. It includes the classification of pseudo-Riemannian symmetric spaces of arbitrary signature, which is already too complicated a problem to expect a simple solution.

Chapter 18 by Antonio J. Di Scala, Thomas Leistner and Thomas Neukirchner contains proofs of some facts about irreducible representations of Lie groups and applications of these results in holonomy theory.

In the chapter by Konrad Waldorf, the notion of holonomy of a line bundle (endowed with a connection) around a loop is extended to the holonomy of a gerbe along a closed oriented surface, which corresponds to the interaction of a string with a three-form gauge field.

Part F. Symmetric spaces and spaces of constant curvature

A pseudo-Riemannian manifold M is called a *symmetric space* if every point $x \in M$ is an isolated fixed point of an involutive isometry. This includes the complete simply connected pseudo-Riemannian manifolds of constant curvature.

Ines Kath reviews the state of the art in the classification of pseudo-Riemannian symmetric spaces. Like for the classification of pseudo-Riemannian holonomy groups, little is known beyond metrics of index 2. She explains various approaches and partial results, for instance under the assumption of additional geometric structures.

Dmitry Alekseevsky discusses in Chapter 21 the classification problem for pseudo-Kähler and para-Kähler symmetric spaces. In particular, he describes some classes of Ricci-flat examples.

Oliver Baues develops the theory of flat pseudo-Riemannian manifolds in the general context of flat affine structures and prehomogeneous affine representations. Flat Riemannian manifolds are well understood by Bieberbach's theorems, but there are still many long standing open problems concerning flat pseudo-Riemannian manifolds of arbitrary signature. For instance, it is not known whether every compact flat pseudo-Riemannian manifold of signature (p, q) (with $p \geq q \geq 2$) is a quotient of the pseudo-Euclidian space $\mathbb{R}^{p,q}$ and also cocompact properly discontinuous groups of pseudo-Euclidian motions are scarcely understood.

Part G. Conformal geometry

A *conformal structure* of signature (p, q) on a smooth manifold M is a ray subbundle $L \subset S^2T^*M$ such that any local section of L defines a pseudo-Riemannian metric of signature (p, q). In particular, any pseudo-Riemannian metric g on M defines a conformal structure $L = \mathbb{R}^+ g$. Conformal geometry is concerned with properties which do not depend on the choice of a section $g \in \Gamma(L)$. The holonomy group of a pseudo-Riemannian manifold (M, g), for instance, is not a conformal invariant.

Helga Baum's contribution is a survey on the holonomy theory of Cartan connections. This theory applies, in particular, to conformal geometry. As explained in her exposition, the conformal holonomy group contains important information about a pseudo-Riemannian manifold. The knowledge of the holonomy group allows one to decide, for example, whether the pseudo-Riemannian manifold admits conformal Killing spinors or an Einstein metric in the conformal class. She describes Lorentzian manifolds with conformal holonomy in $\mathrm{SU}(1, n)$.

Chapter 24 by Yoshinubo Kamishima is also concerned with conformal and related geometric structures. It provides a unified treatment of conformal, CR and quaternionic CR-structures. In the positive definite case, the corresponding model spaces are the boundaries at infinity of the hyperbolic spaces over the real, complex and quaternionic numbers, respectively.

Part H. Other topics of recent interest

The chapter by Christian Bär summarises the analytic theory of linear wave equations on globally hyperbolic Lorentzian manifolds, as developed in his book with Nicolas Ginoux and Frank Pfäffle. In the final chapter, Dan Freed explains the relation between D-branes in string theory and K-theory.

Part A

Special geometry and supersymmetry

Chapter 1

Quaternion-Kähler spaces, hyper-Kähler cones, and the c-map

Martin Roček, Cumrun Vafa, and Stefan Vandoren

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1 Introduction

The moduli spaces of Kähler and complex structure deformations of Calabi–Yau manifolds are naturally related to special Kähler (SK) and quaternion-Kähler (QK) geometry. Consequently, these types of manifolds arise in the low energy effective action of string theory compactifications on Calabi–Yau three-folds.

SK manifolds were discovered in the context of $N = 2$ supergravity[1] theories coupled to vector multiplets [1]. They are described by a holomorphic function $F(X)$ that is homogeneous of degree two in complex coordinates X^I. A more mathematically precise and intrinsic formulation of this special geometry was given in [2], [3].

QK manifolds arose in the context of $N = 2$ supergravity coupled to hypermultiplets [4]. QK manifolds are also described by a single function. This follows from the construction of the Swann bundle [5] over the QK space. This bundle is hyper-Kähler; locally, its metric is determined by a hyper-Kähler potential $\chi(\phi)$, where ϕ are local coordinates on the space. In [6], we called such spaces hyper-Kähler cones (HKC's) because they have a homothety arising from the underlying conformal symmetry. One therefore also uses the terminology conformal hyper-Kähler manifolds, as in [7], [8].

[1] The supersymmetry transformations have eight real components.

In this note, we review the construction of quaternion-Kähler (QK) manifolds from special Kähler (SK) geometry, along the lines of our recent work [9], but with more emphasis on the mathematical structure. One constructs QK spaces from SK manifolds by using the c-map [10]. This maps extends Calabi's construction of hyper-Kähler metrics on cotangent bundles of Kähler manifolds [11], [12] – a construction well known in the mathematics community – to quaternionic geometry.

As we shall see, the hyper-Kähler cones arising from the c-map have additional symmetries: they have an equal number of commuting triholomorphic isometries as their quaternionic dimension. Hyper-Kähler manifolds with such isometries were classified in [13] by performing a Legendre transform on the (hyper-)Kähler potential [14] and writing the result in terms of a contour integral of a meromorphic[2] function H. In our case, because of the conformal symmetry of the HKC, this function H is homogeneous of degree one. As a result, the c-map induces a map from the holomorphic function F, which characterizes the SK geometry and is homogeneous of degree two, to a function H, which characterizes the QK geometry and is homogeneous of degree one. Following [9], we now describe this construction.

2 The c-map

In this section, we introduce our notation and review the c-map [10], [15].

Consider an affine (or rigid) special Kähler manifold[3] of dimension $2(n+1)$. It is characterized by a holomorphic prepotential $F(X^I)$, which is homogeneous of degree two ($I = 1, \ldots, n+1$). The Kähler potential and metric of the rigid special geometry are given by

$$K(X,\bar{X}) = i(\bar{X}^I F_I - X^I \bar{F}_I), \quad \mathrm{d}s^2 = N_{IJ}\,\mathrm{d}X^I \mathrm{d}\bar{X}^J, \quad N_{IJ} = i(F_{IJ} - \bar{F}_{IJ}), \tag{2.1}$$

where F_I is the first derivative of F, *etc.*

The projective (or local) special Kähler geometry is then of real dimension $2n$, with complex inhomogeneous coordinates

$$Z^I = \frac{X^I}{X^1} = \{1, Z^A\}, \tag{2.2}$$

where A runs over n values. Its Kähler potential is given by

$$\mathcal{K}(Z,\bar{Z}) = \ln(Z^I N_{IJ} \bar{Z}^I). \tag{2.3}$$

We further introduce the matrices [1]

$$\mathcal{N}_{IJ} = -i\,\bar{F}_{IJ} - \frac{(NX)_I (NX)_J}{(XNX)}, \tag{2.4}$$

where $(NX)_I \equiv N_{IJ} X^J$, *etc.*

[2] Actually, certain branch cut singularities sometimes arise.

[3] We use the language of local coordinates, but a coordinate free description can be found in [2], [3].

The c-map defines a $4(n+1)$-dimensional quaternion-Kähler metric as follows: One builds a G-bundle, with $2(n+2)$ dimensional fibers coordinatized by ϕ, σ, A^I and B_I, over the projective special Kähler manifold; the real group G is a semidirect product of a Heisenberg group with $\mathbb{R}$, and acts on the coordinates by

$$\begin{aligned} &A^I \to \mathrm{e}^{\beta}(A^I + \epsilon^I), \qquad B_I \to \mathrm{e}^{\beta}(B_I + \epsilon_I), \\ &\phi \to \phi + \beta, \qquad\qquad \sigma \to \mathrm{e}^{2\beta}(\sigma + \alpha - \tfrac{1}{2}\epsilon_I A^I + \tfrac{1}{2}\epsilon^I B_I), \end{aligned} \tag{2.5}$$

Then the explicit G-invariant QK metric is [15]

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{d}\phi^2 - \mathrm{e}^{-\phi}(\mathcal{N} + \bar{\mathcal{N}})_{IJ} W^I \overline{W}^J + \mathrm{e}^{-2\phi}\big(\mathrm{d}\sigma - \tfrac{1}{2}(A^I \mathrm{d}B_I - B_I \mathrm{d}A^I)\big)^2 \\ & - 4\mathcal{K}_{A\bar{B}}\, \mathrm{d}Z^A \mathrm{d}\bar{Z}^{\bar{B}}. \end{aligned} \tag{2.6}$$

The metric is only positive definite in the domain where $(Z\mathcal{N}\bar{Z})$ is positive and hence $\mathcal{K}_{A\bar{B}}$ is negative definite. One can then show that $\mathcal{N} + \bar{\mathcal{N}}$ is negative definite [16]. The one-forms W^I are defined by

$$W^I = (\mathcal{N} + \bar{\mathcal{N}})^{-1\,IJ}\big(2\bar{\mathcal{N}}_{JK}\mathrm{d}A^K - i\,\mathrm{d}B_J\big). \tag{2.7}$$

As shown in [15], such metrics are indeed quaternion-Kähler; they were further studied in [17], [18], including an analysis of their isometries. There are always the $2(n+4)$ manifest isometries (2.5), of which $n+2$ are commuting, *e.g.*,

$$B_I \to B_I + \epsilon_I, \quad \sigma \to \sigma + \alpha - \tfrac{1}{2}\epsilon_I A^I. \tag{2.8}$$

This is one isometry *more* than the quaternionic dimension of the QK.

3 Hyper-Kähler cones and the Legendre transform

The Swann bundle over a QK geometry, *i.e.,* the hyper-Kähler cone (HKC), is a hyper-Kähler manifold with one extra quaternionic dimension. As for special Kähler manifolds, the geometry of the HKC is again affine. In physics terminology, this arises because one adds a compensating hypermultiplet. Adding the compensator to the original hypermultiplets that parametrize the $4(n+1)$-dimensional QK space, one obtains a cone with real dimension $4(n+2)$. This space is hyper-Kähler and admits a homothety as well as an SU(2) isometry group that rotates the three complex structures.

The metric on the HKC can be constructed from a hyper-Kähler potential [5], which is a Kähler potential with respect to *any* of the complex structures. In real local coordinates ϕ^A, the metric and the hyper-Kähler potential $\chi(\phi)$ are related by

$$g_{AB} = D_A \partial_B\, \chi(\phi), \tag{3.1}$$

where D_A is the Levi-Civita connection. As for all Kähler manifolds, in complex coordinates, the hermitian part of (3.1) defines the metric in terms of the complex

hessian of the potential; however, in this case, the vanishing of the holomorphic parts of the metric is an additional constraint on the geometry.

Any QK isometry can be lifted to a triholomorphic isometry on the HKC. In the physics literature, this was shown in [6], [19]. Using the notation of the previous section, we thus have an HKC of real dimension $4(n+2)$ together with $n+2$ commuting triholomorphic isometries determined by (2.8). As mentioned before, hyper-Kähler manifolds of this type were classified in [13]. It is convenient to introduce complex coordinates $v^{\hat{I}}$ and $w_{\hat{I}}$, in such a way that the isometries act as imaginary shifts in $w_{\hat{I}}$. Notice that $\hat{I} = 0, 1, \ldots, n+1$. The hyper-Kähler potential is then a function $\chi(v, \bar{v}, w + \bar{w})$, and can be written as a Legendre transform of a function $\mathcal{L}(v, \bar{v}, G)$ of $3(n+2)$ variables, The Legendre transform with respect to $G^{\hat{I}}$ is defined by

$$\chi(v, \bar{v}, w, \bar{w}) \equiv \mathcal{L}(v, \bar{v}, G) - (w + \bar{w})_{\hat{I}}\, G^{\hat{I}}, \quad w_{\hat{I}} + \bar{w}_{\hat{I}} = \frac{\partial \mathcal{L}}{\partial G^{\hat{I}}}. \tag{3.2}$$

The constraints from hyper-Kähler geometry can be solved by writing $\mathcal{L}$ in terms of a contour integral [20], [21], [13]

$$\mathcal{L}(v, \bar{v}, G) \equiv \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta}\, H(\eta, \zeta), \tag{3.3}$$

with

$$\eta^{\hat{I}} \equiv \frac{v^{\hat{I}}}{\zeta} + G^{\hat{I}} - \bar{v}^{\hat{I}} \zeta. \tag{3.4}$$

These objects have an interpretation in twistor space as sections of an $\mathcal{O}(2)$ bundle. In physics terminology, these are $N = 2$ tensor multiplets. Furthermore, the conditions for a homothetic Killing vector and SU(2) isometries imply that H is a function homogeneous of first degree[4] (in η) and without explicit ζ dependence [6].

Since H is homogeneous of first degree in η, it follows that the hyper-Kähler potential is also homogeneous of first degree in v and $\bar{v}$:

$$\chi(\lambda v, \lambda \bar{v}, w, \bar{w}) = \lambda \chi(v, \bar{v}, w, \bar{w}). \tag{3.5}$$

In addition to a homothety, hyper-Kähler cones also have an SU(2) isometry group that rotates the sphere of complex structures. Under infinitesimal variations with respect to an element of the Lie algebra $\varepsilon^+ T_+ + \varepsilon^- T_- + \varepsilon^3 T_3$, with $\varepsilon^- = (\varepsilon^+)^*$, these act as [6]

$$\delta_{\vec{\varepsilon}}\, v^{\hat{I}} = -i\varepsilon^3 v^{\hat{I}} + \varepsilon^- G^{\hat{I}}(v, \bar{v}, w, \bar{w}), \quad \delta_{\vec{\varepsilon}}\, w_{\hat{I}} = \varepsilon^+ \frac{\partial \mathcal{L}}{\partial \bar{v}^{\hat{I}}}, \tag{3.6}$$

where $G^{\hat{I}}$ has to be understood as the function of the coordinates $v, \bar{v}, w, \bar{w}$ obtained by the Legendre transform defined in (3.2). The coordinates w_I do not transform under ε^3. One can now explicitly check that the hyper-Kähler potential is $\mathrm{SU}(2)_R$

[4] Actually, quasihomogeneity up to terms of the form $\eta \ln(\eta)$ is sufficient [6], but such terms do not seem to arise in the c-map.

invariant,

$$\delta_{\vec{\varepsilon}}\,\chi = \mathcal{L}_{v^{\hat{I}}}\,\delta_{\vec{\varepsilon}}\,v^{\hat{I}} + \mathcal{L}_{\bar{v}^{\hat{I}}}\,\delta_{\vec{\varepsilon}}\,\bar{v}^{\hat{I}} - \delta_{\vec{\varepsilon}}\,(w_{\hat{I}} + \bar{w}_{\hat{I}})\,G^{\hat{I}} = 0. \tag{3.7}$$

(The δG terms cancel identically because χ is a Legendre transform.) For the generators $\varepsilon^{\pm}$ this is immediately obvious; for variations proportional to ε^3 one needs to use the invariance of $\mathcal{L}$, *i.e.*, $v^{\hat{I}}\,\mathcal{L}_{v^{\hat{I}}} = \bar{v}^{\hat{I}}\,\mathcal{L}_{\bar{v}^{\hat{I}}}$.

4 Hyper-Kähler cones from the c-map

The quaternion-Kähler space in the image of the c-map has dimension $4(n+1)$. The hyper-Kähler cone above it has dimension $4(n+2)$. It therefore needs to be described by $n+2$ twistor variables, say η^I and η^0, where $I = 1, \ldots, n+1$. As we shall show, the result for the tree level c-map is given by

$$H(\eta^I, \eta^0) = \frac{F(\eta^I)}{\eta^0}, \tag{4.1}$$

where F is the prepotential of the special Kähler geometry, now evaluated on the twistor variables η. This is our main result. Note that H does not depend explicitly on ζ and, since F is homogeneous of degree two, H is homogeneous of degree one, as required by superconformal invariance.

We now give a detailed proof of (4.1) by explicit calculation [9]. To be precise, we prove that (4.1) leads to (2.6).

4.1 Gauge fixing and the contour integral

As explained in the previous section, any hyper-Kähler cone has an SU(2) symmetry and a homothety. The generators of the homothety and $\mathrm{U}(1) \subset \mathrm{SU}(2)$ give a natural complexified action on the HKC; the remaining two generators of the SU(2) combine to give the roots $T_{\pm}$.

To evaluate the contour integral (3.3), it is convenient to make use of the isometries. In physics terminology, one can impose gauge choices. Mathematically, the isometries fiber the quotient space by the orbits, and a gauge choice is just a choice of section. For the symmetries generated by $T_{\pm}$, whose action is given by (3.6), we choose

$$v^0 = 0. \tag{4.2}$$

In this gauge, we have that $\eta^0 = G^0$ and this simplifies the pole structure in the complex ζ-plane. Then, using as well the homogeneity properties of F, the contour integral (3.3) simplifies to

$$\mathcal{L}(v, \bar{v}, G) = \frac{1}{G^0}\,\mathrm{Im}\oint \frac{\mathrm{d}\zeta}{2\pi i}\,\frac{F(\zeta\eta^I)}{\zeta^3}, \tag{4.3}$$

with

$$\zeta\eta^I = v^I + \zeta G^I - \zeta^2 \bar{v}^I, \quad I = 1, \ldots, n+1, \tag{4.4}$$

which, for nonzero values of v, has no zeroes at $\zeta = 0$. Therefore, assuming F is regular at $\eta = 0$, $F(\zeta\eta)$ has no poles (in ζ) inside the contour around the origin. It is now easy to evaluate the contour integral, because the residue at $\zeta = 0$ replaces all the $\zeta\eta^I$ by v^I. The result is

$$\mathcal{L}(v, \bar{v}, G) = \frac{1}{4G^0}\big(N_{IJ} G^I G^J - 2K(v, \bar{v})\big), \tag{4.5}$$

where $K(v, \bar{v})$ is the Kähler potential of the rigid special geometry given in (2.1), with $F_I(v)$ now the derivative with respect to v^I, *etc.* Notice that the function $\mathcal{L}$ satisfies the Laplace-like equations [20], [21], [13]

$$\mathcal{L}_{G^I G^J} + \mathcal{L}_{v^I \bar{v}^J} = 0. \tag{4.6}$$

The equation is not satisfied for the components $\mathcal{L}_{G^0 G^0}$ and $\mathcal{L}_{G^0 G^I}$, because we have chosen the gauge $v^0 = 0$. It would be interesting to compute $\mathcal{L}$ for arbitrary values of v^0. For a special case, this was done in [22].

4.2 The hyper-Kähler potential

To compute the hyper-Kähler potential χ, we have to Legendre transform $\mathcal{L}$,

$$\chi(v, \bar{v}, w, \bar{w}) = \mathcal{L}(v, \bar{v}, G) + (w + \bar{w})_0 G^0 - (w + \bar{w})_I G^I, \tag{4.7}$$

The hyper-Kähler potential χ, computed by extremizing[5] (4.7) with respect G^0, G^I completely determines the associated hyper-Kähler geometry. In general, it is a function of the $2(n+2)$ complex coordinates v^0, v^I and w_0, w_I, but we work only on the (Kähler but not hyper-Kähler) submanifold $v^0 = 0$. The geometry of the HKC only depends on w through the combination $w + \bar{w}$ which makes manifest the $n+2$ commuting isometries. The Legendre transform of (4.5) gives

$$\begin{aligned} \frac{G^I}{G^0} &= 2N^{IJ}(w + \bar{w})_J, \\ (G^0)^2 &= \frac{K}{2\big((w + \bar{w})_I N^{IJ}(w + \bar{w})_J - (w + \bar{w})_0\big)}. \end{aligned} \tag{4.8}$$

Up to an irrelevant overall sign we find, using (4.5),

$$\chi\big(v, \bar{v}, G(v, \bar{v}, w, \bar{w})\big) = \frac{K(v, \bar{v})}{G^0}, \tag{4.9}$$

where G^0 is determined by (4.8). More explicitly, in terms of the HKC coordinates,

$$\chi(v, \bar{v}, w, \bar{w}) = \sqrt{2}\,\sqrt{K(v, \bar{v})}\,\sqrt{(w + \bar{w})_I N^{IJ}(w + \bar{w})_J - (w + \bar{w})_0}. \tag{4.10}$$

[5] The relative minus signs between the last two terms in (4.7) is purely a matter of convention.

4.3 Twistor space

The twistor space above a $4(n+1)$ dimensional QK has dimension two higher, and is Kähler. It can be seen as a CP^1 bundle over the QK. It can also be obtained from the HKC by a Kähler quotient with respect to $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Equivalently, we define inhomogeneous coordinates, *e.g.*,

$$Z^I = \frac{v^I}{v^1} = \{1, Z^A\}, \tag{4.11}$$

where A runs over n values. As we show below, these inhomogeneous coordinates will be identified with (2.2).

The Kähler potential on the twistor space, denoted by K_T, is given by the logarithm of the hyper-Kähler potential restricted to the submanifold given by $v^1 = 1$ [6]:

$$K_T(Z, \bar{Z}, w, \bar{w}) = \frac{1}{2}\big[\mathcal{K}(Z, \bar{Z}) + \ln\big((w+\bar{w})_I N^{IJ} (w+\bar{w})_J - (w+\bar{w})_0\big)\big] + \ln(\sqrt{2}), \tag{4.12}$$

where $\mathcal{K}(Z, \bar{Z})$ is the *same* as the special Kähler potential (2.3).

On the twistor space, there always exists a holomorphic one-form $\mathcal{X}$ which can be constructed from the holomorphic two-form that any hyper-Kähler manifold admits. In our case this one-form is obtained from the holomorphic HKC two-form $\Omega = \mathrm{d}w_I \wedge \mathrm{d}v^I$. Without going into details, it is given by [6]

$$\mathcal{X} = 2Z^I \mathrm{d}w_I \equiv \mathcal{X}_\alpha \mathrm{d}z^\alpha, \tag{4.13}$$

where the index $\alpha = 1, \ldots, 2(n+1)$ runs over the complete set of holomorphic coordinates w_I, w_0, Z^A on the submanifold[6] of the twistor space given by $v^0 = 0$. In total this gives $2(n+1) + 2 + 2n = 4(n+1)$ – the (real) dimension of the QK. The metric on the QK manifold can then be computed[7]:

$$G_{\alpha\bar{\beta}} = K_{T,\alpha\bar{\beta}} - \mathrm{e}^{-2K_T} \mathcal{X}_\alpha \bar{\mathcal{X}}_{\bar{\beta}}. \tag{4.14}$$

4.4 The quaternionic metric

We now compute the QK metric that follows from the c-map using (4.14). To compare with (2.6) we only need to identify the coordinates w_I, w_0 with those of (2.6), since the Z^A coordinates of the special Kähler manifold can be identified with the ones above. We define

$$\begin{aligned} w_0 &= iA^I A^J F_{IJ} - i\left(\sigma + \frac{1}{2} A^I B_I\right) - \mathrm{e}^{\phi}, \\ w_I &= iF_{IJ} A^J - \frac{i}{2} B_I. \end{aligned} \tag{4.15}$$

[6] This submanifold can be thought of invariantly as a quotient of the original HKC.

[7] Note that the constant term in K_T (4.12) enters in (4.14).

The metric can be written in these coordinates; after considerable calculation [9], up to an irrelevant overall all factor of $-1/8$, we obtain precisely the result (2.6)! From (4.8), we find the following relations between the QK coordinates and the twistor variables $G^{\hat{I}}$ (see (3.4)):

$$2A^I = \frac{G^I}{G^0}, \quad 4\mathrm{e}^{\phi} = \frac{K(v,\bar{v})}{(G^0)^2}. \tag{4.16}$$

This concludes the proof of (4.1).

5 Summary and conclusion

We have constructed the Swann bundle over the quaternion-Kähler manifolds that arise in the c-map. The corresponding hyper-Kähler potential was given in (4.10), and was first derived in [9]. Introducing coordinates

$$X^I(v,\bar{v},w,\bar{w}) \equiv \frac{v^I}{\sqrt{G^0(v,\bar{v},w,\bar{w})}}, \tag{5.1}$$

we can conveniently rewrite the hyper-Kähler potential as follows:

$$\boxed{\chi(v,\bar{v},w,\bar{w}) = K\big(X^I(v,\bar{v},w,\bar{w}), \bar{X}^I(v,\bar{v},w,\bar{w})\big)} \tag{5.2}$$

Here K is the Kähler potential $K(X,\bar{X}) = i(\bar{X}^I F_I - X^I \bar{F}_I)$ of the affine special geometry.

The special hyper-Kähler cones given by the c-map have as many $(n+2)$ commuting triholomorphic isometries as their quaternionic dimension. As explained before, this implies the hyper-Kähler potential can be Legendre transformed to a function $\mathcal{L}$ that can be written in terms of a contour integral over a function $H(\eta)$; equivalently, the twistor space of the HKC can be described in terms of sections of $(n+2)$ $\mathcal{O}(2)$-bundles. These twistor variables η were defined in (3.4) and the function H was determined in (4.1). Defining

$$X^I(\eta) \equiv \frac{\eta^I}{\sqrt{\eta^0}}, \tag{5.3}$$

we can write H as follows:

$$\boxed{H(\eta^I,\eta^0) = F\big[X^I(\eta^I,\eta^0)\big]} \tag{5.4}$$

The function F is well known to be related to the topological string amplitude [23], [24]. Typical examples that appear in the context of Calabi–Yau compactifications are of the form

$$F(X^I) = d_{ABC}\frac{X^A X^B X^C}{X^1}, \tag{5.5}$$

where $X^I = \{X^1, X^A\}$ and the constants d_{ABC} are related to the triple intersection numbers of the Calabi–Yau. To give an explicit example, one can choose specific values for these coefficients such that the local (projective) special Kähler geometry is the symmetric space

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(n-1,2)}{\mathrm{SO}(n-1)\times \mathrm{SO}(2)}. \tag{5.6}$$

After the c-map, the hyper-Kähler cone is based on the function

$$H(\eta^{\hat{I}}) = d_{ABC}\frac{\eta^A\eta^B\eta^C}{\eta^0\eta^1}. \tag{5.7}$$

This corresponds to a homogeneous quaternion-Kähler manifold of the form (see e.g. appendix C in [10], and references therein)

$$\frac{\mathrm{SO}(n+1,4)}{\mathrm{SO}(n+1)\times \mathrm{SO}(4)}. \tag{5.8}$$

Other examples were recently given in [25], were quantum effects were taken into account.

The connection of these geometries with topological strings is very profound, and has important physical implications. For instance, it was recently shown that the topological string amplitude F appears in the study of supersymmetric black holes in string theory [26], [27]. More precisely, the Legendre transform of F is related to the entropy of the black hole. It would be interesting to see if this Legendre transform is related to the one described here; speculations along these lines can be found in [9]. To make progress on this issue, one needs to evaluate the contour integral (3.3) without making use of the special coordinate system in which we can set $v^0 = 0$. We leave this for future research.

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Chapter 2

Differential forms on quaternionic Kähler manifolds

Gregor Weingart

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1 Introduction

Quaternionic Kähler manifolds and hyper-Kähler manifolds have long been studied for their rich geometry and their marvellous implications on the topology of the underlying manifold. In the context of these studies the holonomy decomposition of the bundle of complex valued differential forms into minimal parallel subbundles has been studied extensively for hyper-Kähler manifolds generalizing ideas developed in Kähler geometry. Interestingly there exists an intimate relationship between this construction and a strange property of general representation theory, the existence of a dual Lie algebra governing the decomposition of tensor products of exterior or symmetric algebras. For the complex valued differential forms on hyper-Kähler manifolds this dual Lie algebra is isomorphic to $\mathfrak{so}_{4,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

In this chapter we will define the action of the Lie algebra $\mathfrak{so}_{4,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ in Section 2 entirely from the representation theoretic point of view with only a passing reference on how this algebra can be constructed using wedge products and contractions with Kähler forms. One merit of this approach is that it highlights the close relationship to Weyl's construction of the irreducible representations of the classical matrix groups showing in particular that for a generic hyper-Kähler manifold the resulting decomposition is minimal. Moreover the representation theoretic approach allows us to use the comparatively simple branching formulas from $\mathfrak{so}_{4,1}(\mathbb{R})$ to $\mathbb{R} \oplus \mathfrak{sp}(1)$ instead of the branching formulas from $\mathfrak{so}_{4n}(\mathbb{R})$ to $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ to calculate the parallel decomposition of differential forms on quaternionic Kähler and hyper-Kähler manifolds.

Before presenting the final decomposition formula let us recall that a quaternionic Kähler manifold M carries quite a few geometric vector bundles besides the usual tensor bundles. Eventually this plentitude is due to the fact that the complexified tangent

bundle of a quaternionic Kähler manifold M of dimension $4n$ factorizes parallely into a tensor product $TM \otimes_{\mathbb{R}} \mathbb{C} \cong HM \otimes EM$ of two (locally defined) complex symplectic vector bundles HM and EM of dimensions 2 and $2n$ respectively. All minimal parallel subbundles of the differential forms bundle $\Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$ on a generic quaternionic Kähler manifold M are of the form

$$\mathrm{Sym}^s HM \otimes \Lambda_\circ^{r,\bar r} EM$$

with $n \geq r \geq \bar r \geq 0$ and $s \equiv r + \bar r$ modulo 2 compare [3], where $\Lambda_\circ^{r,\bar r} EM$ is the joint kernel of all possible contractions with the parallel symplectic form on EM inside the Schur functor bundle $\Lambda^{r,\bar r} EM := \ker(\overline{\mathrm{Pl}}\colon \Lambda^r EM \otimes \Lambda^{\bar r} EM \to \Lambda^{r+1} EM \otimes \Lambda^{\bar r - 1} EM)$:

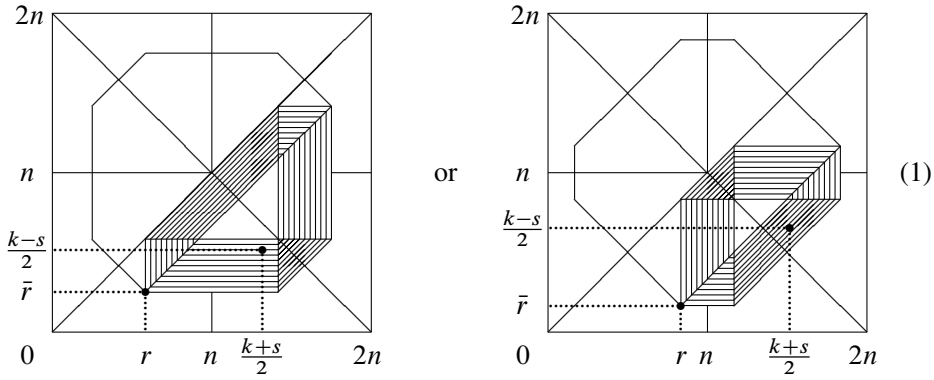
Theorem 1.1 (Parallel decomposition of differential forms). *On a generic quaternionic Kähler manifold M of dimension $4n$ the holonomy decomposition of the bundle of complex valued differential forms into minimal parallel subbundles reads*

$$\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\substack{n \geq r \geq \bar r \geq 0 \\ s \geq 0}} m_{s,r,\bar r}(k) \mathrm{Sym}^s HM \otimes \Lambda_\circ^{r,\bar r} EM,$$

where the multiplicities $m_{s,r,\bar r}(k)$ vanish unless $k \equiv s \equiv r + \bar r$ modulo 2 and

$$s \leq 2n - r - \bar r, \quad \bar r \leq \frac{k-s}{2} \leq 2n - r, \quad r \leq \frac{k+s}{2} \leq 2n - \bar r.$$

A precise and ready to use formula for the multiplicities $m_{s,r,\bar r}(k)$ is given in the specification Theorem 3.1 of Theorem 1.1 in Section 3, it appears inadequate for an introduction to reproduce the threefold case distinction necessary. Suffice it to say at this point that under the conditions of Theorem 1.1 the multiplicities $m_{s,r,\bar r}(k) > 0$ are strictly positive unless $\min\{r - \bar r, s\} = 0$ and $r \not\equiv \frac{k-s}{2}$ modulo 2. It is useful to think of



(1)

$m_{s,r,\bar r}(k)$ as depending on the position of the point $(\frac{k+s}{2}, \frac{k-s}{2})$ in the hexagonal pattern determined by the point $(r, \bar r)$ the different shapes correspond to $3r - \bar r \leq 2n$ and

$3r - \bar{r} \geq 2n$. Essentially the multiplicities $m_{s,r,\bar{r}}(k)$ vanish outside the hexagon and increase from the value 1 on the edges towards the central triangle by steps of 1 every second parallel to their maximal value $\lceil \frac{r-\bar{r}}{2} \rceil$ or $1 + n - r$ respectively taken in the triangle, of course only points $(\frac{k+s}{2}, \frac{k-s}{2})$ satisfying $k \equiv s \equiv r + \bar{r}$ modulo 2 ought to be considered. If r and $\bar{r}$ share the same parity, then this geometric interpretation of the formula for $m_{s,r,\bar{r}}(k)$ given in Theorem 3.1 is not entirely correct, the actual multiplicities may be one less than their geometrically deduced values.

2 The Lie algebra $\mathfrak{so}_{4,1}(\mathbb{R})$ of operators on forms

Usually an orthogonal quaternionic structure on a euclidian vector space T with scalar product g is defined as the choice of three anticommuting, skew-symmetric complex structures I, J and K on T. In this exposition however we prefer an alternative weaker notion, an orthogonal quaternionic structure on T is to be the choice of a subalgebra $Q \subset \operatorname{End} T$ containing the identity of T and isomorphic to $\mathbb{H}$ as an associative algebra with unit over $\mathbb{R}$, such that $g(AX, Y) = g(X, \bar{A}Y)$ for all $A \in Q$ and all $X, Y \in T$. Fixing an isomorphism $Q \cong \mathbb{H}$ brings us back to the standard notion of orthogonal quaternionic structures, this innocuous difference for euclidian vector spaces becomes important for Riemannian manifolds.

In order to discuss orthogonal quaternionic structures on euclidian vector spaces in more detail we recall that a positive quaternionic structure on a complex symplectic vector space E with symplectic form σ is a conjugate linear map $C : E \to E$ satisfying $C^2 = -1$ as well as $\sigma(Ce_1, Ce_2) = \overline{\sigma(e_1, e_2)}$ and $\sigma(Ce, e) > 0$ for all $e \neq 0$. Quite remarkably the complexification $T \otimes_{\mathbb{R}} \mathbb{C}$ of a euclidian vector space T of dimension $4n$ with scalar product g and orthogonal quaternionic structure $Q \subset \operatorname{End} T$ turns out to be the tensor product

$$H \otimes E \cong T \otimes_{\mathbb{R}} \mathbb{C} \tag{2}$$

of two complex symplectic vector spaces H and E of dimension 2 and $2n$ endowed with positive quaternionic structures, such that every $A \in Q$ acts as a Kronecker product $A \otimes \mathrm{id}_E$ on $H \otimes E$, while the complex bilinear extension of g and the real structure on $T \otimes_{\mathbb{R}} \mathbb{C}$ agree with $\sigma \otimes \sigma$ and $C \otimes C$ respectively. One way to justify this convenient description of $T \otimes_{\mathbb{R}} \mathbb{C}$ as a tensor product is to use the representation theory of the unitary symplectic group $\mathrm{Sp}(n)$, $n \geq 1$, which can be interpreted alternatively as the group of orthogonal maps $T \to T$ commuting with Q or as the group of symplectic maps $E \to E$ commuting with C. A somewhat more direct approach is to note that $Q \cong \mathrm{Cl}\mathbb{R}^{2,0}$ is a Clifford algebra with a unique complex module H, which is a complex symplectic vector space of dimension 2 with a positive quaternionic structure, forcing the multiplicity space $E := \operatorname{Hom}_Q(H, T \otimes_{\mathbb{R}} \mathbb{C})$ to have a symplectic form and positive quaternionic structure as well [5].

A particular merit of the description of $T \otimes_{\mathbb{R}} \mathbb{C} = H \otimes E$ is that the algebra Q acts on T essentially via its action on its module H. In particular the choice of a

complex structure in Q corresponds to the choice of decomposition $H = L \oplus CL$ into two conjugated complex lines L and CL, in fact for any such decomposition we can find $I \in Q$ such that L and CL are the eigenspaces for the eigenvalues i and $-i$ respectively. In the same vein every unit vector $q \in H$ with $\sigma(Cq, q) = 1$ determines a unique algebra isomorphism $\varphi_q \colon \mathbb{H} \to Q$ such that the images I, J of $i, j \in \mathbb{H}$ satisfy $Iq = iq$ and $Jq = Cq$, more precisely the matrices of the images $I, J, K \in Q$ with respect to the canonical basis Cq, q of H, read as follows:

$$I = \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{3}$$

Consider now a Riemannian manifold M with scalar product g together with a smooth choice of orthogonal quaternionic structures $Q_pM \subset \operatorname{End} T_pM$ on every tangent space T_pM, $p \in M$. Such a Riemannian manifold M is called a quaternionic Kähler manifold, if the subalgebra bundle $QM \subset \operatorname{End} TM$ is parallel with respect to the Levi-Civita connection ∇, in other words if the parallel transport along every curve γ from p to q conjugates Q_pM with Q_qM. Hyper-Kähler manifolds are special quaternionic Kähler manifolds, their subalgebra bundle $QM \subset \operatorname{End} TM$ is not only parallel, but trivial for the Levi-Civita connection. Hence we can choose a parallel isomorphism $\mathbb{H}M \cong QM$ to obtain a smooth choice of an orthogonal quaternionic structure on every tangent space T_pM in the stronger standard sense.

In order to discuss the decomposition of the bundle of differential forms on a hyper-Kähler or quaternionic Kähler manifold M into minimal parallel subbundles we fix once and for all a point $p \in M$ and decompose the vector space $\Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C}$ of complex-valued alternating forms on the tangent space $T := T_pM$ in p into irreducible subspaces under the holonomy group of parallel transports of curves beginning and ending in p, parallel transport along arbitrary curves will deliver this decomposition in p to all of M. The holonomy group in p is a group of orthogonal transformations γ normalizing the given orthogonal quaternionic structure $Q \subset \operatorname{End} T$ of the euclidian vector space T. In terms of decomposition (2) the group of all such orthogonal transformations agrees with $\operatorname{Sp} H \cdot \operatorname{Sp} E \subset \operatorname{SO} T$, where $\operatorname{Sp} E \cong \operatorname{Sp}(n)$ say is the group of all symplectic transformations of E commuting with C. Of course on a hyper-Kähler manifold the holonomy group actually centralizes the subalgebra $Q \subset \operatorname{End} T$ and the holonomy group becomes a subgroup of $\operatorname{Sp} E$.

Choosing a unit vector $q \in H$ satisfying $\sigma(Cq, q) = 1$ we get a canonical basis p, q of H with $p := Cq$, the resulting algebra isomorphism $Q \cong \mathbb{H}$ allows to replace $\operatorname{Sp} H$ by $\operatorname{Sp}(1)$ in the holonomy group and the complex linear forms on T by $E^* \oplus E^*$ via the isomorphism

$$E^* \oplus E^* \xrightarrow{\cong} T^* \otimes_{\mathbb{R}} \mathbb{C}, \quad \eta \oplus \tilde{\eta} \longmapsto (dq \otimes \eta) + (dp \otimes \tilde{\eta}).$$

This isomorphism extends to an $\operatorname{Sp} E$-equivariant decomposition of the alternating forms

$$\Phi_q \colon \quad \Lambda^\bullet E^* \otimes \Lambda^\bullet E^* \xrightarrow{\cong} \Lambda^\bullet (T^* \otimes_{\mathbb{R}} \mathbb{C}) \cong \Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C} \tag{4}$$

on T (with the product grading on the left indicated by the repeated grading symbol) via

$$\begin{aligned}\Phi_q(&\eta_1 \wedge \cdots \wedge \eta_k \otimes \tilde\eta_1 \wedge \cdots \wedge \tilde\eta_{\bar k})\\ &:= (dq \otimes \eta_1) \wedge \cdots \wedge (dq \otimes \eta_k) \wedge (dp \otimes \tilde\eta_1) \wedge \cdots \wedge (dp \otimes \tilde\eta_{\bar k}).\end{aligned}$$

Evidently the isomorphism Φ_q is an isomorphism of algebras for the twisted multiplication on $\Lambda E^* \otimes \Lambda E^*$ sometimes denoted $\Lambda E^* \widehat{\otimes} \Lambda E^*$. On the other hand the algebra isomorphism $Q \cong \mathbb{H}$ coming along with the choice of unit vector q is characterized by $Iq = iq$ and $Jq = p$. The finer bigrading on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ thus agrees with the Hodge bigrading $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the complex structure I, in other words Φ_q maps $\Lambda^k E^* \otimes \Lambda^{\bar k} E^*$ to the space $\Lambda^{k,\bar k} T^* \otimes_{\mathbb{R}} \mathbb{C}$ of $(k,\bar k)$-forms on T. Interestingly the natural real structure on the algebra $\Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C}$ of complex valued alternating forms on T by conjugation of values

$$\bar\Omega(h_1 \otimes e_1, \ldots, h_r \otimes e_r) := \overline{\Omega(Ch_1 \otimes Ce_1, \ldots, Ch_r \otimes Ce_r)}$$

on real arguments interchanges $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ with $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$. In order to describe its precise relation to Φ_q we extend the quaternionic structures on E and H to quaternionic structures on E^* and H^* via $(C\eta)(e) := -\overline{\eta(Ce)}$. The musical isomorphism $\sharp\colon E \to E^*, e \mapsto \sigma(e,\cdot)$, and its inverse $\flat$ are then both real due to $C(e^\sharp) = -\overline{\sigma(e, C\cdot)} = (Ce)^\sharp$ and the identity

$$(C\alpha \otimes C\eta)(h \otimes e) = \overline{\alpha(Ch)\eta(Ce)} = \overline{(\alpha \otimes \eta)(Ch \otimes Ce)}$$

shows that the real structure $C \otimes C$ on $H^* \otimes E^*$ coincides with the natural real structure. In consequence $Cdp = -dq$ and $Cdq = dp$ tell us that on $\Lambda^k E^* \otimes \Lambda^{\bar k} E^*$ we have as expected:

$$\Phi_q^{-1}(\overline{\Phi_q(\eta \otimes \tilde\eta)}) = (-1)^{\bar k(k+1)} C\tilde\eta \otimes C\eta.$$

Remark 2.1 (Real Structure on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$). The bigraded algebra isomorphism $\Phi_q\colon \Lambda^\bullet E^* \otimes \Lambda^\circ E^* \to \Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ is real with respect to

$$\Lambda^\bullet E^* \otimes \Lambda^\circ E^* \longrightarrow \Lambda^\circ E^* \otimes \Lambda^\bullet E^*, \quad \eta \otimes \tilde\eta \longmapsto (-1)^{|\tilde\eta|(|\eta|+1)} C\tilde\eta \otimes C\eta.$$

For the moment we want to quit discussion of $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ and study the simpler algebra $\Lambda^\bullet E^*$ instead. Note first that for every pair $\{e_\mu\}$ and $\{de_\mu\}$ of dual bases for E and E^* respectively the pairs $\{Ce_\mu\}$ and $\{Cde_\mu\}$ as well as $\{de_\mu^\flat\}$ and $\{-e_\mu^\sharp\}$ are dual bases, too. Bilinear sums over pairs of dual bases are independent of the pair, so we can replace one of these pairs by another in such sums whenever convenient. For example the symplectic form $\sigma = \frac{1}{2}\sum_\mu de_\mu \wedge e_\mu^\sharp$ and $\sigma^\flat = \frac{1}{2}\sum_\mu de_\mu^\flat \wedge e_\mu$ are real, in consequence the two operators

$$\sigma\wedge := \frac{1}{2}\sum_\mu de_\mu \wedge e_\mu^\sharp \wedge, \quad \sigma^\flat \lrcorner := \frac{1}{2}\sum_\mu e_\mu \lrcorner\, de_\mu^\flat \lrcorner$$

on $\Lambda^\bullet E^*$ of bidegree $+2$ and -2 respectively both commute with C. The calculation

$$[\sigma\wedge, e\lrcorner\,] = \frac{1}{2}\sum_\mu \big((de_\mu^\flat)^\sharp \wedge \{e_\mu^\sharp\wedge, e\lrcorner\,\} - \{de_\mu\wedge, e\lrcorner\,\}e_\mu^\sharp\wedge\big) = -e^\sharp\wedge$$

and its analogue for $[\sigma^\flat\lrcorner, \eta\wedge]$ provide us with the fundamental commutation relations:

$$\begin{aligned} [\sigma\wedge, \eta\wedge] &= 0, & [\sigma^\flat\lrcorner, \eta\wedge] &= -\eta^\flat\lrcorner, \\ [\sigma\wedge, e\lrcorner\,] &= -e^\sharp\wedge, & [\sigma^\flat\lrcorner, e\lrcorner\,] &= 0. \end{aligned} \tag{5}$$

On a symplectic vector space E of dimension $2n$ the commutator of $\sigma\wedge$ and $\sigma^\flat\lrcorner$ reads

$$\begin{aligned} [\sigma\wedge, \sigma^\flat\lrcorner\,] &= \frac{1}{2}\sum_\mu \big([\sigma\wedge, e_\mu\lrcorner\,]de_\mu^\flat\lrcorner\, + e_\mu\lrcorner\,[\sigma\wedge, de_\mu^\flat\lrcorner\,]\big) \\ &= -\frac{1}{2}\sum_\mu \big(-de_\mu\wedge e_\mu\lrcorner\, + e_\mu\lrcorner\, de_\mu\wedge\big) =: N - n \end{aligned}$$

where $N := \sum_\mu de_\mu\wedge e_\mu\lrcorner$ is the so called (fermionic) number operator, which multiplies forms in $\Lambda^k E^*$ by k. In other words there is a canonical $\mathfrak{sl}_2$-triple of real operators on $\Lambda^\bullet E^*$

$$H := n - N, \quad X := \sigma^\flat\lrcorner, \quad Y := \sigma\wedge \tag{6}$$

satisfying the classical commutation relations $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$.

Coming back to the description of the alternating forms we note that the factor $\operatorname{Sp} H$ of the quaternionic Kähler holonomy group $\operatorname{Sp} H \cdot \operatorname{Sp} E$ does not act on the source $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ of the isomorphism Φ_q, although it acts on the target $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$. Nevertheless there is a well-defined action of the group $\operatorname{Sp}(1) \subset \mathbb{H}$ of unit quaternions on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ such that for all unit vectors $q \in H$ the isomorphism Φ_q is actually equivariant over the isomorphism

$$\varphi_q \cdot \mathrm{id}\colon\ \operatorname{Sp}(1) \cdot \operatorname{Sp} E \xrightarrow{\cong} \operatorname{Sp} H \cdot \operatorname{Sp} E$$

induced by the algebra isomorphism $\varphi_q\colon \mathbb{H} \to Q$. The characterization $\varphi_q(i)q = iq$ and $\varphi_q(j)q = Cq$ of the isomorphism φ_q implies that the isomorphism φ_{Aq} associated to another unit vector $Aq \in H$ with $A \in \operatorname{Sp} H \subset Q$ is conjugated to φ_q in the sense $\varphi_{Aq} := A\varphi_q A^{-1}$. On the other hand the automorphisms $A \in \operatorname{GL} T$ of T act on the differential forms by their inverse adjoint A^{-*}. For the special isomorphisms $A \in \operatorname{Sp} H$ this representation satisfies

$$A^{-*}\Phi_q(\eta \otimes \tilde\eta) = \Phi_{Aq}(\eta \otimes \tilde\eta)$$

because $C(Aq) = A(Cq) = Ap$ and $\{A^{-*}dp, A^{-*}dq\}$ is the basis dual to Ap, Aq. In consequence the representation $\star$ of $\operatorname{Sp}(1)$ on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ defined to make Φ_q equivariant

$$z \star (\eta \otimes \tilde\eta) := \Phi_q^{-1}\big(\varphi_q(z)^{-*}\Phi_q(\eta \otimes \tilde\eta)\big) \tag{7}$$

over $\varphi_q \cdot \mathrm{id}$ is actually independent of the choice of the unit vector $q \in H$, because

$$\Phi_{Aq}^{-1}\big(\varphi_{Aq}(z)^{-*}\Phi_{Aq}(\eta\otimes\tilde\eta)\big) = \Phi_q^{-1}\big(A^*(A\varphi_q(z)A^{-1})^{-*}A^{-*}\Phi_q(\eta\otimes\tilde\eta)\big)$$

and $(A\varphi_q(z)A^{-1})^{-*} = A^{-*}\varphi_q(z)^{-*}A^*$. For the time being we will only make the infinitesimal representation associated to (7) explicit. Recall that the infinitesimal representation for the representation of $\mathrm{GL}\,T$ on alternating forms by inverse adjoints lets the endomorphism $A \in \mathrm{End}\,T$ act by minus the derivation extension of the adjoint endomorphism A^* with

$$(\mathrm{Der}_{A^*}\,\Omega)(X_1, X_2, \ldots, X_r) = \Omega(AX_1, X_2, \ldots, X_r) + \cdots + \Omega(X_1, X_2, \ldots, AX_r)$$

For the images I, J and K of the imaginary unit quaternions $i, j, k \in \mathbb{H}$ under φ_q we find the special values $I^*dp = -i\,dp$, $I^*dq = i\,dq$, as well as $J^*dp = dq$, $J^*dq = -dp$, and $K^*dp = -i\,dq$, $K^*dq = -i\,dp$, using (3) and conclude for the infinitesimal representation

$$\begin{aligned} i\star &:= \Phi_q^{-1}\circ(-\mathrm{Der}_{I^*})\circ\Phi_q = i\,\bar N - i\,N,\\ j\star &:= \Phi_q^{-1}\circ(-\mathrm{Der}_{J^*})\circ\Phi_q = \mathrm{Pl} + \overline{\mathrm{Pl}},\\ k\star &:= \Phi_q^{-1}\circ(-\mathrm{Der}_{K^*})\circ\Phi_q = i\,\mathrm{Pl} - i\,\overline{\mathrm{Pl}} \end{aligned} \tag{8}$$

where the Pücker differential Pl say is defined by (as usual $\hat\eta_s$ denotes an omitted factor):

$$\mathrm{Pl}(\eta_1\wedge\cdots\wedge\eta_k\otimes\tilde\eta_1\wedge\cdots\wedge\tilde\eta_{\tilde k}) = \sum_{s=1}^{k}(-1)^{k-s}\eta_1\wedge\cdots\wedge\hat\eta_s\wedge\cdots\wedge\eta_k\otimes\eta_s\wedge\tilde\eta_1\wedge\cdots\wedge\tilde\eta_{\tilde k}.$$

The definition of $\overline{\mathrm{Pl}}$ reverses the role of the two sides while keeping $(-1)^{k-s}$, equivalently,

$$\mathrm{Pl} := (-1)^N\circ\Big(\sum_\mu e_\mu\lrcorner\,\otimes\, de_\mu\wedge\Big),\quad \overline{\mathrm{Pl}} := (-1)^N\circ\Big(\sum_\mu de_\mu\wedge\,\otimes\, e_\mu\lrcorner\Big).$$

Defnition 2.2 (Natural operators on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$). Consider the tensor product $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ of two copies of the exterior algebra the alternating forms on a complex symplectic vector space E of dimension $2n$ with symplectic form σ and a pair of dual bases $\{e_\mu\}$ and $\{de_\mu\}$. Using the number operators N and $\bar N$ of the two tensor factors we can define ten natural, bigraded operators on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$, namely two copies

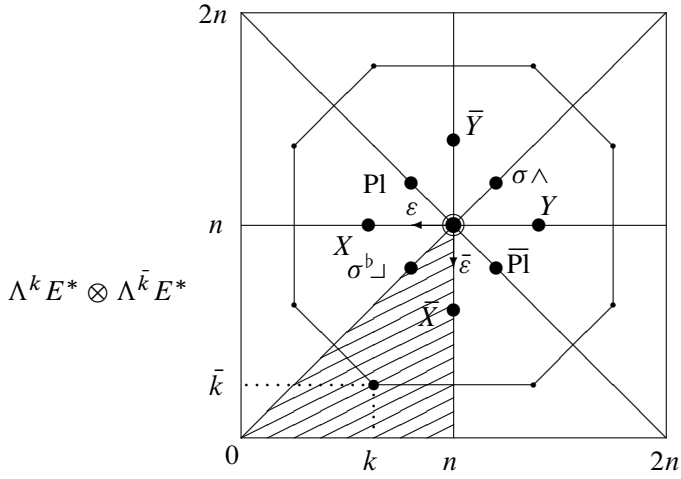
$$\begin{aligned} H &:= n - N, & \bar H &:= n - \bar N,\\ X &:= \frac12\sum_\mu e_\mu\lrcorner\, de_\mu^\flat\lrcorner\,\otimes\mathrm{id}, & \bar X &:= \frac12\sum_\mu \mathrm{id}\otimes e_\mu\lrcorner\, de_\mu^\flat\lrcorner,\\ Y &:= \frac12\sum_\mu de_\mu\wedge e_\mu^\sharp\wedge\,\otimes\mathrm{id}, & \bar Y &:= \frac12\sum_\mu \mathrm{id}\otimes de_\mu\wedge e_\mu^\sharp\wedge \end{aligned}$$

of $\mathfrak{sl}_2(\mathbb{C})$ acting on the left and right tensor factor respectively, and four diagonal operators:

$$\mathrm{Pl} := (-1)^N \circ \Big(\sum_\mu e_\mu \lrcorner \otimes de_\mu \wedge \Big), \quad \sigma\wedge := (-1)^N \circ \Big(\sum_\mu de_\mu \wedge \otimes e_\mu^\sharp \wedge \Big),$$

$$\overline{\mathrm{Pl}} := (-1)^N \circ \Big(\sum_\mu de_\mu \wedge \otimes e_\mu \lrcorner \Big), \quad \sigma^\flat\lrcorner := (-1)^N \circ \Big(\sum_\mu de_\mu^\flat \lrcorner \otimes e_\mu \lrcorner \Big).$$

In due course we are going about to prove that the subspace of endomorphisms of $\Lambda^\bullet E^* \otimes \Lambda^\circ E$ spanned by these ten natural operators is actually a Lie algebra isomorphic to $\mathfrak{so}_5(\mathbb{C})$ or equivalently $\mathfrak{sp}_4(\mathbb{C})$ with maximal torus spanned by H and $\bar{H}$. At this point let us simply point out that H and $\bar{H}$ are linear in the number operators N and $\bar{N}$ so that every bigraded endomorphism is an eigenvector for ad H and ad $\bar{H}$. In terms of the basis $\{\varepsilon, \bar{\varepsilon}\}$ dual to the basis $\{H, \bar{H}\}$ of the maximal torus the eigenvalues of the other operators and the weight spaces $\Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$ of the representation $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ can be read off from the diagram below, where the dashed region



(9)

indicates a preferred Weyl chamber and the octagon the support of the character of the irreducible representation of $\mathfrak{so}_5(\mathbb{C})$ of highest weight

Lemma 2.3 (Action of $\mathfrak{so}_{4,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$). *The subspace of operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ spanned linearly over $\mathbb{C}$ by the ten operators of Definition 2.2 is closed under brackets and thus a complex Lie algebra isomorphic to $\mathfrak{so}_5(\mathbb{C})$. The real structure on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ induces the real structure on this Lie algebra $\mathfrak{so}_5(\mathbb{C})$ of operators indicated by the choice of notation, both $\overline{\sigma\wedge} = -\sigma\wedge$ and $\overline{\sigma^\flat\lrcorner} = -\sigma^\flat\lrcorner$ are imaginary. The real subalgebra fixed by this real structure is isomorphic to $\mathfrak{so}_{4,1}(\mathbb{R})$.*

Proof. According to the calculation (6) of the commutators of the operators $\sigma\wedge$ and $\sigma^\flat\lrcorner$ on $\Lambda^\bullet E^*$ the two $\mathfrak{sl}_2$-triples $\langle H : X : Y\rangle$ and $\langle \bar H : \bar X : \bar Y\rangle$ span commuting $\mathfrak{sl}_2(\mathbb{C})$-subalgebras of operators acting on different tensor factors. All operators in the direct sum $\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$ preserve the parity of the bigrading and so commute with the operator $(-1)^N$, hence brackets with this subalgebra leaves the subspace spanned by Pl, $\overline{\mathrm{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ invariant. More precisely the fundamental commutation relations (5) tell us that

$$\begin{aligned}
[H,\mathrm{Pl}] &= \mathrm{Pl}, & [X,\mathrm{Pl}] &= 0, & [Y,\mathrm{Pl}] &= \sigma\wedge,\\
[H,\overline{\mathrm{Pl}}] &= -\overline{\mathrm{Pl}}, & [X,\overline{\mathrm{Pl}}] &= -\sigma^\flat\lrcorner, & [Y,\overline{\mathrm{Pl}}] &= 0,\\
[H,\sigma\wedge] &= -\sigma\wedge, & [X,\sigma\wedge] &= \mathrm{Pl}, & [Y,\sigma\wedge] &= 0,\\
[H,\sigma^\flat\lrcorner] &= \sigma^\flat\lrcorner, & [X,\sigma^\flat\lrcorner] &= 0, & [Y,\sigma^\flat\lrcorner] &= -\overline{\mathrm{Pl}}.
\end{aligned}$$

Instead of verifying the missing commutation relations with $\bar H$, $\bar X$ and $\bar Y$ directly we can infer these relations using the reality $\overline{[A,B]} = [\bar A,\bar B]$ of the commutator. For this reason we skip this point and proceed to determine the real structure induced by the real structure on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ made explicit in Remark 2.1. This real structure interchanges the tensor factors and hence interchanges the corresponding endomorphisms $H\leftrightarrow\bar H$, $X\leftrightarrow\bar X$, $Y\leftrightarrow\bar Y$, say:

$$X(\overline{\eta\otimes\tilde\eta}) = (-1)^{|\tilde\eta|(|\eta|+1)}C(\sigma^\flat\lrcorner\,\tilde\eta)\otimes C\,\eta = \overline{\bar X(\eta\otimes\tilde\eta)}.$$

The main point of the argument concerning the operators Pl, $\overline{\mathrm{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ is that these operators change the parity of both (sic!) factors, with this in mind we find for example

$$\sigma\wedge(\overline{\eta\otimes\tilde\eta}) = (-1)^{|\tilde\eta||\eta|+1}\sum_\mu (C\,de_\mu)\wedge(C\,\tilde\eta)\otimes(C\,e_\mu^\sharp)\wedge(C\,\eta) = -\overline{\sigma\wedge(\eta\otimes\tilde\eta)}.$$

Last but not least we have to calculate the commutators of the operators Pl, $\overline{\mathrm{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$. To begin with we note that these operators anticommute with $(-1)^N$, hence the $(-1)^N$-factor of square 1 conveniently turns into a minus sign in all the calculations:

$$\begin{aligned}
[\mathrm{Pl},\overline{\mathrm{Pl}}] &= -\sum_{\mu\nu}\big(\{e_\mu\lrcorner, de_\nu\wedge\}\otimes de_\mu\wedge e_\nu\lrcorner - de_\nu\wedge e_\mu\lrcorner\otimes\{de_\mu\wedge, e_\nu\lrcorner\}\big)\\
&= -\mathrm{id}\otimes\bar N + N\otimes\mathrm{id} = \bar H - H.
\end{aligned}$$

We omit the analogous calculation of the remaining commutators and tabulate the result:

$$\begin{aligned}
[\mathrm{Pl},\overline{\mathrm{Pl}}] &= \bar H - H, & [\mathrm{Pl},\sigma\wedge] &= -2\bar Y, & [\mathrm{Pl},\sigma^\flat\lrcorner] &= -2X,\\
[\sigma\wedge,\sigma^\flat\lrcorner] &= \bar H + H, & [\overline{\mathrm{Pl}},\sigma\wedge] &= 2Y, & [\overline{\mathrm{Pl}},\sigma^\flat\lrcorner] &= 2\bar X.
\end{aligned}$$

In order to prove that the algebra of operators is isomorphic to $\mathfrak{so}_5(\mathbb{C})$ we observe that the ten operators of Definition 2.2 all occur in at least one $\mathfrak{sl}_2(\mathbb{C})$-subalgebra

of operators, more precisely the relevant $\mathfrak{sl}_2(\mathbb{C})$-subalgebras are spanned by the $\mathfrak{sl}_2$-triples of operators:

$$\langle H : X : Y\rangle, \quad \langle \bar{H} : \bar{X} : \bar{Y}\rangle, \quad \langle H - \bar{H} : \mathrm{Pl} : -\overline{\mathrm{Pl}}\rangle, \quad \langle H + \bar{H} : \sigma^\flat \lrcorner : -\sigma\wedge\rangle.$$

On the other hand the trace form of every finite-dimensional representation R of $\mathfrak{sl}_2(\mathbb{C})$ satisfies $\mathrm{tr}_R(H^2) = 2\mathrm{tr}_R(XY)$, it is actually sufficient to verify this for the symmetric powers $\mathrm{Sym}^k\mathbb{C}^2$ of the defining representation $\mathbb{C}^2$ of $\mathfrak{sl}_2(\mathbb{C})$. Consulting the weight diagram (9) for the eigenvalues of the diagonalizable operators ad H and ad $\bar{H}$ we find immediately

$$B(H,H) = 12 = B(\bar{H},\bar{H}), \quad B(H,\bar{H}) = 0$$

for the trace form B of the adjoint representation, the so called Killing form, and conclude $B(H+\bar{H}, H+\bar{H}) = 24 = B(H-\bar{H}, H-\bar{H})$. The remaining non-zero values of B are

$$B(X,Y) = 6 = B(\bar{X},\bar{Y}) \quad \text{and} \quad B(\mathrm{Pl},\overline{\mathrm{Pl}}) = -12 = B(\sigma^\flat \lrcorner\, \sigma\wedge),$$

all other tuples of basis vectors are orthogonal/isotropic by equivariance alone. Having a non-degenerate Killing form the Lie algebra of operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ spanned by the ten operators of Definition 2.2 is semisimple and the weight diagram (9) tells us that it is isomorphic to $\mathfrak{so}_5(\mathbb{C})$. An interesting twist to this argument allows us to determine the signature of the Killing form B restricted to the real subalgebra and thus its isomorphism type as well. Evidently the real structure preserves the orthogonal decomposition

$$\mathrm{span}_{\mathbb{C}}\{H,\bar{H}\} \oplus \mathrm{span}_{\mathbb{C}}\{X,\bar{X},Y,\bar{Y}\} \oplus \mathrm{span}_{\mathbb{C}}\{\mathrm{Pl},\overline{\mathrm{Pl}}\} \oplus \mathrm{span}_{\mathbb{C}}\{\sigma\wedge, \sigma^\flat\lrcorner\}$$

into subspaces of signature $(1,1)$, $(2,2)$, $(0,2)$ and $(1,1)$. Restricted to the real subalgebra the Killing form thus has signature $(4,6)$ characterizing the real form $\mathfrak{so}_{4,1}(\mathbb{R})$ of $\mathfrak{so}_5(\mathbb{C})$. □

Remark 2.4 (Explicit isomorphism with $\mathfrak{so}_{4,1}(\mathbb{R})$). Consider the realization $\mathfrak{so}_r(\mathbb{C}) = \{A \in \mathrm{Mat}_{r\times r}\mathbb{C} \mid A^T S + SA = 0\}$ of the complex Lie algebra $\mathfrak{so}_r(\mathbb{C})$ associated to a non-degenerate, real, symmetric $r \times r$-matrix S. The signature of S determines the isomorphism type of the standard real structure $\mathfrak{so}_r(\mathbb{C}) \to \mathfrak{so}_r(\mathbb{C})$, $A \mapsto \bar{A}$, given by conjugation of coefficients. In particular the symmetric matrix of signature $(4,1)$

$$S := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

defines a real structure with real form $\mathfrak{so}_{4,1}(\mathbb{R})$ on $\mathfrak{so}_5(\mathbb{C})$. Sending the operators H, X, Y to

$$\begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 \end{pmatrix},$$

respectively, and the operators Pl, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ similarly to the matrices

$$\begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -i & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix}, \quad \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

defines a real isomorphism between the algebra of operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ and $\mathfrak{so}_5(\mathbb{C})$.

There are other ways to construct the algebra $\mathfrak{so}_{4,1}(\mathbb{R})$ besides Definition 2.2 closer in spirit to the standard procedure in Kähler geometry. Research on this idea and the resulting algebra goes back at least to the extensive work of Bonan [1], [4], who introduced the algebra $\mathfrak{so}_{4,1}(\mathbb{R})$ by considering all ways to wedge and contract with Kähler forms. An orthogonal quaternionic structure $Q \subset \operatorname{End} T$ gives rise to three linearly independent Kähler forms $\omega_I = g(I\cdot,\cdot)$ etc. associated to the images I, J, K of the unit quaternions $i, j, k \in \mathbb{H}$ under an isomorphism $\mathbb{H} \to Q$. Every such isomorphism is induced by the choice of a unit vector $q \in H$ satisfying $\sigma(Cq, q) = 1$ and is in this way related to a canonical basis p, q of H with $p := Cq$. In terms of this basis the scalar product on $T \otimes_{\mathbb{R}} \mathbb{C} \cong H \otimes E$ can be written in the form

$$g = \sigma \otimes \sigma = \sum_\mu \big((dp \otimes de_\mu) \otimes (dq \otimes e_\mu^\sharp) - (dq \otimes de_\mu) \otimes (dp \otimes e_\mu^\sharp)\big).$$

The equivalent formulation $\omega_I = (I^* \otimes \mathrm{id})(g)$ of the definition of the Kähler form together with the explicit values $I^*dp = -i\,dp, I^*dq = i\,dq$ as well as $J^*dp = dq, J^*dq = -dp$ and $K^*dp = -i\,dq, K^*dq = -i\,dp$ allows us to expand the three Kähler forms according to

$$\begin{aligned} \omega_I &= -i \sum_\mu (dp \otimes de_\mu) \wedge (dq \otimes e_\mu^\sharp), \\ \omega_J &= \frac{1}{2} \sum_\mu \big((dp \otimes de_\mu) \wedge (dp \otimes e_\mu^\sharp) + (dq \otimes de_\mu) \wedge (dq \otimes e_\mu^\sharp)\big), \\ \omega_K &= \frac{i}{2} \sum_\mu \big((dp \otimes de_\mu) \wedge (dp \otimes e_\mu^\sharp) - (dq \otimes de_\mu) \wedge (dq \otimes e_\mu^\sharp)\big). \end{aligned}$$

Some additional considerations based on these expansions lead to the description

$$\begin{aligned}
\Phi_q^{-1} \circ (\omega_I \wedge) \circ \Phi_q &= i\sigma\wedge, & \Phi_q^{-1} \circ (\omega_I^\flat \lrcorner) \circ \Phi_q &= i\sigma^\flat \lrcorner,\\
\Phi_q^{-1} \circ (\omega_J \wedge) \circ \Phi_q &= \bar{Y} + Y, & \Phi_q^{-1} \circ (\omega_J^\flat \lrcorner) \circ \Phi_q &= X + \bar{X},\\
\Phi_q^{-1} \circ (\omega_J \wedge) \circ \Phi_q &= i\bar{Y} - iY, & \Phi_q^{-1} \circ (\omega_K^\flat \lrcorner) \circ \Phi_q &= iX - i\bar{X}
\end{aligned}$$

of wedge products and contractions with Kähler forms in terms of the Lie algebra $\mathfrak{so}_{4,1}(\mathbb{R})$, conversely these six operators evidently suffice to generate all $\mathfrak{so}_{4,1}(\mathbb{R})$. Recall now that the isomorphism Φ_q conjugates the representation of $\operatorname{Sp} H \subset \operatorname{SO} T$ on the alternating forms $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ on the euclidian vector space T to a representation of the group $\operatorname{Sp}(1)$ on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$, whose infinitesimal generators we have calculated in equation (8):

$$\begin{aligned}
i\star &= \Phi_q^{-1} \circ (-\operatorname{Der}_{I^*}) \circ \Phi_q = iH - i\bar{H},\\
j\star &= \Phi_q^{-1} \circ (-\operatorname{Der}_{J^*}) \circ \Phi_q = \operatorname{Pl} + \overline{\operatorname{Pl}},\\
k\star &= \Phi_q^{-1} \circ (-\operatorname{Der}_{K^*}) \circ \Phi_q = i\operatorname{Pl} - i\overline{\operatorname{Pl}}.
\end{aligned}$$

Somewhat more general the infinitesimal representation of the Lie algebra $\mathfrak{so}_{4,1}(\mathbb{R})$ of operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ integrates to an actual representation of the unique connected and simply connected Lie group $\operatorname{Spin}^+_{4,1}(\mathbb{R})$ with Lie algebra $\mathfrak{so}_{4,1}(\mathbb{R})$. The weight diagram (9) tells us that the central element $-1 \in \operatorname{Spin}^+_{4,1}(\mathbb{R})$ acts as $(-1)^{k+\bar{k}}$ on $\Lambda^{k,\bar{k}} T^* \otimes_{\mathbb{R}} \mathbb{C} \cong \Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$, hence this representation does not descend to $\operatorname{SO}^+_{4,1}(\mathbb{R})$. Nevertheless the invariant subspace $\Lambda^{\mathrm{ev}} T^* \otimes_{\mathbb{R}} \mathbb{C}$ of forms of even total degree is a genuine representation of $\operatorname{SO}^+_{4,1}(\mathbb{R})$.

Besides the subgroup $\operatorname{Sp}(1) \subset \operatorname{Spin}^+_{4,1}(\mathbb{R})$ conjugated to the factor $\operatorname{Sp} H \subset \operatorname{SO} T$ of the holonomy group we want to mention two other interesting subgroups of $\operatorname{Spin}^+_{4,1}(\mathbb{R})$, namely the double cover $\operatorname{SL}_2(\mathbb{C}) \cong \operatorname{Spin}^+_{3,1}(\mathbb{R})$ of the Lorentz group and the maximal compact subgroup $\operatorname{Spin}_4(\mathbb{R})$. Interestingly there seems to be no natural choice for the latter, and it is tempting to try to relate this oddity to geometrical and topological properties of hyper-Kähler or quaternionic Kähler manifolds. Concerning the former we note that the involutive automorphism of $\mathfrak{so}_5(\mathbb{C})$ given by conjugation with $(-1)^N$ fixes the subalgebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{so}_5(\mathbb{C})$ spanned by the operators H, X, Y and $\bar{H}$, $\bar{X}$, $\bar{Y}$. The resulting symmetric pair of real Lie algebras

$$\mathfrak{so}_{4,1}(\mathbb{R}) = \mathfrak{so}_{3,1}(\mathbb{R}) \oplus \mathbb{R}^{3,1} \supset \mathfrak{so}_{3,1}(\mathbb{R})$$

defines a Lorentz symmetric space, the de Sitter space $\operatorname{Spin}^+_{4,1}(\mathbb{R})/\operatorname{SL}_2(\mathbb{C})$. Incidentally we note that the inclusion of the factor $(-1)^N$ in the definition of the operators Pl, $\overline{\operatorname{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ has the same effect on the symmetric pair $\mathfrak{so}_{4,1}(\mathbb{R}) \supset \mathfrak{so}_{3,1}(\mathbb{R})$ as the multiplication by i in the eigenspace of eigenvalue -1 in the classical construction of the dual symmetric pair.

Interestingly the construction of the algebra $\mathfrak{so}_{4,1}(\mathbb{R})$ acting on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ is a special case of a general property of representation theory related to Howe's Theory of dual pairs. On the tensor product of r copies of the exterior or symmetric algebra of a complex vector space V lives a natural simple Lie algebra of operators isomorphic to $\mathfrak{gl}_r(\mathbb{C})$, consisting of the $r(r-1)$ possible generalizations of the Plücker differentials Pl and $\overline{\mathrm{Pl}}$ and r shifted number operators like H or $\bar{H}$. If either the factors are exterior algebras and the vector space V is symplectic or the factors are symmetric algebras and V is euclidian, then we can extend this algebra by $r(r+1)$ additional operators like X or $\sigma\wedge$ and the resulting algebra is isomorphic to $\mathfrak{sp}_{2r}(\mathbb{C}) = \mathfrak{sp}(r) \otimes_{\mathbb{R}} \mathbb{C}$. The remarkable thing about this construction is that Weyl's construction of the irreducible representations of the classical matrix groups using Schur functors [2] tells us that under rather general assumptions on r and $\dim V$ the multiplicity spaces for the decomposition of the tensor product into irreducible representations of $\mathfrak{gl}_r(\mathbb{C})$ or $\mathfrak{sp}_{2r}(\mathbb{C})$ are irreducible representations of the automorphism group of V!

The algebra $\mathfrak{so}_{4,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sp}_4(\mathbb{C})$ acting on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ is a special example of this construction. The space of highest weight vectors of $\mathfrak{so}_{4,1}(\mathbb{R})$ in $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ given a dominant weight $(n-\bar{k})\bar{\varepsilon} + (n-k)\varepsilon$ in the Weyl chamber indicated in diagram (9) is by definition the intersection of the kernels of the four operators X, $\sigma^\flat\lrcorner$, $\bar{X}$ and $\overline{\mathrm{Pl}}$ corresponding to the positive roots with the weight space $\Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$. Denoting the irreducible representation of $\mathfrak{so}_{4,1}(\mathbb{R})$ of highest weight λ by $R_\lambda^{\mathfrak{so}_{4,1}(\mathbb{R})}$ we can rewrite this definition in the following way

$$\mathrm{Hom}_{\mathfrak{so}_{4,1}(\mathbb{R})}(R^{\mathfrak{so}_{4,1}(\mathbb{R})}_{(n-\bar{k})\bar{\varepsilon}+(n-k)\varepsilon}, \Lambda E^* \otimes \Lambda E^*) = \ker X \cap \ker \overline{\mathrm{Pl}} \cap (\Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*)$$

because the operators X and $\overline{\mathrm{Pl}}$ corresponding to the simple roots generate the subalgebra of operators corresponding to positive roots. According to Weyl's construction of the irreducible representations of the symplectic Lie groups as trace-free Schur functors however the right hand side is exactly the irreducible representation $\Lambda^{k,\bar{k}}_\circ E^* \cong \Lambda^{k,\bar{k}}_\circ E$ of $\mathrm{Sp}\, E$ of highest weight $2\varepsilon_1 + \cdots + 2\varepsilon_{\bar{k}} + \varepsilon_{\bar{k}+1} + \cdots + \varepsilon_k$, where $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ are the weights of the defining representation E of $\mathrm{Sp}\, E$. In consequence the complete decomposition of the tensor product $\Lambda E^* \otimes \Lambda E^*$ of two copies of the exterior algebra of a symplectic vector space E^* into irreducible representations reads for either member of the Howe dual pair $\mathrm{Sp}\, E$ and $\mathfrak{so}_{4,1}(\mathbb{R})$:

$$\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^\bullet E^* \otimes \Lambda^\circ E^* = \bigoplus_{n \geq r \geq \bar{r} \geq 0} R^{\mathfrak{so}_{4,1}(\mathbb{R})}_{(n-\bar{r})\bar{\varepsilon}+(n-r)\varepsilon} \otimes \Lambda^{r,\bar{r}}_\circ E. \qquad (10)$$

A detailed discussion of Weyl's construction is outside the scope of this exposition, the reader is referred to [2] instead, nevertheless we want to give a very brief sketch of Weyl's argument for the irreducibility of $\ker X \cap \ker \overline{\mathrm{Pl}} \subset \Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$ as a representation of $\mathrm{Sp}\, E$. The most difficult aspect of Weyl's argument concerns proving the surjectivity of the restriction

$$\sigma^\flat\lrcorner \colon \Lambda^k_\circ E^* \otimes \Lambda^{\bar{k}}_\circ E^* \longrightarrow \Lambda^{k-1}_\circ E^* \otimes \Lambda^{\bar{k}-1}_\circ E^*$$

of the operator $\sigma^\flat \lrcorner$ to the joint kernel $\Lambda^k_\circ E^* \otimes \Lambda^{\bar k}_\circ E^* \subset \Lambda^k E^* \otimes \Lambda^{\bar k} E^*$ of the two commuting operators X and $\bar X$ in bidegrees $n \geq k, \bar k \geq 0$. Because of surjectivity we can calculate the dimension of the joint kernel $\Lambda^\bullet{}_\circ E^* \otimes_\circ \Lambda^\circ{}_\circ E^*$ of the three operators X, $\bar X$ and $\sigma^\flat \lrcorner$ from the known dimension of $\Lambda^k_\circ E^* \otimes \Lambda^{\bar k}_\circ E^*$. In turn the surjectivity of the restriction

$$\overline{\mathrm{Pl}}\colon \Lambda^k_\circ E^* \otimes_\circ \Lambda^{\bar k}_\circ E^* \longrightarrow \Lambda^{k+1}_\circ E^* \otimes_\circ \Lambda^{\bar k - 1}_\circ E^*$$

of the operator $\overline{\mathrm{Pl}}$ to the joint kernel of X, $\bar X$ and $\sigma^\flat \lrcorner$ for bidegrees $k \geq \bar k$ is a direct consequence of representation theory of $\mathfrak{sl}_2$ so that we eventually end up calculating the dimension of the joint kernel $\Lambda^{k,\bar k}_\circ E^*$ of all four operators X, $\bar X$, $\sigma^\flat \lrcorner$ and $\overline{\mathrm{Pl}}$ in bidegrees $n \geq k \geq \bar k \geq 0$. Choosing on the other hand a complex basis $\eta_1, \ldots, \eta_n \in L$ of a maximal isotropic or Lagrangian subspace $L \subset E^*$ we note that for all $n \geq k \geq \bar k \geq 0$ the form

$$\eta_1 \wedge \cdots \wedge \eta_k \otimes \eta_1 \wedge \cdots \wedge \eta_{\bar k} \in \Lambda^{k,\bar k}_\circ E^*$$

is a highest weight vector in $\Lambda^{k,\bar k}_\circ E^*$ of weight $2\varepsilon_1 + \cdots + 2\varepsilon_{\bar k} + \varepsilon_{\bar k + 1} + \cdots + \varepsilon_k$ for a suitable choice of a maximal torus of $\mathrm{Sp}\, E$ and a suitable ordering of weights. In consequence the joint kernel $\Lambda^{k,\bar k}_\circ E^*$ of X, $\bar X$, $\sigma^\flat \lrcorner$ and $\overline{\mathrm{Pl}}$ contains at least the irreducible representation of $\mathrm{Sp}\, E$ of this specific highest weight, by the dimension part of Weyl's Character Formula however the dimension of this irreducible summand agrees with the dimension of $\Lambda^{k,\bar k}_\circ E^*$ we have calculated before proving the irreducibility of the latter under $\mathrm{Sp}\, E$.

In concluding this section we want to formulate some straightforward consequences of the decomposition (10) into irreducible representations. Say the maximal torus $\mathbb{R} \oplus i\mathbb{R}$ of $\mathfrak{so}_{4,1}(\mathbb{R})$ is generated by $\bar H + H$ and $i\bar H - iH$, which take the values $\bar k + k$ and $i\bar k - ik$ on the weight space $\Lambda^k E^* \otimes \Lambda^{\bar k} E^* \cong \Lambda^{k,\bar k} T^* \otimes_{\mathbb{R}} \mathbb{C}$. Branching from $\mathfrak{so}_{4,1}(\mathbb{R})$ to $\mathbb{R} \oplus i\mathbb{R}$ thus calculates the decomposition of $\Lambda^{k,\bar k} T^* \otimes_{\mathbb{R}} \mathbb{C}$ under the holonomy group $\mathrm{Sp}(n)$ of hyper-Kähler manifolds:

Lemma 2.5 (Differential forms on hyper-Kähler manifolds). *Under the action of* $\mathrm{Sp}\, E \subset \mathrm{SO}\, T$ *the space* $\Lambda^{k,\bar k} T^* \otimes_{\mathbb{R}} \mathbb{C}$ *of complex* $(k,\bar k)$*-forms with respect to the complex structure* I *decomposes into a sum of representations* $\Lambda^{r,\bar r}_\circ E$ *with multiplicities*

$$\Lambda^{k,\bar k} T^* \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{n \geq r \geq \bar r \geq 0} \dim \mathrm{Hom}_{\mathbb{R} \oplus i\mathbb{R}}\Big(R^{\mathbb{R} \oplus i\mathbb{R}}_{(n-\bar k)\bar\varepsilon + (n-k)\varepsilon},\, R^{\mathfrak{so}_{4,1}(\mathbb{R})}_{(n-\bar r)\bar\varepsilon + (n-r)\varepsilon} \Big) \Lambda^{r,\bar r}_\circ E$$

given by the multiplicity of the irreducible representation of highest weight $(n-\bar k)\bar\varepsilon + (n-k)\varepsilon$ *for* $\mathbb{R} \oplus i\mathbb{R}$ *in the irreducible representation of* $\mathfrak{so}_{4,1}(\mathbb{R})$ *of highest weight* $(n-\bar r)\bar\varepsilon + (n-r)\varepsilon$.

On quaternionic Kähler manifolds we are interested instead in decomposing $\Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C}$ under the holonomy group $\mathrm{Sp}\, H \cdot \mathrm{Sp}\, E$ or equivalently $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ un-

der $\mathrm{Sp}(1)\cdot\mathrm{Sp}\,E$ using the isomorphism Φ_q, where the factor $\mathrm{Sp}(1)\subset \mathrm{Spin}^+_{4,1}(\mathbb{R})$ has Lie algebra $\mathfrak{sp}(1)\subset\mathfrak{so}_{4,1}(\mathbb{R})$ generated by $iH-i\bar H$, $\mathrm{Pl}+\overline{\mathrm{Pl}}$ and $i\,\mathrm{Pl}-i\,\overline{\mathrm{Pl}}$, compare equation (8). It is convenient to consider the central extension $\mathbb{R}\times\mathrm{Sp}(1)\subset\mathrm{Spin}^+_{4,1}(\mathbb{R})$ of $\mathrm{Sp}(1)$ with Lie algebra generated by $\mathfrak{sp}(1)$ and $H+\bar H$ in order to keep track of the total degree of a differential form. To wit $e^{t(H+\bar H)}\in\mathbb{R}\subset\mathrm{Spin}^+_{4,1}(\mathbb{R})$ acts by multiplication with $e^{t(2n-k)}$ on forms $\Lambda E^*\otimes\Lambda E^*$ of total degree k similar to its action on the irreducible representation $\mathbb{C}_{(n-\frac{k}{2})(\bar\varepsilon+\varepsilon)}\otimes\mathrm{Sym}^s\mathbb{H}$ of $\mathbb{R}\times\mathrm{Sp}(1)$ of highest weight $(n-\frac{k}{2})(\bar\varepsilon+\varepsilon)+\frac{s}{2}(\bar\varepsilon-\varepsilon)=(n-\frac{k-s}{2})\bar\varepsilon+(n-\frac{k+s}{2})\varepsilon$:

Lemma 2.6 (Differential forms on quaternionic Kähler manifolds). *Under the action of* $\mathrm{Sp}\,H\cdot\mathrm{Sp}\,E\subset\mathrm{SO}\,T$ *the space of complex valued differential forms on a euclidian vector space* T *of dimension* $4n$ *with orthogonal quaternionic structure* $Q\subset\mathrm{End}\,T$ *decomposes into a sum of irreducible representations* $\mathrm{Sym}^s H\otimes\Lambda^{r,\bar r}_\circ E^*$ *with multiplicities*

$$\Lambda^k T^*\otimes_{\mathbb{R}}\mathbb{C}\cong\bigoplus_{\substack{s\geq 0\\ n\geq r\geq\bar r\geq 0}}\dim\mathrm{Hom}_{\mathbb{R}\oplus\mathfrak{sp}(1)}\Big(R^{\mathbb{R}\oplus\mathfrak{sp}(1)}_{(n-\frac{k-s}{2})\bar\varepsilon+(n-\frac{k+s}{2})\varepsilon},R^{\mathfrak{so}_{4,1}(\mathbb{R})}_{(n-\bar r)\bar\varepsilon+(n-r)\varepsilon}\Big)\mathrm{Sym}^s H\otimes\Lambda^{r,\bar r}_\circ E$$

given by the multiplicity of the irreducible representation of $\mathbb{R}\oplus\mathfrak{sp}(1)$ *of highest weight* $(n-\frac{k-s}{2})\bar\varepsilon+(n-\frac{k+s}{2})\varepsilon$ *in the irreducible representation of* $\mathfrak{so}_{4,1}(\mathbb{R})$ *for* $(n-\bar r)\bar\varepsilon+(n-r)\varepsilon$.

3 Quaternionic Kähler decomposition of forms

Of course neither Lemma 2.5 nor Lemma 2.6 are really explicit decomposition formulas, but they reduce the problem of decomposing the differential forms on a hyper-Kähler manifold or quaternionic Kähler manifold of quaternionic dimension n under the holonomy group $\mathrm{Sp}(n)$ or $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ respectively to finding the branching rules from $\mathfrak{so}_{4,1}(\mathbb{R})$ to $\mathbb{R}\oplus i\mathbb{R}$ or $\mathbb{R}\oplus\mathfrak{sp}(1)$. In this section we will describe a rather general strategy to solve this standard problem in representation theory for a pair $\mathfrak{g}\supset\mathfrak{h}$ of real reductive Lie algebras. Using this strategy we will readily turn Lemma 2.5 and Lemma 2.6 into effective decomposition formulas.

Consider for a moment the Lie algebra $\mathfrak{g}$ of a compact group G and for each $X\in\mathfrak{g}$ the adjoint endomorphism $\mathrm{ad}\,X\colon\mathfrak{g}\to\mathfrak{g}$, $Y\mapsto[X,Y]$. With G being compact there exists a G-invariant negative definite scalar product B on $\mathfrak{g}$, for example we can take the Killing form $B(X,Y):=\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\,X\circ\mathrm{ad}\,Y)$ if G is semisimple. On the Zariski open subset of regular elements $X\in\mathfrak{g}$ the rank of $\mathrm{ad}\,X$ becomes maximal, in turn its kernel becomes minimal and defines a maximal abelian subalgebra of $\mathfrak{g}$ also known as a maximal torus $\mathfrak{t}=\{H\in\mathfrak{g}\mid[X,H]=0\}$. With $\mathfrak{g}$ being the Lie algebra of a compact group G the kernel of the exponential map $\exp\colon\mathfrak{g}\to G$, $X\mapsto e^X$, restricted

to $\mathfrak{t}$ is a lattice in $\mathfrak{t}$, whose dual lattice

$$\Lambda := \{\lambda \in i\mathfrak{t}^* \mid \lambda(X) \in 2\pi i\mathbb{Z} \text{ for all } X \in \mathfrak{t} \text{ with } e^X = 1\} \subset i\mathfrak{t}^*$$

is called the weight lattice of G. In general the weight lattice Λ becomes finer if we replace G by a compact covering group $\tilde{G}$, in other words the weight lattice encodes the global structure of the Lie group G and does not only depend on its Lie algebra $\mathfrak{g}$.

A very useful tool for the calculations to come is the group ring $\mathbb{Z}\Lambda$ of the weight lattice Λ considered as an additive group with coefficients in $\mathbb{Z}$. Elements of the group ring $\mathbb{Z}\Lambda$ are appropriately thought of as finite formal sums of terms ce^λ with $c \in \mathbb{Z}$ and $\lambda \in \Lambda$, because the naive multiplication of such formal sums agrees with the standard convolution product in the group ring $\mathbb{Z}\Lambda$. In particular the character of every finite-dimensional representation V of the compact group G finds a natural home in the ring $\mathbb{Z}\Lambda$

$$\operatorname{ch} V := \sum_{\lambda\in\Lambda} (\dim E_\lambda) e^\lambda$$

where E_λ is the generalized complex eigenspace of the action of the maximal torus on V:

$$E_\lambda := \{v \in V \otimes_{\mathbb{R}} \mathbb{C} \mid H \star v = \lambda(H)v\}$$

The character ch defined this way is a ring homomorphism from the representation ring RG of the group G to the group ring $\mathbb{Z}\Lambda$ of the weight lattice, in other words the equalities $\operatorname{ch} V \oplus W = \operatorname{ch} V + \operatorname{ch} W$ and $\operatorname{ch} V \otimes W = (\operatorname{ch} V)(\operatorname{ch} W)$ hold true. For connected G the character ch is injective, moreover its image is precisely the subring $[\mathbb{Z}\Lambda]^{\mathfrak{W}}$ of elements invariant under the Weyl group $\mathfrak{W} = \operatorname{Norm} T/T$ of G acting on $\mathbb{Z}\Lambda$ by the linear extension of $(w, e^\lambda) \mapsto e^{w\lambda}$ with $(w\lambda)(X) := \lambda(\operatorname{Ad}_w^{-1} X)$. Incidentally we note that the automorphism $*\colon \mathbb{Z}\Lambda \to \mathbb{Z}\Lambda$, $e^\lambda \mapsto e^{-\lambda}$, is useful to "dualize" the character $\operatorname{ch}(V^*) = (\operatorname{ch} V)^*$ of a representation V. For group rings like $\mathbb{Z}\Lambda$ the analogue of the residue of complex analysis

$$\operatorname{ev}_\lambda\colon \mathbb{Z}\Lambda \longrightarrow \mathbb{Z}, \quad e^\mu \longmapsto \delta_{\lambda=\mu}$$

can be introduced to pick up the coefficient of the weight $\lambda \in \Lambda$ in a given element of $\mathbb{Z}\Lambda$.

The weights (in the support of the character) of the adjoint representation $\mathfrak{g}$ of G are of special importance and are called roots. Choosing once and for all a regular element $X \in \mathfrak{t}$ in a fixed maximal torus $\mathfrak{t}$ we can classify a root α according to the value of $i\alpha(X) \in \mathbb{R} \setminus \{0\}$ as positive or negative. In turn an imaginary valued linear form $\lambda \in i\mathfrak{t}^*$ on $\mathfrak{t}$ is called dominant, if it has non-negative scalar product $B(\lambda, \alpha) \geq 0$ with every positive root α. The basic example of a dominant weight is the half sum of positive roots:

$$\rho := \frac{1}{2} \sum_{\alpha \text{positive root}} \alpha \in \Lambda.$$

For every connected compact Lie group G the dominant weights are in bijective correspondence to the isomorphism classes of irreducible representations, up to isomorphism there is thus a unique representation R_λ of G of "highest" dominant weight $\lambda \in \Lambda$. The character of this representation is determined by a very useful formula called Weyl's Character Formula

$$A \operatorname{ch} R_\lambda = \sum_{w \in \mathfrak{W}} (-1)^{|w|} e^{w(\lambda+\rho)} \tag{11}$$

where the denominator A can be defined in two ways due to Weyl and Kostant respectively:

$$A := \sum_{w \in \mathfrak{W}} (-1)^{|w|} e^{w\rho} = \prod_{\alpha \text{positive root}} \left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right) \tag{12}$$

A direct consequence of Weyl's Character Formula is the following formula for the multiplicity

$$m_\lambda(R) := \dim \operatorname{Hom}_G(R_\lambda, R) = \operatorname{ev}_{\lambda+\rho}[A \operatorname{ch} R]$$

of the irreducible representation R_λ in an arbitrary finite-dimensional representation R.

Coming back to our branching problem we consider a pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras associated to a pair $G \supset H$ compact Lie groups. In this case we can find a regular element $X \in \mathfrak{h}$ for $\mathfrak{g}$ in $\mathfrak{h}$ and a fortiori get a pair $\mathfrak{t} \subset \mathfrak{s}$ of maximal tori for $\mathfrak{h}$ and $\mathfrak{g}$ respectively. By construction the restriction map $i\mathfrak{s}^* \to i\mathfrak{t}^*$ preserves positivity of linear forms in the sense $i\lambda(X) > 0$ and maps the weight lattice $\Lambda_\mathfrak{g} \subset i\mathfrak{s}^*$ to the weight lattice $\Lambda_\mathfrak{h}$ of $\mathfrak{h}$. In consequence the restriction map induces a ring homomorphism $\operatorname{res}\colon \mathbb{Z}\Lambda_\mathfrak{g} \to \mathbb{Z}\Lambda_\mathfrak{h}$, which maps the characters of representations of G to the characters of their restriction to H. For example the adjoint representations $\mathfrak{g}$ and $\mathfrak{h}$ of G and H both induce characters in $\mathbb{Z}\Lambda_\mathfrak{h}$ satisfying

$$\operatorname{res}(\operatorname{ch}\mathfrak{g}) - \operatorname{ch}\mathfrak{h} = \operatorname{ch}(\mathfrak{g}/\mathfrak{h}) = (\dim \mathfrak{s} - \dim \mathfrak{t}) + \sum_{\substack{\alpha \text{positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \left(e^{\alpha} + e^{-\alpha}\right)$$

where the right hand side is to be understood as a summation with multiplicities. Using Kostant's formula (12) for Weyl's denominator and the restriction homomorphism we find

$$e^{\rho_\mathfrak{h} - \operatorname{res}\rho_\mathfrak{g}} \frac{\operatorname{res} A_\mathfrak{g}}{A_\mathfrak{h}} = e^{\rho_\mathfrak{h} - \operatorname{res}\rho_\mathfrak{g}} \prod_{\substack{\alpha \text{positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right) = \prod_{\substack{\alpha \text{positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \left(1 - e^{-\alpha}\right)$$

and this expression can be inverted in a suitable completion of the group ring $\mathbb{Z}\Lambda_\mathfrak{h}$ to define

$$B_{\mathfrak{g}/\mathfrak{h}} := \left(e^{\operatorname{res}\rho_\mathfrak{g} - \rho_\mathfrak{h}} \frac{A_\mathfrak{h}}{\operatorname{res} A_\mathfrak{g}}\right)^* = \prod_{\substack{\alpha \text{positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \left(1 + e^{\alpha} + e^{2\alpha} + e^{3\alpha} + \cdots\right) \tag{13}$$

the universal branching formula for the pair $\mathfrak{g} \supset \mathfrak{h}$. The standard branching problem to find the multiplicity of $R^\mathfrak{h}_\mu$ in $R^\mathfrak{g}_\lambda$ is solved in terms of this universal branching

formula via

$$\begin{aligned}\dim \operatorname{Hom}_H(R^{\mathfrak{h}}_{\mu}, R^{\mathfrak{g}}_{\lambda}) &= \operatorname{ev}_{\mu+\rho_{\mathfrak{h}}}\big[A_{\mathfrak{h}}\operatorname{ch} R^{\mathfrak{g}}_{\lambda}\big]\\ &= \operatorname{ev}_{\mu+\operatorname{res}\rho_{\mathfrak{g}}}\Big[e^{\operatorname{res}\rho_{\mathfrak{g}}-\rho_{\mathfrak{h}}}\frac{A_{\mathfrak{h}}}{\operatorname{res} A_{\mathfrak{g}}}\sum_{w\in\mathfrak{W}_{\mathfrak{g}}}(-1)^{|w|}e^{\operatorname{res} w(\lambda+\rho_{\mathfrak{g}})}\Big]\\ &= \sum_{w\in\mathfrak{W}_{\mathfrak{g}}}(-1)^{|w|}\operatorname{ev}_{\mu+\operatorname{res}\rho_{\mathfrak{g}}-\operatorname{res} w(\lambda+\rho_{\mathfrak{g}})}\Big[e^{\operatorname{res}\rho_{\mathfrak{g}}-\rho_{\mathfrak{h}}}\frac{A_{\mathfrak{h}}}{\operatorname{res} A_{\mathfrak{g}}}\Big].\end{aligned}$$

For later reference we will write this branching formula in the following way:

$$\dim \operatorname{Hom}_H(R^{\mathfrak{h}}_{\mu}, R^{\mathfrak{g}}_{\lambda}) = \sum_{w\in\mathfrak{W}_{\mathfrak{g}}}(-1)^{|w|}\operatorname{ev}_{\operatorname{res}(w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}})-\mu}\big[B_{\mathfrak{g}/\mathfrak{h}}\big]. \tag{14}$$

According to the results of Section 2 we need to solve the branching problems for the pairs $\mathfrak{so}_{4,1}(\mathbb{R}) \supset \mathbb{R}\oplus i\,\mathbb{R}$ and $\mathfrak{so}_{4,1}(\mathbb{R}) \supset \mathbb{R}\oplus\mathfrak{sp}(1)$ respectively in order to turn Theorem 2.5 and Theorem 2.6 into effective decomposition formulas. Of course none of the algebras involved is the Lie algebra of a compact group, however the compact pairs $\mathrm{Sp}(2) \supset S^1\times S^1$ and $\mathrm{Sp}(2) \supset S^1\times \mathrm{Sp}(1)$ have on the level of Lie algebras isomorphic complexifications

$$\mathfrak{sp}(2)\otimes_{\mathbb{R}}\mathbb{C} \supset (i\,\mathbb{R}\oplus i\,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \quad\text{and}\quad \mathfrak{sp}(2)\otimes_{\mathbb{R}}\mathbb{C} \supset (i\,\mathbb{R}\oplus\mathfrak{sp}(1))\otimes_{\mathbb{R}}\mathbb{C},$$

and this is sufficient to ensure that we can apply the branching formula (14) for the problems at hand as well. The maximal torus $S^1\times S^1 \subset S^1\times \mathrm{Sp}(1) \subset \mathrm{Sp}(2)$ is evidently the same for all groups and so the restriction is simply the identity. The root

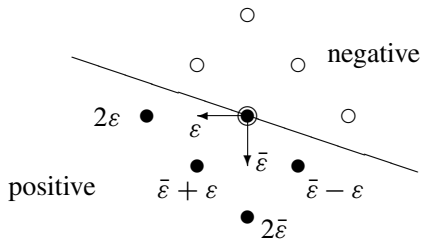


diagram (9) of $\mathfrak{so}_{4,1}(\mathbb{R})$ tells us that the positive roots of the pair $\mathfrak{so}_{4,1}(\mathbb{R}) \supset \mathbb{R}\oplus\mathfrak{sp}(1)$ are $2\bar\varepsilon$, $\bar\varepsilon+\varepsilon$ and 2ε, the positive root $\bar\varepsilon-\varepsilon$ corresponds to $\overline{\mathrm{Pl}} \in \mathfrak{sp}(1)\otimes_{\mathbb{R}}\mathbb{C}$. Hence the branching formula reads as follows:

$$\begin{aligned}&B_{\mathfrak{so}_{4,1}(\mathbb{R})/\mathbb{R}\oplus\mathfrak{sp}(1)}\\ &\quad= \big(1+e^{\bar\varepsilon+\varepsilon}+e^{2\bar\varepsilon+2\varepsilon}+\cdots\big)\big(1+e^{2\varepsilon}+e^{4\varepsilon}+\cdots\big)\big(1+e^{2\bar\varepsilon}+e^{4\bar\varepsilon}+\cdots\big)\\ &\quad= \sum_{k,\bar k\geq 0}\sharp\{r \mid 0\leq r\leq \min\{k,\bar k\} \text{ and } r\equiv k\equiv \bar k \bmod 2\}e^{k\varepsilon+\bar k\bar\varepsilon}.\end{aligned}$$

Evaluation of this power series in $e^{\varepsilon}, e^{\bar{\varepsilon}}$ in the weight $\bar{k}\bar{\varepsilon}+k\varepsilon$ defines the function

$$\begin{aligned} B_{\bar{k},k} &:= \mathrm{ev}_{\bar{k}\bar{\varepsilon}+k\varepsilon}\big[B_{\mathfrak{so}_{4,1}(\mathbb{R})/\mathbb{R}\oplus\mathfrak{sp}(1)}\big] \\ &= \begin{cases} 1+\lfloor\frac{\min\{\bar{k},k\}}{2}\rfloor & \text{if } k,\bar{k}\in\mathbb{N}_0 \text{ and } k\equiv\bar{k} \bmod 2, \\ 0 & \text{for all other arguments.} \end{cases} \end{aligned}$$

Consider now a tuple $(r,\bar{r})$ of integers satisfying $0\le\bar{r}\le r\le n$ with associated dominant weight $\lambda := (n-\bar{r})\bar{\varepsilon}+(n-r)\varepsilon$ in the Weyl chamber indicated in diagram (9). For this choice of Weyl chamber the half sum of positive roots is $\rho_{\mathfrak{so}_{4,1}(\mathbb{R})} = 2\bar{\varepsilon}+\varepsilon$ and the affine Weyl orbit of λ under the Weyl group $\mathfrak{W}_{\mathfrak{so}_{4,1}(\mathbb{R})} = S_2\ltimes\mathbb{Z}_2^2$ of $\mathrm{SO}_{4,1}(\mathbb{R})$ is readily calculated:

$(-1)^{\lvert w\rvert}$	w	$w(\lambda+\rho_{\mathfrak{so}_{4,1}(\mathbb{R})})-\rho_{\mathfrak{so}_{4,1}(\mathbb{R})}$
$+1$	id	$(+n-\bar{r})\bar{\varepsilon}+(+n-r)\varepsilon$
-1	τ_1	$(+n-r-1)\bar{\varepsilon}+(+n-\bar{r}+1)\varepsilon$
-1	τ_2	$(+n-\bar{r})\bar{\varepsilon}+(-n+r-2)\varepsilon$
$+1$	$\tau_1\tau_2$	$(-n+r-3)\bar{\varepsilon}+(+n-\bar{r}+1)\varepsilon$
$+1$	$\tau_2\tau_1$	$(+n-r-1)\bar{\varepsilon}+(-n+\bar{r}-3)\varepsilon$
-1	$\tau_2\tau_1\tau_2$	$(-n+r-3)\bar{\varepsilon}+(-n+\bar{r}-3)\varepsilon$
-1	$\tau_1\tau_2\tau_1$	$(-n+\bar{r}-4)\bar{\varepsilon}+(+n-r)\varepsilon$
$+1$	$\tau_1\tau_2\tau_1\tau_2$	$(-n+\bar{r}-4)\bar{\varepsilon}+(-n+r-2)\varepsilon$

Under the rather superficial assumption $0\le k\le 2n$ on the degree of the differential forms considered the highest weight $\mu = (n-\frac{k-s}{2})\bar{\varepsilon}+(n-\frac{k+s}{2})\varepsilon$ of the corresponding representation $\mathbb{C}_{(n-\frac{k}{2})(\bar{\varepsilon}+\varepsilon)}\otimes\mathrm{Sym}^s\mathbb{H}$ of $\mathbb{R}\oplus\mathfrak{sp}(1)$ has $\bar{\varepsilon}$-coefficient and coefficient sum positive or zero. For this reason the five Weyl group elements w, which have $\bar{\varepsilon}$-coefficient or coefficient sum of $w(\lambda+\rho_{\mathfrak{so}_{4,1}(\mathbb{R})})-\rho_{\mathfrak{so}_{4,1}(\mathbb{R})}$ negative, can not contribute to the summation in the branching formula (14) and we are left with the three summands corresponding to id and the reflections $\tau_1\colon\bar{\varepsilon}\mapsto\varepsilon$, $\varepsilon\mapsto\bar{\varepsilon}$ and $\tau_2\colon\bar{\varepsilon}\mapsto\bar{\varepsilon}$, $\varepsilon\mapsto-\varepsilon$ along the simple roots:

$$\begin{aligned} &\dim\mathrm{Hom}_{\mathbb{R}\oplus\mathfrak{sp}(1)}\Big(R^{\mathbb{R}\oplus\mathfrak{sp}(1)}_{(n-\frac{k-s}{2})\bar{\varepsilon}+(n-\frac{k+s}{2})\varepsilon}, R^{\mathfrak{so}_{4,1}(\mathbb{R})}_{(n-\bar{r})\bar{\varepsilon}+(n-r)\varepsilon}\Big) \\ &\quad = B_{\frac{k-s}{2}-\bar{r},\frac{k+s}{2}-r} - B_{\frac{k-s}{2}-r-1,\frac{k+s}{2}-\bar{r}+1} - B_{\frac{k-s}{2}-\bar{r},\frac{k+s}{2}+r-2n-2}. \end{aligned} \tag{15}$$

The other branching problem from $\mathfrak{so}_{4,1}(\mathbb{R})$ to $\mathbb{R}\oplus i\mathbb{R}$ can be treated similarly, the main difference is the positive root $\bar{\varepsilon}-\varepsilon$. The evaluation of the universal branching formula

$$B_{\mathfrak{so}_{4,1}(\mathbb{R})/\mathbb{R}\oplus i\mathbb{R}} = \big(1+e^{\bar{\varepsilon}-\varepsilon}+e^{2\bar{\varepsilon}-2\varepsilon}+\cdots\big)\,B_{\mathfrak{so}_{4,1}(\mathbb{R})/\mathbb{R}\oplus\mathfrak{sp}(1)}$$

at a weight $\bar{k}\bar{\varepsilon} + k\varepsilon$ however defines a significantly more involved function

$$\widehat{B}_{\bar{k},k} = \begin{cases} \lfloor \frac{(\bar{k}+2)^2}{4} \rfloor & \text{if } k \geq \bar{k} \geq 0 \text{ and } k \equiv \bar{k} \bmod 2, \\ \lfloor \frac{(\bar{k}+2)^2}{4} \rfloor - \frac{(\bar{k}-k)(\bar{k}-k+2)}{8} & \text{if } \bar{k} \geq k \geq 0 \text{ and } k \equiv \bar{k} \bmod 2, \\ \lfloor \frac{(\bar{k}+2)^2}{4} \rfloor + \frac{(\bar{k}+k+2)(\bar{k}+k+4)}{8} & \text{if } 0 \geq k \geq -\bar{k} \text{ and } k \equiv \bar{k} \bmod 2, \\ 0 & \text{in all other cases,} \end{cases}$$

which essentially describes the coefficients of the denominator of Weyl's Character Formula for $\mathfrak{so}_{4,1}(\mathbb{R})$ and is closely related to Kostant's polynomial [2]. Note in particular that $\widehat{B}_{\bar{k},k}$ vanishes in case the sum $k + \bar{k}$ of its two arguments is negative. For a weight μ with non-negative coefficient sum we can thus ignore the four Weyl group elements w in the branching formula (14), which see $w(\lambda+\rho_{\mathfrak{so}_{4,1}(\mathbb{R})})-\rho_{\mathfrak{so}_{4,1}(\mathbb{R})}$ having negative coefficient sum:

$$\begin{aligned} &\dim \operatorname{Hom}_{\mathbb{R}\oplus i\mathbb{R}}\Big(R^{\mathbb{R}\oplus i\mathbb{R}}_{(n-\bar{k})\bar{\varepsilon}+(n-k)\varepsilon}, R^{\mathfrak{so}_{4,1}(\mathbb{R})}_{(n-\bar{r})\bar{\varepsilon}+(n-r)\varepsilon}\Big) \\ &\quad = \widehat{B}_{\bar{k}-\bar{r},k-r} - \widehat{B}_{\bar{k}-r-1,k-\bar{r}+1} - \widehat{B}_{\bar{k}-\bar{r},k+r-2n-2} + \widehat{B}_{\bar{k}+r-2n-3,k-\bar{r}+1} \end{aligned} \tag{16}$$

Since $\mathbb{R} \oplus i\,\mathbb{R}$ is essentially the maximal torus of $\mathfrak{so}_{4,1}(\mathbb{R})$, this formula may be seen as a formula for the dimension of the weight spaces in an arbitrary representation of $\mathfrak{so}_{4,1}(\mathbb{R})$. In the context of hyper-Kähler manifolds formula (16) together with Lemma 2.5 calculates the multiplicity of the irreducible representation $\Lambda^{r,\bar{r}}_{\circ} E$ of $\operatorname{Sp} E$ in the differential forms of bidegree $(k, \bar{k})$ provided $0 \leq k + \bar{k} \leq 2n$. Instead of persuing this idea any further we turn to:

Theorem 3.1 (Explicit formulae for the multiplicities). *The complete decomposition of the complexified differential forms $\Lambda^k T^* \otimes_{\mathbb{R}} \mathbb{C} \cong \Lambda^k(H^* \otimes E^*)$ up to middle dimension $2n \geq k \geq 0$ on the tangent representation T of the holonomy group $\operatorname{Sp} H \cdot \operatorname{Sp} E$ of quaternionic Kähler geometry into irreducible subrepresentations reads*

$$\Lambda^k(H^* \otimes E^*) = \bigoplus_{\substack{s\geq 0 \\ n\geq r\geq \bar{r}\geq 0}} m_{s,r,\bar{r}}(k) \operatorname{Sym}^s H \otimes \Lambda^{r,\bar{r}}_{\circ} E$$

where the multiplicities $m_{s,r,\bar{r}}(k)$ are zero unless both $s \equiv k \equiv r + \bar{r}$ modulo 2 and the parameters $s, r, \bar{r}$ satisfy the defining inequalities of at least one of the following three cases.

First case. *The multiplicities in the first case defined by $\bar{r} \leq \frac{k-s}{2} \leq r \leq \frac{k+s}{2} < 2n + 2 - r$ are*

$$m_{s,r,\bar{r}}(k) = 1 + \left\lfloor \frac{k - s - 2r + 2\min\{s, r - \bar{r}\}}{4} \right\rfloor > 0.$$

Second case. *The second case is characterized by $\frac{k-s}{2} > r$, depending on whether the common parity of $k \equiv s \equiv r - \bar{r}$ mod 2 is even or odd the multiplicities in this*

case are given by

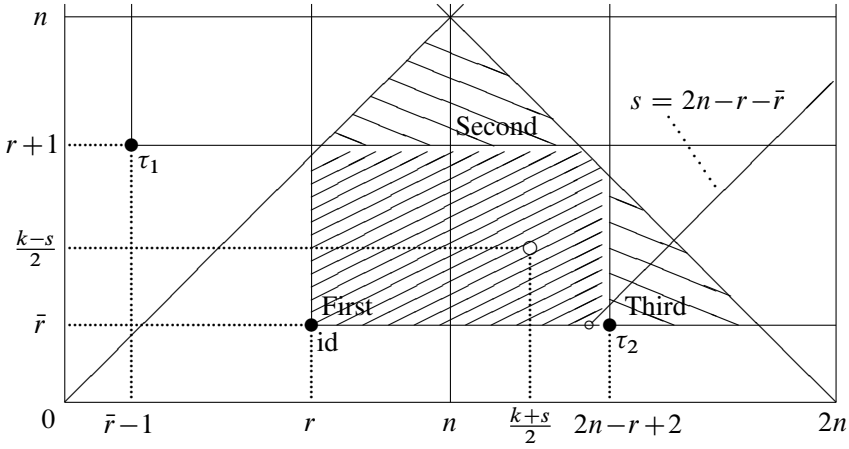
$$m_{s,r,\bar r}(k) = \frac{\min\{r-\bar r, s\}}{2} + \delta_{\frac{k-s}{2}\equiv r(2)} \quad or \quad m_{s,r,\bar r}(k) = \frac{\min\{r-\bar r, s\}+1}{2}$$

respectively, where the δ-summand equals 1 *if* $\frac{k-s}{2} \equiv r \bmod 2$ *and* 0 *otherwise. These multiplicities are strictly positive unless k is even, $s = 0$ or $r = \bar r$ and* $\frac{k-s}{2} \not\equiv r \bmod 2$.

Third case. *Characterized by* $\frac{k+s}{2} \geq 2n-r \geq s+\bar r$ *the third case has strictly positive multiplicities:*

$$m_{s,r,\bar r}(k) = 1 + n - r + \frac{\min\{r-\bar r, s\} - s}{2} > 0.$$

The theorem is a good example of how a seemingly simple sum (15) of three terms may lead to a messy case distinction. Although it is natural to believe that there is a different, simpler way to present the result, no such simplification has been found by the author and the hexagonal pattern (1) of the multiplicities $m_{s,r,\bar r}(k)$ suggests that no simpler presentation exists. The reader interested in the details of the proof is invited to study the diagram first which gives a geometric interpretation of the inequalities in Theorem 3.1. The black points represent the images of the highest weight $\lambda = (n-\bar k)\bar\varepsilon + (n-k)\varepsilon$ under the Weyl group elements id, τ_1



(17)

and τ_2 contributing to (15). The support of the corresponding B-summand considered as functions of $(\frac{k+s}{2}, \frac{k-s}{2})$ is a translate of the first quadrant based at this point. The case distinction thus arises from sectors, where some parts of formula (15) vanish.

Proof of Theorem 3.1. To begin with we note that the arguments of the B-summands in formula (15) are integers of the same parity if and only f $k \equiv s \equiv r + \bar r$ modulo 2, hence this condition is certainly necessary to have a positive multiplicity. Fixing a degree $0 \leq k \leq 2n$ and $s \geq 0$ we note that the corresponding point $(\frac{k+s}{2}, \frac{k-s}{2})$ lies in

the big triangle of diagram (17) or below, hence this point lies in the support of some of the summands of formula (15) if and only if $\frac{k+s}{2} \geq r$ and $\frac{k-s}{2} \geq \bar{r}$. The stronger assumptions of the first case $r \leq \frac{k+s}{2} < 2n - r + 2$ and $\bar{r} \leq \frac{k-s}{2} \leq r$ correspond exactly to the rectangle denoted "First" in diagram (17). In this rectangle only the summand corresponding to id contributes to (15) and we obtain

$$\begin{aligned} B_{\frac{k-s}{2}-\bar{r},\frac{k+s}{2}-r} &= 1 + \left\lfloor \frac{\min\{k-s-2\bar{r}, k+s-2r\}}{4} \right\rfloor \\ &= 1 + \left\lfloor \frac{k-s-2r+2\min\{r-\bar{r}, s\}}{4} \right\rfloor. \end{aligned}$$

Under our standing assumptions $0 \leq k \leq 2n$ and $s \geq 0$ the assumption $\frac{k-s}{2} > r$ characterizing the second case corresponds to the triangle denoted "Second" in diagram (17), which lies in the support of the two summands corresponding to id and τ_1 in formula (15):

$$\begin{aligned} &B_{\frac{k-s}{2}-\bar{r},\frac{k+s}{2}-r} - B_{\frac{k-s}{2}-r-1,\frac{k+s}{2}-\bar{r}+1} \\ &\quad = \left\lfloor \frac{k-s-2r+2\min\{r-\bar{r}, s\}}{4} \right\rfloor - \frac{k-s-2r-2-2\delta_{\frac{k-s}{2}\equiv r(2)}}{4} \\ &\quad = \left\lfloor \frac{\min\{r-\bar{r}, s\}+1+\delta_{\frac{k-s}{2}\equiv r(2)}}{2} \right\rfloor. \end{aligned}$$

Evidently the δ-term will have no bearing on the result in case $s \equiv r - \bar{r}$ is odd, for even $s \equiv r - \bar{r}$ on the other hand it will take us to the next even integer or not. Changing the assumptions of the remaining third case slightly we observe that the conditions $0 \leq k \leq 2n$ and $\frac{k+s}{2} \geq 2n - r + 2$ characterize points in the triangle denoted "Third" or below. In this region the two non-trivial summands in formula (15) will cancel each other in points below the diagonal $s = 2n - r - \bar{r}$. According to the branching formula (15) the multiplicities in the points $(\frac{k+s}{2}, \frac{k-s}{2})$ on or above this diagonal $s \leq 2n - r - \bar{r}$ can be calculated as

$$\begin{aligned} &B_{\frac{k-s}{2}-\bar{r},\frac{k+s}{2}-r} - B_{\frac{k-s}{2}-\bar{r},\frac{k+s}{2}-2n+r-2} \\ &\quad = \left\lfloor \frac{k-s-2r+2\min\{r-\bar{r}, s\}}{4} \right\rfloor - \frac{k+s-4n+2r-4-2\delta_{\frac{k+s}{2}\not\equiv r(2)}}{4} \\ &\quad = n+1-r+\left\lfloor \frac{\min\{r-\bar{r}, s\}-s+\delta_{\frac{k+s}{2}\not\equiv r(2)}}{2} \right\rfloor \end{aligned}$$

However $s \equiv r - \bar{r}$ modulo 2 by our standing assumption, hence $\min\{r - \bar{r}, s\} - s$ is even and we can safely trade the δ-summand for skipping to round down. □

Of course the stated positivity of the multiplicities in the different cases of Theorem 3.1 is of particular importance in applications, because in this way it is possibly to locate exactly those degrees, in which a representation $\mathrm{Sym}^s H \otimes \Lambda_\circ^{r,\bar{r}} E$ definitely

does occur in the differential forms. In particular the exception k even, $s = 0$ or $r = \bar{r}$ and $\frac{k-s}{2} \not\equiv r \bmod 2$ in the second case is relevant for the Betti numbers of quaternionic Kähler manifolds, in that it allows the Betti numbers to increase in steps of 4 only instead of the expected 2.

More precisely it has been shown in [3] that on compact quaternionic Kähler manifolds of positive scalar curvature $\kappa > 0$ every harmonic form is a sum of harmonic forms of types $\Lambda^{r,r}_{\circ} E$ with $n \geq r \geq 0$, while on a compact quaternionic Kähler manifold with negative scalar curvature $\kappa < 0$ the basic harmonic forms can be of the two different types $\Lambda^{r,r}_{\circ} E, n \geq r \geq 0$, and $\mathrm{Sym}^{2n-r-\bar{r}} H \otimes \Lambda^{r,\bar{r}}_{\circ} E$ with $n \geq r \geq \bar{r} \geq 0$. In this context Theorem 3.1 confirms the central conclusion of [3] about the degrees and the multiplicities of general differential forms and thus about the degrees of harmonic forms of type $\Lambda^{r,r}_{\circ} E$ and $\mathrm{Sym}^{2n-r-\bar{r}} H \otimes \Lambda^{r,\bar{r}}_{\circ} E$ in the differential forms. The representation $\Lambda^{r,r}_{\circ} E$ occurs with multiplicity 1 in the forms of degrees $k = 2r, 2r+4, 2r+8, \ldots, 4n-2r$, while the representation $\mathrm{Sym}^{2n-r-\bar{r}} H \otimes \Lambda^{r,\bar{r}}_{\circ} E$ occurs with multiplicity 1 in the forms of degrees $k = 2n-r+\bar{r}, 2n-r+\bar{r}+2, \ldots, 2n+r-\bar{r}$.

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Chapter 3

Sasakian geometry, holonomy, and supersymmetry

Charles P. Boyer and Krzysztof Galicki

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1 Introduction

Supersymmetry has emerged in physics as an attempt to unify the way physical theories deal with bosonic and fermionic particles. Since its birth around the early '70s it has come to dominate theoretical high energy physics (for a historical perspective see [86] with the introduction by Kane and Shifman, and for a mathematical treatment see [132]). This dominance is still ongoing in spite of the fact that after almost 40 years there is no single experimental evidence that would directly and convincingly "prove" or "discover" the existence of supersymmetry in nature. On the other hand, especially in the last 20 years, supersymmetry has given birth to many beautiful mathematical theories. Gromov–Witten theory, Seiberg–Witten theory, Rozansky–Witten theory as well as the Mirror Duality Conjecture are just a few of the more famous examples of important and deep mathematics having its origins in the physics of various supersymmetric theories.

Various supersymmetric field theories naturally include both Riemannian and pseudo-Riemannian manifolds. The latter is necessary in order to incorporate the physical space-time into the picture while the former typically describes the geometry associated with 'invisible' extra dimensions. It is mainly in such a context that Sasaki–

Einstein manifolds appear in physics: they are compact Einstein manifolds of positive scalar curvature that occur in abundance in the physically interesting dimensions five and seven. Moreover, when they are simply connected they admit real Killing spinors. It is this last property that vitally connects them to supergravity, superstring, and M-theory.

The main purpose of this chapter is to describe geometric properties of Sasaki–Einstein manifolds which make them interesting in modern theoretical physics. In spite of the fact that it is *supersymmetry* that connects Sasaki–Einstein spaces to physics, it is not the purpose of this review to describe what this concept really means to either physicists or mathematicians. There have been many recent attempts to frame these important notions of theoretical physics in precise mathematical terms. This enormous task is far beyond the scope of this chapter, so we refer the reader to recent monographs and references therein [50], [132], [81], [2]. Here we content ourselves with providing the main theorems and results concerning Killing spinors.

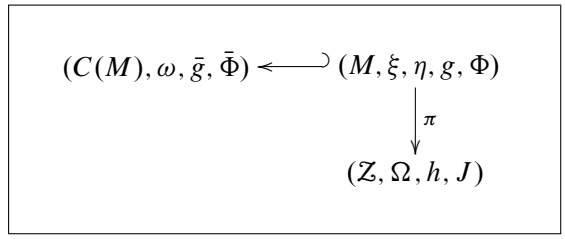
It is most remarkable that, even though Sasaki–Einstein manifolds always have holonomy $\mathrm{SO}(TM)$, *i.e.*, the holonomy of any generic Riemannian metric, they are far from being generic. In fact, the most interesting thing about this geometry is that it naturally relates to several different Riemannian geometries with reduced holonomies. It is this point that we will try to stress throughout this chapter. For more detailed exposition we refer the interested reader to our recent monograph on *Sasakian geometry* [26].

The key to understanding the importance of Sasakian geometry is through its relation to Kählerian geometry. Before we define Sasakian manifolds and describe some of their elementary properties in Section 3 let us motivate things in the more familiar context of contact and symplectic manifolds. These two provide the mathematical foundations of Lagrangian and Hamiltonian Mechanics. Let (M, η, ξ) be a contact manifold where η is a contact form on M and ξ is its Reeb vector field. It is easy to see that the cone $(C(M) = \mathbb{R}_+ \times M, \omega = d(t\eta))$ is symplectic. Likewise, the Reeb field defines a foliation of M and the transverse space $\mathcal{Z}$ is also symplectic. When the foliation is regular the transverse space is a smooth symplectic manifold giving a projection π called Boothby–Wang fibration, and $\pi^*\Omega = d\eta$ relates the contact and the symplectic structures as indicated by

$$\begin{array}{ccc} (C(M), \omega) & \hookleftarrow & (M, \eta, \xi) \\ & & \downarrow \pi \\ & & (\mathcal{Z}, \Omega) \end{array}$$

We do not have any Riemannian structure yet. It is quite reasonable to ask if there is a Riemannian metric g on M which "best fits" into the above diagram. As the preferred metrics adapted to symplectic forms are Kähler metrics one could ask for

the Riemannian structure which would make the cone with the metric $\bar{g} = dt^2 + t^2 g$ together with the symplectic form ω into a Kähler manifold. Then $\bar{g}$ and ω define a complex structure $\bar{\Phi}$. Alternatively, one could ask for a Riemannian metric g on M which would define a Kähler metric h on $\mathcal{Z}$ via a Riemannian submersion. Surprisingly, in both cases the answer to these questions leads naturally and uniquely to *Sasakian geometry*. Our diagram becomes



From this point of view it is quite clear that Kählerian and Sasakian geometries are inseparable, Sasakian geometry being naturally *sandwiched* between two different types of Kählerian geometry.

2 Cones, holonomy, and Sasakian geometry

As we have just described Sasakian manifolds can and will be (cf. Theorem–Definition 10) defined as bases of metric cones which are Kähler. Let us begin with the following more general

Definition 1. For any Riemannian metric g_M on M, the *warped product metric* on $C(M) = \mathbb{R}^+ \times M$ is the Riemannian metric defined by

$$g = dr^2 + \phi^2(r) g_M,$$

where $r \in \mathbb{R}^+$ and $\phi = \phi(r)$ is a smooth function, called the *warping function*. If $\phi(r) = r$ then $(C(M), g)$ is simply called the *Riemannian cone* or *metric cone* on M. If $\phi(r) = \sin r$ then $(C(M), g)$ is called the *sine-cone* on M.

The relevance of sine-cones will become clear later while the importance of metric cones in relation to the Einstein metrics can be summarized by the following fundamental

Lemma 2. *Let (M, g) be a Riemannian manifold of dimension n, and consider $(C(M) = M \times \mathbb{R}^+, \bar{g})$ the cone on M with metric $\bar{g} = dr^2 + r^2 g$. Then if $\bar{g}$ is Einstein, it is Ricci-flat, and $\bar{g}$ is Ricci-flat if and only if g is Einstein with Einstein constant $n - 1$.*

Interestingly, there is a similar lemma about sine-cone metrics.

Lemma 3. *Let (M^n, g) be an Einstein manifold with Einstein constant $n-1$ and consider $(C_s(M) = M \times (0,\pi), \bar{g}_s)$ the sine-cone on M with metric $\bar{g}_s = dr^2 + (\sin^2 r)g$. Then $\bar{g}_s$ is Einstein with Einstein constant n.*

It is well known that one characterization of Kählerian geometry is via the holonomy reduction. We now recall some basic facts about the holonomy groups of irreducible Riemannian manifolds. Let (M, g) be a Riemannian manifold and consider parallel translation defined by the Levi-Civita connection and its associated holonomy group which is a subgroup of the structure group $O(n, \mathbb{R})$ ($\mathrm{SO}(n, \mathbb{R})$ in the oriented case). Since this connection ∇^g is uniquely associated to the metric g, we denote it by $\mathrm{Hol}(g)$, and refer to it as the *Riemannian holonomy* group or just the *holonomy group* when the context is clear. Indeed, it is precisely this Riemannian holonomy that plays an important role here. Now on a Riemannian manifold (M, g) there is a canonical epimorphism $\pi_1(M) \to \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$, in particular, if $\pi_1(M) = 0$ then $\mathrm{Hol}(g) = \mathrm{Hol}^0(g)$. In 1955 Berger proved the following theorem [16] concerning Riemannian holonomy:

Theorem 4. *Let (M, g) be an oriented Riemannian manifold which is neither locally a Riemannian product nor locally symmetric. Then the restricted holonomy group $\mathrm{Hol}^0(g)$ is one of the groups listed in the following table.*

Table 1. *Berger's Riemannian holonomy groups.*

$\mathrm{Hol}^0(g)$	$\dim(M)$	*Geometry of M*	*Comments*
$\mathrm{SO}(n)$	n	*orientable Riemannian*	*generic Riemannian*
$\mathrm{U}(n)$	$2n$	*Kähler*	*generic Kähler*
$\mathrm{SU}(n)$	$2n$	*Calabi–Yau*	*Ricci-flat Kähler*
$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$	$4n$	*quaternionic Kähler*	*Einstein*
$\mathrm{Sp}(n)$	$4n$	*hyper-Kähler*	*Ricci-flat*
G_2	7	G_2*-manifold*	*Ricci-flat*
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$*-manifold*	*Ricci-flat*

Originally Berger's list included $\mathrm{Spin}(9)$, but Alekseevsky proved that any manifold with such holonomy must be symmetric [5]. In the same paper Berger also claimed a classification of all holonomy groups of torsion-free affine (linear) connections that act irreducibly. He produced a list of possible holonomy representations up to what he claimed was a finite number of exceptions. But his classification had some gaps discovered 35 years later by Bryant [34]. An infinite series of exotic holonomies was found in [42] and finally the classification in the non-Riemannian affine case was completed by Merkulov and Schwachhöfer [107]. We refer the reader to [107] for

the proof, references and the history of the general affine case. In the Riemannian case a new geometric proof of Berger's Theorem is now available [115]. An excellent review of the subject just prior to the Merkulov and Schwachhöfer's classification can be found in [35]. We should add that one of the first non-trivial results concerning manifolds with the exceptional holonomy groups of the last two rows of Table 1 is due to Bonan [22] who established Ricci-flatness of manifolds with parallel spinors.

Manifolds with reduced holonomy have always been very important in physics. Partly because Calabi–Yau, hyper-Kähler, quaternionic Kähler, G_2 and Spin(7) manifolds are automatically Einstein. In addition, all of these spaces appear as σ-model geometries in various supersymmetric models. What is perhaps less known is that all of these geometries are also related, often in more than one way, to Sasakian structures of various flavors. Let us list all such known relations.

- **SO(n)-holonomy.** As remarked this is holonomy group of a generic metric on an oriented Riemannian manifold (M^n, g). As we shall see Sasaki–Einstein metrics necessarily have maximal holonomy.
- **U(n)-holonomy and Kähler geometry**
  - (i) Metric cone on a *Sasakian* manifold is *Kähler*.
  - (ii) Transverse geometry of a *Sasakian* manifold is *Kähler*.
  - (iii) Transverse geometry of a *positive Sasakian* manifold is *Fano*.
  - (iv) Transverse geometry of a *Sasaki–Einstein* manifold is *Fano* and *Kähler–Einstein* of *positive* scalar curvature.
  - (v) Transverse geometry of a *negative Sasakian* manifold is *canonical* in the sense that the transverse canonical bundle is ample.
  - (vi) Transverse geometry of a 3-*Sasakian* manifold is a *Kähler–Einstein* with a *complex contact structure*, *i.e., twistor geometry*.
- **SU(n)-holonomy and Calabi–Yau geometry**
  - (i) Metric cone on a *Sasaki–Einstein* manifold is *Calabi–Yau*.
  - (ii) Transverse geometry of a *null Sasakian* manifold is *Calabi–Yau*.
- **Sp(n)Sp(1)-holonomy and Quaternionic Kähler geometry**
  - (i) Transverse geometry of the 3-dimensional foliation of a 3-*Sasakian* manifold is *quaternionic-Kähler* of *positive* scalar curvature.
  - (ii) 3-*Sasakian* manifolds occur as conformal infinities of complete *quaternionic Kähler* manifolds of *negative* scalar curvature.
- **Sp(n)-holonomy and hyper-Kähler geometry**
  - (i) Metric cone on a 3-*Sasakian* manifold is *hyper-Kähler*.
  - (ii) Transverse geometry of a *null Sasakian* manifold with some additional structure is *hyper-Kähler*.

- **G_2-holonomy**
  - (i) The *'squashed' twistor space of a* 3-*Sasakian* 7-manifold is *nearly Kähler*; hence, the metric cone on it has holonomy inside G_2.
  - (ii) *Sine-cone* on a Sasaki–Einstein 5-manifold is *nearly Kähler*; hence, its *metric cone* has holonomy inside G_2.
- **Spin(7)-holonomy**
  - (i) The *'squashed'* 3-*Sasakian* 7-manifold has a nearly parallel G_2-structure; hence, its metric cone has holonomy in Spin(7).
  - (ii) *Sine-cone* on a *'squashed' twistor space of a* 3-*Sasakian* 7-manifold has a *nearly parallel G_2 structure*; hence, its *metric cone* has holonomy inside Spin(7).
  - (iii) *Sine cone on a sine cone* on a 5-dimensional Sasaki–Einstein base has a *nearly parallel G_2-structure*; hence, its *metric cone* has holonomy inside Spin(7).

Note that Sasakian manifolds are related to various other geometries in two very distinct ways. On one hand we can take a Sasakian (Sasaki–Einstein, 3-Sasakian, etc.) manifold and consider its metric or sine-cone. These cones frequently have interesting geometric properties and reduced holonomy. On the other hand, a Sasakian manifold is always naturally foliated by one-dimensional leaves (three-dimensional leaves in addition to the one-dimensional canonical foliation when the manifold is 3-Sasakian) and we can equally well consider the transverse geometries associated to such fundamental foliations. These too have remarkable geometric properties including reduced holonomy. In particular, Sasakian manifolds are not just related to all of the geometries on Berger's holonomy list, but more importantly, they provide a *bridge* between the different geometries listed there. We will investigate some of these bridges in the next two sections.

3 Sasakian and Kählerian geometry

Definition 5. A $(2n+1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η, called a *contact* 1-*form*, on M such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A *contact structure* on M is an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.

Lemma 6. *On a contact manifold (M, η) there is a unique vector field ξ, called the* ***Reeb vector field****, satisfying the two conditions*

$$\xi \lrcorner \eta = 1, \quad \xi \lrcorner d\eta = 0.$$

Definition 7. An *almost contact structure* on a differentiable manifolds M is a triple (ξ, η, Φ), where Φ is a tensor field of type $(1,1)$ (*i.e.*, an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfy

$$\eta(\xi) = 1 \quad \text{and} \quad \Phi \circ \Phi = -\mathbb{1} + \xi \otimes \eta,$$

where $\mathbb{1}$ is the identity endomorphism on TM. A smooth manifold with such a structure is called an *almost contact manifold.*

Remark 8. The reader will notice from Definitions 5 and 7 that an almost contact structure actually has more structure than a contact structure! This is in stark contrast to the usual relationship between a structure and its 'almost structure'; however, we feel that the terminology is too well ensconced in the literature to be changed at this late stage.

Let (M, η) be a contact manifold with a contact 1-form η and consider $\mathcal{D} = \ker \eta \subset TM$. The subbundle $\mathcal{D}$ is maximally non-integrable and it is called the *contact distribution*. The pair $(\mathcal{D}, \omega)$, where ω is the restriction of $d\eta$ to $\mathcal{D}$ gives $\mathcal{D}$ the structure of a symplectic vector bundle. We denote by $\mathcal{J}(\mathcal{D})$ the space of all almost complex structures J on $\mathcal{D}$ that are compatible with ω, that is the subspace of smooth sections J of the endomorphism bundle $\mathrm{End}(\mathcal{D})$ that satisfy

$$J^2 = -\mathbb{1}, \quad d\eta(JX, JY) = d\eta(X, Y), \quad d\eta(JX, X) > 0 \tag{1}$$

for any smooth sections X, Y of $\mathcal{D}$. Notice that each $J \in \mathcal{J}(\mathcal{D})$ defines a Riemannian metric $g_{\mathcal{D}}$ on $\mathcal{D}$ by setting

$$g_{\mathcal{D}}(X, Y) = d\eta(JX, Y). \tag{2}$$

One easily checks that $g_{\mathcal{D}}$ satisfies the compatibility condition $g_{\mathcal{D}}(JX, JY) = g_{\mathcal{D}}(X, Y)$. Furthermore, the map $J \mapsto g_{\mathcal{D}}$ is one-to-one, and the space $\mathcal{J}(\mathcal{D})$ is contractible. A choice of J gives M an almost CR structure. Moreover, by extending J to all of TM one obtains an almost contact structure. There are some choices of conventions to make here. We define the section Φ of $\mathrm{End}(TM)$ by $\Phi = J$ on $\mathcal{D}$ and $\Phi\xi = 0$, where ξ is the Reeb vector field associated to η. We can also extend the transverse metric $g_{\mathcal{D}}$ to a metric g on all of M by

$$g(X, Y) = g_{\mathcal{D}} + \eta(X)\eta(Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y) \tag{3}$$

for all vector fields X, Y on M. One easily sees that g satisfies the compatibility condition $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$.

Definition 9. A contact manifold M with a contact form η, a vector field ξ, a section Φ of $\mathrm{End}(TM)$, and a Riemannian metric g which satisfy the conditions

$$\eta(\xi) = 1, \quad \Phi^2 = -\mathbb{1} + \xi \otimes \eta,$$
$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

is known as a *metric contact structure* on M.

Definition–Theorem 10. *A Riemannian manifold (M, g) is called a* ***Sasakian manifold*** *if any one, hence all, of the following equivalent conditions hold:*

(i) *There exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = -\nabla_X \xi$, satisfies the condition*

$$(\nabla_X \Phi)(Y) = g(X,Y)\xi - g(\xi,Y)X$$

for any pair of vector fields X and Y on M.

(ii) *There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition*

$$R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi,$$

for any pair of vector fields X and Y on M.

(iii) *The metric cone $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ is Kähler.*

We refer to the quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ as a *Sasakian structure* on M, where η is the 1-form dual vector field ξ. It is easy to see that η is a contact form whose Reeb vector field is ξ. In particular $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a special type of *metric contact structure*.

The vector field ξ is nowhere vanishing, so there is a 1-dimensional foliation $\mathcal{F}_\xi$ associated with every Sasakian structure, called the *characteristic foliation*. We will denote the space of leaves of this foliation by $\mathcal{Z}$. Each leaf of $\mathcal{F}_\xi$ has a holonomy group associated to it. The dimension of the closure of the leaves is called the *rank* of $\mathcal{S}$. We shall be interested in the case $\mathrm{rk}(\mathcal{S}) = 1$. We have

Definition 11. The characteristic foliation $\mathcal{F}_\xi$ is said to be *quasi-regular* if there is a positive integer k such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times. Otherwise $\mathcal{F}_\xi$ is called *irregular*. If $k = 1$ then the foliation is called *regular*, and we use the terminology *non-regular* to mean quasi-regular, but not regular.

Let $(M, \mathcal{S})$ be a Sasakian manifold, and consider the contact subbundle $\mathcal{D} = \ker \eta$. There is an orthogonal splitting of the tangent bundle as

$$TM = \mathcal{D} \oplus L_\xi, \tag{4}$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ. The contact subbundle $\mathcal{D}$ is just the normal bundle to the characteristic foliation $\mathcal{F}_\xi$ generated by ξ. It is naturally endowed with both a complex structure $J = \Phi|_{\mathcal{D}}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, d\eta)$ gives M a *transverse Kähler* structure with Kähler form $d\eta$ and metric $g_{\mathcal{D}}$ defined as in (2) which is related to the Sasakian metric g by $g = g_{\mathcal{D}} \oplus \eta \otimes \eta$ as in (3). We have [23] the following fundamental structure theorem:

Theorem 12. *Let (M, ξ, η, Φ, g) be a compact quasi-regular Sasakian manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation.*

Then the leaf space $\mathcal{Z}$ is a Hodge orbifold with Kähler metric h and Kähler form ω which defines an integral class $[\omega]$ in $H^2_{\rm orb}(\mathcal{Z},\mathbb{Z})$ so that $\pi\colon (M,g)\to(\mathcal{Z},h)$ is an orbifold Riemannian submersion. The fibers of π are totally geodesic submanifolds of M diffeomorphic to S^1.

and its converse:

Theorem 13. *Let $(\mathcal{Z},h)$ be a Hodge orbifold. Let $\pi\colon M\to\mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $[\omega]$, and let η be a connection 1-form in M whose curvature is $2\pi^*\omega$, then M with the metric $\pi^*h+\eta\otimes\eta$ is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle S^1, the total space M is a smooth Sasakian manifold.*

Irregular structures can be understood by the following result of Rukimbira [121]:

Theorem 14. *Let (ξ,η,Φ,g) be a compact irregular Sasakian structure on a manifold M. Then the group $\mathfrak{Aut}(\xi,\eta,\Phi,g)$ of Sasakian automorphisms contains a torus T^k of dimension $k\geq 2$. Furthermore, there exists a sequence $(\xi_i,\eta_i,\Phi_i,g_i)$ of quasi-regular Sasakian structures that converge to (ξ,η,Φ,g) in the C^∞ compact-open topology.*

The orbifold cohomology groups $H^p_{\rm orb}(\mathcal{Z},\mathbb{Z})$ were defined by Haefliger [74]. In analogy with the smooth case a *Hodge orbifold* is then defined to be a compact Kähler orbifold whose Kähler class lies in $H^2_{\rm orb}(\mathcal{Z},\mathbb{Z})$. Alternatively, we can develop the concept of basic cohomology which works equally well in the irregular case, but only has coefficients in $\mathbb{R}$. It is nevertheless quite useful in trying to extend the notion of $\mathcal{Z}$ being Fano to both the quasi-regular and the irregular situation. This can be done in several ways. Here we will use the notion of basic Chern classes. Recall [130] that a smooth p-form α on M is called *basic* if

$$\xi\lrcorner\,\alpha=0,\quad \pounds_\xi\alpha=0, \tag{5}$$

and we let Λ^p_B denote the sheaf of germs of basic p-forms on M, and by Ω^p_B the set of global sections of Λ^p_B on M. The sheaf Λ^p_B is a module over the ring, Λ^0_B, of germs of smooth basic functions on M. We let $C^\infty_B(M)=\Omega^0_B$ denote global sections of Λ^0_B, i.e. the ring of smooth basic functions on M. Since exterior differentiation preserves basic forms we get a de Rham complex

$$\cdots\longrightarrow\Omega^p_B\xrightarrow{\ d\ }\Omega^{p+1}_B\longrightarrow\cdots \tag{6}$$

whose cohomology $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology* of $(M,\mathcal{F}_\xi)$. The basic cohomology ring $H^*_B(\mathcal{F}_\xi)$ is an invariant of the foliation $\mathcal{F}_\xi$ and hence, of the Sasakian structure on M. It is related to the ordinary de Rham cohomology $H^*(M,\mathbb{R})$ by the long exact sequence [130]

$$\cdots\longrightarrow H^p_B(\mathcal{F}_\xi)\longrightarrow H^p(M,\mathbb{R})\xrightarrow{\ j_p\ }H^{p-1}_B(\mathcal{F}_\xi)\xrightarrow{\ \delta\ }H^{p+1}_B(\mathcal{F}_\xi)\longrightarrow\cdots \tag{7}$$

where δ is the connecting homomorphism given by $\delta[\alpha] = [d\eta \wedge \alpha] = [d\eta] \cup [\alpha]$, and j_p is the composition of the map induced by $\xi \lrcorner$ with the well-known isomorphism $H^r(M, \mathbb{R}) \approx H^r(M, \mathbb{R})^{S^1}$ where $H^r(M, \mathbb{R})^{S^1}$ is the S^1-invariant cohomology defined from the S^1-invariant r-forms $\Omega^r(M)^{S^1}$. We also note that $d\eta$ is basic even though η is not. Next we exploit the fact that the transverse geometry is Kähler. Let $\mathcal{D}_{\mathbb{C}}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is, $\mathcal{D}_{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the sheaf Λ^1_B of basic one forms on M, namely

$$\Lambda^1_B \otimes \mathbb{C} = \Lambda^{1,0}_B \oplus \Lambda^{0,1}_B.$$

We let $\mathcal{E}^{p,q}_B$ denote the sheaf of germs of basic forms of type (p, q), and we obtain a splitting

$$\Lambda^r_B \otimes \mathbb{C} = \bigoplus_{p+q=r} \mathcal{E}^{p,q}_B. \tag{8}$$

The basic cohomology groups $H^{p,q}_B(\mathcal{F}_\xi)$ are fundamental invariants of a Sasakian structure which enjoy many of the same properties as the ordinary Dolbeault cohomology of a Kähler structure.

Consider the complex vector bundle $\mathcal{D}$ on a Sasakian manifold (M, ξ, η, Φ, g). As such $\mathcal{D}$ has Chern classes $c_1(\mathcal{D}), \ldots, c_n(\mathcal{D})$ which can be computed by choosing a connection $\nabla^{\mathcal{D}}$ in $\mathcal{D}$ [93]. Let us choose a local foliate unitary transverse frame $(X_1, \ldots, X_n)$, and denote by Ω^T the transverse curvature 2-form with respect to this frame. A simple calculation shows that Ω^T is a basic $(1, 1)$-form. Since the curvature 2-form Ω^T has type $(1, 1)$ it follows as in ordinary Chern–Weil theory that

Theorem 15. *The k^{th} Chern class $c_k(\mathcal{D})$ of the complex vector bundle $\mathcal{D}$ is represented by the basic (k, k)-form γ_k determined by the formula*

$$\det\left(\mathbb{1}_n - \frac{1}{2\pi i}\Omega^T\right) = 1 + \gamma_1 + \cdots + \gamma_k.$$

Since γ_k is a closed basic (k, k)-form it represents an element in $H^{k,k}_B(\mathcal{F}_\xi) \subset H^{2k}_B(\mathcal{F}_\xi)$ that is called the basic k^{th} Chern class and denoted by $c_k(\mathcal{F}_\xi)$.

We now concentrate on the first Chern classes $c_1(\mathcal{D})$ and $c_1(\mathcal{F}_\xi)$. We have

Definition 16. A Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is said to be *positive* (*negative*) if $c_1(\mathcal{F}_\xi)$ is represented by a positive (negative) definite $(1, 1)$-form. If either of these two conditions is satisfied $\mathcal{S}$ is said to be *definite*, and otherwise $\mathcal{S}$ is called *indefinite*. $\mathcal{S}$ is said to be *null* if $c_1(\mathcal{F}_\xi) = 0$.

Notice that irregular structures cannot occur for negative or null Sasakian structures, since the dimension of $\mathfrak{Aut}(\xi, \eta, \Phi, g)$ is greater than one. In analogy with common terminology of smooth algebraic varieties we see that a positive Sasakian structure is

a *transverse Fano structure*[1], while a null Sasakian structure is a *transverse Calabi–Yau structure*. The negative Sasakian case corresponds to the canonical bundle being ample.

4 Sasaki–Einstein and 3-Sasakian geometry

Definition 17. A Sasakian manifold $(M, \mathcal{S})$ is *Sasaki–Einstein* if the metric g is also Einstein.

For any 2n+1-dimensional Sasakian manifold $\mathrm{Ric}(X, \xi) = 2n\eta(X)$ implying that any Sasaki–Einstein metric must have positive scalar curvature. Thus any complete Sasaki–Einstein manifold must have a finite fundamental group. Furthermore the metric cone $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ on M is Kähler Ricci-flat (Calabi–Yau).

The following theorem [23] is an orbifold version of the famous Kobayashi bundle construction of Einstein metrics on bundles over positive Kähler–Einstein manifolds [19], [92].

Theorem 18. *Let $(\mathcal{Z}, h)$ be a compact Fano orbifold with $\pi_1^{\mathrm{orb}}(\mathcal{Z}) = 0$ and Kähler–Einstein metric h. Let $\pi : M \to \mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$. Suppose further that the local uniformizing groups of $\mathcal{Z}$ inject into S^1. Then with the metric $g = \pi^* h + \eta \otimes \eta$, M is a compact simply connected Sasaki–Einstein manifold.*

Here $\mathrm{Ind}(\mathcal{Z})$ is the *orbifold Fano index* [23] defined to be the largest positive integer such that $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$ defines a class in the orbifold cohomology group $H^2_{\mathrm{orb}}(\mathcal{Z}, \mathbb{Z})$. A very special class of Sasaki–Einstein spaces is naturally related to several quaternionic geometries.

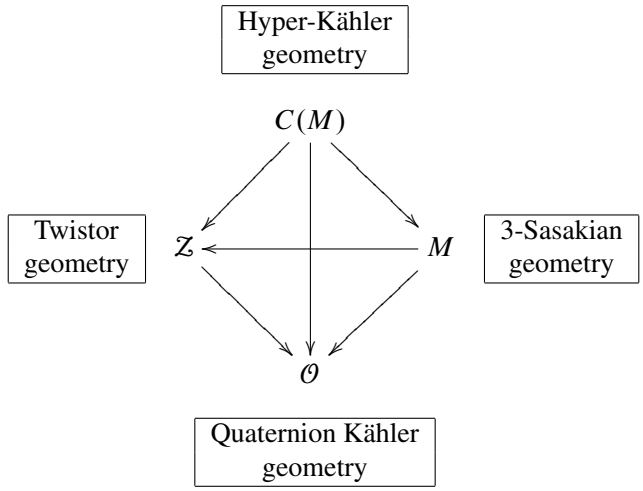
Definition 19. Let (M, g) be a Riemannian manifold of dimension m. We say that (M, g) is 3-*Sasakian* if the metric cone $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ on M is hyper-Kähler.

We emphasize the important observation of Kashiwada [89] that a 3-Sasakian manifold is automatically Einstein. We denote a Sasakian manifold with a 3-Sasakian structure by $(M, \boldsymbol{\mathcal{S}})$, where $\boldsymbol{\mathcal{S}} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a triple or a 2-sphere of Sasakian structures $\mathcal{S}_i = (\eta_i, \xi_i, \Phi_i, g)$.

Remark 20. In the 3-Sasakian case there is an extra structure, *i.e.*, the transverse geometry $\mathcal{O}$ of the 3-dimensional foliation which is quaternionic-Kähler. In this case,

[1] For a more algebro-geometric approach to positivity and fundamentals on log Fano orbifolds see [26].

the transverse space $\mathcal{Z}$ is the twistor space of $\mathcal{O}$ and the natural map $\mathcal{Z} \to \mathcal{O}$ is the orbifold twistor fibration [124]. We get the following diagram which we denote by $\diamondsuit(M, \mathcal{S})$ [29], [30]:



(9)

Remark 21. The table below summarizes properties of the cone and transverse geometries associated to various metric contact structures.

Cone geometry of $C(M)$	M	Transverse geometry of $\mathcal{F}_\xi$
Symplectic	Contact	Symplectic
Kähler	Sasakian	Kähler
Kähler	positive Sasakian	Fano, $c_1(\mathcal{Z}) > 0$
Kähler	null Sasakian	Calabi–Yau, $c_1(\mathcal{Z}) = 0$
Kähler	negative Sasakian	ample canonical bundle, $c_1(\mathcal{Z}) < 0$
Calabi–Yau	Sasaki–Einstein	Fano, Kähler–Einstein
Hyper-Kähler	3-Sasakian	$\mathbb{C}$-contact, Fano, Kähler–Einstein

For numerous examples and constructions of Sasaki–Einstein and 3-Sasakian manifolds see [26]. We finish this section with a remark that both the 3-Sasakian metric on M and the twistor space metric on $\mathcal{Z}$ admit 'squashings' which are again Einstein. More generally, let $\pi\colon M \to B$ be an orbifold Riemannian submersion with g the Riemannian metric on M. Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal subbundles of the tangent bundle TM. For each real number $t > 0$ we construct a one parameter family g_t of Riemannian metrics on M by defining

$$g_t|_{\mathcal{V}} = t g|_{\mathcal{V}}, \quad g_t|_{\mathcal{H}} = g|_{\mathcal{H}}, \quad g_t(\mathcal{V}, \mathcal{H}) = 0. \tag{10}$$

So for each $t > 0$ we have an orbifold Riemannian submersion with the same base space. Furthermore, if the fibers of g are totally geodesic, so are the fibers of g_t. We apply the canonical variation to the orbifold Riemannian submersion $\pi : M \to \mathcal{O}$ and $\pi : \mathcal{Z} \to \mathcal{O}$.

Theorem 22. *Every 3-Sasakian manifold M admits a second Einstein metric of positive scalar curvature. Furthermore, the twistor space $\mathcal{Z}$ also admits a second orbifold Einstein metric which is Hermitian-Einstein, but not Kähler–Einstein.*

5 Toric Sasaki–Einstein 5-manifolds

Examples of Sasaki–Einstein manifolds are plentiful and we refer the interested reader to our monograph for a detailed exposition [26]. Here we would like to consider the toric Sasaki–Einstein structures in dimension 5 again referring to [26] for all necessary details. Toric Sasaki–Einstein 5-manifolds recently emerged from physics in the context of *supersymmetry* and the so-called AdS/CFT duality conjecture which we will discuss in the last section. It is known that, in dimension 5, toric Sasaki–Einstein structures can only occur on the k-fold connected sums $k(S^2 \times S^3)$ [26]. The first inhomogeneous toric Sasaki–Einstein structures on $S^2 \times S^3$ were constructed by Gauntlett, Martelli, Sparks, and Waldram. It follows that $S^2 \times S^3$ admits infinitely many distinct quasi-regular and irregular toric Sasaki–Einstein structures [66]. Toric geometry of these examples was further explored in [102], [104], [105]. We will now describe a slightly different approach to a more general problem.

Consider the symplectic reduction of $\mathbb{C}^n$ (or equivalently the Sasakian reduction of S^{2n-1}) by a k-dimensional torus T^k. Every complex representation of a T^k on $\mathbb{C}^n$ can be described by an exact sequence

$$0 \longrightarrow T^k \xrightarrow{f_\Omega} T^n \longrightarrow T^{n-k} \longrightarrow 0.$$

The monomorphism f_Ω can be represented by the diagonal matrix

$$f_\Omega(\tau_1, \dots, \tau_k) = \operatorname{diag}\Big(\prod_{i=1}^{k} \tau_i^{a_1^i}, \dots, \prod_{i=1}^{k} \tau_i^{a_n^i}\Big),$$

where $(\tau_1, \dots, \tau_k) \in S^1 \times \cdots \times S^1 = T^k$ are the complex coordinates on T^k, and $a_\alpha^i \in \mathbb{Z}$ are the coefficients of a $k \times n$ integral *weight* matrix $\Omega \in \mathcal{M}_{k,n}(\mathbb{Z})$. We have ([26]):

Proposition 23. *Let $X(\Omega) = (\mathbb{C}^n \setminus \mathbf{0}) /\!\!/ T^k(\Omega)$ denote the Kähler quotient of the standard flat Kähler structure on $(\mathbb{C}^n \setminus \mathbf{0})$ by the weighted Hamiltonian T^k-action*

with an integer weight matrix Ω. *Consider the Kähler moment map*

$$\mu^i_\Omega(\mathbf{z}) = \sum_{\alpha=1}^{n} a^i_\alpha |z_\alpha|^2, \quad i = 1, \ldots, k. \tag{11}$$

If all minor $k \times k$ *determinants of* Ω *are non-zero then* $X(\Omega) = C(Y(\Omega))$ *is a cone on a compact Sasakian orbifold* $Y(\Omega)$ *of dimension* $2(n-k)-1$ *which is the Sasakian reduction of the standard Sasakian structure on* S^{2n-1}. *In addition, the projectivization of* $X(\Omega)$ *defined by* $\mathcal{Z}(\Omega) = X(\Omega)/\mathbb{C}^*$ *is a Kähler reduction of the complex projective space* $\mathbb{CP}^{n-1}$ *by a Hamiltonian* T^k*-action defined by* Ω *and it is the transverse space of the Sasakian structure on* $Y(\Omega)$ *induced by the quotient. If*

$$\sum_\alpha a^i_\alpha = 0 \quad \textit{for all } i = 1, \ldots, k, \tag{12}$$

then $c_1(X(\Omega)) = c_1(\mathcal{D}) = 0$. *In particular, the orbibundle* $Y(\Omega) \to \mathcal{Z}(\Omega)$ *is anti-canonical. Moreover, the cone* $C(Y(\Omega))$, *its Sasakian base* $Y(\Omega)$, *and the transverse space* $\mathcal{Z}(\Omega)$ *are all toric orbifolds.*

Remark 24. The conditions on the matrix Ω that assure that $Y(\Omega)$ is a smooth manifold are straightforward to work out. They involve gcd conditions on certain minor determinants of Ω.

This proposition is nicely summarized by the 'reduction' diagram

$$\begin{array}{ccccc} \mathbb{CP}^{n-1} & \longleftarrow & S^{2n-1} & \longleftarrow & \mathbb{C}^n \setminus (\mathbf{0}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathcal{Z}(\Omega) & \longleftarrow & Y(\Omega) & \longleftarrow & C(Y(\Omega)). \end{array} \tag{13}$$

Both the toric geometry and the topology of $Y(\Omega)$ depend on Ω. Furthermore, $Y(\Omega)$ comes equipped with a family of Sasakian structures. When $n-k=3$, assuming that $Y(\Omega)$ is simply connected (which is an additional condition on Ω), we must have $m(S^2 \times S^3)$ for some $m \leq k$. We will be mostly interested in the case when $m = k$.

Gauntlett, Martelli, Sparks, and Waldram [66] gave an explicit construction of a Sasaki–Einstein metric for $\Omega = (p, p, -p+q, -p-q)$, where p and q are relatively prime nonnegative integers with $p > q$. (The general case for $k = 1$ was treated later in [49], [102], see Remark 27 below). To connect with the original notation we write $Y(\Omega) = Y_{p,q}$. Then we get:

One can check that $Y_{1,0}$ is just the homogeneous metric on $S^2 \times S^3$ which is both toric and regular. The next simplest example is $Y_{2,1}$ which, as a toric contact (Sasakian) manifold, is a circle bundle over the blow up of $\mathbb{CP}^2$ at one point $F_1 = \mathbb{CP}^2 \,\#\, \overline{\mathbb{CP}^2}$ [103]. As F_1 cannot admit any Kähler–Einstein metric, Kobayashi's bundle construction cannot give a compatible Sasaki–Einstein structure. But there is a choice of a Reeb vector field in the torus which makes it possible to give $Y_{2,1}$

a Sasaki–Einstein metric. The Sasaki–Einstein structure on $Y_{2,1}$ is not quasi-regular and this was the first such example in the literature. Hence, $S^2 \times S^3$ admits infinitely many toric quasi-regular Sasaki–Einstein structures and infinitely many toric irregular Sasaki–Einstein structures of rank 2. We have the following generalization of the $Y_{p,q}$ metrics due to [60], [44]:

Theorem 25. *Let $Y(\Omega)$ be as in Proposition 23. Then $Y(\Omega)$ admits a toric Sasaki–Einstein structure which is unique up to a transverse biholomorphism.*

This existence of a Sasaki–Einstein metric is proved in [60] although the authors do not draw all the conclusions regarding possible toric Sasaki–Einstein manifolds that can be obtained. They give one interesting example of an irregular Sasaki–Einstein structure which generalizes the $Y_{2,1}$ example of [102] in the following sense: One considers a regular positive Sasakian structure on the anticanonical circle bundle over the del Pezzo surface $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ which gives a toric Sasakian structure on $2(S^2 \times S^3)$. The regular Sasakian structure on $2(S^2 \times S^3)$ cannot have any Sasaki–Einstein metric. However, as it is with $Y_{2,1}$ Futaki, Ono and Wang [60] show that one can deform the regular structure to a unique irregular Sasaki–Einstein structure. A slightly different version of Theorem 25 is proved in [44] where uniqueness is also established. Cho, Futaki and Ono work with toric diagrams rather than with Kähler (Sasakian) quotients which amounts to the same thing by Delzant's construction. We should add that the results of [44] apply to the toric Sasaki–Einstein manifolds in general dimension and not just in dimension 5.

Corollary 26. *The manifolds $k(S^2 \times S^3)$ admit infinite families of toric Sasaki–Einstein structures for each $k \geq 1$.*

As in the $k = 1$ case one would expect infinitely many quasi-regular and infinitely many irregular such Sasaki–Einstein structures for each Ω satisfying all the condition.

Remark 27. The general anticanonical circle reduction was considered independently in two recent papers, [49], [102]. There it was shown that for $\Omega = \boldsymbol{p} = (p_1, p_2, -q_1, -q_2)$, with $p_i, q_i \in \mathbb{Z}^+$, $p_1 + p_2 = q_1 + q_2$, and $\gcd(p_i, q_j) = 1$ for all $i, j = 1, 2$, the 5-manifold $Y(\Omega) \approx S^2 \times S^3$ admits a Sasaki–Einstein structure which coincides with that on $Y_{p,q}$ when $p_1 = p_2 = p$ and $q_1 = p - q, q_2 = p + q$. In [49] this family is denoted by $L^5(a, b, c)$, where $\boldsymbol{p} = (a, b, -c, -a - b + c)$ and they write the metric explicitly. However, in this case it appears to be harder (though, in principle, possible) to write down the condition under which the Sasaki–Einstein Reeb vector field $\xi = \xi(a, b, c)$ is quasi-regular. A priori, it is not even clear whether the quasi-regularity condition has any additional solutions beyond those obtained for the subfamily $Y_{p,q}$. Moreover, it follows from [44] that the metrics of [49], [102] describe all possible toric Sasaki–Einstein structures on $S^2 \times S^3$.

There have been similar constructions of a two-parameter family $X_{p,q}$ of toric Sasaki–Einstein metrics on $2(S^2 \times S^3)$ [75], and another two-parameter family, called

$Z_{p,q}$, on $3(S^2 \times S^3)$ [117]. All these examples, and many more, can be obtained as special cases of Theorem 25 as they are all $Y(\Omega)$ for some choice of Ω. The $Y_{p,q}$, $L^5(a,b,c)$, $X_{p,q}$ and $Z_{p,q}$ metrics have received a lot of attention because of the role such Sasaki–Einstein manifolds play in the AdS/CFT Duality Conjecture. They created an avalanche of papers studying the properties of these metrics from the physics perspective [8], [117], [118], [116], [90], [77], [41], [18], [40], [126], [75], [37], [15], [14], [119], [76]. The AdS/CFT duality will be discussed in the last section.

6 The Dirac operator and Killing spinors

We begin with a definition of spinor bundles and the bundle of Clifford algebras of a vector bundle [98], [55]. Recall that the *Clifford algebra* $\mathrm{Cl}(\mathbb{R}^n)$ over $\mathbb{R}^n$ can be defined as the quotient algebra of the tensor algebra $\mathcal{T}(\mathbb{R}^n)$ by the two-sided ideal $\mathcal{I}$ generated by elements of the form $v \otimes v + q(v)$ where q is a quadratic form on $\mathbb{R}^n$.

Definition 28. Let E be a vector bundle with inner product $\langle \cdot, \cdot \rangle$ on a smooth manifold M, and let $\mathcal{T}(E)$ denote the tensor bundle over E. The *Clifford bundle* of E is the quotient bundle $\mathrm{Cl}(E) = \mathcal{T}(E)/\mathcal{I}(E)$ where $\mathcal{I}$ is the bundle of ideals (two-sided) generated pointwise by elements of the form $v \otimes v + \langle v, v \rangle$ with $v \in E_x$. A *real spinor bundle* $S(E)$ of E is a bundle of modules over the Clifford bundle $\mathrm{Cl}(E)$. Similarly, a complex spinor bundle is a bundle of complex modules over the complexification $\mathrm{Cl}(E) \otimes \mathbb{C}$.

As vector bundles $\mathrm{Cl}(E)$ is isomorphic to the exterior bundle $\Lambda(E)$, but their algebraic structures are different. The importance of $\mathrm{Cl}(E)$ is that it contains the spin group $\mathrm{Spin}(n)$, the universal (double) covering group of the orthogonal group $\mathrm{SO}(n)$, so one obtains all the representations of $\mathrm{Spin}(n)$ by studying representations of $\mathrm{Cl}(E)$. We assume that the vector bundle E admits a spin structure, so $w_2(E) = 0$. We are interested mainly in the case when (M, g) is a Riemannian spin manifold and $E = TM$ in which case we write $S(M)$ instead of $S(TM)$. The Levi-Civita connection ∇ on TM induces a connection, also denoted ∇, on any of the spinor bundles $S(M)$, or more appropriately on the sections $\Gamma(S(M))$.

Definition 29. Let (M^n, g) be a Riemannian spin manifold and let $S(M)$ be any spinor bundle. The *Dirac operator* is the first order differential operator $D\colon \Gamma(S(M)) \to \Gamma(S(M))$ defined by

$$D\psi = \sum_{j=1}^{n} E_j \cdot \nabla_{E_j} \psi,$$

where $\{E_j\}$ is a local orthonormal frame and $\cdot$ denotes Clifford multiplication.

The Dirac operator, of course originating with the famous Dirac equation describing fermions in theoretical physics, was brought into mathematics by Atiyah and Singer

in [9]. Then Lichnerowicz [99] proved his famous result that a Riemannian spin manifold with positive scalar curvature must have vanishing $\hat{A}$-genus. An interesting question on any spin manifold is: what are the eigenvectors of the Dirac operator. In this regard the main objects of interest consists of special sections of certain spinor bundles called *Killing spinor fields* or just *Killing spinors* for short. Specifically (cf. [13], [55]):

Definition 30. Let (M, g) be a complete n-dimensional Riemannian spin manifold, and let $S(M)$ be a spin bundle (real or complex) on M and ψ a smooth section of $S(M)$. We say that ψ is a *Killing spinor* if for every vector field X there is $\alpha \in \mathbb{C}$, called *Killing number*, such that

$$\nabla_X \psi = \alpha X \cdot \psi.$$

Here $X \cdot \psi$ denotes the Clifford product of X and ψ. We say that ψ is *imaginary* when $\alpha \in \mathrm{Im}(\mathbb{C}^*)$, ψ is *parallel* if $\alpha = 0$ and ψ is *real*[2] if $\alpha \in \mathrm{Re}(\mathbb{C}^*)$.

We shall see shortly that the three possibilities for the Killing number α: real, imaginary, or 0, are the only possibilities. The name Killing spinor derives from the fact that if ψ is a non-trivial Killing spinor and α is real, the vector field

$$X_\psi = \sum_{j=1}^{n} g(\psi, E_j \cdot \psi) E_j \tag{14}$$

is a Killing vector field for the metric g (which, of course, can be zero). If ψ is a Killing spinor on an n-dimensional spin manifold, then

$$D\psi = \sum_{j=1}^{n} E_j \cdot \nabla_{E_j} \psi = \sum_{j=1}^{n} \alpha E_j \cdot E_j \cdot \psi = -n\alpha\psi. \tag{15}$$

So Killing spinors are eigenvectors of the Dirac operator with eigenvalue $-n\alpha$. In 1980 Friedrich [54] proved the following remarkable theorem:

Theorem 31. *Let (M^n, g) be a Riemannian spin manifold which admits a non-trivial Killing spinor ψ with Killing number α. Then (M^n, g) is Einstein with scalar curvature $s = 4n(n-1)\alpha^2$.*

A proof of this is a straightforward curvature computation which can be found in either of the books [13], [55]. It also uses the fact that a non-trivial Killing spinor vanishes nowhere. It follows immediately from Theorem 31 that α must be one of the three types mentioned in Definition 30. So if the Killing number is real then (M, g) must be a positive Einstein manifold. In particular, if M is complete, then it is compact. On the other hand if the Killing number is pure imaginary, Friedrich shows that M must be non-compact.

[2]Here the standard terminology real and imaginary Killing spinors can be somewhat misleading. The Killing spinor ψ is usually a section of a complex spinor bundle. So a real Killing spinor just means that α is real.

The existence of Killing spinors not only puts restrictions on the Ricci curvature, but also on both the Riemannian and the Weyl curvature operators [13].

Proposition 32. *Let (M^n, g) be a Riemannian spin manifold. Let ψ be a Killing spinor on M with Killing number α and let $\mathcal{R}, \mathcal{W}\colon \Lambda^2 M \to \Lambda^2 M$ be the Riemann and Weyl curvature operators, respectively. Then for any vector field X and any 2-form β we have*

$$\mathcal{W}(\beta)\cdot\psi = 0; \tag{16}$$

$$(\nabla_X \mathcal{W})(\beta)\cdot\psi = -2\alpha\big(X \lrcorner \mathcal{W}(\beta)\big)\cdot\psi; \tag{17}$$

$$(\mathcal{R}(\beta) + 4\alpha^2\beta)\cdot\psi = 0; \tag{18}$$

$$(\nabla_X \mathcal{R})(\beta)\cdot\psi = -2\alpha\big(X \lrcorner \mathcal{R}(\beta) + 4\alpha^2\beta(X)\big)\cdot\psi. \tag{19}$$

These curvature equations can be used to prove (see [13] or [55])

Theorem 33. *Let (M^n, g) be a connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha \neq 0$. Then (M, g) is locally irreducible. Furthermore, if M is locally symmetric, or $n \leq 4$, then M is a space of constant sectional curvature equal to $4\alpha^2$.*

Friedrich's main objective in [54] was an improvement of Lichnerowicz's estimate in [99] for the eigenvalues of the Dirac operator. Indeed, Friedrich proves that the eigenvalues λ of the Dirac operator on any compact manifold satisfy the estimate

$$\lambda^2 \geq \frac{1}{4}\frac{n s_0}{n-1}, \tag{20}$$

where s_0 is the minimum of the scalar curvature on M. Thus, Killing spinors ψ are eigenvectors that realize equality in equation (20). Friedrich also proves the converse that any eigenvector of D realizing the equality must be a Killing spinor with

$$\alpha = \pm\frac{1}{2}\sqrt{\frac{s_0}{n(n-1)}}. \tag{21}$$

Example 34 (Spheres). In the case of the round sphere (S^n, g_0) equality in equation (20) is always attained. So normalizing such that $s_0 = n(n-1)$, and using Bär's Correspondence Theorem 38 below the number of corresponding real Killing spinors equals the number of constant spinors on $\mathbb{R}^{n+1}$ with the flat metric. The latter is well known (see the appendix of [120]) to be $2^{\lfloor n/2 \rfloor}$ for each of the values $\alpha = \pm\frac{1}{2}$, where $\lfloor n/2 \rfloor$ is the largest integer less than or equal to $n/2$.

Remark 35. Actually (without making the connection to Sasakian geometry) already in [54] Friedrich gives a non-spherical example of a compact 5-manifold with a real Killing spinor: $M = \mathrm{SO}(4)/\mathrm{SO}(2)$ with its homogeneous Kobayashi–Tanno Sasaki–Einstein structure.

We now wish to relate Killing spinors to the main theme of this chapter, Sasakian geometry. First notice that if a Sasakian manifold M^{2n+1} admits a Killing spinor, Theorem 31 says it must be Sasaki–Einstein, so the scalar curvature $s_0 = 2n(2n+1)$, and equation (21) implies that $\alpha = \pm\frac{1}{2}$. We have the following result of Friedrich and Kath [57]

Theorem 36. *Every simply connected Sasaki–Einstein manifold admits non-trivial real Killing spinors. Furthermore,*

(i) *if M has dimension $4m+1$ then (M, g) admits exactly one Killing spinor for each of the values $\alpha = \pm\frac{1}{2}$,*

(ii) *if M has dimension $4m+3$ then (M, g) admits at least two Killing spinors for one of the values $\alpha = \pm\frac{1}{2}$.*

Outline of proof. (Details can be found in [57] or the book [13].) Every simply connected Sasaki–Einstein manifold is known to be spin, so M has a spin bundle $S(M)$. Given a fixed Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ we consider two subbundles $\mathcal{E}_\pm(\mathcal{S})$ of $S(M)$ defined by

$$\mathcal{E}_\pm(\mathcal{S}) = \{\psi \in S(M) \mid (\pm 2\Phi X + \pounds_\xi X)\cdot\psi = 0 \text{ for all } X \in \Gamma(TM)\}. \tag{22}$$

Set $\nabla^\pm_X = \nabla_X \pm \frac{1}{2}X\cdot$. A straightforward computation shows that $\nabla^\pm$ preserves the subbundles $\mathcal{E}_\pm$ and defines a connection there. Moreover, by standard curvature computations it can be shown that the connection $\nabla^\pm$ is flat in $\mathcal{E}_\pm(\mathcal{S})$. So it has covariantly constant sections which are precisely the Killing spinors. One then uses some representation theory of $\mathrm{Spin}(2n+1)$ to compute the dimensions of $\mathcal{E}_+(\mathcal{S})$ and $\mathcal{E}_-(\mathcal{S})$ proving the result. □

We have the following:

Corollary 37. *Let (M, g) be a Sasaki–Einstein manifold of dimension $2m+1$. Then (M, g) is locally symmetric if and only if (M, g) is of constant curvature. Moreover,* $\mathrm{Hol}(g) = \mathrm{SO}(2m+1)$ *and (M, g) is locally irreducible as a Riemannian manifold.*

Proof. If necessary, go to the universal cover $\widetilde{M}$. This is a compact simply connected Sasaki–Einstein manifold; hence, it admits a non-trivial Killing spinor by Theorem 36. The first statement then follows from Theorem 33. The second statement follows from the Berger Theorem 4. Since M has dimension $2m+1$ the only possibilities for $\mathrm{Hol}(g)$ are $\mathrm{SO}(2m+1)$ and G_2. But the latter is Ricci flat, so it cannot be Sasaki–Einstein. □

Friedrich and Kath began their investigation in dimension 5 [56] where they showed that a simply-connected compact 5-manifold which admits a Killing spinor must be Sasaki–Einstein. In dimension 7 they showed that there are exactly three possibilities: weak G_2-manifolds, Sasaki–Einstein manifolds which are not 3-Sasakian, and 3-Sasakian manifolds [57]. Later Grunewald gave a description of 6-manifolds admitting Killing spinors [73]. We should add an earlier result of Hijazi who showed

that the only 8-dimensional manifold with Killing spinors must be the round sphere [78]. By 1990 a decade of research by many people slowly identified all the ingredients of a classification of such manifolds in terms of their underlying geometric structures. The pieces of the puzzle consisting of round spheres in any dimension, Sasaki–Einstein manifolds in odd dimensions, nearly Kähler manifolds in dimension 6, and weak G_2-holonomy manifolds in dimension 7 were all in place with plenty of interesting examples to go around [13]. What remained at that stage was to show that in even dimensions greater than 8 there is nothing else but the round spheres, while in odd dimensions greater than 7 the only such examples must be Sasaki–Einstein. The missing piece of the puzzle was finally uncovered by Bär: real Killing spinors on M correspond to parallel spinors on the cone $C(M)$ [12]. A bit earlier Wang [133] had shown that on a simply connected complete Riemannian spin manifold the existence of parallel spinors corresponds to reduced holonomy. This led Bär to an elegant description of the geometry of manifolds admitting real Killing spinors (in any dimension) in terms of special holonomies of the associated cones. We refer to the correspondence between real Killing spinors on M and parallel spinors on the cone $C(M)$ (equivalently reduced holonomy) as *Bär's correspondence*. In particular, this correspondence not only answered the last remaining open questions, but also allowed for simple unified proofs of most of the theorems obtained earlier.

7 Real Killing spinors, holonomy and Bär's correspondence

As already mentioned the Bär correspondence relates real Killing spinors on a compact Riemannian spin manifold (M, g) to parallel spinors on the Riemannian cone $(C(M), \bar{g})$. We now make this statement precise.

Theorem 38. *Let (M^n, g) be a complete Riemannian spin manifold and $(C(M^n), \bar{g})$ be its Riemannian cone. Then there is a one to one correspondence between real Killing spinors on (M^n, g) with $\alpha = \pm\frac{1}{2}$ and parallel spinors on $(C(M^n), \bar{g})$.*

Proof. The existence of a parallel spinor on $(C(M^n), \bar{g})$ implies that $\bar{g}$ is Ricci flat by Theorem 31. Then by Lemma 2 (M^n, g) is Einstein with scalar curvature $s = n(n-1)$. So any Killing spinors must have $\alpha = \pm\frac{1}{2}$ by equation (21). As in the proof of Theorem 36, $\nabla^{\pm}_X = \nabla_X \pm \frac{1}{2}X\cdot$ defines a connection in the spin bundle $S(M)$. The connection 1-forms $\omega^{\pm}$ of $\nabla^{\pm}$ are related to the connection 1-form ω of the Levi-Civita connection by $\omega^{\pm} = \omega \pm \frac{1}{2}\beta$, where β is a 1-form called the *soldering form*. This can be interpreted as a connection with values in the Lie algebra $\mathfrak{spin}(n+1) = \mathfrak{spin}(n) \oplus \mathbb{R}^n$, and pulls back to the Levi-Civita connection in the spin bundles on the cone $(C(M^n), \bar{g})$. So parallel spinors on the cone correspond to parallel spinors on (M, g) with respect to the connection $\nabla^{\pm}$ which correspond precisely to real Killing spinors with respect to the Levi-Civita connection. □

Now we have the following definition:

Definition 39. We say that a Riemannian spin manifold (M, g) is of *type* (p, q) if it carries exactly p linearly independent real Killing spinors with $\alpha > 0$ and exactly q linearly independent real Killing spinors with $\alpha < 0$.

The following theorem has an interesting history. As mentioned above it was Bär [12] who recognized the correspondence between real Killing spinors on (M, g) and parallel spinors on the Riemannian cone $(C(M), \bar{g})$. The relation between parallel spinors and reduced holonomy was anticipated in the work of Hitchin [79] and Bonan [22], but was formalized in the 1989 paper of Wang [133]. It has also been generalized to the non-simply connected case in [134], [110].

Theorem 40. *Let (M^n, g) be a complete simply connected Riemannian spin manifold, and let* $\mathrm{Hol}(\bar{g})$ *be the holonomy group of the Riemannian cone $(C(M), \bar{g})$. Then (M^n, g) admits a non-trivial real Killing spinor with (M^n, g) of type (p, q) if and only if* $\big(\dim M, \mathrm{Hol}(\bar{g}), (p, q)\big)$ *is one of the* 6 *possible triples listed in the table below.*

Table 2

$\dim(M)$	$\mathrm{Hol}(\bar{g})$	*type* (p, q)
n	id	$(2^{\lfloor n/2 \rfloor}, 2^{\lfloor n/2 \rfloor})$
$4m + 1$	$\mathrm{SU}(2m + 1)$	$(1, 1)$
$4m + 3$	$\mathrm{SU}(2m + 2)$	$(2, 0)$
$4m + 3$	$\mathrm{Sp}(m + 1)$	$(m + 2, 0)$
7	$\mathrm{Spin}(7)$	$(1, 0)$
6	G_2	$(1, 1)$

Here $m \geq 1$, and $n > 1$.

Outline of proof. Since (M, g) is complete and has a non-trivial real Killing spinor, it is compact by Theorem 31. It then follows from a theorem of Gallot [64] that if the Riemannian cone $(C(M), \bar{g})$ has reducible holonomy it must be flat. So we can apply Berger's Theorem 4. Now Wang [133] used the spinor representations of the possible irreducible holonomy groups on Berger's list to give the correspondence between these holonomy groups and the existence of parallel spinors. First he showed that the groups listed in Table 4 that are not on the above table do not admit parallel spinors. Then upon decomposing the spin representation of the group in question into irreducible pieces, the number of parallel spinors corresponds to the multiplicity of the trivial representation. Wang computes this in all but the first line of the table when $(C(M), \bar{g})$ is flat. In this case (M, g) is a round sphere as discussed in Example 34, so the number of linearly independent constant spinors is $(2^{\lfloor n/2 \rfloor}, 2^{\lfloor n/2 \rfloor})$. By Bär's Correspondence

Theorem 38 real Killing spinors on (M, g) correspond precisely to parallel spinors on $(C(M), \bar{g})$. Note that the hypothesis of completeness in Wang's theorem [133] is not necessary, so that the correspondence between the holonomy groups and parallel spinors holds equally well on Riemannian cones. However, the completeness assumption on (M, g) guarantees the irreducibility of the cone $(C(M), \bar{g})$ as mentioned above. □

Let us briefly discuss the types of geometry involved in each case of this theorem. As mentioned in the above proof the first line of the table corresponds to the round spheres. The next three lines correspond to Sasaki–Einstein geometry, so Theorem 40 generalizes the Friedrich–Kath Theorem 36 in this case. The last of these three lines corresponds precisely to 3-Sasakian geometry by Definition 19. Finally the two cases whose cones have exceptional holonomy will be discussed in more detail in Section 8.1 below. Suffice it here to mention that it was observed by Bryant and Salamon [36] that a cone on a nearly parallel G_2 manifold has its own holonomy in Spin(7). It is interesting to note that Theorem 40 generalizes the result of Hijazi in dimension eight mentioned earlier as well as part of the last statement in Theorem 33, namely

Corollary 41. *Let (M^{2n}, g) be a complete simply connected Riemannian spin manifold of dimension $2n$ with $n \neq 3$ admitting a non-trivial real Killing spinor. Then M is isometric to the round sphere.*

We end this section with a brief discussion of the non-simply connected case. Here we consider two additional cases for $\mathrm{Hol}(\bar{g})$, namely $\mathrm{SU}(2m+2)\rtimes\mathbb{Z}_2$ and $\mathrm{Sp}(2)\times\mathbb{Z}_d$. See [134], [110] for the list of possibilities.

Example 42. $\mathrm{Hol}(\bar{g}) = \mathrm{SU}(2m) \rtimes \mathbb{Z}_2$. Consider the $(4m-1)$-dimensional Stiefel manifold $V_2(\mathbb{R}^{2m+1})$ with its homogeneous Sasaki–Einstein metric. The quotient manifold M_σ^{4m-1} of $V_2(\mathbb{R}^{2m+1})$ by the free involution σ induced from complex conjugation has an Einstein metric which is "locally Sasakian". The cone $C(M_\sigma^{4m-1})$ is not Kähler and its holonomy is $\mathrm{Hol}(\bar{g}) = \mathrm{SU}(2m+2) \rtimes \mathbb{Z}_2$. According to Wang [134] $C(M_\sigma^{4m-1})$ admits a spin structure with precisely one parallel spinor if and only if m is even, and according to Moroianu and Semmelmann [110] $C(M_\sigma^{4m-1})$ admits exactly two spin structures each with precisely one parallel spinor if m is even. Thus, by Theorem 38 M_σ^{4m-1} admits exactly two spin structures each with exactly one Killing spinor if and only if m is even.

Example 43. Consider a 3-Sasakian manifold $(M^{4n-1}, \mathcal{S})$ and choose a Reeb vector field $\xi(\boldsymbol{\tau})$. Let C_m be the cyclic subgroup of order $m > 2$ of the circle group generated by $\xi(\boldsymbol{\tau})$. Assume that m is relatively prime to the order $\upsilon(\mathcal{S})$ of $\mathcal{S}$ and that the generic fibre of the fundamental 3-dimensional foliation $\mathcal{F}_Q$ is SO(3), so that C_m acts freely on M^{4n-1}. This last condition on the generic fibre is easy to satisfy; for example, it holds for any of the 3-Sasakian homogeneous spaces other than the standard round sphere, as well as the bi-quotients described in [30]. (To handle the case when the

generic fibre is Sp(1) we simply need to divide m by two when it is even). Since C_m is not in the center of SO(3), the quotient M^{4n-1}/C_m is not 3-Sasakian. However, C_m does preserve the Sasakian structure determined by $\xi(\tau)$, so M^{4n-1}/C_m is Sasaki–Einstein. The cone $C(M^{4n-1}/C_m)$ has holonomy $\mathrm{Sp}(n)\times\mathbb{Z}_m$, and admits precisely $\frac{n+1}{m}$ parallel spinors if and only if m divides $n+1$ [134], [110]. Thus, by Theorem 38 M^{4n-1}/C_m admits precisely $\frac{n+1}{m}$ Killing spinors when m divides $n+1$.

8 Geometries associated with 3-Sasakian 7-manifolds

It is most remarkable that to each $4n$-dimensional positive QK metric $(\mathcal{O}, g_{\mathcal{O}})$ (even just locally) one can associate *nine* other Einstein metrics in dimensions $4n+k$, $k=1,2,3,4$. Alternatively, one could say that each 3-Sasakian metric (M,g) canonically defines an additional nine Einstein metrics in various dimensions. We have already encountered all of these metrics. First there are the four geometries of the diamond diagram $\diamondsuit(M,\mathcal{S})$. Then M and $\mathcal{Z}$ admit additional "squashed" Einstein metrics discussed in Theorem 22. Thus we get five Einstein metrics with positive Einstein constants: $(\mathcal{O}, g_{\mathcal{O}})$, (M,g), (M',g'), $(\mathcal{Z},h)$, $(\mathcal{Z}',h')$. Of course $M\simeq M'$ and $\mathcal{Z}\simeq\mathcal{Z}'$ as smooth manifolds (orbifolds) but they are different as Riemannian manifolds (orbifolds), hence, the notation. Let us scale all these metrics so that the Einstein constant equals the dimension of the total space minus 1. Note that any 3-Sasakian metric already has this property. In the other four cases this is a choice of scale which is quite natural due to Lemma 2. However, note that this is not the scale one gets for $(\mathcal{Z},h)$, and $(\mathcal{O}, g_{\mathcal{O}})$ via the Riemannian submersion from (M,g). Now, in each case one can consider its Riemannian cone which will be Ricci-flat by Lemma 2. We thus obtain five Ricci-flat metrics on the corresponding Riemannian cones. In addition, one can also take (iterated) sine-cone metrics defined in (1) on the same five bases. These metrics are all Einstein of positive scalar curvature (cf. Lemma 3). Let us summarize all this with the following extension of $\diamondsuit(M,\mathcal{S})$:

$$\begin{array}{ccccc} C(\mathcal{Z}') & \hookleftarrow & \mathcal{Z}' & & \\ & \searrow & & \swarrow & \\ & & \mathcal{O} & \hookrightarrow & C(\mathcal{O}) \\ & \nearrow & & \nwarrow & \\ C(\mathcal{Z}) & \hookleftarrow & \mathcal{Z} & \longleftarrow & M \hookrightarrow C(M). \end{array} \qquad (23)$$

There would perhaps be nothing special about all these 10 (and many more by iterating the sine-cone construction) geometries beyond what has already been discussed in the previous sections. This is indeed true when $\dim(M)>7$. However, when $\dim(M)=7$, or, alternatively, when $\mathcal{O}$ is a positive self-dual Einstein orbifold

metric (more generally, just a local metric of this type) some of the metrics occurring in diagram (23) have additional properties. We shall list all of them first. For the moment, let us assume that (M, g) is a compact 3-Sasakian 7-manifold, then the following hold:

(1) $(\mathcal{O}, g_{\mathcal{O}})$ is a positive self-dual Einstein manifold (orbifold). We will think of it as the *source* of all the other geometries.

(2) $(C(\mathcal{O}), dt^2 + t^2 g_{\mathcal{O}})$ is a 5-dimensional Ricci-flat cone with base $\mathcal{O}$.

(3) $(\mathcal{Z}, h)$ is the orbifold twistor space of $\mathcal{O}$.

(4) $(\mathcal{Z}', h')$ is a nearly-Kähler manifold (orbifold).

(5) (M, g) is the 3-Sasakian manifold.

(6) (M', g') is a 7-manifold with weak G_2 structure.

(7) $(C(\mathcal{Z}'), dt^2 + t^2 h')$ is a 7-manifold with holonomy inside G_2.

(8) $(C_s(\mathcal{Z}'), dt^2 + (\sin^2 t)h')$ is a 7-manifold with weak G_2 structure.

(9) $(C(\mathcal{Z}), dt^2 + t^2 h)$ is a 7-dimensional Ricci-flat cone with base $\mathcal{Z}$.

(10) $(C(M), dt^2 + t^2 g)$ is hyper-Kähler with holonomy contained in Sp(2).

(11) $(C(M'), dt^2 + t^2 g')$ has holonomy contained in Spin(7).

The cases (2) and (8) do not appear to have any special properties other than Ricci-flatness. The cases (1), (3), (5), and (10) are the four geometries of $\diamondsuit(M, \mathcal{S})$. The five remaining cases are all very interesting from the point of view of the classification of Theorem 40. Indeed $\mathcal{Z}'$ and $C(\mathcal{Z}')$ are examples of the structures listed in the last row of the table while $C_2(\mathcal{Z}')$, M' and $C(M')$ give examples of the structures listed in the fifth row. In particular, our diagram (23) provides for a cornucopia of orbifold examples in the first case and smooth manifolds in the latter.

8.1 Nearly parallel G_2-structures and Spin(7) holonomy cones

Recall, that geometrically G_2 is defined to be the Lie group acting on the imaginary octonions $\mathbb{R}^7$ and preserving the 3-form

$$\begin{aligned}\varphi = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge (\alpha_4 \wedge \alpha_5 - \alpha_6 \wedge \alpha_7) \\ & + \alpha_2 \wedge (\alpha_4 \wedge \alpha_6 - \alpha_7 \wedge \alpha_5) + \alpha_3 \wedge (\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6),\end{aligned} \tag{24}$$

where $\{\alpha_i\}_{i=1}^7$ is a fixed orthonormal basis of the dual of $\mathbb{R}^7$. A G_2 structure on a 7-manifold M is, by definition, a reduction of the structure group of the tangent bundle to G_2. This is equivalent to the existence of a global 3-form $\varphi \in \Omega^3(M)$ which may be written locally as (24). Such a 3-form defines an associated Riemannian metric, an orientation class, and a spinor field of constant length.

Definition 44. Let (M, g) be a complete 7-dimensional Riemannian manifold. We say that (M, g) is a *nearly parallel*[3] G_2 structure if there exist a global 3-form $\varphi \in \Omega^3(M)$ which locally can be written in terms of a local orthonormal basis as in (24), and $d\varphi = c \star \varphi$, where $\star$ is the Hodge star operator associated to g and $c \neq 0$ is a constant whose sign is fixed by an orientation convention.

The case $c = 0$ in Definition 44 is somewhat special. In particular, it is known [123] that the condition $d\varphi = 0 = d \star \varphi$ is equivalent to the condition that φ be parallel, *i.e.*, $\nabla\varphi = 0$ which is equivalent to the condition that the metric g has holonomy group contained in G_2. The following theorem provides the connection with the previous discussion on Killing spinors [12]

Theorem 45. *Let (M, g) be a complete 7-dimensional Riemannian manifold with a nearly parallel G_2 structure. Then the holonomy* $\mathrm{Hol}(\bar{g})$ *of the metric cone $(C(M), \bar{g})$ is contained in* $\mathrm{Spin}(7)$. *In particular, $C(M)$ is Ricci-flat and M is Einstein with positive Einstein constant $\lambda = 6$.*

Remark 46. The sphere S^7 with its constant curvature metric is isometric to the isotropy irreducible space $\mathrm{Spin}(7)/G_2$. The fact that G_2 leaves invariant (up to constants) a unique 3-form and a unique 4-form on $\mathbb{R}^7$ implies immediately that this space has a nearly parallel G_2 structure.

Definition 47. Let (M, g) be a complete 7-dimensional Riemannian manifold. We say that g is a *proper G_2-metric* if $\mathrm{Hol}(\bar{g}) = \mathrm{Spin}(7)$.

We emphasize here that G_2 is the structure group of M, not the Riemannian holonomy group. Specializing Theorem 40 to dimension 7 gives the following theorem due to Friedrich and Kath [57].

Theorem 48. *Let (M^7, g) be a complete simply-connected Riemannian spin manifold of dimension 7 admitting a non-trivial real Killing spinor with $\alpha > 0$ or $\alpha < 0$. Then there are four possibilities:*

(i) *(M^7, g) is of type $(1, 0)$ and it is a proper G_2-manifold.*

(ii) *(M^7, g) is of type $(2, 0)$ and it is a Sasaki–Einstein manifold, but (M^7, g) is not 3-Sasakian.*

(iii) *(M^7, g) is of type $(3, 0)$ and it is 3-Sasakian.*

(iv) *$(M^7, g) = (S^7, g_{\mathrm{can}})$ and is of type $(8, 8)$.*

[3] It had become customary to refer to this notion as 'weak holonomy G_2', a terminology introduced by Gray [71]. However, it was pointed out to us by the anonymous referee that this terminology is misleading due to the fact that Gray's paper contains errors rendering the concept of weak holonomy useless as discovered by Alexandrov [3]. Hence, the term 'nearly parallel' used in [58] is preferred.

Conversely, if (M^7, g) is a compact simply-connected proper G_2-manifold then it carries precisely one Killing spinor with $\alpha > 0$. If (M^7, g) is a compact simply-connected Sasaki–Einstein 7-manifold which is not 3-Sasakian then M carries precisely two linearly independent Killing spinors with $\alpha > 0$. Finally, if (M^7, g) is a 3-Sasakian 7-manifold, which is not of constant curvature, then M carries precisely three linearly independent Killing spinors with $\alpha > 0$.

Remark 49. The four possibilities of Theorem 48 correspond to the sequence of inclusions

$$\mathrm{Spin}(7) \supset \mathrm{SU}(4) \supset \mathrm{Sp}(2) \supset \mathbb{1}.$$

All of the corresponding cases are examples of nearly parallel G_2 metrics. If we exclude the trivial case when the associated cone is flat, we have three types of nearly parallel G_2 geometries. Following [58] we use the number of linearly independent Killing spinors to classify these geometries, and call them type I, II, and III corresponding to cases (i), (ii), and (iii) of Theorem 48, respectively.

We are now ready to describe the G_2 geometry of the $M' \hookrightarrow C(M')$ part of the diagram (23) [63], [58]:

Theorem 50. *Let $(M, \mathcal{S})$ be a 7-dimensional 3-Sasakian manifold. Then the 3-Sasakian metric g is a nearly parallel G_2 metric. Moreover, the second Einstein metric g' given by Theorem 22 and scaled so that the Einstein constant $\lambda = 6$ is a nearly parallel G_2 metric; in fact, it is a proper G_2 metric.*

Proof. For the second Einstein metric g' we have three mutually orthonormal 1-forms $\alpha^1 = \sqrt{t}\eta^1$, $\alpha^2 = \sqrt{t}\eta^2$, $\alpha^3 = \sqrt{t}\eta^3$, where t is the parameter of the canonical variation. Let $\{\alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ be local 1-forms spanning the annihilator of the vertical subbundle $\mathcal{V}_3$ in $T^*\mathcal{S}$ such that

$$\begin{aligned}
\bar{\Phi}^1 &= 2(\alpha^4 \wedge \alpha^5 - \alpha^6 \wedge \alpha^7),\\
\bar{\Phi}^2 &= 2(\alpha^4 \wedge \alpha^6 - \alpha^7 \wedge \alpha^5),\\
\bar{\Phi}^3 &= 2(\alpha^4 \wedge \alpha^7 - \alpha^5 \wedge \alpha^6).
\end{aligned}$$

Then the set $\{\alpha^1, \ldots, \alpha^7\}$ forms a local orthonormal coframe for the metric g'. Let

$$\Upsilon = \eta_1 \wedge \eta_2 \wedge \eta_3, \quad \Theta = \sum_a \eta_a \wedge \bar{\Phi}_a = \sum_a \eta_a \wedge d\eta_a + 6\Upsilon \tag{25}$$

In terms of the 3-forms Υ and Θ we have $\varphi = \frac{1}{2}\sqrt{t}\Theta + \sqrt{t}^3\Upsilon$. One easily sees that this is of the type of equation (24) and, therefore, defines a compatible G_2-structure. Moreover, a straightforward computation gives

$$d\varphi = \frac{1}{2}\sqrt{t}\Omega + \sqrt{t}(t+1)d\Upsilon, \quad \star\varphi = -\frac{1}{2}t\,d\Upsilon - \frac{1}{24}\Omega.$$

Thus, $d\varphi = c \star \varphi$ is solved with $\sqrt{t} = 1/\sqrt{5}$, and $c = -12/\sqrt{5}$. So g' is nearly parallel. That g' is a proper G_2 metric is due to [58]. The idea is to use Theorem 48. Looking at the four possibilities given in that theorem, we see that it suffices to show that g' is not Sasaki–Einstein. The details are in [58]. □

Example 51. 3-Sasakian 7-manifolds are plentiful [26]. All of them give, by Theorem 50, examples of type I and type III geometries. Examples of simply connected type I geometries that do not arise via Theorem 50 are the homogeneous Aloff–Wallach spaces $M^7_{m,n}$, $(m,n) \neq (1,1)$ which, as special cases of Eschenburg bi-quotients [42], [13], are together with an isotropy irreducible homogeneous space defined as follows: Consider the space $\mathcal{H}_2$ of homogeneous polynomials of degree 2 in three real variables (x_1, x_2, x_3). As $\dim(\mathcal{H}_2) = 5$ it gives rise to the embedding $\mathrm{SO}(3) \subset \mathrm{SO}(5)$. We take $M = \mathrm{SO}(5)/\mathrm{SO}(3)$. This example was used by Bryant to get the first 8-dimensional metric with holonomy Spin(7) [33]. Examples of type II geometries (Sasaki–Einstein) are equally rich [26]. In particular, there are hundreds of examples of type II nearly parallel G_2 metrics on each of the 28 homotopy spheres in dimension 7.

Remark 52. According to [42] the Aloff–Wallach manifold $M^7_{1,1}$ has three Einstein metrics. One is the homogeneous 3-Sasakian metric. The second is the proper G_2 metric of Theorem 50. The third Einstein metric is also nearly parallel most likely being of type I, but we could not positively exclude type II as a possibility.

Open Problem 53. Classify all compact 7-manifolds with nearly parallel G_2 structures of type I, II, or III, respectively.

The classification of type III consists of the classification of all compact 3-Sasakian 7-manifolds. This is probably very hard. The case of 3-Sasakian 7-manifolds with vanishing $\mathfrak{aut}(M, \mathcal{S})$ appears quite difficult. The type II classification (7-dimensional Sasaki–Einstein manifolds which are not 3-Sasakian) is clearly completely out of reach at the moment. A classification of proper nearly parallel G_2 structures on a compact manifold that do not arise via Theorem 50 would be very interesting and it is not clear how hard this problem really is.

Remark 54. The holonomy Spin(7) cone metrics are plentiful but never complete. However, some of these metrics can be deformed to complete holonomy Spin(7) ones on non compact manifolds. The first example was obtained by Bryant and Salamon who observed that the spin bundle over S^4 with its canonical metric carries a complete metric with holonomy Spin(7) [36]. Locally the metric was later considered also in [69]. More generally, spin orbibundles over positive QK orbifolds also carry such complete orbifold metrics as observed by Bryant and Salamon in [36]. Other complete examples were constructed later by physicists [47], [48], [87], [88]. Finally, the first compact examples were obtained in 1996 by Joyce [82], [85]. See Joyce's book [84] for an excellent detailed exposition of the methods and the discussion of examples.

Open Problem 55 (Complete metrics on cones). Let $(M^7, \mathcal{S})$ be any 3-Sasakian 7-manifold and let (M^7, g') be the associated proper nearly parallel G_2 squashed metric. Consider the two Riemannian cones for these metrics.

(i) When does the metric cone $(C(M), dt^2 + t^2 g')$ admit complete holonomy Spin(7) deformations?

(ii) When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete holonomy Sp(2) (hyper-Kähler) deformations?

In other dimensions one also could ask the following more general questions:

(iii) Let $(M^{4n+3}, \mathcal{S})$ be a compact 3-Sasakian manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete hyper-Kähler (or just Calabi–Yau) deformations?

(iv) Let $(M^{2n+1}, \mathcal{S})$ be a compact Sasaki–Einstein manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete Calabi–Yau deformations?

(v) Let (M^7, g) be a compact nearly parallel G_2-manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete holonomy Spin(7) deformations?

(vi) Let (M^6, g) be a compact strict nearly Kähler manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete holonomy G_2 deformations?

The metric on the spin bundle $S(S^4)$ by Bryant and Salamon is a deformation of the Spin(7) holonomy metric on the cone over the squashed metric on S^7 [47], [48], so there are examples of such deformations regarding question (i). Regarding (ii), we recall that every compact 3-Sasakian 3-manifold is isometric to S^3/Γ and the metric cone is the flat cone $\mathbb{C}^2/\Gamma$. Hence, one could think of (ii) as a 7-dimensional analogue of a similar problem whose complete solution was given by Kronheimer [96]. There are non-trivial examples also in the higher dimensional cases. The metric cone on the homogeneous 3-Sasakian manifold $\mathcal{S}(1,1,1)$ of [30] admits complete hyper-Kähler deformations, namely the Calabi metric on $T^*\mathbb{CP}^2$. We do not know of any other example at the moment. In case (iv) it is not known whether there are any *complete* Calabi–Yau deformations. However, in [59] Futaki showed that there are deformations to complete constant scalar curvature metrics with scalar curvature either zero or negative.

8.2 Nearly Kähler 6-manifolds and G_2 holonomy cones

In this section we explain the geometry of the $\mathcal{Z}' \hookrightarrow C(\mathcal{Z}')$ part of the diagram (23). Before we specialize to dimension 6 we begin with a more general introduction. Nearly Kähler manifolds were first studied by Tachibana in [129] and they appear under the name of almost Tachibana spaces in Chapter VIII of the book [138]. They were then rediscovered by Gray [70] and given the name nearly Kähler manifolds which by now is the accepted name.

Definition 56. A *nearly Kähler manifold* is an almost Hermitian manifold (M, g, J, ω) such that $(\nabla_X J)X = 0$ for all tangent vectors X, where ∇ is the Levi-Civita connection and J is the almost complex structure. One says that a nearly Kähler manifold is *strict* if it is not Kähler.

This definition is equivalent to the condition

$$(\nabla_X J)Y + (\nabla_Y J)X = 0 \tag{26}$$

for all vector fields X, Y, which is to say that J is a *Killing tensor* field. An alternative characterization of nearly Kähler manifolds is given by

Proposition 57. *An almost Hermitian manifold (M, g, J, ω) is nearly Kähler if and only if*

$$\nabla\omega = \frac{1}{3}d\omega.$$

In particular, a strict nearly Kähler structure is never integrable.

Any nearly Kähler manifold can be locally decomposed as the product of a Kähler manifold and a strict nearly Kähler manifold. Such a decomposition is global in the simply connected case [112]. Hence, the study of nearly Kähler manifolds reduces to the case of strict ones. In addition every nearly Kähler manifold in dimension 4 must be Kähler so that the first interesting dimension is six.

The following theorem establishes relationship between the twistor space $\mathcal{Z} \to \mathcal{O}$ of a quaternionic Kähler manifold (orbifold) and nearly Kähler geometry.

Theorem 58. *Let $\pi : (\mathcal{Z}, h) \to (\mathcal{O}, g_{\mathcal{O}})$ be the twistor space of a positive QK manifold with its Kähler structure (J, h, ω_h). Then $\mathcal{Z}$ admits a strict nearly Kähler structure (J_1, h_1, ω_{h_1}). If $TM = \mathcal{V} \oplus \mathcal{H}$ is the natural splitting induced by π then*

$$h|_{\mathcal{V}} = 2h_1|_{\mathcal{V}}, \quad h|_{\mathcal{H}} = h_1|_{\mathcal{H}} = \pi^*(g_{\mathcal{O}}), \tag{27}$$

$$J|_{\mathcal{V}} = -J_1|_{\mathcal{V}}, \quad J|_{\mathcal{H}} = J_1|_{\mathcal{H}}\,. \tag{28}$$

Theorem 58 is due to Eells and Salamon [51] when $\mathcal{O}$ is 4-dimensional. The higher dimensional analogue was established in [4] (see also [112]).

Remark 59. Observe that the metric of the nearly Kähler structure of Theorem 58, in general, is *not* Einstein. In particular, h_1 is not the squashed metric h' introduced in the diagram (23), unless $\dim(\mathcal{Z}) = 6$. In six dimensions, we can scale h_1 so that it has scalar curvature $s = 30$ and then indeed $h_1 = h'$ as one can easily check.

Definition 60. Let $M = G/H$ be a homogeneous space. We say that M is 3-*symmetric* if G has an automorphism σ of order 3 such that $G_0^\sigma \subset H \subset G^\sigma$, where G^σ is the fixed point set of σ and G_0^σ is the identity component in G_0^σ.

We have the following two theorems concerning nearly Kähler homogeneous Riemannian manifolds. The first is due to Wolf and Gray in all dimensions but 6 [136], [137]. They also conjectured that the result is true for strict nearly Kähler 6-manifolds. The Wolf–Gray conjecture was proved quite recently by Butruille [38], [39] which is the second theorem below.

Theorem 61. *Every compact homogeneous strict nearly Kähler manifold M of dimension different than* 6 *is* 3*-symmetric.*

Theorem 62. *Let (M, g) be a strict nearly Kähler* 6*-dimensional Riemannian homogeneous manifold. Then M is isomorphic as a homogeneous space to a finite quotient of G/H, where G and H are one of the following:*

(1) $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ *and* $H = \{\mathrm{id}\}$*;*
(2) $G = G_2$ *and* $H = \mathrm{SU}(3)$*, where metrically* $G/H = S^6$ *the round sphere;*
(3) $G = \mathrm{Sp}(2)$ *and* $H = \mathrm{SU}(2)\mathrm{U}(1)$*, where* $G/H = \mathbb{CP}^3$ *with its nearly Kähler metric determined by Theorem* 58*;*
(4) $G = \mathrm{SU}(3)$ *and* $H = T^2$*, where G/H is the flag manifold with its nearly Kähler metric determined by Theorem* 58*.*

Each of these manifolds carries a unique invariant nearly Kähler structure, up to homothety.

In every dimension, the only known compact examples of nearly Kähler manifolds are 3-symmetric. On the other hand, Theorem 58 can be easily generalized to the case of orbifolds so that there are plenty examples of compact inhomogeneous strict nearly Kähler orbifolds in every dimension.

Theorem 63. *Let M be a compact simply-connected strict nearly Kähler manifold. Then, in all dimensions, as a Riemannian manifold M decomposes as a product of*

(1) 3*-symmetric spaces,*
(2) *twistor spaces of positive QK manifolds $\mathcal{Q}$ such that $\mathcal{Q}$ is not symmetric,*
(3) 6*-dimensional strict nearly Kähler manifold other than the ones listed in Theorem* 62*.*

This theorem is due to Nagy [111], but our formulation uses the result of Butruille together with the fact that the twistor spaces of all symmetric positive QK manifolds are 3-symmetric. The LeBrun–Salamon conjecture can now be phrased as follows

Conjecture 64. *Any compact simply connected strict irreducible nearly Kähler manifold (M, g) of dimension greater than* 6 *must be a* 3*-symmetric space.*

In particular, the Conjecture 64 is automatically true in dimensions $4n$ because of Nagy's classification theorem and also true in dimensions 10 and 14 because all positive QK manifolds in dimension 8 and 12 are known. The third case leads to an important

Open Problem 65. Classify all compact strict nearly Kähler manifolds in dimension 6.

Dimension 6 is special not just because of the rôle it plays in Theorem 63. They have several remarkable properties which we summarize in the following theorem.

Theorem 66. *Let (M, J, g, ω_g) be a compact strict nearly Kähler 6-manifold. Then*

(i) *The metric g is Einstein of positive scalar curvature.*

(ii) *$c_1(M) = 0$ and $w_2(M) = 0$.*

(iii) *If g is scaled so that it has Einstein constant $\lambda = 5$ then the metric cone $(C(M), dt^2 t + t^2 g)$ has holonomy contained in G_2. In particular, $C(M)$ is Ricci-flat.*

The first property is due to Matsumoto [106] while the second is due to Gray [72]. The last part is due to Bär [12]. In fact, nearly Kähler 6-manifolds is the geometry of the last row of the table of Theorem 40. More precisely we have the following theorem proved by Grunewald [73]:

Theorem 67. *Let (M^6, g) be a complete simply connected Riemannian spin manifold of dimension 6 admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. Then there are two possibilities:*

(i) *(M, g) is of type $(1, 1)$ and it is a strict nearly Kähler manifold,*

(ii) *$(M, g) = (S^6, g_{\text{can}})$ and is of type $(8, 8)$.*

Conversely, if (M, g) is a compact simply-connected strict nearly Kähler 6-manifold of non-constant curvature then M is of type $(1, 1)$.

Compact strict nearly Kähler manifolds with isometries were investigated in [108] where it was shown that

Theorem 68. *Let (M, J, g, ω_g) be a compact strict nearly Kähler 6-manifold. If M admits a unit Killing vector field, then up to finite cover M is isometric to $S^3 \times S^3$ with its standard nearly Kähler structure.*

Remark 69. The first example of a non-trivial G_2 holonomy metric was found by Bryant [33], who observed that a cone on the complex flag manifold $\mathrm{U}(3)/T^3$ carries an incomplete metric with G_2-holonomy. The flag $\mathrm{U}(3)/T^3$ is the twistor space of the complex projective plane $\mathbb{CP}^2$ and as such it also has a strict nearly Kähler structure. As explained in this section, this therefore is just one possible example. One gets such non-trivial metrics also for the cones with bases $\mathbb{CP}^3$ and $S^3 \times S^3$ with their homogeneous strict nearly Kähler structures. Interestingly, in some cases there exist complete metrics with G_2 holonomy which are smooth deformations of the asymptotically conical ones. This fact was noticed by Bryant and Salamon [36] who constructed complete examples of G_2 holonomy metrics on bundles of self-dual

2-forms over $\mathbb{C}\mathbb{P}^2$ and S^4. Replacing the base with any positive QK orbifold $\mathcal{O}$ gives complete (in the orbifold sense) metrics on orbibundles of self-dual 2-forms over $\mathcal{O}$. Locally some of these metrics were considered in [125]. More complete examples of explicit G_2 holonomy metrics on non-compact manifolds were obtained by Salamon [122]. G_2 holonomy manifolds with isometric circle actions were investigated by Apostolov and Salamon [7]. The first compact examples are due to the ground breaking work of Joyce [83].

9 Geometries associated with Sasaki–Einstein 5-manifolds

Like 3-Sasakian manifolds Sasaki–Einstein 5-manifolds are naturally associated to other geometries introduced in the previous section. Of course, each such space $(M^5, \mathcal{S})$ comes with its Calabi–Yau cone $(C(M), \bar{g})$ and, if the Sasaki–Einstein structure $\mathcal{S}$ is quasi-regular, with its quotient log del Pezzo surface $(\mathcal{Z}, h)$. But as it turns out, there are *two* more Einstein metrics associated to g. The examples of this section also illustrate how Theorem 40 and Bär's correspondence break down when (M, g) is a manifold with Killing spinors which is, however, *not complete*.

We begin by describing a relation between 5-dimensional Sasaki–Einstein structures and six-dimensional nearly Kähler structures which was uncovered recently in [52]. This relation involves the sine-cones of Definition 1. We use the notation $\bar{g}_s$ to distinguish the sine-cone metric from the usual Riemannian cone metric $\bar{g}$. Of course this metric is not complete, but one can compactify M obtaining a very tractable stratified space $\bar{M} = N \times [0, \pi]$ with conical singularities at $t = 0$ and $t = \pi$. Observe the following simple fact which shows that the Riemannian cone on a sine cone is always a Riemannian product.

Lemma 70. *Let (M, g) be a Riemannian manifold. Then the product metric $ds^2 = dx^2 + dy^2 + y^2 g$ on $\mathbb{R} \times C(M)$ can be identified with the iterated cone metric on $C(C_s(M))$.*

Proof. Consider the map $\mathbb{R}^+ \times (0, \pi) \to \mathbb{R} \times \mathbb{R}^+$ given by polar coordinate change $(r, t) \mapsto (x, y) = (r \cos t, r \sin t)$, where $r > 0$ and $t \in (0, \pi)$. We get

$$ds^2 = dx^2 + dy^2 + y^2 g = dr^2 + r^2 dt^2 + r^2 \sin^2 t g = dr^2 + r^2(dt^2 + \sin^2 t g). \quad \square$$

So the iterated Riemannian cone $(C(C_s(M)), ds^2)$ has reducible holonomy $1 \times \mathrm{Hol}(C(M))$. This leads to

Corollary 71. *Let (N, g) be a Sasaki–Einstein manifold of dimension $2n + 1$. Then the sine-cone $C_s(N)$ with the metric $\bar{g}_s = dr^2 + (\sin^2 r)g$ is Einstein with Einstein constant $2n + 1$.*

We are particularly interested in the case $n = 2$. Compare Lemma 70 with the following result in [84], Propositions 11.1.1-2:

Proposition 72. *Let* (M^4, g_4) *and* (M^6, g_6) *be Calabi–Yau manifolds, and let* $(\mathbb{R}^3, ds^2 = dx^2 + dy^2 + dz^2)$ *and* $(\mathbb{R}, ds^2 = dx^2)$ *be the Euclidean spaces. Then*

(1) $(\mathbb{R}^3 \times M^4, g = ds^2 + g_4)$ *has a natural* G_2 *structure and* g *has holonomy* $\mathrm{Hol}(g) \subset \mathbb{1}_3 \times \mathrm{SU}(2) \subset G_2$,

(2) $(\mathbb{R} \times M^6, g = ds^2 + g_6)$ *has a natural* G_2 *structure and* g *has holonomy* $\mathrm{Hol}(g) \subset 1 \times \mathrm{SU}(3) \subset G_2$.

As long as (M^4, g_4) and (M^6, g_6) are simply connected then the products $\mathbb{R}^3 \times M^4$ and $\mathbb{R} \times M^6$ are simply connected G_2-holonomy manifolds with reducible holonomy groups and parallel Killing spinors. Note that this does not violate Theorem 40 as these spaces are not Riemannian cones over complete Riemannian manifolds. Using (ii) of Proposition 72 we obtain the following corollary of Theorem 3 first obtained in [52]

Corollary 73. *Let* (N^5, g) *be a Sasaki–Einstein manifold. Then the sine cone* $C_s(N^5) = N^5 \times (0, \pi)$ *with metric* $\bar{g}_s$ *is nearly Kähler of Einstein constant* $\lambda = 5$. *Furthermore* $\bar{g}_s$ *approximates pure* SU(3) *holonomy metric near the cone points.*

Using Corollary 73 we obtain a host of examples of nearly Kähler 6-manifolds with conical singularities by choosing N^5 to be any of the Sasaki–Einstein manifolds constructed in [32], [31], [24], [95], [94], [66], [65], [49], [60], [44]. For example, in this way we obtain nearly-Kähler metrics on $N \times (0, \pi)$ where N is any Smale manifold with a Sasaki–Einstein metric such as S^5 or $k(S^2 \times S^3)$, etc. Note that every simply connected strict nearly Kähler manifold has exactly two real Killing spinors. So as long as N^5 is simply connected $C_s(N^5)$ will have two real Killing spinors. Using Theorem 3 the Sasaki–Einstein metrics constructed in [23], [27], [28], [68], [25] in all odd dimensions also give new Einstein metrics on $C_s(N^{2n+1})$. For example, one obtains many positive Einstein metrics on $\Sigma^{2n+1} \times (0, \pi)$ where Σ^{2n+1} is any odd dimensional homotopy sphere bounding a parallelizable manifold. Of course, there are no Killing spinors unless $n = 2$. Returning to the case of dimension 6, a somewhat more general converse has been obtained in [52], namely

Theorem 74. *Any totally geodesic hypersurface* N^5 *of a nearly Kähler* 6*-manifold* M^6 *admits a Sasaki–Einstein structure.*

The method in [52] uses the recently developed notion of hypo SU(2) structure due to Conti and Salamon [45]. The study of sine cones appears to have originated in the physics literature [20], [1], but in one dimension higher. Now recall the following result of Joyce (cf. [84], Propositions 13.1.2-3)

Proposition 75. *Let* (M^6, g_6) *and* (M^7, g_7) *be Calabi–Yau and* G_2*-holonomy manifolds, respectively. Let* $(\mathbb{R}^2, ds^2 = dx^2 + dy^2)$ *and* $(\mathbb{R}, ds^2 = dx^2)$ *be Euclidean spaces. Then*

(1) $(\mathbb{R}^2 \times M^6, g = ds^2 + g_6)$ *has a natural* Spin(7) *structure and* g *has holonomy* $\mathrm{Hol}(g) \subset \mathbb{1}_2 \times \mathrm{SU}(3) \subset \mathrm{Spin}(7)$,

(2) $(\mathbb{R} \times M^7, g = ds^2 + g_7)$ *has a natural* Spin(7) *structure and* g *has holonomy* $\mathrm{Hol}(g) \subset 1 \times G_2 \subset \mathrm{Spin}(7)$.

Again, if (M^6, g_6) and (M^7, g_7) are simply connected so are the Spin(7)-manifolds $\mathbb{R}^2 \times M^6$ and $\mathbb{R} \times M^7$ so that they have parallel spinors. Not surprisingly, in view of Lemma 70 and Proposition 75, the sine cone construction now relates strict nearly Kähler geometry in dimension 6 to nearly parallel G_2 geometry in dimension 7. More precisely [20]:

Theorem 76. *Let* (N^6, g) *be a strict nearly Kähler* 6*-manifold such that* g *has Einstein constant* $\lambda_6 = 5$. *Then the manifold* $C_s(N) = N^6 \times (0, \pi)$ *with its sine cone metric* $\bar{g}_s$ *has a nearly parallel* G_2 *structure with Einstein constant* $\lambda_7 = 6$ *and it approximates pure* G_2 *holonomy metric near the cone points.*

Proof. Just as before, starting with (N^6, g_6) we consider its metric cone $C(N^6)$ with the metric $\bar{g} = dy^2 + y^2 g_6$ and the product metric g_8 on $\mathbb{R} \times C(N^6)$. With the above choice of the Einstein constant we see that $g_8 = dx^2 + dy^2 + y^2 g_6$ must have holonomy $\mathrm{Hol}(g_8) \subset 1 \times G_2 \subset \mathrm{Spin}(7)$. By Lemma 70 g_8 is a metric cone on the metric $g_7 = dt^2 + \sin^2 t g_6$, which must, therefore, have weak G_2 holonomy and the Einstein constant $\lambda_7 = 6$. □

Again, any simply connected weak G_2-manifold has at least one Killing spinor. That real Killing spinor on $C_s(N^6)$ will lift to a parallel spinor on $C(C_s(N^6)) = \mathbb{R} \times C(N^6)$ which is a non-complete Spin(7)-manifold of holonomy inside $1 \times G_2$. One can iterate the two cases by starting with a compact Sasaki–Einstein 5-manifold N^5 and construct either the cone on the sine cone of N^5 or the sine cone on the sine cone of N^5 to obtain a nearly parallel G_2 manifold. We list the Riemannian manifolds coming from this construction that are irreducible.

Proposition 77. *Let* (N^5, g_5) *be a compact Sasaki–Einstein manifold which is not of constant curvature. Then the following have irreducible holonomy groups:*

(1) *the manifold* $C(N^5)$ *with the metric* $g_6 = dt^2 + t^2 g_5$ *has holonomy* SU(3)*;*

(2) *the manifold* $C_s(N^5) = N^5 \times (0, \pi)$ *with metric* $g_6 = dt^2 + \sin^2 t \; g_5$ *is strict nearly Kähler;*

(3) *the manifold* $C_s(C_s(N^5)) = N^5 \times (0, \pi) \times (0, \pi)$ *with the metric* $g_7 = d\alpha^2 + \sin^2 \alpha (dt^2 + \sin^2 t \; g_5)$ *has a nearly parallel* G_2 *structure.*

In addition we have the reducible cone metrics: $C(C_s(N^5)) = \mathbb{R} \times C(N^5)$ has holonomy in $1 \times \mathrm{SU}(3) \subset G_2$ and $C(C_s(C_s(N^5))) = \mathbb{R} \times C(C_s(N^5)) = \mathbb{R} \times \mathbb{R} \times C(N^5)$ has holonomy $\mathbb{1}_2 \times \mathrm{SU}(3) \subset 1 \times G_2 \subset \mathrm{Spin}(7)$. If N^5 is simply connected then g_5, g_6 and g_7 admit two Killing spinors. For a generalization involving conformal factors see [109].

Remark 78. Recall Remark 49. Note that when a nearly parallel G_2 metric is not complete then the type I-III classification is no longer valid. The group Spin(7) has other subgroups than the ones listed there and we can consider the following inclusions of (reducible) holonomies

$$\mathrm{Spin}(7) \supset G_2 \times 1 \supset \mathrm{SU}(3) \times \mathbb{1}_2 \times \supset \mathrm{SU}(2) \times \mathbb{1}_3 \supset \mathbb{1}_8.$$

According to the Friedrich–Kath Theorem 48 the middle three cannot occur as holonomies of Riemannian cones of complete 7-manifolds with Killing spinors. But as the discussion of this section shows, they most certainly can occur as holonomy groups of Riemannian cones of incomplete nearly parallel G_2 metrics. These metrics can be still separated into three types depending on the holonomy reduction: say the ones that come from strict nearly Kähler manifolds are generically of type I_s while the ones that come from Sasaki–Einstein 5-manifolds via the iterated sine cone construction are of type II_s and of type III_s when $H \subset \mathrm{SU}(3)$ is some proper non-trivial subgroup. On the other hand, it is not clear what is the relation between the holonomy reduction and the actual number of Killing spinors one gets in each case.

10 Geometric structures on manifolds and supersymmetry

The intricate relationship between supersymmetry and geometric structures on manifolds was recognized along the way the physics of supersymmetry slowly evolved from its origins: first globally supersymmetric field theories ('70s) arose, later came supergravity theory ('80s), which evolved into superstring theory and conformal field theory (late '80s and '90s), and finally into M-theory and the supersymmetric branes of today. At every step the "first" theory would quickly lead to various generalizations creating many different new ones: so it is as if after discovering plain vanilla ice cream one would quickly find oneself in an Italian ice cream parlor confused and unable to decide which flavor was the right choice for the hot afternoon. This is a confusion that is possibly good for one's sense of taste, but many physicists believe that there should be just one theory, the Grand Unified Theory which describes our world at any level.[4] An interesting way out of this conundrum is to suggest that even if two theories appear to be completely different, if both are consistent and admissible, they actually *do* describe the same physical world and, therefore, *they should be dual* to one another in a certain sense. This gave rise to various duality conjectures such as the Mirror Symmetry Conjecture or the AdS/CFT Duality Conjecture.

[4]Actually, string theory of today appears to offer a rather vast range of vacua (or possible universes). Such possible predictions have been nicknamed the *string landscape* [127]. This fact has been seen as a drawback by some, but not all, physicists (see more recent discussion on *landscape* and *swampland* in [131], [114]). The insistence that the universe we experience, and this on such a limited scale at best, *is the only Universe*, is largely a matter of 'philosophical attitude' towards science. See the recent book of Leonard Susskind on the anthropic principle, string theory and the cosmic landscape [128].

The first observation of how supersymmetry can restrict the underlying geometry was due to Zumino [139] who discovered that globally $N = 1$ supersymmetric σ-models in $d = 4$ dimensions require that the bosonic fields (particles) of the theory are local coordinates on a Kähler manifold. Later Alvarez-Gaumé and Friedman observed that $N = 2$ supersymmetry requires that the σ-model manifold be not just Kähler but hyper-Kähler [6]. This relation between globally supersymmetric σ-models and complex manifolds was used by Lindström and Roček to discover the hyper-Kähler quotient construction in [100], [80].

The late seventies witnessed a series of attempts to incorporate gravity into the picture which quickly led to the discovery of various supergravity theories. Again the $N = 1$ supergravity-matter couplings in $d = 4$ dimensions require bosonic matter fields to be coordinates on a Kähler manifold with some special properties [135] while $N = 2$ supergravity demands that the σ-model manifold be quaternionic Kähler [11]. The quaternionic underpinnings of the matter couplings in supergravity theories led to the discovery of quaternionic Kähler reduction in [61], [62].

At the same time manifolds with Killing spinors emerged as important players in the physics of the supergravity theory which in $D = 11$ dimensions was first predicted by Nahm [113] and later constructed by Cremmer, Julia and Scherk [46]. The well-known Kaluza–Klein trick applied to a $D = 11$ supergravity model is a way of constructing various limiting *compactifications* which would better describe the apparently four-dimensional physical world we observe. The geometry of such a compactification is simply a Cartesian product $\mathbb{R}^{3,1} \times M^7$, where $\mathbb{R}^{3,1}$ is the Minkowski space-time (or some other Lorentzian 4-manifold) and M^7 is a compact manifold with so small a radius that its presence can only be felt and observed at the quantum level. Many various models for M^7 were studied in the late seventies which by the eighties had already accrued into a vast physics literature (cf. the extensive three-volume monograph by Castellani, D'Auria and Fré [43]). Most of the models assumed a homogeneous space structure on $M^7 = G/H$ (see Chapter V.6 in [43], for examples). Two things were of key importance in terms of the required physical properties of the compactified theory. First, the compact space M^7, as a Riemannian manifold, had to be Einstein of positive scalar curvature. Second, although one could consider any compact Einstein space for the compactification, the new theory would no longer be supersymmetric unless (M^7, g) admitted Killing spinor fields, and the number of them would be exactly the number of *residual* supersymmetries of the compactified theory. For that reason compactification models involving (S^7, g_0) were quite special as they gave the maximally supersymmetric model. However, early on it was realized that there are other, even homogeneous, 7-manifolds of interest. The Sp(2)-invariant Jensen metric on S^7, or as physicists correctly nicknamed it, the *squashed 7-sphere* is one of the examples. Indeed, Jensen's metric admits exactly one Killing spinor field since it has a nearly parallel G_2 structure. Of course, any of the Einstein geometries in the table of Theorem 40 can be used to obtain such supersymmetric models.

The $D = 11$ supergravity theory only briefly looked liked it was the Grand Theory of Einstein's dream. It was soon realized that there are difficulties with getting from

$D = 11$ supergravity to the standard model. The theory which was to solve these and other problems was superstring theory and later M-theory (which is yet to be constructed). With the arrival of superstring theory and M-theory, supersymmetry continues its truly remarkable influence on many different areas of mathematics and physics: from geometry to analysis and number theory. For instance, once again five-, six-, and seven-dimensional manifolds admitting real Killing spinors have become of interest because of the so called AdS/CFT duality. Such manifolds have emerged naturally in the context of p-brane solutions in superstring theory. These so-called p-branes, "near the horizon" are modelled by the pseudo-Riemannian geometry of the product $\mathrm{AdS}_{p+2} \times M$, where AdS_{p+2} is the $(p+2)$-dimensional anti-de-Sitter space (a Lorentzian version of a space of constant sectional curvature) and (M, g) is a Riemannian manifold of dimension $d = D - p - 2$. Here D is the dimension of the original supersymmetric theory. In the most interesting cases of M2-branes, M5-branes, and D3-branes D equals either 11 (Mp-branes of M-theory) or 10 (Dp-branes in type IIA or type IIB string theory). String theorists are particularly interested in those vacua of the form $\mathrm{AdS}_{p+2} \times M$ that preserve some residual supersymmetry. It turns out that this requirement imposes constraints on the geometry of the Einstein manifold M which is forced to admit real Killing spinors. Depending on the dimension d, the possible geometries of M are as follows:

d	Geometry of M	$(\mu, \bar{\mu})$
any	round sphere	$(1, 1)$
7	nearly parallel G_2	$(\frac{1}{8}, 0)$
	Sasaki–Einstein	$(\frac{1}{4}, 0)$
	3-Sasakian	$(\frac{3}{8}, 0)$
6	nearly Kähler	$(\frac{1}{8}, \frac{1}{8})$
5	Sasaki–Einstein	$(\frac{1}{4}, \frac{1}{4})$

where the notation $(\mu, \bar{\mu})$, which is common in the physics literature, represents the ratio of the number of real Killing spinors of type (p, q) to the maximal number of real Killing spinors that can occur in the given dimension. This maximum is, of course, realized by the round sphere of that dimension. So this table is just a translation of the table of Theorem 40 for the special dimensions that occur in the models used by the physicists.

Furthermore, given a p-brane solution of the above type, the interpolation between $\mathrm{AdS}_{p+2} \times M$ and $\mathbb{R}^{p,1} \times C(M)$ leads to a conjectured duality between the supersymmetric background of the form $\mathrm{AdS}_{p+2} \times M$ and a $(p+1)$-dimensional superconformal field theory of n coincident p-branes located at the conical singularity of the $\mathbb{R}^{p,1} \times C(M)$ vacuum. This is a generalized version of the Maldacena or AdS/CFT Conjecture [101]. In the case of D3-branes of string theory the relevant near

horizon geometry is that of $\mathrm{AdS}_5 \times M$, where M is a Sasaki–Einstein 5-manifold. The D3-brane solution interpolates between $\mathrm{AdS}_5 \times M$ and $\mathbb{R}^{3,1} \times C(M)$, where the cone $C(M)$ is a Calabi–Yau threefold. In its original version the Maldacena conjecture (also known as AdS/CFT duality) states that the 't Hooft large n limit of $N = 4$ supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(n)$ is dual to type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ [101]. This conjecture was further examined by Klebanov and Witten [91] for the type IIB theory on $\mathrm{AdS}_5 \times T^{1,1}$, where $T^{1,1}$ is the other homogeneous Sasaki–Einstein 5-manifold $T^{1,1} = S^2 \times S^3$ and the Calabi–Yau 3-fold $C(T^{1,1})$ is simply the quadric cone in $\mathbb{C}^4$. Using the well-known fact that $C(T^{1,1})$ is a Kähler quotient of $\mathbb{C}^4$ (or, equivalently, that $S^2 \times S^3$ is a Sasaki–Einstein quotient of S^7), a dual super Yang–Mills theory was proposed, representing D3-branes at the conical singularities. In the framework of D3-branes and the AdS/CFT duality the question of what are all the possible near horizon geometries M and $C(M)$ might be of importance. Much of the interest in Sasaki–Einstein manifolds is precisely due to the fact that each such explicit metric, among other things, provides a useful model to test the AdS/CFT duality. We refer the reader interested in the mathematics and physics of the AdS/CFT duality to the recent book in the same series [21]. In particular, in this context, Sasaki–Einstein geometry is discussed in one of the articles there [67].

Remark 79 (G_2 holonomy manifolds, unification scale and proton decay). Until quite recently the interest in 7-manifolds with G_2 holonomy as a source of possible physical models was tempered by the fact the Kaluza–Klein compactifications on smooth and complete manifolds of this type led to models with no charged particles. All this has dramatically changed in the last few years largely because of some new developments in M-theory. Perhaps the most compelling reasons for reconsidering such 7-manifolds was offered by Atiyah and Witten who considered the dynamics on manifolds with G_2 holonomy which are asymptotically conical [10]. The three models of cones on the homogeneous nearly Kähler manifolds mentioned earlier are of particular interest, but Atiyah and Witten consider other cases which include orbifold (quotient) singularities. Among other things they point to a very interesting connection between Kronheimer's quotient construction of the ALE metrics [96], [97] and asymptotically conical manifolds with G_2-holonomy. To explain the connection, consider Kronheimer's construction for $\Gamma = \mathbb{Z}_{n+1}$. Suppose one chooses a circle $S^1_{k,l} \simeq \mathrm{U}(1) \subset K(\mathbb{Z}_{n+1}) = \mathrm{U}(1)^n$ and then one considers a 7-manifold obtained by performing Kronheimer's HK quotient construction with zero momentum level ($\boldsymbol{\xi} = 0$) while "forgetting" the three moment map equations corresponding to this particular circle. An equivalent way of looking at this situation is to take the Kronheimer quotient with nonzero momentum $\boldsymbol{\xi} = a \in \mathfrak{sp}(1)$ but only for the moment map of the chosen circle $S^1_{k,l}$ (such $\boldsymbol{\xi}$ is never in the "good set") and then consider the fibration of singular Kronheimer quotients over a 3-dimensional base parameter space. Algebraically this corresponds to a partial resolution of the quotient singularity and this resolution depends on the choice of $S^1_{k,l}$, hence $\boldsymbol{\xi}$. This example was first introduced in [10]. It can be shown that the 7-manifold is actually a cone on the complex weighted projective 3-space with weights (k, k, l, l),

where $k + l = n + 1$. It then follows from the physical model considered that such a cone should admit a metric with G_2 holonomy. However, unlike the homogeneous cones over the four homogeneous strict nearly Kähler manifolds of Theorem 62, the metric in this case is not known explicitly. This construction appears to differ from all previous geometric constructions of metrics with G_2 holonomy. One can consider similar constructions for other choices of $S^1 \subset K(\Gamma)$ [17].

In [53] using specific models of M-theory compactifications on manifolds with G_2 holonomy, Friedman and Witten address the fundamental questions concerning the unification scale (*i.e.*, the scale at which the Standard Model of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ unifies in a single gauge group) and proton decay. The authors point out that the results obtained are model dependent, but some of the calculations and conclusions apply to a variety of different models.

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Chapter 4

Special geometry for arbitrary signatures

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1 Introduction

Special Kähler geometry is the geometry of the manifold spanned by the scalars of vector multiplets of $D = 4$, $N = 2$ supersymmetry. The *rigid* version, that occurs in supersymmetry without gravity, appeared first in the references [1] and [2]. The construction for $N = 2$ supergravity appeared in [3], and it is called *projective special Kähler geometry*. It played an important role in several developments of string theory.

These first formulations offered a local point of view. For the rigid case, the condition for a Kähler geometry to be 'special' is the existence of a preferred set of holomorphic coordinates z^i, called *special coordinates*, in which the Kähler potential $\mathcal{K}$ can be expressed in terms of a holomorphic function, the *prepotential* $\mathcal{F}$,

$$g_{i\bar{\jmath}} = \frac{\partial \mathcal{K}}{\partial z^i \partial \bar{z}^{\bar{\jmath}}}, \quad \mathcal{K} = 2\Im\left(\frac{\partial \mathcal{F}}{\partial z^k}\bar{z}^{\bar{k}}\right). \tag{1}$$

It is seen then as a further restriction on the metric, compatible with the complex structure. For the projective case, the original construction was based on superconformal tensor calculus and involves a projectivization of the manifold due to the extra vector field, the graviphoton, which does not have associated a scalar. In simple words, one has a rigid special manifold with a dilation symmetry and a non physical scalar, which is projected out by fixing the symmetry.

The property of being a special Kähler manifold is then a purely geometrical one, and can be formulated independently of supersymmetry. It is given though in terms of a preferred set of coordinates. Although this local formulation is not incomplete (there has to exists an *open cover* of the manifold by special coordinates), it remains the intriguing question if there is a way of defining what is a special Kähler manifold with global statements, independent of coordinates. The first attempts were made in references [4], [5], [6]. A set of equivalent definitions was found in [7], and later on, a mathematical formulation appeared in [8].

One fundamental ingredient in the global approach is the existence of a certain flat *symplectic bundle*. Peculiar to Freed's formulation [8] is that the symplectic bundle is recognised as the tangent bundle, so the construction is *intrinsic*. In fact, the rigid case (see Definition 3.1) comes out very elegantly, and for this part we will follow closely Freed's work (with the exception of the pseudo-Riemannian case, which we will mention later). The projective case is much more involved. We define a projective special manifold in terms of a rigid special manifold with a *homothetic Killing vector* (see Definition 4.2). In this way, the definition is not only intrinsic but directly related to the way in which it is obtained in supergravity [3]. The point of contact of this definition with Freed's work is in his Proposition 4.6.

So far as for Riemannian, special Kähler manifolds. Pseudo-Riemannian special Kähler manifolds[1] are very relevant in supergravity. A physically sensible supergravity

[1]Note that all discussions on the signature in this work concern the signature of the Kähler manifold, i.e. the target manifold of the supergravity theory. This is unrelated to the signature of spacetime, which we keep Minkowskian to have the standard special geometries. Discussions on generalizations to Euclidean spacetime signature are in references [9], [10], [11].

theory must have a positive definite target-space metric. From the conformal calculus approach it is known that in order to get such positive definite metric the rigid Kähler manifold before projection has to have signature $(2, 2n)$. But pseudo-Riemannian special Kähler manifolds present an additional complication. Special coordinates are complex coordinates constructed from a set of flat Darboux coordinates (q^i, p_i) by taking the holomorphic extension of the q^i's (or, alternatively, of the $p_i's$). They have then the prepotential property (1). When the signature of the metric is indefinite, this holomorphic extension does not always result in a set of n independent holomorphic coordinates. There is a subclass of Darboux systems that have this property. It is important thought that one can always make a constant symplectic rotation to coordinates (q'^i, p'_i) such that the q'^i's extend to special coordinates, so there is still a covering of the manifold by special coordinates. But not all flat Darboux systems are suitable to obtain special coordinates. As a consequence, the structure group of the bundle is reduced to a subgroup of the symplectic group. This was first observed in [12]. Nevertheless, flat Darboux coordinates which do not lead to special coordinates nor prepotential are very relevant. They were used to prove that one can break $N = 2$ supersymmetry partially to $N = 1$ [13] and not necessarily to $N = 0$, as it was thought before. This is an extremely important property for phenomenological applications. It is then one of the main motives of this work (which was missing in [8]) to generalize the construction of special geometry to arbitrary signatures.

In another context it has been recently shown ([14], [15], [16]) that relating flat Darboux coordinates with the real central charges and attractor equations would have a simplifying role in the description of the attractor mechanism of black holes in $N = 2$ supergravity (see references [17], [18], [19] for the attractor mechanism).

Pseudo-Riemannian, projective special Kähler manifolds appear also as dimensional reductions of supergravity theories in eleven dimensions and exotic signatures, obtained by duality transformations from the standard Minkowskian signature. These are the theories M^* and M', in signatures (9,2) and (6,5) proposed in [20]. The pseudo-Riemannian special manifolds arising in $D = 4$ are discussed in [21].

An important part of the work in dealing with projective special Kähler manifolds concerns in fact a more general class of Kähler manifolds, the so-called Kähler–Hodge manifolds.[2] It was found in [22] that the Kähler geometries of $N = 1$ supergravity should be Kähler–Hodge. We propose an intrinsic definition of projective Kähler manifolds (see Definition 4.2), inspired in the conformal calculus approach used in physics. Then we show that they have integer Kähler cohomology class, so they are Kähler–Hodge.

This chapter is as much self-contained as possible, so we have included vast review material. On the other hand, having in mind the connection to physics, we have tried to work everything out in coordinates, as to have the sometimes difficult translation between two languages, the physicist's and the mathematician's one, each of them with its own advantages. We have also taken time in explaining some examples, which may clarify the abstract definitions.

[2] A Kähler–Hodge manifold is a Kähler manifold with integer Kähler cohomology class.

The chapter is organized as follows.

In Section 2 we review basic material on complex and Kähler manifolds and Hermitian bundles. It is used extensively in the exposition, so with it we set the basic notation. The reader can also skip it and come back to it punctually when some concept is called for.

Section 3 is devoted to the rigid case. We start with some geometric preliminaries not included in Section 2 and then we take up the definition of rigid special Kähler manifolds. We follow the lines of reference [8], explaining carefully how the formulas in coordinates are obtained from the abstract definition. Then we treat the pseudo-Riemannian case, giving some clarifying simple examples. We come back to Freed's treatment for the holomorphic cubic form, which will be used later on.

Section 4 is dedicated to projective Kähler manifolds as a previous step towards projective special geometry. We introduce some concepts on affine transformations and homothetic Killing vectors and derive some of its properties. This material is not new, but perhaps not so widely known, so it is fundamental to have it at hand. Then we define projective Kähler manifolds starting from a Kähler manifold $\widetilde{\mathcal{M}}$ (with arbitrary signature) which has an action of $\mathbb{C}^\times$ (as well as other properties, see Definition 4.2). We then consider $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{C}^\times$ and construct the symplectic and the line bundles over it. The line bundle has a Hermitian metric induced from the Kähler metric on $\widetilde{\mathcal{M}}$ (here the importance of the intrinsic approach), whose Ricci form turns out to be closed and non degenerate, defining then a symplectic structure (actually, a Kähler one). Since it is the first Chern class of a line bundle, the manifold is Kähler–Hodge.

We then propose an alternative and beautiful way of understanding the Levi-Civita connection in $\mathcal{M}$, directly induced from the one in $\widetilde{\mathcal{M}}$. It is a bit involved, but it really gives precious insight into the geometry of $\mathcal{M}$.

Section 5 is a brief excursion on the origin of projective Kähler geometry as it is seen from a model in physics. It is the simplest one to consider, and it does not include supersymmetry. Indeed, the ideas of conformal calculus are more general than their applications to supergravity.

In Section 6 we impose on $\widetilde{\mathcal{M}}$ the condition to be rigid special Kähler, then $\mathcal{M}$ will be a projective special Kähler manifold. The precise definition is Definition 6.1, and the consequences are analysed in the sequel. In particular, we obtain the holomorphic cubic form and then the formula for the curvature. We conclude with some examples, in particular the pseudo-Riemannian space

$$\frac{\mathrm{SU}(1,2)}{\mathrm{SU}(1,1)\times\mathrm{U}(1)}.$$

2 Kähler manifolds

This first section recapitulates the basic definitions on complex manifolds and Kähler manifolds in particular. It is essentially a summary of part of Chapter IX in [23].

It can be skipped by readers familiar with Kähler manifolds or used just to set the conventions.

2.1 Generalities on quasicomplex and complex manifolds

Let $\mathcal{M}$ be a quasicomplex (or almost complex) manifold of dimension $2n$, with $J : T\mathcal{M} \mapsto T\mathcal{M}$ the quasicomplex structure, $J^2 = -1$.

Remark 2.1. Suppose that $\mathcal{M}$ is a complex manifold and that $(z^1, \ldots, z^n)$ are complex coordinates on an open set $U \subset \mathcal{M}$, $z^j = x^j + \mathrm{i}y^j$. Then $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ is a coordinate system in U and we have that

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}, \quad J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}, \quad j = 1, \ldots, n.$$

Let $T_m^c\mathcal{M}$ denote the complexification of the tangent space at $m \in \mathcal{M}$. We denote by $T_m^{1,0}\mathcal{M}$ and $T_m^{0,1}\mathcal{M}$ the eigenspaces of J at m with eigenvalues i and $-$i respectively. Then

$$Z = X - \mathrm{i}JX \in T_m^{1,0}\mathcal{M}, \quad \overline{Z} = X + \mathrm{i}JX \in T_m^{0,1}\mathcal{M}$$

for any real vector $X \in T_m\mathcal{M}$. The operation $Z \mapsto \overline{Z}$ is a real linear endomorphism called *complex conjugation*. From now on we will denote the (complexified) tangent space simply as $T_m = T_m\mathcal{M}$.

Let T_m^{*c} denote the complexification of the cotangent space at m and T^{*c} the complexified cotangent bundle of $\mathcal{M}$. Let $\omega \in T^*$. The pull back, at each point m, of ω through J,

$$J^*\omega_m(X) = \omega_m(JX) \quad \text{for all } X \in T_m,$$

defines an endomorphism

$$J^* : T^* \to T^*$$

with $(J^*)^2 = -1$, which extends in the obvious way to the complexified cotangent space. The eigenspaces of eigenvalues i and $-$i of J^* at m are denoted as

$$\Omega_m^{1,0} = \Lambda^{1,0}T_m^{*c} \quad \text{and} \quad \Omega_m^{0,1} = \Lambda^{0,1}T_m^{*c}$$

respectively. One has that

$$\Omega_m^{0,1} = \{\omega_m \in T_m^{*c} \mid \omega_m(Z) = 0 \text{ for all } Z \in T_m^{1,0}\},$$
$$\Omega_m^{1,0} = \{\omega_m \in T_m^{*c} \mid \omega_m(Z) = 0 \text{ for all } Z \in T_m^{0,1}\}.$$

Since the exterior product space, $\Omega_m = \sum_{r=0}^{2n} \Lambda^r T_m^{*c}$, is generated by $\Omega_m^{0,0}$, $\Omega_m^{1,0}$ and $\Omega_m^{0,1}$, Ω_m has a bigrading

$$\Omega_m = \sum_{p,q=0}^{n} \Omega_m^{p,q},$$

and so has the space of complex forms

$$\Omega = \sum_{p,q=0}^{n} \Omega^{p,q}.$$

Remark 2.2. If $\mathcal{M}$ is a complex manifold as in Remark 2.1, then

$$T_m^{1,0} = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z^j}\Big|_m = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial y^j} \right)\Big|_m \right\}_{j=1}^{n},$$
$$T_m^{0,1} = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}^{\bar{j}}}\Big|_m = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \mathrm{i}\frac{\partial}{\partial y^j} \right)\Big|_m \right\}_{j=1}^{n}.$$

For the complex forms we have

$$\Omega_m^{1,0} = \operatorname{span}_{\mathbb{C}} \left\{ \mathrm{d}z^j|_m = (\mathrm{d}x^j + \mathrm{i}\mathrm{d}y^j)|_m \right\}_{j=1}^{n},$$
$$\Omega_m^{0,1} = \operatorname{span}_{\mathbb{C}} \left\{ \mathrm{d}\bar{z}^{\bar{j}}|_m = (\mathrm{d}x^j - \mathrm{i}\mathrm{d}y^j)|_m \right\}_{j=1}^{n}.$$

The set of forms

$$\left\{ \mathrm{d}z^{j_1} \wedge \mathrm{d}z^{j_2} \wedge \cdots \wedge \mathrm{d}z^{j_p} \wedge \mathrm{d}\bar{z}^{\bar{k}_1} \wedge \mathrm{d}\bar{z}^{\bar{k}_2} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{k}_q} \right\}$$

with $1 \le j_1 \le \cdots \le j_p \le n$, $1 \le \bar{k}_1 \le \cdots \le \bar{k}_q \le n$ is a local basis of $\Omega^{p,q}$.

For a complex manifold one can prove [23] that the differential

$$\mathrm{d}\Omega^{p,q} \subset \Omega^{p+1,q} + \Omega^{p,q+1}.$$

Then we can define $\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}$ as

$$\mathrm{d} = \partial + \bar{\partial},$$

and since $\mathrm{d}^2 = 0$ we have

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0.$$

A form $\omega \in \Omega^{p,0}$ is said to be holomorphic if $\bar{\partial}\omega = 0$. A form $\omega \in \Omega^{0,p}$ is said to be antiholomorphic if $\partial\omega = 0$. A function is holomorphic if

$$\frac{\partial}{\partial \bar{z}^{\bar{j}}} f = 0, \quad j = 1, \ldots, n$$

(respectively, antiholomorphic). A holomorphic vector field Z is a complex vector field of type $(p, 0)$ such that Zf is holomorphic for every holomorphic f. Locally,

$$Z = \sum_{j=1}^{n} f^j \frac{\partial}{\partial z^j}$$

with all the f^j holomorphic.

2.2 Hermitian metrics and Kähler metrics

A *Hermitian metric* on a quasicomplex manifold $\mathcal{M}$ with quasicomplex structure J is a Riemannian metric g such that

$$g(JX, JY) = g(X, Y) \quad \text{for all } X, Y \in T\mathcal{M}.$$

Every paracompact quasicomplex manifold admits a Hermitian metric. This is because for a given Riemannian metric h and a quasicomplex structure J we can obtain a Hermitian metric by setting

$$g(X, Y) = h(X, Y) + h(JX, JY) \quad \text{for all } X, Y \in T\mathcal{M};$$

g is extended to T^c by linearity. It is easy to check that

1. $g(Z, W) = 0$ for Z, W of type $(1, 0)$,
2. $g(Z, \bar{Z}) > 0$,
3. $g(\bar{Z}, \bar{W}) = \overline{g(Z, W)}$.

The fundamental 2-form of a Hermitian metric is

$$\Phi(X, Y) = g(X, JY) \quad \text{for all } X, Y \in T\mathcal{M}.$$

It is non degenerate at each point of the manifold.

Remark 2.3 (Almost complex linear connections). The *torsion* of a quasicomplex structure J is the tensor field (1-covariant, 2-contravariant)

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\}.$$

A quasicomplex structure is said to be integrable if it has no torsion. This is equivalent to saying that the commutator of two vector fields of type $(1, 0)$ (alternatively $(0, 1)$) is a vector field of type $(1, 0)$ (alternatively $(0, 1)$). To see this, let Z, W be such that $JZ = \mathrm{i}Z$ and $JW = \mathrm{i}W$, then if $N(Z, W) = 0$ it is immediate that $J[Z, W] = \mathrm{i}[Z, W]$. In the other direction, a real vector field can be always written as the sum $X = Z + \bar{Z}$, where Z is $(1, 0)$ and $\bar{Z}$ is $(0, 1)$. Let also $Y = W + \bar{W}$. Then it is immediate to prove that $N(X, Y) = 0$.

A quasicomplex structure is a complex structure if and only if it is integrable. This is the Newlander–Nirenberg theorem [24].

We say that a linear connection is quasicomplex if the covariant derivative of the quasicomplex structure is zero (which is equivalent to being a connection in the bundle of complex linear frames). Every quasicomplex manifold admits a quasicomplex affine connection whose torsion T is proportional to the torsion N of the quasicomplex structure.

In general, the Riemannian connection associated to a Hermitian metric is not quasicomplex. If it is so, then the quasicomplex structure has no torsion and the fundamental form is closed. The converse is also true: for a complex manifold,

the Riemannian connection of a Hermitian metric is quasicomplex if and only if the fundamental 2-form Φ is closed. (The proof of these statements can be read in [23], Chapter IX.)

A quasicomplex manifold, with a Hermitian metric is a *quasi-Kähler* (or almost Kähler) manifold if the fundamental form is closed.

Let $\mathcal{M}$ be a differential manifold. A *symplectic structure* on $\mathcal{M}$ is a 2-form Φ such that

i. it is closed, $\mathrm{d}\Phi = 0$,

ii. it is non degenerate: for every $X \in T\mathcal{M}$, there exists $Y \in T\mathcal{M}$ such that $\Phi(X, Y) \neq 0$.

The couple $(\mathcal{M}, \Phi)$ is a *symplectic manifold*, and $\mathcal{M}$ has always even dimension. In any symplectic manifold, we have local *Darboux coordinates* defined by the following theorem.

Theorem 2.4 (Darboux). *If $\mathcal{M}$ is a symplectic manifold,* $\dim \mathcal{M} = 2n$*, for each $m \in \mathcal{M}$ there is a chart $(U, \varphi\colon U \to \mathbb{R})$ such that $\varphi(m) = 0$ and for $u \in U$, $\varphi(u) = \big(x^1(u), \ldots, x^n(u), y_1(u), \ldots, y_n(u)\big)$ and Φ on the open set U is*

$$\Phi\big|_U = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y_i. \qquad \square$$

Notice that a *quasi-Kähler* manifold is a *symplectic manifold*, since the fundamental 2-form is non degenerate. If, in addition, the manifold is complex then it is a *Kähler manifold*. Moreover, let D be a Riemannian connection, so $D_X g = 0$ for every vector field X on $\mathcal{M}$. We have that

$$D_X \Phi(Y, Y') = D_X g(Y, J\,Y') + g(Y, (D_X J)Y') = 0, \tag{2}$$

which means that the Riemannian connection is trivially a *symplectic connection*.

The holonomy of a Kähler manifold of complex dimension n is a subgroup of $\mathrm{U}(n) \simeq \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C})$, since the Riemannian connection is quasicomplex. Here $\mathrm{GL}(n, \mathbb{C})$ is taken in its real representation

$$A + \mathrm{i}B \longrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

One can prove that if the manifold is Ricci flat then the *restricted holonomy group* (that is, considering only parallel displacements along paths that are homotopic to a point) is contained in $\mathrm{SU}(n)$.

Kähler manifolds in coordinates. On a quasicomplex manifold, we can consider the principal bundle of unitary frames, that is the bundle of complex frames that are orthonormal with respect to the Hermitian metric. Its structural group is $\mathrm{U}(n)$. We will denote this bundle by $\mathrm{U}(\mathcal{M})$. We want to give the metric, connection and curvature of a Kähler manifold in coordinates.

Let $\mathcal{M}$ be a complex manifold, with Hermitian metric g and complex structure J. We use the notation of Remarks 2.1, and 2.2 and denote

$$Z_i = \frac{\partial}{\partial z^i}, \quad \bar{Z}_i = Z_{\bar{\imath}} = \frac{\partial}{\partial \bar{z}^i}.$$

It is easy to see that

$$g_{ij} = g(Z_i, Z_j) = 0, \quad g_{\bar{\imath}\bar{\jmath}} = g(\bar{Z}_i, \bar{Z}_j) = 0, \tag{3}$$

and the only non zero components are of the form $g_{i\bar{\jmath}} = g(Z_i, \bar{Z}_j) = g_{\bar{\jmath}i}$, so[3]

$$g = g_{i\bar{\jmath}}(\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{\jmath}} + \mathrm{d}\bar{z}^{\bar{\jmath}} \otimes \mathrm{d}z^i) = 2g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{\jmath}}.$$

Since $J(Z_i) = \mathrm{i}Z_i$ and $J(\bar{Z}_{\bar{\imath}}) = -\mathrm{i}\bar{Z}_{\bar{\imath}}$, the fundamental 2-form is

$$\Phi = -2\mathrm{i}\, g_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}. \tag{4}$$

If $\mathcal{M}$ is a Kähler manifold, the fundamental form is closed, so

$$\mathrm{d}\Phi = -2\mathrm{i}\left(\frac{g_{i\bar{\jmath}}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} + \frac{g_{i\bar{\jmath}}}{\partial \bar{z}^{\bar{k}}}\mathrm{d}\bar{z}^{\bar{k}} \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}\right) = 0,$$

which implies

$$\frac{g_{i\bar{\jmath}}}{\partial z^k} - \frac{g_{k\bar{\jmath}}}{\partial z^i} = 0, \quad \frac{g_{i\bar{\jmath}}}{\partial \bar{z}^{\bar{k}}} - \frac{g_{i\bar{k}}}{\partial \bar{z}^{\bar{\jmath}}} = 0. \tag{5}$$

These equations are the integrability condition for the existence of a real valued function $\mathcal{K}$ such that

$$g_{i\bar{\jmath}} = \frac{\partial \mathcal{K}}{\partial z^i \bar{z}^{\bar{\jmath}}}.$$

For any real function $\mathcal{K}$, the tensor $g_{i\bar{\jmath}}$ satisfies $(g_{i\bar{\jmath}})^* = g_{j\bar{\imath}}$ and property (5). If it is positive definite, then it is a Kähler metric on $\mathcal{M}$. So any Kähler metric can be written locally in this way. Notice also that $\mathcal{K}$ is defined modulo a holomorphic function f,

$$\mathcal{K} \to \mathcal{K} + f(z) + f(\bar{z}).$$

$\mathcal{K}$ is the *Kähler potential*

We will denote by I an arbitrary index in $\{1, 2, \ldots, 2n\}$ and by $\{x^I\}$ arbitrary coordinates in $\mathcal{M}$. Let $Y = Y^I \frac{\partial}{\partial x^I} = Y^I \partial_I$ be a vector field on $\mathcal{M}$. The covariant derivative of Y with respect to a linear connection can be written as

$$(D_J Y)^I = \frac{\partial Y^I}{\partial x^J} + \Gamma^I_{JK} Y^K,$$

where the *Christoffel symbols* Γ are

$$\Gamma^I_{JK} = (D_J \partial_K)^I.$$

[3]Note the conventions with factors for symmetric products and for forms. A symmetric product of forms $\alpha\beta$ is $\frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. Similarly a wedge product is taken as $\alpha \wedge \beta = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)$.

The Levi-Civita connection is the only torsionfree connection satisfying $Dg = 0$. The Christoffel symbols are

$$\Gamma^K_{JI} = \frac{1}{2} g^{KL} \Big(\frac{\partial g_{LI}}{\partial x^J} + \frac{\partial g_{JL}}{\partial x^I} - \frac{\partial g_{JI}}{\partial x^L} \Big). \tag{6}$$

For a complex manifold, we can extend the covariant differentiation to complex vector fields by linearity. We can then consider $I = i$ for $I = 1, \dots, n$ and $I = n + \bar{\imath}$ for $I = n + 1, \dots, 2n$. The Christoffel symbols become complex, and it is easy to see that

$$\overline{\Gamma}^I_{JK} = \Gamma^{\bar{I}}_{\bar{J}\bar{K}},$$

where we have denoted $\bar{I} = I + n$ for $I = i$ and $\bar{I} = I - n$ for $I = n + i$.

A linear connection is quasicomplex if the complex structure is parallel. For $J = \mathrm{id} z^j \otimes \partial_j - \mathrm{id} \bar{z}^{\bar{\jmath}} \otimes \partial_{\bar{\jmath}}$ this means

$$(D_A J)^B_C = \Gamma^B_{AD} J^D_C - \Gamma^D_{AC} J^B_D = 0 \implies \Gamma^d_{A\bar{c}} = \Gamma^{\bar{d}}_{Ac} = 0.$$

If the connection is torsionfree we have,

$$T(X, Y) = D_X Y - D_Y X - [X, Y] = 0 \implies \Gamma^A_{BC} = \Gamma^A_{CB},$$

so the only non zero Christoffel symbols of the quasicomplex connection are

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \quad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = \Gamma^{\bar{\imath}}_{\bar{k}\bar{\jmath}}. \tag{7}$$

If the Riemannian connection is quasicomplex then the manifold is a Kähler manifold, and we have from (6) and (5)

$$\Gamma^i_{jk} = g^{i\bar{\ell}} \partial_j g_{\bar{\ell}k}, \quad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = g^{\bar{\imath}\ell} \partial_{\bar{\jmath}} g_{\ell\bar{k}}. \tag{8}$$

The curvature tensor associated to a linear connection is a 3-contravariant 1-covariant tensor given by

$$R(X, Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z, \quad X, Y, Z \in T\mathcal{M},$$

and in components

$$R^I{}_{JKL} = \Big(\frac{\partial \Gamma^I_{LJ}}{\partial x^K} - \frac{\partial \Gamma^I_{KJ}}{\partial x^L} \Big) + \sum_M \big(\Gamma^M_{LJ} \Gamma^I_{KM} - \Gamma^M_{KJ} \Gamma^I_{LM} \big).$$

It satisfies $R^I{}_{JKL} = -R^I{}_{JLK}$.

If the connection is torsionfree, the curvature tensor satisfies the *Bianchi identities*

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0, \tag{9}$$

$$D_X R(Y, Z) + D_Z R(X, Y) + D_Y R(Z, X) = 0. \tag{10}$$

(If the torsion is not zero, then the Bianchi identities are modified by terms containing the torsion, see [23], volume I, page 135.)

It is immediate to see that for a quasicomplex connection we have

$$R^i{}_{\bar{\jmath}k\ell} = 0, \quad R^i{}_{\bar{\jmath}\bar{k}\bar{\ell}} = 0, \quad R^i{}_{j\bar{k}\bar{\ell}} = 0, \quad R^i{}_{\bar{\jmath}k\bar{\ell}} = 0.$$

From (8) one finds that for a Kähler metric

$$R^i{}_{jk\ell} = 0,$$

and the only components that can be different from zero are

$$\begin{aligned} R^i{}_{jk\bar{\ell}} &= -\partial_{\bar{\ell}}\Gamma^i_{kj}, & R^i{}_{j\bar{k}\ell} &= \partial_{\bar{k}}\Gamma^i_{j\ell}, \\ R^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}\ell} &= -\partial_{\ell}\Gamma^{\bar{\imath}}_{\bar{k}\bar{\jmath}}, & R^{\bar{\imath}}{}_{\bar{\jmath}k\bar{\ell}} &= \partial_{k}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{\ell}}, \end{aligned} \tag{11}$$

and those obtained using the symmetry property $R^I{}_{JKL} = -R^I{}_{JLK}$. The upper and lower line are related by complex conjugation.

The Ricci tensor is the contraction $R_{AB} = R^C{}_{ACB}$. We have that

$$R_{ij} = R_{\bar{\imath}\bar{\jmath}} = 0, \quad R_{i\bar{\jmath}} = \bar{R}_{\bar{\imath}j} = -\partial_{\bar{\jmath}}\Gamma^k_{ik} = -\partial_{\bar{\jmath}}(g^{k\bar{\ell}}\partial_i g_{k\bar{\ell}}).$$

Let $G = \det g_{i\bar{\jmath}}$, then

$$\partial_i G = G g^{k\bar{\ell}}\partial_i g_{k\bar{\ell}} \implies R_{i\bar{\jmath}} = -\partial_i\partial_{\bar{\jmath}}\log|G|. \tag{12}$$

Example 2.5 (The complex projective space $\mathbb{C}\mathrm{P}^1$). We consider the complex projective space of 1-dimensional subspaces in $\mathbb{C}^2$. Let z^1, z^2 be the natural coordinate system in $\mathbb{C}^2$, $z^i : \mathbb{C}^2 \to \mathbb{C}$. They are complex linear maps. Let U_1 be the set of subspaces S such that $z^1|_S \neq 0$. Then $z^1|_S$ spans the dual space to S, so we may write

$$z^2|_S = t^1 z^1|_S, \quad t^1 \in \mathbb{C}. \tag{13}$$

Each equation as (13) defines a subspace in U_1, so t^1 is a complex coordinate in U_1. In the same way we can define U_2 as the set of subspaces S such that $z^2|_S \neq 0$. Then we have that

$$z^1|_S = t^2 z^2|_S, \quad t^2 \in \mathbb{C}$$

and t^2 is a complex coordinate in U_2. $\{(U_1, t^1), (U_2, t^2)\}$ is a complex atlas of $\mathbb{C}\mathrm{P}^1$. In the intersection $U_1 \cap U_2$ the gluing condition is

$$t^2 = \frac{1}{t^1}.$$

We want to define a Kähler metric on $\mathbb{C}\mathrm{P}^1$. On U_1 and U_2 we consider, respectively, the following real-valued functions:

$$f_1 = (t^1\bar{t}^1 + 1), \quad f_2 = (t^2\bar{t}^2 + 1).$$

It is easy to see that the two 2-forms defined by

$$\Phi_1 = 4\mathrm{i}\partial\bar{\partial}\ln f_1 = -4\mathrm{i}\frac{1}{f_1^2}\mathrm{d}t^1 \wedge \mathrm{d}\bar{t}^1, \quad \Phi_2 = 4\mathrm{i}\partial\bar{\partial}\ln f_2 = -4\mathrm{i}\frac{1}{f_2^2}\mathrm{d}t^2 \wedge \mathrm{d}\bar{t}^2$$

coincide in the intersection, so they define globally a closed 2-form Φ. The Kähler metric is then

$$g(X, Y) = \Phi(JX, Y).$$

One can see that it is positive definite by computing it in an open set:

$$\mathrm{d}s^2 = \frac{4}{(1 + t^1\bar{t}^1)^2}\mathrm{d}t^1\mathrm{d}\bar{t}^1. \qquad \square$$

2.3 Hermitian line bundles and fiber metrics

Hermitian fiber metrics are introduced here and will be used later, in Section 4. The definitions and statements in this section can be found in references [23], [25].

Let $E \to \mathcal{M}$ be a rank k complex vector bundle over the complex manifold $\mathcal{M}$. Then the fiber at $m \in \mathcal{M}$, E_m, is a complex vector space of complex dimension k.

Let us assume that the total space E has a complex structure, that the projection $\pi : E \to \mathcal{M}$ is a *holomorphic map*[4] between complex manifolds and that there is a local trivialization $\{\mathcal{U}_A\}_{A\in I}$ such that the maps

$$\pi^{-1}(\mathcal{U}_A) \to \mathcal{U}_A \times \mathbb{C}^k$$

and their inverses are holomorphic with $\pi^{-1}(m) = E_m \approx \mathbb{C}^k$. Then we say that E is a *holomorphic vector bundle* over $\mathcal{M}$.

Example 2.6 (The tangent bundle of a complex manifold). Let $\mathcal{M}$ be a complex manifold. Let (x^j, y^j), $j = 1, \ldots, n$ be coordinates on a neighbourhood $\mathcal{U}$ of $m \in \mathcal{M}$ such that the complex structure on $\mathcal{M}$ is given by

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}, \quad J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}, \quad j = 1, \ldots, n. \tag{14}$$

A vector on m is of the form

$$V_m = X^j_m \frac{\partial}{\partial x^j} + Y^j_m \frac{\partial}{\partial y^j}.$$

The components X^j_m, Y^j_m are coordinates on $T_m(\mathcal{M})$. On $\pi^{-1}(\mathcal{U})$ we have coordinates (x^j, y^j, X^j, Y^j), and a quasicomplex structure on $T\mathcal{M}$ is given by (14) and

$$J\left(\frac{\partial}{\partial X^j}\right) = \frac{\partial}{\partial Y^j}, \quad J\left(\frac{\partial}{\partial Y^j}\right) = -\frac{\partial}{\partial X^j}, \quad j = 1, \ldots, n.$$

The quasicomplex structure is integrable and we have complex coordinates on $\pi^{-1}(\mathcal{U})$:

$$(z^j = x^j + \mathrm{i}y^j,\ Z^j = X^j + \mathrm{i}Y^j).$$

$T\mathcal{M}$ is a complex manifold and a holomorphic vector bundle over $\mathcal{M}$.

[4] That is, a map preserving the complex structures.

A *fiber metric* on a vector bundle $E \to \mathcal{M}$ is a smooth assignment of an inner product on each fiber

$$h_m \colon E_m \times E_m \longrightarrow \mathbb{R}.$$

If the fiber has a complex structure J_m we require that the inner product is Hermitian,

$$h(JX, JY) = h(X, Y) \quad \text{for all } X, Y \in E_m,$$

and we say that E is a *Hermitian vector bundle*.

A connection ∇ on E is *metric* if $\nabla h = 0$ (the connection is extended to $E^* \otimes E^*$). If the bundle is holomorphic, we can ask the covariant derivative of a holomorphic section to be holomorphic,

$$\nabla s \in \Omega^{1,0}(E) \quad \text{for } s \text{ holomorphic.} \tag{15}$$

There is a unique metric connection satisfying (15); it is the *Hermitian connection* of the Hermitian vector bundle (see for example [23]).

Let $\{s_a\}_{a=1,\dots,k}$ be a holomorphic frame of the bundle E on a neighbourhood $\mathcal{U}$ of $m \in \mathcal{M}$ (that is, k independent local sections) and $\{\alpha^a\}$ the dual coframe. The connection 1-form on $\mathcal{U}$, for a connection satisfying (15), is

$$(\nabla s_b) = \Gamma^a{}_b s_a = \Gamma_i{}^a{}_b \mathrm{d}z^i \otimes s_a, \quad i = 1, \dots, n,\ a, b = 1, \dots, k,$$

so the covariant derivative of a holomorphic section $s = a^a s_a$ is

$$\nabla s = (\partial_i a^a + \Gamma_i{}^a{}_b a^b) \mathrm{d}z^i \otimes s_a.$$

The hermiticity of the fiber metric means

$$h = 2h_{a\bar{b}} \alpha^a \bar{\alpha}^b,$$

and the condition for the connection to be metric is

$$\partial_i h_{a\bar{b}} - \Gamma_i{}^c{}_a h_{c\bar{b}} = 0 \implies \Gamma_i{}^a{}_b = h^{a\bar{c}} \partial_i h_{\bar{c}b}, \tag{16}$$

where $h^{a\bar{c}}$ is the inverse matrix of $h_{a\bar{c}}$. The curvature is then

$$R^a{}_{bi\bar{j}} = -h^{a\bar{c}} \partial_i \partial_{\bar{j}} h_{b\bar{c}} + h^{a\bar{c}} h^{d\bar{e}} \partial_i h_{b\bar{e}} \partial_{\bar{j}} h_{d\bar{c}}.$$

We can define the *Ricci form* of the Hermitian bundle as the trace of the curvature tensor,

$$\rho = -2\mathrm{i} R^a{}_{ai\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

If E is a Hermitian *line bundle*, that is, it has rank 1, then the metric is just

$$h = \theta(z, \bar{z}) \alpha \bar{\alpha},$$

and the Ricci form becomes

$$\rho = -2\mathrm{i} \bar{\partial}\partial \log|\theta| = -2\mathrm{i} \partial_i \partial_{\bar{j}} \log|\theta| \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}. \tag{17}$$

3 Rigid special Kähler manifolds

In this section we will deal with rigid special Kähler geometry, or simply special Kähler geometry, as opposed to projective special Kähler geometry, which will be the subject of Section 6.

3.1 Some geometric preliminaries

This part is inspired in the second chapter of [26].

Let E be a vector bundle over $\mathcal{M}$ with a connection ∇. For every vector field X of $\mathcal{M}$ (section of $T\mathcal{M}$), ∇ sends sections of E to sections of E,

$$\nabla_X \colon \Gamma(E) \longrightarrow \Gamma(E).$$

Let $\Omega^p_{\mathcal{M}}(E) = \Lambda^p(\mathcal{M}) \otimes \Gamma(E)$ be the space of *E-valued p-forms on $\mathcal{M}$*, with $\Omega^0_{\mathcal{M}}(E) = \Gamma(E)$. We are going to define the *covariant differential* $\mathrm{d}_\nabla \colon \Omega^p_{\mathcal{M}}(E) \to \Omega^{p+1}_{\mathcal{M}}(E)$. For 0-forms we define

$$\begin{aligned} \Omega^0_{\mathcal{M}}(E) &\xrightarrow{\mathrm{d}_\nabla} \Omega^1_{\mathcal{M}}(E), \\ F &\longmapsto \mathrm{d}_\nabla F, \end{aligned} \qquad \text{such that } \mathrm{d}_\nabla F(X) = \nabla_X F,\ X \in T\mathcal{M}.$$

This definition can be extended to $\Omega^p_{\mathcal{M}}(E)$,

$$\begin{aligned} \Omega^p_{\mathcal{M}}(E) &\xrightarrow{\mathrm{d}_\nabla} \Omega^{p+1}_{\mathcal{M}}(E), \\ F &\longmapsto \mathrm{d}_\nabla F, \end{aligned}$$

assuming the condition

$$\mathrm{d}_\nabla(\alpha \wedge F) = \mathrm{d}\alpha \wedge F + (-1)^p \alpha \wedge \mathrm{d}_\nabla F$$

for $\alpha \in \Lambda^p(\mathcal{M})$ and $F \in \Omega^q_{\mathcal{M}}(E)$. For example, if $F \in \Omega^1_{\mathcal{M}}(E)$, locally $F = \mathrm{d}x^i \otimes \alpha_i = \mathrm{d}x^i \wedge \alpha_i$ with $\alpha_i \in \Omega^0_{\mathcal{M}}(E)$, then

$$\mathrm{d}_\nabla F(X, Y) = -\mathrm{d}x^i \wedge \mathrm{d}_\nabla \alpha_i(X, Y) = -\left(X^i \nabla_Y \alpha_i - Y^i \nabla_X \alpha_i\right).$$

If $A^I_J = A^I_{\mu J} \mathrm{d}x^\mu$ is the 1-form connection matrix in an open set $U \subset \mathcal{M}$, then

$$(\mathrm{d}_\nabla F)^I = \mathrm{d}F^I + A^I_J \wedge F^J,$$

from which it is easy to deduce the standard transformation rule under a local fiber (gauge) transformation

$$F' = UF, \quad (\mathrm{d}_\nabla F)' = U \mathrm{d}_\nabla F \implies A' = -\mathrm{d}U U^{-1} + UAU^{-1},$$

where $(\mathrm{d}_\nabla F)' \equiv \mathrm{d}F' + A'F'$. Differently than for the ordinary differential, d_∇^2 is not zero in general. In fact,

$$(\mathrm{d}_\nabla^2 F)^I = (\mathrm{d}A^I_K + A^I_J \wedge A^J_K) \wedge F^K = R^I_K \wedge F^K,$$

where R is the (Lie algebra valued) curvature 2-form associated to the connection. A flat connection then defines a complex. The de Rham complex is associated to the trivial connection on the trivial bundle $E = \mathcal{M} \times V$.

It is easy to check the Bianchi identity $\mathrm{d}_\nabla R = 0$. In the associated bundle with typical fiber the Lie algebra, the group acts with the adjoint representation. The covariant differential in such bundle is then

$$\mathrm{d}_\nabla R^I_J = \mathrm{d}R^I_J + A^I_K \wedge R^K_J - A^K_J \wedge R^I_K = 0.$$

3.2 Definition of rigid special Kähler manifolds

Here we follow the first section of reference [8].

Definition 3.1. Let $\mathcal{M}$ be a Kähler manifold with Kähler form Φ and complex structure J. A special Kähler structure on $\mathcal{M}$ is a real, flat, torsionfree, symplectic connection ∇ satisfying

$$\mathrm{d}_\nabla J = 0. \tag{18}$$

J is seen here as a 1-form with values in the tangent bundle $T\mathcal{M}$, and the covariant differential must be interpreted in the sense described in Section 3.1.

As we have seen, a Kähler manifold is always symplectic, being the Kähler form Φ its symplectic form. On a symplectic manifold, a linear connection ∇ is said to be symplectic if

$$\nabla\Phi = 0. \tag{19}$$

We want to see what is the meaning of the ingredients in this definition. We first examine the implications of the existence of a flat, torsionfree connection.

Let U be an open set, with coordinates $\{x^I\}_{I=1}^{2n}$ and a (matrix-valued) connection 1-form $A^K_L = A^K_{ML}\mathrm{d}x^M$.

Due to the flatness condition, $\mathrm{d}_\nabla^2 = 0$, so a d_∇-closed form is locally d_∇-exact.

Let $\mathbb{1}$ be the identity endomorphism in $T\mathcal{M}$. It can be seen as a $T\mathcal{M}$ valued 1-form, $\mathbb{1} = \partial_I \otimes dx^I$. The torsionfree condition can be expressed as

$$\mathrm{d}_\nabla \mathbb{1} = (\mathrm{d}_\nabla \mathbb{1})^K \partial_K = A^K_{LM}\mathrm{d}x^L \wedge \mathrm{d}x^M \otimes \partial_K = 0 \implies A^K_{LM} = A^K_{ML}. \tag{20}$$

A local frame on $T\mathcal{M}$ on $U \subset_{\text{open}} \mathcal{M}$ is a set $\{\xi_\alpha = \xi^J_\alpha \partial_J\}_{\alpha=1}^{2n}$ of $2n$ local sections of $TU \subset T\mathcal{M}$ that are linearly independent for each point $x \in U$. Since the connection is flat (the curvature tensor is zero), there exists a *flat frame*, that is,

$$\nabla_I \xi_\alpha = 0 \ \text{ for } I = 1, \ldots, 2n, \quad \text{or equivalently,} \quad \mathrm{d}_\nabla \xi_\alpha = 0. \tag{21}$$

This is because the integrability condition of (21) is

$$\mathrm{d}_\nabla^2 \xi_\alpha = R^I_K \wedge \xi^K_\alpha \partial_I = 0$$

for the $2n$ independent sections ξ_α, which implies necessarily that $R^I_K = 0$.

Let $\theta^\alpha = \theta^\alpha_I \mathrm{d}x^I$ be the dual coframe, that is $\theta^\alpha(\xi_\beta) = \delta^\alpha_\beta$. We have that

$$\theta^\alpha_I \xi^I_\beta = \delta^\beta_\alpha \iff \xi^J_\alpha \theta^\alpha_I = \delta^J_I .$$

Then we can express

$$\mathbb{1} = \theta^\alpha \otimes \xi_\alpha = \theta^\alpha \wedge \xi_\alpha ,$$

and

$$\mathrm{d}_\nabla \mathbb{1} = 0 \implies \mathrm{d}_\nabla \theta^\alpha \wedge \xi_\alpha - \theta^\alpha \wedge \mathrm{d}_\nabla \xi_\alpha = 0 \implies \mathrm{d}_\nabla \theta^\alpha = \mathrm{d}\theta^\alpha = 0.$$

This means that $\theta^\alpha = \mathrm{d}t^\alpha$ for some functions t^α. Then $\xi_\alpha = \partial/\partial t_\alpha$ and t^α are local coordinates on U called *flat coordinates*.

Up to here, we used the fact that the connection is real, flat and torsionfree. We introduce now the additional condition that the connection is symplectic, that is, $\nabla_I \Phi = 0$. We denote the symplectic matrix as

$$P = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}. \tag{22}$$

The coordinates t^α are *Darboux* coordinates if

$$\Phi(\xi_\alpha, \xi_\beta) = P_{\alpha\beta}, \quad \text{so} \quad \Phi = \frac{1}{2} P_{\alpha\beta} \mathrm{d}t^\alpha \wedge \mathrm{d}t^\beta .$$

It is possible to choose the flat coordinates t^α in such a way that they are Darboux. This is because

$$\partial_I(\Phi(\xi_\alpha, \xi_\beta)) = \nabla_I(\Phi)(\xi_\alpha, \xi_\beta) - \Phi(\nabla_I \xi_\alpha, \xi_\beta) - \Phi(\xi_\alpha, \nabla_I \xi_\beta) = 0,$$

so $\Phi(\xi_\alpha, \xi_\beta)$ is a constant (antisymmetric, non degenerate) matrix which can always be brought to the form (22) by a linear change of coordinates.

We see that the existence of a flat, torsionfree, symplectic connection on $\mathcal{M}$ is equivalent to having a covering by *flat Darboux coordinates* (it is also said that $\mathcal{M}$ has a flat symplectic structure). If $\{q^\alpha\}_{\alpha=1}^{2n}$ are also flat Darboux coordinates, we have that the transition functions satisfy

$$\mathrm{d}q^\alpha = \frac{\partial q^\alpha}{\partial t^\beta} \mathrm{d}t^\beta, \quad \nabla_{t^\nu} \mathrm{d}q^\alpha = \frac{\partial q^\alpha}{\partial t^\beta \partial t^\gamma} \mathrm{d}t^\beta = 0$$

since

$$\nabla_{t^\nu} \mathrm{d}t^\beta = \nabla_{t^\nu} \theta^\beta = 0.$$

This implies

$$\frac{\partial q^\alpha}{\partial t^\beta \partial t^\gamma} = 0 \implies q^\alpha = A^\alpha_\beta t^\beta + c^\alpha ,$$

with A^α_β and c^α constant. It follows that $A \in \mathrm{Sp}(2n)$.

Let us now consider the condition (18). In arbitrary coordinates $\{x^I\}_{I=1}^{2n}$ it becomes

$$J = J^I \partial_I = J^I_K \partial_I \otimes \mathrm{d}x^K, \quad (\mathrm{d}_\nabla J)^I = \mathrm{d}J^I + A^I_L \wedge J^L = 0, \tag{23}$$

which in components reads

$$\frac{1}{2}(\partial_M J^I_N - \partial_N J^I_M) + \frac{1}{2}(A^I_{ML} J^L_N - A^I_{NL} J^L_M) = 0.$$

(The factor $1/2$ appears when M and N are arbitrary, so each *strict component* is counted twice). This implies, assuming that the connection is torsionfree, that

$$A^I_{i\bar{j}} = A^I_{ij} = 0. \tag{24}$$

The connection ∇ is a linear connection (a connection on the frame bundle of $\mathcal{M}$), so one can compute

$$(\nabla_I J)^K_L = \partial_I J^K_L + A^K_{IM} J^M_L - A^M_{IL} J^K_M. \tag{25}$$

The condition $\nabla_I J = 0$ together with the torsionfreeness implies, in addition to (24) that

$$A^i_{\bar{m}\bar{n}} = A^{\bar{l}}_{mn} = 0.$$

If the connection is torsionfree and $\nabla_I J = 0$ then we have that $\mathrm{d}_\nabla J = 0$, but the converse is not necessarily true. Then the flat symplectic connection is not necessarily complex.

The complex structure can be written locally in terms of the complex coordinates $\{z^j\}_{j=0}^n$ as

$$J = \mathrm{i}(\partial_{z^j} \otimes \mathrm{d}z^j - \partial_{\bar{z}^{\bar{j}}} \otimes \mathrm{d}\bar{z}^{\bar{j}}) = \mathrm{i}(\pi^{(1,0)} - \pi^{(0,1)}),$$

where

$$\pi^{(1,0)} = \partial_{z^j} \otimes \mathrm{d}z^j \quad \text{and} \quad \pi^{(0,1)} = \partial_{\bar{z}^{\bar{j}}} \otimes \mathrm{d}\bar{z}^{\bar{j}} \tag{26}$$

are the projectors onto the $T\mathcal{M}^{(1,0)}$ and $T\mathcal{M}^{(0,1)}$ spaces respectively. The condition $\mathrm{d}_\nabla J = 0$, together with the torsionfreeness, is equivalent to

$$\mathrm{d}_\nabla \pi^{(1,0)} = 0.$$

Indeed, one can also write

$$\mathbb{1} = \pi^{(1,0)} + \pi^{(0,1)}, \tag{27}$$

and the torsionfree condition was expressed as $\mathrm{d}_\nabla \mathbb{1} = 0$.

Using the Poincaré lemma, $\mathrm{d}_\nabla \pi^{(1,0)} = 0$ implies that locally there exists a complex vector field χ such that

$$\nabla \chi = \mathrm{d}_\nabla \chi = \pi^{(1,0)},$$

which is unique up to a flat complex vector field.

Let $\{x^j, y_j\}_{j=1}^n$ be a flat Darboux coordinate system, that is,

$$\Phi = \mathrm{d}x^j \wedge \mathrm{d}y_j, \quad \text{and} \quad \mathrm{d}_\nabla\left(\frac{\partial}{\partial x^j}\right) = 0, \quad \mathrm{d}_\nabla\left(\frac{\partial}{\partial y_j}\right) = 0. \tag{28}$$

In this coordinate system we denote

$$\chi = \frac{1}{2}\left(\eta^i \frac{\partial}{\partial x^i} - \lambda_j \frac{\partial}{\partial y_j}\right), \tag{29}$$

where η^j, λ_j are complex functions (χ is a complex vector field). Taking the covariant differential and using (28) we obtain

$$\pi^{(1,0)} = \mathrm{d}_\nabla \chi = \frac{1}{2}\left(\mathrm{d}\eta^j \otimes \frac{\partial}{\partial x^j} - \mathrm{d}\lambda_j \otimes \frac{\partial}{\partial y_j}\right). \tag{30}$$

$\pi^{(1,0)}$ is a (1,0) tensor, so it follows that η^j and λ_j are holomorphic functions. Taking the real part of this equation we have, using (27),

$$\Re(\pi^{(1,0)}) = \frac{1}{2}\mathbb{1} = \frac{1}{2}\left(\mathrm{d}x^j \otimes \frac{\partial}{\partial x^j} + \mathrm{d}y_j \otimes \frac{\partial}{\partial y_j}\right), \tag{31}$$

so we can identify

$$\Re(\mathrm{d}\eta^j) = \mathrm{d}x^j, \quad \Re(\mathrm{d}\lambda_j) = -\mathrm{d}y_j. \tag{32}$$

Together with the condition that η^j and λ_j are holomorphic, we have that

$$\mathrm{d}\eta^j = \mathrm{d}x^j - \mathrm{i}J^*\mathrm{d}x^j, \quad \mathrm{d}\lambda_j = -\mathrm{d}y_j + \mathrm{i}J^*\mathrm{d}y_j. \tag{33}$$

We want to see under what conditions the sets $\{\eta^j\}$ and $\{\lambda_j\}$ are sets of complex coordinates. Let $\{z^1, \ldots, z^n\}$ be complex coordinates with

$$\begin{gathered} z^l = \xi^l + \mathrm{i}\omega^l, \quad \mathrm{d}z^l = \mathrm{d}\xi^l + \mathrm{i}\mathrm{d}\omega^l, \quad \frac{\partial}{\partial z^l} = \frac{1}{2}\left(\frac{\partial}{\partial \xi^l} - \mathrm{i}\frac{\partial}{\partial \omega^l}\right), \\ \frac{\partial}{\partial \xi^l} = \frac{\partial}{\partial z^l} + \frac{\partial}{\partial \bar{z}^l}, \quad \frac{\partial}{\partial \omega^l} = \mathrm{i}\left(\frac{\partial}{\partial z^l} - \frac{\partial}{\partial \bar{z}^l}\right). \end{gathered} \tag{34}$$

We have

$$\mathrm{d}\eta^j = \alpha^j{}_l(z)\mathrm{d}z^l, \quad \mathrm{d}\lambda_j = \beta_{jl}(z)\mathrm{d}z^l. \tag{35}$$

$\pi^{(1,0)}$ is the projector on the space of holomorphic vectors. This means that it kills all the antiholomorphic vectors and its image is the set of all the holomorphic vectors. Using (30), (34) and (35) we have

$$\pi^{(1,0)}\left(\frac{\partial}{\partial \xi^j}\right) = \frac{1}{2}\left(\alpha^l{}_j \frac{\partial}{\partial x^l} - \beta_{lj}\frac{\partial}{\partial y_l}\right), \tag{36}$$

$$\pi^{(1,0)}\left(\frac{\partial}{\partial \omega^j}\right) = \frac{\mathrm{i}}{2}\left(\alpha^l{}_j \frac{\partial}{\partial x^l} - \beta_{lj}\frac{\partial}{\partial y_l}\right). \tag{37}$$

The set $\{\partial/\partial\xi^l, \partial/\partial\omega^l\}$ forms a basis of the tangent space, so the vectors in (36) and (37) are linearly independent. This implies that the $2n \times n$ matrix

$$(\alpha^\top, -\beta^\top) \tag{38}$$

has rank n (the superindex $\top$ indicates the usual transpose). From the set of $2n$ functions (η^j, λ_j) we can always select n independent holomorphic functions that form a set of holomorphic coordinates.

From the fact that the symplectic form $\Phi = \mathrm{d}x^i \wedge \mathrm{d}y_j$ is of type $(1,1)$ and using (32) and (35) in (28), and compare with (4), we obtain as conditions for α and β

$$\begin{aligned} -\alpha^\top \beta + \beta^\top \alpha &= 0, \\ \beta^\top \bar{\alpha} - \alpha^\top \bar{\beta} &= 8\mathrm{i}\, g, \end{aligned} \tag{39}$$

where g is the $n \times n$ matrix $g_{i\bar{j}}$.

These are the equations that we can obtain in general where we have not used any information on the signature of the metric.

3.3 The signature of the metric

Let us first assume that the metric is positive definite (Riemannian metric). We want to show that α itself has rank n. Suppose that $\mathrm{rank}(\alpha) < n$. Then there exists a holomorphic vector c such that

$$\alpha^j{}_l(z) c^l(z) = 0.$$

But then $\beta_{jl} c^l(z) \neq 0$, since otherwise the total rank of the matrix (38) would be lower than n. This means that there exists a non zero, holomorphic linear combination of the vectors $\{\partial/\partial y^k\}$, namely

$$\gamma = \sum \tilde{c}_k(z) \frac{\partial}{\partial y_k} \neq 0, \quad \tilde{c}_j(z) = \beta_{jk} c^k.$$

Then, as γ has only y-components,

$$0 = \Phi(\gamma, \bar{\gamma}) = g(\gamma, J\bar{\gamma}) = g(\gamma, -\mathrm{i}\bar{\gamma}) = -\mathrm{i}\, g(\gamma, \bar{\gamma}).$$

For a Kähler manifold (with positive definite metric)

$$g(\gamma, \bar{\gamma}) = 0 \iff \gamma = 0, \tag{40}$$

so we have a contradiction and α must have rank n. Looking now to (35) we can conclude that $\{\eta_j\}$ is a set of holomorphic coordinates.

In the same way we can prove that β has rank n so $\{\lambda_j\}$ is also a set of holomorphic coordinates.

Remark 3.2 (Symplectic transformations (Riemannian case)). One can see independently that a real symplectic transformation cannot change the rank of α and β, provided they satisfy the following conditions:

1. α and β have rank n.
2. $-\alpha^\top \beta + \beta^\top \alpha = 0$,

3. $\beta^\top \bar{\alpha} - \alpha^\top \bar{\beta} = 8\mathrm{i} g$, where g is the $n \times n$ matrix $g_{i\bar{\jmath}}$.

In our case, conditions 2. and 3. were obtained in (39).

Let us first introduce the vielbein for the metric

$$\begin{pmatrix} 0 & g \\ g^\top & 0 \end{pmatrix} = E \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} E^\top, \quad E = \begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}, \quad g = e\bar{e}^\top.$$

We can define

$$\alpha = 2\alpha' e^\top, \quad \beta = 2\beta' e^\top,$$

so we can express 2. and 3. as

2. $-\alpha'^\top \beta' + \beta'^\top \alpha' = 0$,
3. $\beta'^\top \bar{\alpha}' - \alpha'^\top \bar{\beta}' = 2\mathrm{i}\mathbb{1}$.

Let us denote $\alpha' = \alpha'_0 + \mathrm{i}\alpha'_1$, $\beta' = \beta'_0 + \mathrm{i}\beta'_1$ with α'_i and β'_i real. We define the matrix

$$S_0 = \begin{pmatrix} \alpha'_0 & \alpha'_1 \\ \beta'_0 & \beta'_1 \end{pmatrix}.$$

Then properties 2. and 3. are equivalent to

$$S_0^\top \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} S_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{41}$$

Equation (41) means that S_0 is a symplectic matrix, $S_0 \in \mathrm{Sp}(2n, \mathbb{R})$.

We are ready now to prove the statement above. We have that

$$S_0 \begin{pmatrix} \mathbb{1} \\ \mathrm{i}\mathbb{1} \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}.$$

We assume that $\mathrm{rank}(\alpha') = \mathrm{rank}(\beta') = n$ and that (41) holds. We want to prove that for

$$\hat{S} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}),$$

the matrix

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \hat{S} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$$

is such that $\mathrm{rank}(\hat{\alpha}) = \mathrm{rank}(\hat{\beta}) = n$. Let us write

$$\hat{S} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \hat{S}\, S_0\, S_0^{-1} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = S \begin{pmatrix} \mathbb{1} \\ \mathrm{i}\mathbb{1} \end{pmatrix}, \quad \text{with } S = \hat{S}\, S_0.$$

S is an arbitrary matrix in $\mathrm{Sp}(2n, \mathbb{R})$, so all we have to prove is that

$$S \begin{pmatrix} \mathbb{1} \\ \mathrm{i}\mathbb{1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ \mathrm{i}\mathbb{1} \end{pmatrix} = \begin{pmatrix} A + \mathrm{i}B \\ C + \mathrm{i}D \end{pmatrix}$$

is such that $\operatorname{rank}(A + \mathrm{i}B) = \operatorname{rank}(C + \mathrm{i}D) = n$. We consider the matrices

$$M = \mathrm{i}(A + \mathrm{i}B), \quad N = (C + \mathrm{i}D).$$

We have that

$$\mathcal{A} \equiv M^\dagger N = \mathbb{1} - \mathrm{i}\mathcal{A}_H, \quad \mathcal{A}_H = A^\top C + B^\top D,$$

since S is a symplectic matrix,

$$A^\top C = C^\top A, \quad B^\top D = D^\top B, \quad A^\top D - C^\top B = \mathbb{1}.$$

The matrix $\mathcal{A}_H$ is therefore also symmetric, and can be diagonalized such that $\mathcal{A}$ is diagonalized with eigenvalues of the form $(1 + \mathrm{i}a) \neq 0$. The determinant of $\mathcal{A}$ is the product of its eigenvalues, so it is different from zero. This implies that $\det M \neq 0$, $\det N \neq 0$, so our statement is proven.

Remark 3.3 (Symplectic transformations (pseudo-Riemannian case)). If g has pseudo-Riemannian signature, there are symplectic transformation changing the rank of α and β satisfying 1. to 3. in Remark 3.2. It is enough to give one of such symplectic matrices. First we realize that, as before, conditions 2. and 3. can be put as

2. $-\alpha'^\top \beta' + \beta'^\top \alpha' = 0,$
3. $\beta'^\top \bar{\alpha}' - \alpha'^\top \bar{\beta}' = 2\mathrm{i}\eta,$

where η is the flat pseudo-Riemannian metric. For definiteness, let us assume that the signature of η is $(n-1, 1)$ (the other cases can be obtained in the same way). We take η in the standard form

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$\eta^2 = \mathbb{1}$ and the vielbein is defined accordingly. We have that the matrix

$$S_0 = \begin{pmatrix} \alpha'_0 & \alpha'_1 \eta \\ \beta'_0 & \beta'_1 \eta \end{pmatrix}$$

is a symplectic matrix, condition that is equivalent to 2 and 3. Also, we have that

$$S_0 \begin{pmatrix} \mathbb{1} \\ \mathrm{i}\eta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix},$$

so we can bring $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$ to the standard form $\begin{pmatrix} \mathbb{1} \\ \mathrm{i}\eta \end{pmatrix}$ with the symplectic transformation

S_0^{-1}. It is enough to consider the $n = 2$ case. The symplectic matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

has the property

$$S \begin{pmatrix} \mathbb{1} \\ \mathrm{i}\eta \end{pmatrix} = S \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathrm{i} & 0 \\ 0 & \mathrm{i} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha'' \\ \beta'' \end{pmatrix},$$

with the property that $\det \alpha' = \det \beta' = 0$, as we wanted to show. The proof in the remark 3.2 is not valid here because the real part of $\mathcal{A}$, determined by $(A + \mathrm{i}B\eta)$ and $(C + \mathrm{i}D\eta)$, would be zero.

We consider now a pseudo-Riemannian metric (pseudo-Kähler manifold). Notice that in this case (40) is not true since we can have null vectors. In fact, assume that we have a holomorphic vector field γ such that $\Phi(\gamma, \bar{\gamma}) = 0$ and let us consider the vectors

$$N_\pm = \gamma \pm \bar{\gamma}.$$

Since g is of type (1,1) and $\Phi(\gamma, \bar{\gamma}) = -\mathrm{i}g(\gamma, \bar{\gamma}) = 0$, each one of the terms below is separately 0,

$$g(N_\pm, N_\pm) = g(\gamma, \gamma) + g(\bar{\gamma}, \bar{\gamma}) \pm 2g(\gamma, \bar{\gamma}) = 0,$$

and then N_+ and $\mathrm{i}N_-$ are null, real vectors. On the other hand, if N is a null, real vector, its holomorphic and antiholomorphic extensions $\gamma = N - \mathrm{i}JN$ satisfy $g(\gamma, \bar{\gamma}) = 0$ and thus $\Phi(\gamma, \bar{\gamma}) = 0$. We will treat the case of special pseudo-Kähler manifolds in Section 3.5.

We have thus proven in this section that for a positive definite metric, the matrices α and β are each of rank n. When the metric is not positive definite, this proof breaks down due to null vectors that may be zero modes of these matrices. However, these matrices might even then still be invertible (see the example in Section 3.6). In fact, in [7] it is proven that with a symplectic rotation we can always bring α to be non degenerate. A sketch of the proof is given in Appendix A.

3.4 The prepotential

We come back to the positive definite metric, or, at least that α and β are invertible. Then $\{\eta^j\}_{j=1}^n$ and $\{\lambda_j\}_{j=1}^n$ are called *conjugate* coordinate systems. Equations (26) and (35) then imply

$$\pi^{(1,0)} = \mathrm{d}\eta^j \otimes \frac{\partial}{\partial \eta^j},$$

from which, comparing with (30),

$$\frac{\partial}{\partial \eta^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \tau_{jk}\frac{\partial}{\partial y_k}\right), \quad \tau_{jk} = \frac{\partial \lambda_k}{\partial \eta^j}. \tag{42}$$

The Kähler form is

$$\Phi = \mathrm{d}x^j \wedge \mathrm{d}y_j = -\frac{1}{4}(\mathrm{d}\eta^j + \mathrm{d}\bar{\eta}^{\bar{j}}) \wedge (\tau_{jk}\mathrm{d}\eta^k + \tau_{\bar{j}\bar{k}}\mathrm{d}\bar{\eta}^{\bar{k}}).$$

Since it is of type (1,1), it follows that

$$\tau_{ij} = \tau_{ji}, \tag{43}$$

so

$$\Phi = \frac{1}{4}\mathrm{d}\eta^i \wedge \mathrm{d}\bar{\eta}^{\bar{j}}(\tau_{ij} - \bar{\tau}_{\bar{i}\bar{j}}).$$

Comparing to (4), we see the metric and the Kähler form become

$$g_{i\bar{j}} = -\frac{1}{4}\Im(\tau_{ij}), \quad \Phi = \frac{1}{2}\mathrm{i}\Im(\tau_{ij})\mathrm{d}\eta^i \wedge \mathrm{d}\bar{\eta}^{\bar{j}}. \tag{44}$$

Because of (43), there exists a local holomorphic function, determined up to a constant, such that

$$\lambda_j = -8\frac{\partial \mathcal{F}}{\partial \eta^j}, \quad \tau_{ij} = -8\frac{\partial^2 \mathcal{F}}{\partial \eta^i \partial \eta^j}. \tag{45}$$

$\mathcal{F}$ is called the *holomorphic prepotential*. In terms of it, the Kähler potential becomes

$$\mathcal{K} = -\frac{1}{4}\Im(\lambda_k \bar{\eta}^k) = 2\Im\left(\frac{\partial \mathcal{F}}{\partial \eta^k}\bar{\eta}^k\right). \tag{46}$$

The coordinate system $\{\eta^j\}_{j=1}^n$ is a *special coordinate system*.

In the particular case in which $\tau_{jk} = \mathrm{i}\delta_{jk}$, then $\eta^j = x^j + \mathrm{i}y_j$, and

$$\Phi = \frac{\mathrm{i}}{2}\mathrm{d}\eta^i \wedge \mathrm{d}\bar{\eta}^{\bar{i}},$$

so the manifold is locally isometric to $\mathbb{C}^n$.

Recovering the flat connection. A structure of special geometry can be given, in an open set, by a holomorphic function $\mathcal{F}(\eta)$ such that $\Im(\tau_{ij})$ with τ_{ij} as in (45), is a non singular, negative definite matrix. The holomorphic coordinates are declared to be special coordinates. From the knowledge of $\mathcal{F}$ we can recover the flat symplectic coordinates

$$x^i = \Re(\eta^i), \quad y_j = -\Re(\lambda_j) = 8\Re\left(\frac{\partial \mathcal{F}}{\partial \eta^j}\right)$$

(up to a constant) and also reconstruct the symplectic section

$$\chi = \frac{1}{2}\left(\eta^i \frac{\partial}{\partial x^i} - \lambda_j \frac{\partial}{\partial y_j}\right).$$

In the flat coordinates the coefficients of the flat connection are zero (that is, the covariant derivatives are usual derivatives). If we want to use the holomorphic coordinates, these coefficients are not zero anymore. We want to compute them in the coordinates (η^j). In order to do this, we perform the coordinate change so the connection transforms as

$$A' = A + \Lambda^{-1}\mathrm{d}\Lambda$$

with $A = 0$ and

$$\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \Lambda\begin{pmatrix}\mathrm{d}\eta\\ \mathrm{d}\bar\eta\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathbb{1} & \mathbb{1}\\ -\tau & -\bar\tau\end{pmatrix}\begin{pmatrix}\mathrm{d}\eta\\ \mathrm{d}\bar\eta\end{pmatrix}.$$

A and A' are considered here as matrices $A^I{}_J$, $\bar\tau$ is $\tau_{\bar\imath\bar\jmath}$ and Λ_{IJ} is written in terms of blocks of size $n\times n$. We have

$$\Lambda^{-1} = 2\begin{pmatrix}\beta\bar\tau & \beta\\ -\beta\tau & -\beta\end{pmatrix},\quad \beta = (\bar\tau-\tau)^{-1} = -\frac{1}{8}\mathrm{i}g^{-1}.$$

Then

$$A' = \Lambda^{-1}\mathrm{d}\Lambda = \begin{pmatrix}-\beta & 0\\ \beta & 0\end{pmatrix}\mathrm{d}\tau + \begin{pmatrix}0 & -\beta\\ 0 & \beta\end{pmatrix}\mathrm{d}\bar\tau,\quad \mathrm{d}\tau_{ij} = \frac{\partial\tau_{ij}}{\partial\eta^k}\mathrm{d}\eta^k, \tag{47}$$

$$\text{with}\quad A' = \begin{pmatrix}A'^{j}_{k} & A'^{j}_{\bar k}\\ A'^{\bar\jmath}_{k} & A'^{\bar\jmath}_{\bar k}\end{pmatrix}.$$

From this expression one can check that conditions (24) are satisfied.

Let us compute the covariant differential of a vector with only holomorphic components, $H = H^i\partial/\partial\eta^i$. Notice that, acting on such vector, only the first term in (47) contributes, so

$$\begin{aligned}\nabla H &= (\nabla_i H^j)\mathrm{d}\eta^i\otimes\frac{\partial}{\partial\eta^j} + (\nabla_{\bar\imath}H^j)\mathrm{d}\bar\eta^{\bar\imath}\otimes\frac{\partial}{\partial\eta^j} \qquad (48)\\ &= \partial_i H^j\mathrm{d}\eta^i\otimes\frac{\partial}{\partial\eta^j} + A'^{j}_{k}H^k\otimes\frac{\partial}{\partial\eta^j} + A'^{\bar\jmath}_{k}H^k\otimes\frac{\partial}{\partial\bar\eta^{\bar\jmath}} + \partial_{\bar\imath}H^j\mathrm{d}\bar\eta^{\bar\imath}\otimes\frac{\partial}{\partial\eta^j}\\ &= \partial_i H^j\mathrm{d}\eta^i\otimes\frac{\partial}{\partial\eta^j} + \partial_{\bar\imath}H^j\mathrm{d}\bar\eta^{\bar\imath}\otimes\frac{\partial}{\partial\eta^j} - \frac{1}{2}\frac{\partial\tau_{jl}}{\partial\eta^k}H^j\mathrm{d}\eta^k\otimes\frac{\partial}{\partial y_l},\end{aligned}$$

since

$$\frac{\partial}{\partial y_k} = 2\left(\beta^{jk}\frac{\partial}{\partial\eta^j} - \beta^{jk}\frac{\partial}{\partial\bar\eta^{\bar\jmath}}\right).$$

Example 3.4 (The flat metric on $\mathbb{C}^n$). Let $z^1,\dots,z^n$ be the standard coordinates in $\mathbb{C}^n$. To have a rigid special Kähler structure it is enough to give a holomorphic function $\mathcal{F}(z^1,\dots,z^n)$ such that the matrix

$$-\Im\left(\frac{\partial^2\mathcal{F}}{\partial z^i\partial z^j}\right)$$

is positive definite and non degenerate. If we take

$$\mathcal{F} = \frac{1}{4}\mathrm{i}\big((z^1)^2 + \cdots + (z^n)^2\big),$$

we obtain the flat metric on $\mathbb{C}^n$

$$g_{i\bar{j}} = \delta_{ij}. \qquad \square$$

Example 3.5 (The upper half plane). In one complex dimension, we consider the holomorphic prepotential

$$\mathcal{F} = -\frac{1}{24}\eta^3,$$

giving the metric

$$\mathrm{d}s^2 = \Im(\eta)\mathrm{d}\eta\mathrm{d}\bar{\eta},$$

which is positive definite and non degenerate on

$$\{\eta \in \mathbb{C}/\Im(\eta) > 0\}.$$

From $\mathcal{F}$ we can recover the symplectic coordinates

$$x = \Re(\eta), \quad y = 8\Re\left(\frac{\partial\mathcal{F}}{\partial\eta}\right) = -\Re(\eta^2).$$

denoting $\eta = x + \mathrm{i}p$, we have that $y = p^2 - x^2$ and

$$\Im(\eta) = p = +\sqrt{y + x^2}, \quad y > -x^2,$$

from which the metric reads

$$\mathrm{d}s^2 = \frac{4(y + 2x^2)\mathrm{d}x^2 + \mathrm{d}y^2 + 4x\mathrm{d}x\mathrm{d}y}{4\sqrt{y + x^2}}. \tag{49}$$

This metric is not the Poincaré metric on the upper half plane. From (12) we can see that it has non constant curvature

$$R = \frac{1}{4(\Im(\eta))^3}. \qquad \square$$

3.5 The pseudo-Kähler case

As we have seen at the end of Section 3.3, in the pseudo-Kähler case we cannot conclude the independence of the η^i, so they may not form a complex coordinate system. Nevertheless, the $2n \times n$ matrix $(\alpha^\top, -\beta^\top)$ has still rank n, so at each point we can always perform a linear transformation A

$$\begin{pmatrix}\mathrm{d}\eta' \\ \mathrm{d}\lambda'\end{pmatrix} = A\begin{pmatrix}\mathrm{d}\eta \\ \mathrm{d}\lambda\end{pmatrix} = \begin{pmatrix}\alpha' \\ \beta'\end{pmatrix}(\mathrm{d}z),$$

such that the matrix

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

has α' of rank n. Moreover, the linear transformation A can be chosen as a transformation of the symplectic group. A proof of this fact is given in Lemma A1 in [7]. We reproduce a sketch of the proof and some further comments in Appendix A, Lemma A.1 and Remark A.2.

The conclusion is that there exists a locally finite covering by flat Darboux coordinates such that in each open set the matrix α has rank n and then the functions η_i are a system of complex coordinates. These will be also called *special coordinates*. The calculation of the prepotential in these coordinates follows as in Section 3.4.

The lesson to learn here is that, unlike the Kähler case, in the pseudo-Kähler case not all the Darboux coordinates are suitable to construct special complex coordinates, but one can equally cover the manifold with special coordinates. These systems of *special Darboux coordinates* transform in the intersections between charts with matrices belonging to a subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, the subgroup that preserves the maximal rank of the block α in the $2n \times n$ matrix

$$V = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

It is easy to determine this subgroup. First, we notice that the matrices

$$M_0 = \begin{pmatrix} A & 0 \\ C & (A^\top)^{-1} \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})$$

form a subgroup, and this subgroup is maximal (we relegate the proof to the Appendix, Lemma A.3). For matrices of this form, we have that $\det A \neq 0$, so the rank of α is preserved. On the other hand, as we proved in Remark 3.3, there exists always a symplectic transformation that does not preserve the rank of α. The conclusion is that the flat symplectic structure of the tangent bundle is reduced to the subgroup of matrices

$$\left\{ \begin{pmatrix} A & 0 \\ C & (A^\top)^{-1} \end{pmatrix} \right\} \subset \mathrm{Sp}(n, \mathbb{R}).$$

3.6 A special pseudo-Kähler manifold

Let (z^1, z^2) be holomorphic coordinates on $\mathbb{C}^2$ and consider the prepotential

$$\mathcal{F} = -\frac{1}{8} \mathrm{i} z^1 z^2,$$

then

$$\tau_{ij} = -8 \frac{\partial^2 \mathcal{F}}{\partial z^i \partial z^j} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$g_{i\bar{\jmath}} = -\frac{1}{4}\Im(\tau_{ij}) = -\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathrm{d}s^2 = 2g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^j = -\frac{1}{2}\Im(\tau_{ij})\mathrm{d}z^i\mathrm{d}\bar{z}^j = -\frac{1}{2}\left(\mathrm{d}z^1\mathrm{d}\bar{z}^2 + \mathrm{d}z^2\mathrm{d}\bar{z}^1\right),$$

which clearly has signature $(2,2)$ (null vectors always come in pairs, one holomorphic and one antiholomorphic). The Kähler form is

$$\Phi = -2\mathrm{i}g_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = \frac{\mathrm{i}}{2}\Im(\tau_{ij})\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = \frac{\mathrm{i}}{2}(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^2 + \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^1).$$

Let us denote

$$z^1 = x^1 + \mathrm{i}y_2, \quad z^2 = x^2 + \mathrm{i}y_1, \tag{50}$$

the real and imaginary parts of the complex holomorphic coordinates. These are the Darboux coordinates of (28). Then the Kähler form takes the standard form

$$\Phi = \mathrm{d}x^1 \wedge \mathrm{d}y_1 + \mathrm{d}x^2 \wedge \mathrm{d}y_2,$$

so (x^i, y_i) are symplectic coordinates. For these symplectic coordinates, there is associated a special holomorphic system of coordinates, just as in the Riemannian case.

We want to show now that not all the symplectic coordinate systems have this property when the metric is pseudo-Riemannian. Let us make the following symplectic change of coordinates:

$$\begin{aligned} &x'^1 = x^1, \quad x'^2 = y_2, \quad y'_1 = y_1, \quad y'_2 = -x^2, \\ &\text{with} \quad z^1 = x'^1 + \mathrm{i}x'^2, \quad z^2 = -y'_2 + \mathrm{i}y'_1. \end{aligned} \tag{51}$$

We have

$$\begin{aligned} \pi^{(1,0)} &= \mathrm{d}z^i \otimes \frac{\partial}{\partial z^i} = \mathrm{d}z^1 \otimes \frac{1}{2}\left(\frac{\partial}{\partial x^1} - \mathrm{i}\frac{\partial}{\partial y^2}\right) + \mathrm{d}z^2 \otimes \frac{1}{2}\left(\frac{\partial}{\partial x^2} - \mathrm{i}\frac{\partial}{\partial y^1}\right) \\ &= \frac{1}{2}\left(\mathrm{d}z^1 \otimes \frac{\partial}{\partial x^1} + \mathrm{d}z^2 \otimes \frac{\partial}{\partial x^2} - \mathrm{i}\mathrm{d}z^2 \otimes \frac{\partial}{\partial y^1} - \mathrm{i}\mathrm{d}z^1\frac{\partial}{\partial y^2}\right). \end{aligned} \tag{52}$$

Comparing this equation with (30)

$$\mathrm{d}\eta^1 = \mathrm{d}z^1, \quad \mathrm{d}\lambda_1 = \mathrm{i}\mathrm{d}z^2,$$
$$\mathrm{d}\eta^2 = \mathrm{d}z^2, \quad \mathrm{d}\lambda_2 = \mathrm{i}\mathrm{d}z^1,$$

and thus, following (35),

$$\alpha^\top = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta^\top = \mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can use the new variables (x', y') defined in (51) to calculate (η', λ'):

$$\pi^{(1,0)} = \mathrm{d}z^1 \otimes \frac{1}{2}\left(\frac{\partial}{\partial x'^1} - \mathrm{i}\frac{\partial}{\partial x'^2}\right) + \mathrm{d}z^2 \otimes \frac{1}{2}\left(-\frac{\partial}{\partial y'_2} - \mathrm{i}\frac{\partial}{\partial y'_1}\right). \tag{53}$$

Comparing this equation with (30)

$$\begin{aligned} \mathrm{d}\eta'^1 &= \mathrm{d}z^1, \qquad \mathrm{d}\lambda'^1 = \mathrm{i}\mathrm{d}z^2, \\ \mathrm{d}\eta'^2 &= -\mathrm{i}\mathrm{d}z^1, \quad \mathrm{d}\lambda'^2 = \mathrm{d}z^2, \end{aligned}$$

then

$$\alpha'^{\top} = \begin{pmatrix} 1 & -\mathrm{i} \\ 0 & 0 \end{pmatrix}, \quad \beta'^{\top} = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 1 \end{pmatrix}.$$

We compute now the null vector, following the general case explained at the beginning of Section 3.3. Let $c = (c_1, c_2)$ be such that

$$c\alpha'^{\top} = (c_1, c_2) \begin{pmatrix} 1 & -\mathrm{i} \\ 0 & 0 \end{pmatrix} = (0, 0) \implies c_1 = 0.$$

For any $c = (c_1, c_2)$ we have that

$$c\beta'^{\top} = (c_1, c_2) \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 1 \end{pmatrix} = c_2(\mathrm{i}, 1),$$

so $(0, c_2)$ is not a null vector of $\beta'^{\top}$. The vector γ and its complex conjugate $\bar{\gamma}$ are then

$$\gamma = \mathrm{i}\frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^2} \quad \text{and} \quad \bar{\gamma} = -\mathrm{i}\frac{\partial}{\partial \bar{z}^1} + \frac{\partial}{\partial \bar{z}^2}.$$

Then

$$\Phi(\gamma, \bar{\gamma}) = \frac{1}{2}\mathrm{i}(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^2 + \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^1)(\gamma \otimes \bar{\gamma}) = 0. \tag{54}$$

3.7 The holomorphic cubic form

Let $\mathcal{M}$ be a rigid special Kähler manifold. We want to compute the difference between the Levi-Civita connection D and the flat connection ∇. Using the same notation as in reference [8], we define the tensor $B_{\mathbb{R}}$ as

$$B_{\mathbb{R}} \equiv \nabla - D, \quad B_{\mathbb{R}} \in \Omega^1(\mathcal{M}, \mathrm{End}_{\mathbb{R}} T\mathcal{M}).$$

Since both connections are symplectic, $D\Phi = 0$ and $\nabla\Phi = 0$ (see (2) and (19)), we have that

$$\begin{aligned} \partial_u(\Phi(v, w)) &= \nabla_u(\Phi(v, w)) = \Phi(\nabla_u(v), w) + \Phi(v, \nabla_u(w)), \\ \partial_u(\Phi(v, w)) &= D_u(\Phi(v, w)) = \Phi(D_u(v), w) + \Phi(v, D_u(w)), \\ 0 &= \Phi((B_{\mathbb{R}})_u(v), w) + \Phi(v, (B_{\mathbb{R}})_u(w)). \end{aligned} \tag{55}$$

This says that the endomorphism $(B_{\mathbb{R}})_u$, for arbitrary u, is in the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ defined by Φ. In components, using (7) and (24), we get

$$\begin{aligned} B_{\mathbb{R}} = \nabla - D = (A^k_{ij} - \Gamma^k_{ij})\mathrm{d}\eta^i \otimes \mathrm{d}\eta^j \otimes \partial_k + A^{\bar{k}}_{ij}\mathrm{d}\eta^i \otimes \mathrm{d}\eta^j \otimes \partial_{\bar{k}} \\ + (A^{\bar{k}}_{\bar{i}\bar{j}} - \Gamma^{\bar{k}}_{\bar{i}\bar{j}})\mathrm{d}\bar{\eta}^{\bar{i}} \otimes \mathrm{d}\bar{\eta}^{\bar{j}} \otimes \partial_{\bar{k}} + A^k_{\bar{i}\bar{j}}\mathrm{d}\bar{\eta}^{\bar{i}} \otimes \mathrm{d}\bar{\eta}^{\bar{j}} \otimes \partial_k. \end{aligned}$$

Let u, v, w vectors of type (1,0). Then $(B_{\mathbb{R}})_u(\bar{w}) = 0$, and the last line of (55), with w replaced by $\bar{w}$, implies that $\Phi((B_{\mathbb{R}})_u(v), \bar{w}) = 0$. In components, this means

$$(A^k_{ij} - \Gamma^k_{ij}) = 0, \quad (A^{\bar{k}}_{\bar{i}\bar{j}} - \Gamma^{\bar{k}}_{\bar{i}\bar{j}}) = 0,$$

where the second one follows by complex conjugation. One can define an element B in $\Omega^{1,0}(\mathrm{Hom}(T\mathcal{M}, \overline{T\mathcal{M}}))$ such that

$$B = A^{\bar{k}}_{ij}\, \mathrm{d}\eta^i \otimes \mathrm{d}\eta^j \otimes \partial_{\bar{k}}, \quad \text{so } B_{\mathbb{R}} = B + \bar{B}.$$

Lowering the antiholomorphic index with the metric, we can define locally a holomorphic 3-tensor:

$$\Xi_{ijk} = -2\mathrm{i} g_{i\bar{\ell}} A^{\bar{\ell}}_{jk} \iff A^{\bar{i}}_{jk} = \frac{1}{2}\mathrm{i} g^{\bar{i}\ell} \Xi_{\ell jk}. \tag{56}$$

Using (47) and the fact that $g^{\bar{j}\ell} = 8\mathrm{i}\beta^{\bar{j}\ell}$ we get

$$\Xi_{ijk} = -\frac{1}{4}\frac{\partial \tau_{ij}}{\partial \eta^k} = 2\frac{\partial^3 \mathcal{F}}{\partial\eta^i \partial\eta^j \partial\eta^k}, \tag{57}$$

from which it follows that Ξ is holomorphic and symmetric.

In [8] the following global definition is given for this tensor:

$$\Xi(X, Y, Z) = \Phi(\pi^{(1,0)} X, (\nabla_Y \pi^{(1,0)}) Z). \tag{58}$$

In fact, since $D\pi^{(1,0)} = 0$, we can substitute ∇ by $B_{\mathbb{R}}$ in (58) so

$$\begin{aligned} \Xi(X, Y, Z) &= \Phi(\pi^{(1,0)} X, [(B_{\mathbb{R}})_Y, \pi^{(1,0)}] Z) \\ &= \Phi(\pi^{(1,0)} X, [(B + \bar{B})_Y, \pi^{(1,0)}] Z) \\ &= \Phi(\pi^{(1,0)} X, B_Y \pi^{(1,0)} Z), \end{aligned}$$

which in components, using (4), gives (56).

It is then clear that given the flat connection ∇ we can determine the cubic form Ξ. Conversely, assume that we are given a holomorphic symmetric cubic form Ξ on a Kähler manifold. We can determine a tensor $B_{\mathbb{R}} = B + \bar{B}$ from (56). Then a new connection is defined by $\nabla = D + B_{\mathbb{R}}$. The symmetry of Ξ guarantees that ∇ is torsionfree, symplectic and satisfies (18), as it follows straightforwardly from (56). The flatness condition imposes some restrictions on Ξ. We have to impose $\mathrm{d}^2_\nabla = 0$, with

$$\mathrm{d}_\nabla F = \mathrm{d}_D F + B \wedge F + \bar{B} \wedge F,$$

for $F \in \Omega^p_{\mathcal{M}}(T\mathcal{M})$. Then, if R is the curvature of the Levi-Civita connection, $\frac{1}{2} R^I{}_{JKL} \mathrm{d}x^K \wedge \mathrm{d}x^L$, then

$$\mathrm{d}^2_\nabla = 0 \iff R + \mathrm{d}_D B + \mathrm{d}_D \bar{B} + B \wedge \bar{B} + \bar{B} \wedge B = 0.$$

Analysing the holomorphic and antiholomorphic components in this equation, we obtain that the following expressions have to cancel separately:

$$R + B \wedge \bar{B} + \bar{B} \wedge B = 0, \tag{59}$$

$$\mathrm{d}_D B = 0, \tag{60}$$

$$\mathrm{d}_D \bar{B} = 0. \tag{61}$$

Equations (60) and (61) are the complex conjugate of each other. (59) imposes a constraint on the curvature of the Kähler manifold. It should be expressed solely in terms of the holomorphic cubic form. In coordinates this means

$$R^{\ell}{}_{ij\bar{k}} = A^{\bar{p}}_{ji} A^{\ell}_{\bar{p}\bar{k}} = \frac{1}{4} g^{\ell\bar{\ell}'} g^{p'\bar{p}} \Xi_{p'ji} \bar{\Xi}_{\bar{p}\bar{\ell}'\bar{k}}. \tag{62}$$

(60) imposes a constraint on the metric and the cubic tensor. In components we have

$$B^{\bar{k}}_{j} = A^{\bar{k}}_{ij} \, \mathrm{d}\eta^i = -\frac{1}{2\mathrm{i}} g^{\bar{k}\ell} \Xi_{\ell ij} \, \mathrm{d}\eta^i,$$

so

$$-2\mathrm{i}\mathrm{d}_D B^{\bar{k}}_{j} = \mathrm{d}_D g^{\bar{k}\ell} \wedge \Xi_{\ell ij} \, \mathrm{d}\eta^i + g^{\bar{k}\ell} \mathrm{d}_D (\Xi_{\ell ij} \, \mathrm{d}\eta^i) = 0.$$

Here, $g^{\bar{k}\ell}$ is understood as the components of an element of $\Omega^0(\mathcal{M}, T\mathcal{M} \otimes \overline{T\mathcal{M}})$, so its covariant differential is the covariant derivative and it is 0. Only the last term survives, so we have

$$\mathrm{d}_D (\Xi_{\ell ij} \, \mathrm{d}\eta^i) = (\partial_k \Xi_{lij} - \Gamma^{\ell'}_{k\ell} \Xi_{\ell' ij} - \Gamma^{j'}_{kj} \Xi_{\ell ij'}) \, \mathrm{d}\eta^k \wedge \mathrm{d}\eta^i = 0. \tag{63}$$

It is easy to see that for a metric as in (44) and a cubic form as in (58) this equation is satisfied identically. What this argument proves is that (59) and (60), or the equivalent statements (62) and (63), are sufficient conditions to have a flat symplectic connection satisfying the requirements of a rigid special Kähler structure.

Indeed, given a Hermitian metric and an arbitrary holomorphic cubic form with components $\Xi_{ijk} = \Xi_{(ijk)}$, one can construct a torsionfree, symplectic connection as $\nabla = D + B_{\mathbb{R}}$, where the connection coefficients for $B_{\mathbb{R}}$ are determined by (56). This connection, by construction, satisfies $d_\nabla J = 0$. Then (59), (60) are equivalent to the statement that ∇ is flat, ($\mathrm{d}^2_\nabla = 0$). So given a Hermitian metric and a holomorphic cubic form, they will in this case define a special Kähler structure.

4 Projective Kähler (Kähler–Hodge) manifolds

4.1 Affine transformations, isometries and homothetic Killing vectors

For the results in this section see [23], Chapter VI.

An *affine transformation* of a manifold $\mathcal{M}$ with linear connection ∇ is a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ whose tangent map $Tf : T\mathcal{M} \to T\mathcal{M}$ maps any parallel

vector field along a curve γ into a parallel vector field along the curve $f(\gamma)$. The push-forward by f of a vector field X on $\mathcal{M}$ is

$$f_*(X) = Tf \circ X \circ f^{-1},$$

or, in components,

$$f_* X(x) = \frac{\partial f^I}{\partial x^J} X^J(f^{-1}(x))\partial_I. \tag{64}$$

If Y and Z are two vector fields on $\mathcal{M}$ and f is an affine transformation, then

$$(f_* \nabla_Y Z) = \nabla_{f_* Y}(f_* Z). \tag{65}$$

Let K be a vector field on $\mathcal{M}$ and let $\varphi_t : U \to \mathcal{M}$ be the flow of K on a neighbourhood U of $x \in \mathcal{M}$, $t \in]-\epsilon, \epsilon[$. φ_t is a local uniparametric group of transformations, and for each $x \in \mathcal{M}$, $\varphi_t(x)$ is an integral curve of K:

$$\frac{\mathrm{d}\varphi_t(x)}{\mathrm{d}t} = K(\varphi_t(x)).$$

We say that K is an *infinitesimal affine transformation* of $\mathcal{M}$ if φ_t is an affine transformation of U (the connection being the restriction of ∇ to U). Specifying $f = \varphi_t$ in (65) and taking a derivative with respect to t and putting $t = 0$ one obtains[5]

$$\mathcal{L}_K \circ \nabla_Y - \nabla_Y \circ \mathcal{L}_K = \nabla_{[K,Y]}, \quad \text{for every vector field } Y \text{ on } \mathcal{M}, \tag{66}$$

which characterizes K as an infinitesimal affine transformation. (Here $\mathcal{L}_K$ stands for the Lie derivative with respect to K). In components, this condition reads

$$K^J \partial_J \nabla_I Z^L - \nabla_\sigma \left(K^J \partial_J Z^L\right) + Z^J \nabla_I \partial_J K^L + (\partial_I K^J)\nabla_J Z^L = 0. \tag{67}$$

The infinitesimal affine transformations form a subalgebra of the Lie algebra of vector fields on $\mathcal{M}$.

For torsionfree connections, (66) reduces to

$$R(K, Y)Z + \nabla_Y \nabla_Z K - \nabla_{\nabla_Y Z} K = 0, \tag{68}$$

or, in components (as Z is arbitrary),

$$K^J R_{JIK}{}^L + \nabla_I \nabla_K K^L = 0, \tag{69}$$

which was used in [27] as the definition of symmetry of the physical sigma model, independently of the action (in fact, such action may not exist).

A vector field X on $\mathcal{M}$ is *complete* if each integral curve $\varphi_t(x)$ extends to $t \in]-\infty, +\infty[$. This means that the local uniparametric group extends to a global uniparametric group

$$\mathbb{R} \times \mathcal{M} \longrightarrow \mathcal{M},$$

$$(t, x) \longmapsto \varphi_t(x).$$

[5] One uses here $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{t*}X\big|_{t=0} = -[K, X] = -\mathcal{L}_K X$, where $\varphi_t(x)|_{t=0} = x$. The first can be derived from (64) with $f = \varphi_t$.

We say that ∇ is a *complete linear connection* if every geodesic extends to $t \in]-\infty, +\infty[$.

The Lie algebra of the Lie group[6] of affine transformations consists of all infinitesimal affine transformations that are complete. Moreover, if ∇ is *complete* then all the infinitesimal affine transformations are complete.

Let $\mathcal{M}$ be a manifold with Riemannian metric g and Riemannian connection ∇. An *isometry* of $\mathcal{M}$ is a transformation that leaves the metric invariant. An isometry is necessarily an affine transformation of $\mathcal{M}$ with respect to the Riemannian connection.

A vector field X is an *infinitesimal isometry* (or *Killing vector*) if the uniparametric group of transformations generated by X in a neighbourhood of $x \in \mathcal{M}$ (for arbitrary x) consists of local isometries. An infinitesimal isometry X is characterized by

$$\mathcal{L}_X g = 0,$$

which implies that the set of infinitesimal isometries is a Lie algebra.

As in the case of affine transformations, the Lie algebra of the Lie group of isometries is the Lie algebra of all complete infinitesimal isometries, and if $\mathcal{M}$ is complete all the infinitesimal isometries are complete.

We say that a transformation ϕ of a (pseudo) Riemannian manifold $\mathcal{M}$ is *homothetic* if there is a positive constant a^2 (which depends on ϕ) such that

$$\begin{gathered}(\phi^* g)_x(X, Y) = g_{\phi(x)}(T\phi(X), T\phi(Y)) = a^2 g_x(X, Y), \\ \text{for all } X, Y \in T\mathcal{M} \text{ and } x \in \mathcal{M}.\end{gathered} \tag{70}$$

Notice that the Christoffel symbols for the metrics g and $a^2 g$ are the same, so the covariant derivatives are the same. It is then easy to see that a homothetic transformation is an affine transformation of the Levi-Civita connection.

An infinitesimal transformation K of a (pseudo) Riemannian manifold is homothetic if its flow is a homothetic transformation in a neighbourhood of each point $x \in \mathcal{M}$. Infinitesimal homothetic transformations are also called *homothetic Killing vectors* and can be characterized as

$$\mathcal{L}_K g = cg, \tag{71}$$

for a constant c. This can be seen by substituting $\phi = \varphi_t$, the flow of X, in (70) and taking the derivative with respect to t at $t = 0$. We obtain also

$$c = \frac{\mathrm{d}a^2}{\mathrm{d}t}\Big|_{t=0}.$$

If D is the Levi-Civita connection, then (71) is equivalent to the statement that

$$g(X, D_Y K) + g(Y, D_X K) = cg(X, Y) \quad \text{for all } X, Y \in T\mathcal{M}, \tag{72}$$

since $D_K X - \mathcal{L}_K X = D_X K$. In components we have

$$D_I K_J + D_J K_I = c g_{IJ}.$$

[6] It is necessary to assume that $\mathcal{M}$ has a finite number of connected components.

Let us consider the 1-form $g_K(X) = g(K, X)$. If ∇ is a torsionfree connection we have that

$$\mathrm{d}g_K(X, Y) = \nabla_X(g_K)(Y) - \nabla_Y(g_K)(X).$$

This is true for any 1-form. In our case,

$$\nabla_X(g_K)(Y) = \nabla_X(g)(K, Y) + g(\nabla_X K, Y).$$

If the connection is compatible with the metric, $\nabla_X(g) = 0$, we have

$$\mathrm{d}g_K(X, Y) = g(\nabla_X K, Y) - g(\nabla_Y K, X),$$

so

$$\mathrm{d}g_K = 0 \iff g(\nabla_X K, Y) - g(\nabla_Y K, X) = 0 \quad \text{for all } X, Y \in T\mathcal{M}, \tag{73}$$

in components

$$\nabla_I K_J - \nabla_J K_I = 0.$$

We say that K is a *closed homothetic Killing vector* if it is a homothetic Killing vector such that g_K is a closed 1-form.

If K is a closed, homothetic Killing vector and D is the Levi-Civita connection, then equations (72) and (73) imply that

$$D_Y K = \tfrac{1}{2} c Y \quad \text{for all } Y \in T\mathcal{M}. \tag{74}$$

This condition is also sufficient. In components we have that

$$D_I K^J = \frac{1}{2} c \delta_I{}^J.$$

Observe that the statement (74) involves only the connection, so we can use it to generalize the concept of closed homothetic Killing vector to any linear connection. For a torsionfree connection ∇, we will say that a vector field is a closed homothetic Killing vector if

$$\nabla_Y K = \tfrac{1}{2} c Y \quad \text{for all } Y \in T\mathcal{M}. \tag{75}$$

We would like to see if such a vector is in fact an infinitesimal affine transformation for the linear connection. For an arbitrary torsionfree connection (66) is reduced to (68). Using (75) the last two terms of (68) vanish, so the condition for a closed homothetic Killing vector to be an infinitesimal affine transformation is

$$R(K, Y)Z = 0. \tag{76}$$

On the other hand, for an arbitrary connection, the integrability condition of (75) is $R(Y, Z)K = 0$ for all Y and Z, which implies

$$R(K, Y)Z - R(K, Z)Y = 0$$

by using the Bianchi identity (9). The symmetric combination in Y and Z is not zero in general.

We conclude that in general for a torsionfree connection, a closed homothetic Killing vector is not necessarily an infinitesimal affine transformation. For a flat, torsionfree connection, (76) is trivial and thus in this case the closed homothetic Killing vector is an infinitesimal affine transformation.

For the Levi-Civita connection (not necessarily flat), we have seen that any homothetic transformation is an affine transformation. In fact, because of the symmetries of the curvature tensor

$$R(K,Y)Z = 0 \iff R(Y,Z)K = 0,$$

so K is an infinitesimal affine transformation, even if the connection is not flat.

Let $\mathcal{M}$ be a Kähler manifold with complex structure J and let g be the Hermitian metric. Let H be a holomorphic vector field. The equations above can be extended to the complexified tangent space. We assume that H is a homothetic Killing vector. In components this reads

$$\mathcal{L}_H g_{\alpha\bar\beta} = g_{\gamma\bar\beta} D_\alpha H^\gamma = c g_{\alpha\bar\beta} \iff D_Y H = cY, \quad Y \text{ holomorphic.} \tag{77}$$

As before, the last expression does not involve explicitly the metric and can be used as a generalization of holomorphic homothetic Killing vector for any linear connection. Since the metric is Hermitian, it is easy to see that $\bar H$ is homothetic with the same constant c (real). It follows that $K = \frac{1}{2}(H + \bar H)$ is also a homothetic Killing vector with constant c while $\hat K = -JK = \frac{1}{2\mathrm{i}}(H - \bar H)$ is a Killing vector,

$$\mathcal{L}_K g = \frac{1}{2}(\mathcal{L}_H g + \mathcal{L}_{\bar H} g) = cg, \tag{78}$$

$$\mathcal{L}_{\hat K} g = \frac{1}{2\mathrm{i}}(\mathcal{L}_H g - \mathcal{L}_{\bar H} g) = 0. \tag{79}$$

Notice that (79) can be written in components as

$$g_{\gamma\bar\beta} D_\alpha H^\gamma - g_{\bar\gamma\alpha} D_{\bar\beta} H^{\bar\gamma} = 0,$$

which is just the requirement that K is closed, so if H is a holomorphic, homothetic Killing vector then $K = \frac{1}{2}(H + \bar H)$ is a closed homothetic Killing vector.

The converse is also true: if the metric has a closed homothetic Killing vector K, then JK is a Killing vector. It also implies the presence of the holomorphic homothetic Killing vector $H = (1 - \mathrm{i}J)K$, i.e. satisfying (77).

Example 4.1 (Euclidean space). We consider $\mathbb{C}^n$ with the metric

$$\mathrm{d}s^2 = \mathrm{d}z^\alpha \mathrm{d}\bar z^\alpha.$$

We take $H = z^\beta \partial/\partial z^\beta$. Then $\mathcal{L}_H g = g$, so H is an holomorphic, homothetic Killing vector with $c = 1$. We define $K = \frac{1}{2}(H + \bar H) = \frac{1}{2}(z^\beta \partial/\partial z^\beta + \bar z^\beta \partial/\partial \bar z^\beta)$. Then

$$g_K = \frac{1}{2}(z^\alpha \mathrm{d}\bar z^\alpha + \bar z^\alpha \mathrm{d}z^\alpha) \implies \mathrm{d}g_K = 0. \qquad \square$$

4.2 Definition of projective Kähler manifolds

Let $\widetilde{\mathcal{M}}$ be a complex manifold and let H be a holomorphic vector field. Then $[H, \bar{H}] = 0$, so $\{H, \bar{H}\}$ form an integrable distribution on $T\widetilde{\mathcal{M}}$. By Frobenius theorem, there is a foliation of $\widetilde{\mathcal{M}}$ whose leaves[7] are complex submanifolds of $\widetilde{\mathcal{M}}$ whose tangent space is generated by H and $\bar{H}$. If $H \neq 0$ at all points of $\widetilde{\mathcal{M}}$ the foliation is *regular*; then all the leaves have complex dimension 1. We can define an equivalence relation on $\widetilde{\mathcal{M}}$ by declaring as equivalent two points if they belong to the same leaf. Then, if the foliation is regular, the quotient of $\widetilde{\mathcal{M}}$ by this relation (the set of all equivalence classes) is a manifold.

Let K and $\hat{K}$ be, as above, the real and imaginary parts of H, respectively, so $H = K + \mathrm{i}\hat{K}$. Let φ_τ and $\hat{\varphi}_\sigma$ the flows of K and $\hat{K}$ respectively,

$$\frac{\mathrm{d}\varphi_\tau(x)}{\mathrm{d}\tau} = K(\varphi_\tau(x)), \quad \frac{\mathrm{d}\hat{\varphi}_\sigma(x)}{\mathrm{d}\sigma} = \hat{K}(\hat{\varphi}_\sigma(x)), \quad \varphi_0(x) = \hat{\varphi}_0(x) = x.$$

Since $[K, \hat{K}] = 0$, K is invariant under the flow of $\hat{K}$ and viceversa. This in turn implies that

$$\varphi_\tau \circ \hat{\varphi}_\sigma = \hat{\varphi}_\sigma \circ \varphi_\tau.$$

Let us define $\lambda = \tau - \mathrm{i}\sigma$ and $\rho_\lambda = \varphi_\tau \circ \hat{\varphi}_\sigma$; then it is easy to see that

$$\frac{\mathrm{d}\rho_\lambda(x)}{\mathrm{d}\lambda} = \frac{\mathrm{d}\rho_\lambda(x)}{\mathrm{d}\tau} + \mathrm{i}\frac{\mathrm{d}\rho_\lambda(x)}{\mathrm{d}\sigma} = H(\rho_\lambda(x)), \quad \rho_0(x) = x. \tag{80}$$

ρ_λ is a local, complex, 1-dimensional group of transformations,

$$\rho_\lambda \circ \rho_{\lambda'} = \rho_{\lambda+\lambda'}.$$

H is the *fundamental vector field* of the action of G.

We consider now on $\widetilde{\mathcal{M}}$ a holomorphic action of $\mathbb{C}^\times$:

$$\begin{aligned} \mathbb{C}^\times \times \mathcal{M} &\longrightarrow \mathcal{M}, \\ (b, x) &\longmapsto \Re_b(x), \end{aligned} \tag{81}$$

with $b \in \mathbb{C}^\times$. Locally, $b = \exp\lambda$ and $\Re_b(x) = \rho_\lambda(x)$ with ρ_λ as in (80). Let H be the (holomorphic) fundamental vector field of this action (80). The orbits of the action are the integral submanifolds of the foliation defined by H. We assume also that the action is free, so the orbits are diffeomorphic to $\mathbb{C}^\times$. Since the group is abelian, the left action is also a right action, so $\widetilde{\mathcal{M}}$ is a principal $\mathbb{C}^\times$-bundle over the orbit space $\widetilde{\mathcal{M}}/\mathbb{C}^\times$.

Definition 4.2 (Projective Kähler manifold). Let $\widetilde{\mathcal{M}}$ be a (pseudo) Kähler manifold with metric $\tilde{g}$. We assume that on $\widetilde{\mathcal{M}}$ there is a free holomorphic action of $\mathbb{C}^\times$ such that the fundamental vector field H is a non null, holomorphic homothetic Killing

[7] The leaves of a foliation are disjoint sets whose union is the whole manifold.

vector of the metric $\tilde{g}$ (or the Levi-Civita connection $\widetilde{D}$),

$$\mathcal{L}_H \tilde{g} = c\tilde{g} \iff \widetilde{D}_Y H = cY \quad \text{for all } Y \in T^{1,0}\widetilde{\mathcal{M}},$$

such that $\tilde{g}(H, \bar{H}) \neq 0$. Then we say that $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{C}^\times$ is a (pseudo) projective Kähler manifold.

We are going to show that $\mathcal{M}$ is a Kähler manifold itself, of a particular class. In order to do that, we need to construct a Hermitian line bundle on $\mathcal{M}$. It is in fact inherited from the tangent bundle on $T\widetilde{\mathcal{M}}$.

The symplectic and line bundles and the fiber metric. $\widetilde{\mathcal{M}}$ has the structure of a principal $\mathbb{C}^\times$-bundle over $\mathcal{M}$, $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$. As in (81) we denote the finite right action of $\mathbb{C}^\times$ on $\widetilde{\mathcal{M}}$

$$\begin{aligned} \widetilde{\mathcal{M}} &\longrightarrow \widetilde{\mathcal{M}}, \\ \tilde{m} &\longmapsto \Re_b(\tilde{m}), \quad b \in \mathbb{C}^\times, \end{aligned}$$

with $\Re_1 \tilde{m} = \tilde{m}$.

Let $T\widetilde{\mathcal{M}}$ be the tangent bundle. The tangent of the action above gives an action on $T\widetilde{\mathcal{M}}$

$$\begin{aligned} T\widetilde{\mathcal{M}} &\longrightarrow T\widetilde{\mathcal{M}}, \\ (\tilde{m}, v_{\tilde{m}}) &\longmapsto (\Re_b(\tilde{m}), \mathrm{d}\Re_b v_{\tilde{m}}). \end{aligned}$$

$T_{\tilde{m}}\widetilde{\mathcal{M}}$ is a complex vector space, so we also have an action of $\mathbb{C}^\times$ on it. We will simply denote it by multiplication,

$$\begin{aligned} T_{\tilde{m}}\widetilde{\mathcal{M}} &\longrightarrow T_{\tilde{m}}\widetilde{\mathcal{M}}, \\ (\tilde{m}, v_{\tilde{m}}) &\longmapsto (\tilde{m}, b v_{\tilde{m}}). \end{aligned}$$

We are going to define an associated bundle to $\widetilde{\mathcal{M}}$ using these actions. We identify elements in $T\widetilde{\mathcal{M}}$ related by

$$(\tilde{m}, v_{\tilde{m}}) \sim (\Re_b(\tilde{m}), b^{-1}\mathrm{d}\Re_b v_{\tilde{m}}). \tag{82}$$

It is easy to see that this is an equivalence relation. The quotient space is a complex vector bundle over $\mathcal{M}$ of rank $n+1$, with $\dim_{\mathbb{C}} \mathcal{M} = n$. We will denote it by $\mathcal{H} = T\widetilde{\mathcal{M}}/\sim$. It is a bundle over $\mathcal{M}$ associated to the $\mathbb{C}^\times$-principal bundle $\widetilde{\mathcal{M}} \to \mathcal{M}$ (see [23]), so there is an action of $\mathbb{C}^\times$ on it. Also, the underlying real vector bundle of $\mathcal{H}$ (and its complexification) inherit the action of the symplectic group $\mathrm{Sp}(2n+2, \mathbb{R})$ from $T\widetilde{\mathcal{M}}$.

A vector in the kernel of the projection $\ker(T\pi|_{\tilde{m}}) \approx \mathrm{span}_{\mathbb{C}}\{H(\tilde{m})\}$ is a vertical vector. We can consider the subbundle of $T\widetilde{\mathcal{M}}$ consisting of vertical vectors. It is a trivial line subbundle of $T\widetilde{\mathcal{M}}$, and it projects to a line bundle on $\mathcal{M}$. We will denote it by L. Two different trivializations $(\tilde{m}, \lambda H(\tilde{m}))$ and $(\tilde{m}', \lambda' H(\tilde{m}'))$ (with $\pi(\tilde{m}') = \pi(\tilde{m})$)

are related, according to (82), by

$$(\tilde{m}, \lambda H(\tilde{m})) \sim (\tilde{m}', \lambda' H(\tilde{m}')) \implies \tilde{m}' = \Re_b(\tilde{m}),\ \lambda' = b^{-1}\lambda,$$

since (80) implies for the finite global transformation that $\mathrm{d}\Re_b H(\tilde{m}) = H(\Re_b(\tilde{m}))$. The transition functions of the bundle are of the form b^{-1}.

On L we can define a fiber metric. Let $(\tilde{m}, \lambda H(\tilde{m}))$ be a representative of the equivalence class $(m, v_m) \in L$, and the same for $(\tilde{m}, \sigma H(\tilde{m}))$ in $(m, u_m) \in L$. We set

$$h_m(v_m, \bar{u}_m) = \tilde{g}_{\tilde{m}}(\lambda H(\tilde{m}), \bar{\sigma}\bar{H}(\tilde{m})) = (\lambda\bar{\sigma})\tilde{g}_{\tilde{m}}(H(\tilde{m}), \bar{H}(\tilde{m})).$$

We remind that $H(\tilde{m})$ is non null by assumption. We just have to check that this definition is independent of the representatives that we have used, so acting with $b \in \mathbb{C}^\times$ we have

$$\begin{aligned}
&\tilde{g}_{\tilde{m}b}(b^{-1}\lambda H(\Re_b(\tilde{m})), \bar{b}^{-1}\bar{\sigma}\bar{H}(\Re_b(\tilde{m})))\\
&\quad = (\lambda\bar{\sigma})(b\bar{b})^{-1}\tilde{g}_{\Re_b(\tilde{m})}(H(\Re_b(\tilde{m})), \bar{H}(\Re_b(\tilde{m})))\\
&\quad = (\lambda\bar{\sigma})\tilde{g}_{\tilde{m}}(H(\tilde{m}), \bar{H}(\tilde{m})),
\end{aligned}$$

as we wanted to show. The last equality follows from (70), taking $b\bar{b} = a^2$.

We can now define the Kähler structure on $\mathcal{M}$.

The metric and the Kähler potential. Let α be a local basis of L^* (a coframe) dual to the frame $\{H\}$ of L, so $\alpha(H) = 1$. Using the formulae from the end of Section 2.3 we have (the index a runs only over one value and can be omitted)

$$h = \theta\alpha\bar{\alpha}, \quad \theta = 2\tilde{g}_{\tilde{m}}(H(\tilde{m}), \bar{H}(\tilde{m})). \tag{83}$$

We want to compute θ using convenient coordinates in $\tilde{\mathcal{M}}$. Let $z^i, i = 1, \ldots, n$ be complex coordinates on an open set $U \subset \mathcal{M}$. Let $s\colon U \to \pi^{-1}(U)$ be a local section on $\mathcal{M}$. Then we can choose the local trivialization $\pi^{-1}(U) \approx U \times \mathbb{C}^\times$ given by

$$\tilde{m} = (m, \hat{y}s(m)), \quad m = \pi(\tilde{m}),\ \hat{y} \in \mathbb{C}^\times.$$

$(\hat{z}^i, \hat{y})$ are local coordinates on $\pi^{-1}(U) \overset{\text{open}}{\subset} \tilde{\mathcal{M}}$. We define *homogeneous coordinates* $\hat{\eta}^\alpha$ on $\pi^{-1}(U)$ as

$$\hat{\eta}^0 = \hat{y}, \quad \hat{\eta}^\alpha = \hat{y}\hat{z}^i \ \text{ for } \alpha = i. \tag{84}$$

The action of $\mathbb{C}^\times$ on $\tilde{\mathcal{M}}$ defined in equation (81), expressed in these coordinates, is simply

$$\Re_b(\hat{z}^i, y) = (\hat{z}^i, b^p\hat{y}), \quad \Re_b(\hat{\eta}^\alpha) = b^p\hat{\eta}^\alpha,$$

for an arbitrary $p \in \mathbb{R}$. So the fundamental vector field is

$$H = \hat{\eta}^\alpha\frac{\partial}{\partial\hat{\eta}^\alpha} = \hat{y}\frac{\partial}{\partial\hat{y}}. \tag{85}$$

(One may choose a multiple of it, which by the definition (77) amounts to a rescaling of c.) The homothety condition is

$$\mathcal{L}_H \tilde{g}_{\alpha\bar{\beta}} = \hat{\eta}^\gamma \frac{\partial}{\partial \hat{\eta}^\gamma} \tilde{g}_{\alpha\bar{\beta}} + \frac{\partial \hat{\eta}^\gamma}{\partial \hat{\eta}^\alpha} \tilde{g}_{\gamma\bar{\beta}} = c\tilde{g}_{\alpha\bar{\beta}}.$$

We make the change of variables

$$\eta^\alpha = (\hat{\eta}^\alpha)^c \implies H = c\eta^\alpha \frac{\partial}{\partial \eta^\alpha}.$$

In these coordinates the homothety condition is

$$\mathcal{L}_H \tilde{g}_{\alpha\bar{\beta}} = c\eta^\gamma \frac{\partial}{\partial \eta^\gamma} \tilde{g}_{\alpha\bar{\beta}} + c\frac{\partial \eta^\gamma}{\partial \eta^\alpha} \tilde{g}_{\gamma\bar{\beta}} = c\tilde{g}_{\alpha\bar{\beta}}.$$

This condition becomes simply

$$\eta^\gamma \frac{\partial}{\partial \eta^\gamma} \tilde{g}_{\alpha\bar{\beta}} = 0. \tag{86}$$

Together with its complex conjugate, (86) implies the following property of the metric:

$$\tilde{g}_{\alpha\bar{\beta}}(\lambda\eta, \bar{\lambda}\bar{\eta}) = \tilde{g}_{\alpha\bar{\beta}}(\eta, \bar{\eta}). \tag{87}$$

If we choose $p = 1/c$, the action $\Re_b$ in the coordinates η^α is

$$\Re_b \eta^\alpha = b\eta^\alpha. \tag{88}$$

If we denote

$$\eta^0 = y, \quad \eta^i = yz^i, \tag{89}$$

then the z^i are coordinates on $\mathcal{M}$. The coordinates on $\widetilde{\mathcal{M}}$ are also homogeneous coordinates, which we will further use, and from now on $\partial_\alpha = \frac{\partial}{\partial \eta^\alpha}$. We have that

$$H = c\eta^\alpha \partial_\alpha = cy\frac{\partial}{\partial y}. \tag{90}$$

The metric in these coordinates can be written in terms of a Kähler potential

$$\tilde{g}_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}. \tag{91}$$

The transformation (87) leads to

$$\mathcal{K}(\eta, \bar{\eta}) = \mathcal{K}'(\eta, \bar{\eta}) + f(\eta) + f'(\bar{\eta}),$$

with

$$\mathcal{K}'(\lambda\eta, \bar{\lambda}\bar{\eta}) = (\lambda\bar{\lambda})\mathcal{K}(\eta, \bar{\eta}).$$

Since $\mathcal{K}$ is real, $f = \bar{f}'$ and with a Kähler transformation

$$\mathcal{K} \longrightarrow \mathcal{K} - f - \bar{f}$$

we can take $\mathcal{K}'$ as the Kähler potential. We will denote it as $\mathcal{K}$ from now on, so we have

$$\mathcal{K}(\lambda\eta, \bar{\lambda}\bar{\eta}) = (\lambda\bar{\lambda})\mathcal{K}(\eta, \bar{\eta}). \tag{92}$$

In particular, this implies

$$\eta^\gamma \partial_\gamma \mathcal{K} = \mathcal{K}, \quad \bar{\eta}^\gamma \partial_{\bar{\gamma}} \mathcal{K} = \mathcal{K}, \quad \eta^\gamma \bar{\eta}^\delta \partial_\gamma \partial_{\bar{\delta}} \mathcal{K} = \mathcal{K}, \tag{93}$$

so the definition (83) gives

$$\theta = \tilde{g}_{\alpha\bar{\beta}} H^\alpha \bar{H}^{\bar{\beta}} = c^2 \eta^\alpha \bar{\eta}^{\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}} \mathcal{K} = c^2 \mathcal{K}. \tag{94}$$

Let us consider the exact (1,1)-form

$$\tilde{\rho} = 2\tilde{\rho}_{\alpha\bar{\beta}} \mathrm{d}\eta^\alpha \wedge \mathrm{d}\bar{\eta}^{\beta} = 2\Big(-\frac{1}{\mathcal{K}^2} \frac{\partial \mathcal{K}}{\partial \eta^\alpha} \frac{\partial \mathcal{K}}{\partial \bar{\eta}^{\beta}} + \frac{1}{\mathcal{K}} \frac{\partial^2 \mathcal{K}}{\partial \eta^\alpha \partial \bar{\eta}^{\beta}} \Big) \mathrm{d}\eta^\alpha \wedge \mathrm{d}\bar{\eta}^{\beta}, \tag{95}$$

and let us denote by $\mathrm{i}\rho$ its pull-back by the section s,

$$s^* \tilde{\rho} = 2\tilde{\rho}_{\alpha\bar{\beta}} \frac{\partial \eta^\alpha}{\partial z^i} \frac{\partial \bar{\eta}^{\beta}}{\partial \bar{z}^{\bar{\jmath}}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} = \mathrm{i}\rho.$$

Using (92), we can see that the result is independent of the section s used. In fact, we have that $\tilde{\rho} = \mathrm{i}\pi^* \rho$,

$$\tilde{\rho}_{\alpha\bar{\beta}} = \mathrm{i}\rho_{i\bar{\jmath}} \frac{\partial z^i}{\partial \eta^\alpha} \frac{\partial \bar{z}^{\bar{\imath}}}{\partial \bar{\eta}^{\beta}}, \tag{96}$$

where $z^i(\eta^\alpha)$ is the expression of the projection map $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ in coordinates.

The tensor $\tilde{\rho}_{\alpha\bar{\beta}}$ is degenerate. Indeed, $H^\alpha = c\eta^\alpha$ is a zero eigenvector due to the identities (93). We want to show that there is no other zero mode, under the assumption that $\tilde{g}_{\alpha\bar{\beta}}$ is non-degenerate. Let us write it as

$$\tilde{\rho}_{\alpha\bar{\beta}} = \frac{1}{\mathcal{K}} \tilde{g}_{\alpha\bar{\beta}} - \mu_\alpha \bar{\mu}_\beta = \partial_\alpha \partial_{\bar{\beta}} \log|\mathcal{K}|, \quad \mu_\alpha \equiv \frac{1}{\mathcal{K}} \partial_\alpha \mathcal{K} = \partial_\alpha \log|\mathcal{K}|.$$

We assume now that there is a vector v^α such that $v^\alpha \tilde{\rho}_{\alpha\bar{\beta}} = 0$, then we find that

$$v^\alpha = \mathcal{K}(v^\gamma \mu_\gamma) \tilde{g}^{\bar{\beta}\alpha} \bar{\mu}_\beta,$$

where $\tilde{g}^{\bar{\beta}\alpha}$ is the inverse of $\tilde{g}_{\alpha\bar{\beta}}$. Hence any zero eigenvector is proportional to $\tilde{g}^{\bar{\beta}\alpha}\bar{\mu}_\beta$, and thus there is only one zero mode. In particular, we also obtain

$$H^\alpha = \nu \bar{\mu}_\beta \tilde{g}^{\bar{\beta}\alpha},$$

for some undetermined function $\nu(\eta, \bar{\eta})$.

The vectors $\partial_i = \eta^0 \partial/\partial\eta^i$ are transversal to H, thus the matrix

$$\rho_{i\bar{\jmath}} = -\mathrm{i}\tilde{\rho}_{\alpha\bar{\beta}} \frac{\partial \eta^\alpha}{\partial z^i} \frac{\partial \bar{\eta}^{\beta}}{\partial \bar{z}^{\bar{\jmath}}}$$

is non degenerate. This matrix (or a matrix proportional to it) can therefore be taken to be the metric on $\mathcal{M}$.

We define therefore the metric on $\mathcal{M}$

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} [\pm \log |\mathcal{K}|] = \pm \mathrm{i} \rho_{i\bar{\jmath}}, \quad \pm = \operatorname{sign} \mathcal{K}. \tag{97}$$

The reason for the $\pm$ convention will be explained below.

The Ricci form of the Hermitian bundle agrees according to (17) with

$$\begin{aligned} \rho &= 2\rho_{i\bar{\jmath}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} = -2\mathrm{i} \frac{\partial^2 \log |\mathcal{K}|}{\partial z^i \partial \bar{z}^{\bar{\jmath}}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} \\ &= -2\mathrm{i} \Big(- \frac{1}{\mathcal{K}^2} \frac{\partial \mathcal{K}}{\partial z^i} \frac{\partial \mathcal{K}}{\partial \bar{z}^{\bar{\jmath}}} + \frac{1}{\mathcal{K}} \frac{\partial^2 \mathcal{K}}{\partial z^i \partial \bar{z}^{\bar{\jmath}}} \Big) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}. \end{aligned}$$

We can also compute the signature of the matrix $\tilde{\rho}_{\alpha\bar{\beta}}$. A vector $V = V^\alpha \partial_\alpha$ is orthogonal to H if $\mu_\alpha V^\alpha = 0$ and the space of such vectors has dimension n. For two such vectors, V and V' we have

$$\tilde{g}(V, \bar{V}') = \mathcal{K} \tilde{\rho}(V, \bar{V}'),$$

so the signature of $\tilde{\rho}$ in the space orthogonal to H the same than the signature of $\tilde{g}$ in such space up to a sign. Furthermore, the sign in the remaining direction of $\tilde{g}$ is the sign of $\mathcal{K}$ as it follows from (93).

We can choose a section s such that the vectors ∂_i have a lift $s_* \partial_i$ orthogonal to H. Then

$$\rho_{i\bar{\jmath}} = -\mathrm{i} s^* \tilde{\rho}(\partial_i, \partial_{\bar{\jmath}}),$$

which is actually independent of the section. So the signature of the metric g in $\mathcal{M}$, (97), is the same than the signature of $\tilde{g}$ in the space orthogonal to H.

We conclude that ρ defines a symplectic structure compatible with the complex structure, so $\mathcal{M}$ is a Kähler manifold with Kähler metric as in (97). The Kähler form is in the first Chern class of a line bundle. This implies that the Kähler form is integer. Such manifolds are called *Kähler–Hodge manifolds* in the literature. When $\mathcal{M}$ is compact, this condition implies that $\mathcal{M}$ is a projective variety, so it is embedded in projective space. This is the *Kodaira embedding theorem*, see for example [28], page 181.

What we have proven here is that a projective Kähler manifold is a Kähler–Hodge manifold.

4.3 The Levi-Civita connection on a Kähler–Hodge manifold

The previous part leads to consider (97) as the metric on $\mathcal{M}$. One can compute its Levi-Civita connection. However, there is a natural way of inducing a connection on $\mathcal{M}$ from the Levi-Civita connection in $\widetilde{\mathcal{M}}$, which gives the same result. It clarifies the geometrical meaning of the metric in the quotient manifold $\mathcal{M}$. We will perform two projections of the connection, first to the bundle $\mathcal{H} = T\widetilde{\mathcal{M}}/\sim$ and then to the tangent bundle $T\mathcal{M}$.

Projecting down to the symplectic bundle. Let X be a vector field on $\mathcal{M}$ and ξ a section of $\mathcal{H}$. Let $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ and $p\colon T\widetilde{\mathcal{M}} \to \mathcal{H}$ the natural projections. Let $\widetilde{\mathcal{D}}$ denote a linear connection on $T\widetilde{\mathcal{M}}$.

The idea is to find adequate lifts $\widetilde{X}$ of X and $\tilde{\xi}$ of ξ, both vector fields on $\widetilde{\mathcal{M}}$, in such way that the covariant derivative $\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi}(\tilde{m})$ projects through p to the same vector on $\mathcal{H}$, independently of the point $\tilde{m}$ in the fiber $\pi^{-1}(m)$ where it has been computed. This will define immediately a covariant derivative on $\mathcal{H}$ as

$$\mathcal{D}_X\xi(m) = p\big(\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi}(\tilde{m})\big), \quad \pi(\tilde{m}) = m. \tag{98}$$

Let us first define the respective lifts. A local section ξ of $\mathcal{H}$ is specified by associating an equivalence class $[(\tilde{m}, v_{\tilde{m}})]$ to any point m, with $\pi(\tilde{m}) = m$. We can choose an arbitrary $\tilde{m} \in \pi^{-1}(m)$ and set $\tilde{\xi}(\tilde{m}) = v_{\tilde{m}}$. Then $\tilde{\xi}$ is a vector field on $\widetilde{\mathcal{M}}$ satisfying (see equation (82))

$$\tilde{\xi}(\Re_b(\tilde{m})) = b^{-1}\mathrm{d}\Re_b\tilde{\xi}(\tilde{m}) \iff \Re_{b*}\tilde{\xi} = b\tilde{\xi}. \tag{99}$$

There is a one to one correspondence between the set of local sections of $\mathcal{H}$ and the set of local sections of $T\widetilde{\mathcal{M}}$ satisfying (99). So $\tilde{\xi}$ is a natural lift of ξ.

Notice that (98) means just that $\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi}$ is a vector field on $\widetilde{\mathcal{M}}$ satisfying (99), so it defines a section of $\mathcal{H}$. For any affine transformation $\Re_b$ of the connection $\widetilde{\mathcal{D}}$, we have that (65)

$$\Re_{b*}(\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi}) = \widetilde{\mathcal{D}}_{\Re_{b*}\widetilde{X}}\big(\Re_{b*}\tilde{\xi}\big),$$

so all we need to complete the definition (98) are lifts satisfying

$$\Re_{b*}\tilde{\xi} = b\tilde{\xi}, \tag{100}$$

$$\Re_{b*}\widetilde{X} = \widetilde{X}. \tag{101}$$

(100) is already guaranteed. There are many lifts of the vector field X to $T\widetilde{\mathcal{M}}$, but we have a connection on the principal bundle $\widetilde{\mathcal{M}}$ (or on its associated bundle L), so it is natural to consider the *horizontal lift*. Horizontal lifts satisfy (101), so this will show the existence of the induced connection on $\mathcal{H}$. Note that for the Levi-Civita connection or for an arbitrary flat connection, $\Re_b$ are affine transformations, so the result applies for these cases of special interest.

To understand the horizontal lift we introduce the definition of connection on a principal bundle as a Lie algebra valued 1-form. The relation with the standard covariant derivative in the associated vector bundles can be found in many places (see for example [23]). For completeness we give a brief outline in the Appendix B.

Definition 4.3. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A connection on a principal G-bundle $P \xrightarrow{\pi} \mathcal{M}$ can be given by a $\mathfrak{g}$-valued 1-form ω on P such that:

(i) If A is a fundamental vector field, generating the action of G on the fibre, associated to $\hat{A} \in \mathfrak{g}$ then $\omega(A) = \hat{A}$.

(ii) $\Re_b{}^*\omega = \mathrm{Ad}_{b^{-1}}\omega = b^{-1}\omega b$, $b \in G$. (Ad is the adjoint representation of G).

A horizontal vector X_u is a vector in T_uP satisfying $\omega(X_u) = 0$. In fact, ω defines a distribution of *horizontal spaces* on TP, denoted by TP^h. At each point u with $\pi(u) = m$, the horizontal space is mapped isomorphically to $T_m\mathcal{M}$. If $TP^v = \ker(T\pi|_u)$ is the set of *vertical vectors* tangent to the fiber, then $T_uP = T_uP^h \oplus T_uP^v$. Moreover, (ii) implies that the distribution is equivariant, that is

$$T_{\Re_b(u)}P^h = T\Re_b(T_uP^h). \tag{102}$$

Let X be a vector field on $T\mathcal{M}$. One can prove that there is a unique vector field $\widetilde{X}$ on TP such that $T\pi(\widetilde{X}) = X$ and $\widetilde{X}(\tilde{m})$ is horizontal for every $\tilde{m}$. It is the *horizontal lift* of X.

The equivariance of horizontal subspaces, (102), implies (101) as we wanted to show. One can also show that any horizontal vector field on $\widetilde{\mathcal{M}}$ satisfying the invariance condition (101) is the lift of a vector field on $\mathcal{M}$.

One can prove that if $\widetilde{X}$ and $\widetilde{Y}$ are horizontal lifts of X and Y respectively, then $[\widetilde{X}, \widetilde{Y}]$ is the horizontal lift of $[X, Y]$. So if $\widetilde{\mathcal{D}}$ is a flat connection (as $\widetilde{\nabla}$) then the induced connection $\mathcal{D}$ on $\mathcal{H}$ is also a flat connection.

Example 4.4 (Horizontal lift in $\widetilde{\mathcal{M}}$). As an example that we will use in the following, we are going to compute the horizontal lift of a holomorphic vector X on $T\mathcal{M}$ to $T\widetilde{\mathcal{M}}$ for the Hermitian connection.

In the coordinates (89) we have

$$\Re_b(m, y) = (m, by), \quad \mathrm{d}\Re_b = \mathrm{d}z^i \otimes \partial_i + b\mathrm{d}y \otimes \partial_y. \tag{103}$$

The connection 1-form and its pull back are

$$\omega = \omega_y \mathrm{d}y + \kappa_i \mathrm{d}z^i, \quad s^*\omega = \kappa_i \mathrm{d}z^i.$$

κ_i is determined by the pull back, which from (16) and (94) is

$$\kappa_i = \mathcal{K}^{-1}\partial_i\mathcal{K} = \partial_i \log|\mathcal{K}|.$$

The other component, ω_y, is determined by conditions (i) and (ii) in Definition 4.3. Since $A = y\partial_y$ and $\widehat{A} = 1$, (i) implies $\omega_y = y^{-1}$. (ii) is then satisfied. The connection 1-form is then

$$\omega = y^{-1}\mathrm{d}y + \kappa_i \mathrm{d}z^i = y^{-1}\mathrm{d}y + \partial_i \log|\mathcal{K}|\,\mathrm{d}z^i. \tag{104}$$

A vector on $T\widetilde{\mathcal{M}}$, $v = v^i\partial_i + v^y\partial_y$ is horizontal if and only if

$$y^{-1}v^y + \kappa_i v^i = 0.$$

If v is arbitrary, then $v = v^h + v^v$ with

$$v^h = v^i\partial_i - y\kappa_i v^i\partial_y, \quad v^v = (v^y + y\kappa_i v^i)\partial_y. \tag{105}$$

v^h is the *horizontal projection* of v.

A vector $\widetilde{X} = \widetilde{X}^y \partial_y + \widetilde{X}^i \partial_i$ is the horizontal lift of $X = X^i \partial_i$ if

$$T\pi(\widetilde{X}) = X, \quad \text{i.e.,} \ \widetilde{X}^i = X^i,$$
$$\omega(\widetilde{X}) = 0, \quad \text{i.e.,} \ \widetilde{X}^y = -y\kappa_i \widetilde{X}^i,$$

so

$$\widetilde{X} = X^i \partial_i - y\kappa_i X^i \partial_y. \tag{106}$$

Projecting down to the tangent bundle. Let us consider the subbundle of $\mathcal{H}$ formed by equivalence classes $[(\tilde{m}, v_{\tilde{m}})]$ such that $v_{\tilde{m}}$ is a horizontal vector. Notice that, due to (102) $b^{-1}\mathrm{d}\Re_b v_{\tilde{m}}$ is horizontal if so is $v_{\tilde{m}}$. We will denote this bundle by $\mathrm{hor}(\mathcal{H})$. We have the following lemma:

Lemma 4.5. $\mathrm{hor}(\mathcal{H}) \approx L \otimes T\mathcal{M}$.

Proof. Let $[(\tilde{m}, v^h_{\tilde{m}})]$ be an element of $\mathrm{hor}(\mathcal{H})$. We can map it to $T_m\mathcal{M}$ with the projection $v_m = \mathrm{d}\pi v^h_{\tilde{m}} \in T_m\mathcal{M}$. If we choose another representative of the same equivalence class, $(\Re_b(\tilde{m}), v'^h_{\Re_b(\tilde{m})})$, with $v'^h_{\Re_b(\tilde{m})} = b^{-1}\mathrm{d}\Re_b v^h_{\tilde{m}}$ we obtain another vector on $T_m\mathcal{M}$, $v'_m = b^{-1}v_m$. The natural projection applied to $\mathrm{hor}(\mathcal{H})$ defines then a section of $L \otimes T\mathcal{M}$.

In the other direction, let $X_m \in T_m\mathcal{M}$ and $\sigma \in L$. We consider the horizontal lift of $\sigma \otimes X_m$ to $L \otimes T\widetilde{\mathcal{M}}$ for some choice of $\tilde{m} \in \pi^{-1}(m)$ and we denote it by $\sigma \otimes X^h_{\tilde{m}}$. Then we consider the equivalence class $[(\tilde{m}, \sigma \otimes X^h_{\tilde{m}})] \in L \otimes \mathrm{hor}(\mathcal{H})$. Let $\tilde{m}' = \Re_b(\tilde{m})$ another choice and $\sigma' = b\sigma$. Then we have the equivalence class $[(\tilde{m}', \sigma' \otimes X^h_{\tilde{m}'})]$, with $X^h_{\tilde{m}'} = \mathrm{d}\Re_b X^h_{\tilde{m}}$. Since $(\tilde{m}, X^h_{\tilde{m}}) \sim (\tilde{m}', b^{-1}X^h_{\tilde{m}'})$, then $(\tilde{m}', \sigma' \otimes X^h_{\tilde{m}'}) \sim (\tilde{m}, \sigma \otimes X^h_{\tilde{m}})$, as we wanted to show. □

Let $\mathcal{D}$ be a connection on $\mathcal{H}$ and let $p_h \colon \mathcal{H} \to \mathrm{hor}(\mathcal{H})$ be the natural projection. We can define a connection on $\mathrm{hor}(\mathcal{H})$ as

$$\widehat{\mathcal{D}}_X \xi = p_h(\mathcal{D}_X \xi), \quad \text{with } X \in T\mathcal{M}, \text{ and } \xi \text{ a section in } \mathrm{hor}(\mathcal{H}) \subset \mathcal{H}.$$

We want to compute $\widehat{\mathcal{D}}$ in coordinates. As before, let s be a local section of $\widetilde{\mathcal{M}}$, so $\tilde{m} = (m, ys(m)) \in \widetilde{\mathcal{M}}$ and let $\{z^i\}$ be local coordinates on $\mathcal{M}$. Then $\{y, z^i\}$ are coordinates on $\widetilde{\mathcal{M}}$. We need to compute the horizontal projection of an arbitrary section χ of $\mathcal{H}$, $\chi^h = p_h(\chi)$. The section χ has a lift $\tilde{\chi}$ to $T\widetilde{\mathcal{M}}$ satisfying (99). In coordinates, using the action of $\Re_b$ as in (103), these equations imply the following y-dependence:

$$\tilde{\chi}(y, z) = y^{-1}\chi^i(z)\partial_i + \chi^y(z)\partial_y,$$

and according to (105), the horizontal projection is

$$\tilde{\chi}^h(y, z) = \chi^i(z)\left(y^{-1}\partial_i - \kappa_i(z)\partial_y\right).$$

Let ξ be a section of $\mathrm{hor}(\mathcal{H})$, so

$$\tilde{\xi} = \xi^i(z)\left(y^{-1}\partial_i - \kappa_i(z)\partial_y\right). \tag{107}$$

We have to compute the horizontal projection of $\mathcal{D}_X\xi$. Lifting to $\widetilde{\mathcal{M}}$, we know that $(\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi})^h$ must be of the same form

$$\big(\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi}\big)^h = \chi^i\big(y^{-1}\partial_i - \kappa_i(z)\partial_y\big),$$

for some χ^i. So we can identify

$$\chi^i = \big(\widehat{\mathcal{D}}_X\xi\big)^i = X^j(\partial_j\xi^i + \hat{\Gamma}^i_{jk}\xi^k).$$

For vectors of the form (107) and (106)

$$\begin{aligned}\widetilde{\mathcal{D}}_{\widetilde{X}}\tilde{\xi}^j &= y^{-1}X^i\partial_i\xi^j + y^{-1}\tilde{\Gamma}^j_{ik}\xi^k X^i - \tilde{\Gamma}^j_{i0}X^i\kappa_k\xi^k\\ &\quad + y^{-1}\kappa_i X^i\xi^j - \tilde{\Gamma}^j_{0k}\xi^k X^i\kappa_i + y\kappa_i X^i\tilde{\Gamma}^j_{00}\xi^k\kappa_k\\ &= y^{-1}\widehat{\mathcal{D}}_X\xi^j,\end{aligned}$$

where we used the coordinates $\{y, z^i\}$ in $\mathcal{M}$ and used the index 0 for the components with respect to y. We obtain therefore for the connection coefficients:

$$\hat{\Gamma}^j_{ik} = \tilde{\Gamma}^j_{ik} - y\tilde{\Gamma}^j_{i0}\kappa_k - y\tilde{\Gamma}^j_{0k}\kappa_i + y^2\kappa_i\kappa_k\tilde{\Gamma}^j_{00} + \kappa_i\delta^j_k. \tag{108}$$

The last term is just the connection on L, while the rest defines a connection on $T\mathcal{M}$,

$$\Gamma^j_{ik} = \tilde{\Gamma}^j_{ik} - y\tilde{\Gamma}^j_{i0}\kappa_k - y\tilde{\Gamma}^j_{0k}\kappa_i + y^2\kappa_i\kappa_k\tilde{\Gamma}^j_{00}. \tag{109}$$

We have then written $\widehat{\mathcal{D}}$ as a connection on $L\otimes T\mathcal{M}$.

We can now compute explicitly the Levi-Civita connection of $\tilde{g}$ in terms of the Levi-Civita connection of g, and check that the formula (108) is satisfied in this case. Due to (92), $\mathcal{K}$ is $y\bar{y}$ times a function that depends only on z and $\bar{z}$. The relation between $\tilde{g}_{\alpha\bar{\beta}}$ and $g_{i\bar{\jmath}}$, given by (97), is

$$\begin{aligned}\tilde{g}_{i\bar{\jmath}} &= \mathcal{K}\left(\pm g_{i\bar{\jmath}} + \kappa_i\bar{\kappa}_{\bar{\jmath}}\right), \quad \tilde{g}_{0\bar{\imath}} = \frac{\mathcal{K}}{y}\bar{\kappa}_{\bar{\imath}}, \quad \tilde{g}_{0\bar{0}} = \frac{\mathcal{K}}{y\bar{y}},\\ \tilde{g}^{\bar{\imath}j} &= \pm\frac{1}{\mathcal{K}}g^{\bar{\imath}j}, \quad \tilde{g}^{\bar{\imath}0} = \mp\frac{y}{\mathcal{K}}g^{\bar{\imath}j}\kappa_j, \quad \tilde{g}^{\bar{0}0} = \frac{y\bar{y}}{\mathcal{K}}\left(1 \pm \kappa_i\bar{\kappa}_{\bar{\jmath}}g^{\bar{\jmath}i}\right),\end{aligned} \tag{110}$$

where $\kappa_i = \partial_i \log|\mathcal{K}|$. This leads to the Levi-Civita connection coefficients $\tilde{\Gamma}^\gamma_{\alpha\beta}$:

$$\begin{aligned}\tilde{\Gamma}^j_{ik} &= g^{j\bar{m}}\left(\frac{1}{\mathcal{K}}\partial_i\partial_{\bar{m}}\partial_k\mathcal{K} - \frac{1}{\mathcal{K}^2}\partial_{\bar{m}}\mathcal{K}\partial_i\partial_k\mathcal{K}\right) = \Gamma^j_{ik}(g) + \kappa_i\delta^j_k + \kappa_k\delta^j_i,\\ \tilde{\Gamma}^j_{i0} &= y^{-1}g^{j\bar{m}}\partial_{\bar{m}}\partial_i\log|\mathcal{K}| = y^{-1}\delta^j_i,\\ \tilde{\Gamma}^0_{ij} &= y\Gamma^k_{ij}(g)\kappa_k + 2y\kappa_i\kappa_j \pm \frac{y}{\mathcal{K}}\partial_i\partial_j\mathcal{K},\\ \tilde{\Gamma}^j_{00} &= \tilde{\Gamma}^0_{0i} = \tilde{\Gamma}^0_{00} = 0,\end{aligned}$$

where $\Gamma(g)$ is the Levi-Civita connection of the metric on $\mathcal{M}$ We thus find that indeed Γ^j_{ik} as determined in (109) are the Christoffel symbols of the Levi-Civita connection on $\mathcal{M}$, as we wanted to show.

4.4 Examples of Kähler–Hodge manifolds

Example 4.6 (Complex Grassmannian as a Kähler–Hodge manifold). We consider the Grassmannian manifold of complex p-planes in $\mathbb{C}^{p+q}$, denoted by $G(p,q)$. We take

$$\widetilde{\mathcal{M}} = \{Z \mid Z \text{ is a } (p+q)\times p \text{ matrix of rank } p\}\,.$$

We will write

$$Z = \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix}$$

with Z_0 a $p\times p$ matrix and Z_1 a $q\times p$ matrix. Each Z defines a p-plane in $\mathbb{C}^{p+q}$ as the span of the column vectors. Taking linear combinations of these vectors gives the same plane. Then there is a right action of $\mathrm{GL}(p,\mathbb{C})$ on $\widetilde{\mathcal{M}}$ which does not change the p-plane. $\widetilde{\mathcal{M}} \to G(p,q)$ is a principal bundle with structure group $\mathrm{GL}(p,\mathbb{C})$.

The group $\mathrm{SL}(p+q)$ acts transitively on $G(p,q)$, but also the action of $\mathrm{SU}(p+q)$ is transitive, with little group $\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)$, so we have that $G(p,q)$ is the Hermitian symmetric space

$$G(p,q) = \frac{\mathrm{SU}(p+q)}{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}.$$

$G(p,q)$ is a Kähler manifold and we are going to show that it is in fact a Kähler–Hodge manifold.

An open cover of $G(p,q)$ is given by the open sets with some fixed minor of order p of Z different from zero. Notice that this property is not changed by the right action of $\mathrm{GL}(p,\mathbb{C})$, so it is well defined on the equivalence classes For concreteness, let us fix

$$U_0 = \left\{Z = \left(\begin{smallmatrix} Z_0 \\ Z_1 \end{smallmatrix}\right) \in G(p,q) \mid \det Z_0 \neq 0\right\}.$$

A p-plane in U_0 can be characterized by a $q\times p$ matrix T such that a vector $(z_1,\ldots,z_p,z_{p+1},\ldots,z_{p+q})$ satisfies

$$\begin{pmatrix} z_{p+1} \\ \vdots \\ z_{p+q} \end{pmatrix} = T \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}.$$

In fact, a matrix Z with $\det Z_0 \neq 0$ is a collection of p column vectors satisfying the above property, so

$$Z_1 = TZ_0 \iff T = Z_1 Z_0^{-1}.$$

An arbitrary matrix T defines a p-plane in U_0, so we have

$$U_0 \approx \mathcal{M}_{q\times p}(\mathbb{C}) \approx \mathbb{C}^{pq},$$

and the entries of T are local coordinates on U_0.

The tautological bundle $H \to G(p,q)$, is the vector bundle with the fiber at each point of $G(p,q)$ the plane that it represents. It is a rank p subbundle of the trivial bundle $G(p,q) \times \mathbb{C}^{p+q}$. It is a bundle associated to the principal bundle $\tilde{\mathcal{M}}$.

On the trivial bundle there is a fiber metric

$$\langle \zeta, \zeta' \rangle = \zeta^1 \bar{\zeta}'^1 + \cdots + \zeta^{p+q} \bar{\zeta}'^{p+q} \tag{111}$$

for ζ, ζ' vectors at a point in $G(p,q)$. It induces a fiber metric on the tautological bundle. A local section on H is given by functions $\zeta^1, \ldots, \zeta^p$, so that T determines the plane:

$$\zeta(T) = \begin{pmatrix} \zeta^1 \\ \vdots \\ \zeta^p \\ \zeta^{p+1} \\ \vdots \\ \zeta^{p+q} \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ T \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \vdots \\ \zeta^p \end{pmatrix}. \tag{112}$$

The Hermitian inner product on the fiber is

$$\begin{aligned} \langle \zeta(T), \zeta'(T) \rangle &= (\bar{\zeta}'^1, \ldots, \bar{\zeta}'^p)(\mathbb{1}, T^\dagger) \begin{pmatrix} \mathbb{1} \\ T \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \vdots \\ \zeta^p \end{pmatrix} \\ &= (\bar{\zeta}'^1, \ldots, \bar{\zeta}'^p)(\mathbb{1} + T^\dagger T) \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_p \end{pmatrix}. \end{aligned}$$

If $\alpha, \beta = 1, \ldots, p$, then we have the fiber metric

$$\langle \zeta(T), \zeta'(T) \rangle = h_{\alpha\bar{\beta}} \zeta^\alpha \bar{\zeta}'^\beta, \quad h^{\mathrm{t}} = \mathbb{1} + T^\dagger T. \tag{113}$$

We can write the Hermitian fiber metric as

$$h = h_{\alpha\bar{\beta}} \mathrm{d}z^\alpha \mathrm{d}\bar{z}^\beta.$$

We consider now the line bundle $\Lambda^p(H)$ with fiber at a point $x \in G(p,q)$ $\Lambda^p H_x \approx \Lambda^p \mathbb{C}^p \approx \mathbb{C}$, i.e. the determinant. The structure group is

$$\mathrm{GL}(p,\mathbb{C})/\mathrm{SL}(p,\mathbb{C}) \approx \mathbb{C}^\times.$$

Let $\{s_\alpha(T)\}$ be a local frame on H, with $h_{\alpha\bar{\beta}} = \langle s_\alpha, s_\beta \rangle$. (To compare with (113) it is enough to take $s_\alpha(T) = \zeta(T)$ as in (112) with $\zeta^\alpha = 1$ and the rest 0). Then a local section on $\Lambda^p(H)$ is of the form

$$U(T) = u(T)\, s_1 \wedge \cdots \wedge s_p.$$

There is an induced fiber metric on this bundle given by

$$H = \det(h_{\alpha\bar{\beta}}) \mathrm{d}u \mathrm{d}\bar{u}.$$

As in (17) we get for the Ricci form associated to the Hermitian connection on the line bundle

$$\rho_{i\bar{\jmath}} = -\mathrm{i}\partial_{\bar{\jmath}}\partial_i \log\det(h_{\alpha\bar{\beta}}) = -\mathrm{i}\partial_{\bar{\jmath}}\partial_i \log\det(\mathbb{1} + T^\dagger T), \tag{114}$$

where $i, j = 1, \ldots, pq$ run over all the entries of the matrix T.

Example 4.7 (Non compact "Grassmannian" as a Kähler–Hodge manifold). In the example above, let us change the fiber metric (111) to a pseudo-Euclidean one with signature (p, q),

$$\langle \zeta, \zeta' \rangle = \zeta^1 \bar{\zeta}'^1 + \cdots + \zeta^p \bar{\zeta}'^p - \zeta^{p+1}\bar{\zeta}'^{p+1} - \cdots - \zeta^{p+q}\bar{\zeta}'^{p+q}.$$

Then, instead of (113) we have

$$h^{\mathrm{t}} = \mathbb{1} - T^\dagger T,$$

so on the points where the matrix $\mathbb{1} - T^\dagger T$ is positive definite we have a positive definite, non degenerate fiber metric. The space of matrices satisfying this property is a domain in $\mathbb{C}^{pq}$. It is the Hermitian symmetric space

$$D(p, q) = \frac{\mathrm{SU}(p, q)}{\mathrm{SU}(p) \times \mathrm{SU}(q) \times \mathrm{U}(1)}.$$

The corresponding expression for the Ricci curvature is proportional, as before, to the standard Kähler metric on this symmetric space.

5 Conformal calculus

The ideas described in the previous section originate in physics as a property of certain sigma models of scalar fields coupled to gravity with a scaling symmetry. It is in fact a simplification of what occurs in supergravity (see for example references [29], [30]), but the essential idea can be grasped in this simplification. We first consider the version with real scalars and then we move to Kähler manifolds.

5.1 Real manifold

We consider a nonlinear sigma model of n real scalar fields ϕ^I with Lagrangian

$$\mathcal{L}_{\mathbb{R},0} = -\tfrac{1}{2}\sqrt{g}\, g^{\mu\nu} G_{IJ}\, \partial_\mu \phi^I\, \partial_\nu \phi^J.$$

Here $g_{\mu\nu}$ is the metric of space time, gravitational field, $g^{\mu\nu}$ its inverse, and $g = |\det g_{\mu\nu}|$. The target space is a real Riemannian manifold with coordinates ϕ^I and $G_{IJ}(\phi)$ is the Riemannian metric.

We will be interested in the case that the Lagrangian has a dilatation symmetry given at the infinitesimal level by a vector $K = K^I \partial_I$. Let D be the dimension of

space-time. We assume that the vector K is a homothetic Killing vector of the metric G_{IJ},

$$\mathcal{L}_K G_{IJ} = K^L \partial_L G_{IJ} + \partial_I K^L G_{LJ} + \partial_J K^L G_{IL} = c\, G_{IJ}.$$

We fix

$$c = D - 2. \tag{115}$$

Then the Lagrangian $\mathcal{L}_{\mathbb{R},0}$ is invariant under the infinitesimal transformations

$$\begin{aligned} \delta_\epsilon \phi^I &= \epsilon K^I, \\ \delta_\epsilon g_{\mu\nu} &= -2\epsilon g_{\mu\nu}, \end{aligned} \tag{116}$$

for an infinitesimal parameter ϵ independent of the point x in spacetime.

A simple example is when the metric on the target space is such that

$$G_{IJ}(\lambda\phi) = \lambda^{D-4} G_{IJ}(\phi), \tag{117}$$

which means that the vector $K = \phi^I \partial_I$ is a homothetic Killing vector (71) with $c = D - 2$. Then the Lagrangian is invariant under the set of transformations

$$\begin{aligned} \phi^I &\longmapsto \lambda \phi^I, \\ g_{\mu\nu} &\longmapsto \lambda^{-2} g_{\mu\nu}, \quad \lambda \in \mathbb{R}, \end{aligned} \tag{118}$$

for a constant parameter λ (independent of x). We obtain the infinitesimal transformations (116) by writing $\lambda = 1 + \epsilon + \mathcal{O}(\epsilon^2)$.

Let us now consider transformations of the form (116) but with a parameter $\epsilon(x)$ depending on the point. Then

$$\delta_\epsilon \mathcal{L} = -\sqrt{g} G_{IJ} (\partial_\mu \epsilon) K^I \partial_\nu \phi^J g^{\mu\nu}.$$

The Lagrangian $\mathcal{L}_{\mathbb{R},0}$ is not invariant, but this can be remedied if we add an Einstein–Hilbert-like term for the spacetime metric

$$\mathcal{L}_{\mathbb{R}} = -\tfrac{1}{2}\sqrt{g} G_{IJ} \partial_\mu \phi^I \partial_\nu \phi^J g^{\mu\nu} - \tfrac{1}{2} a (G_{IJ} K^I K^J) \sqrt{g} R(g), \tag{119}$$

where $R(g)$ is the spacetime curvature and

$$a = \frac{1}{(D-1)(D-2)}. \tag{120}$$

In addition, we ought to assume that K is a *closed* homothetic Killing vector (73). In this case we have that

$$\delta_\epsilon \left(\sqrt{g} R(g)\right) = \epsilon(2 - D)\sqrt{g} R(g) + 2(D-1)\partial_\nu \left(\sqrt{g} g^{\mu\nu} \partial_\mu \epsilon\right),$$

and using the identity

$$G_{IJ} K^I \partial_\nu \phi^J = \frac{1}{D-2} \partial_\nu (G_{IJ} K^I K^J),$$

one can prove that $\mathcal{L}_{\mathbb{R}}$ transforms into a total derivative.

In order to have a positive definite energy for the gravitational field we must have $G_{IJ}K^I K^J < 0$, so one of the scalars is a ghost.

One can fix this gauge invariance by taking

$$G_{IJ}K^I K^J = -\frac{1}{a\kappa^2}, \tag{121}$$

where κ^2 is the gravitational coupling constant. Then the second term of (119) is just the Einstein–Hilbert action and the first term is a sigma model defined now on the surface (121).

5.2 Kähler manifolds

We can consider the same kind of model for $n+1$ complex scalar fields X^α, which are coordinates in a Kähler manifold with metric $G_{\alpha\bar\beta}$. We assume now that this metric has a closed homothetic Killing vector K. As mentioned at the end of Section 4.1, this implies the presence of the holomorphic homothetic Killing vector $H = (1 - \mathrm{i}J)K$. The Lagrangian density has the form

$$\mathcal{L}_{\mathbb{C},0} = -\sqrt{g}g^{\mu\nu}G_{\alpha\bar\beta}\partial_\mu X^\alpha \partial_\nu \bar X^{\bar\beta} - \tfrac{1}{4}aG_{\alpha\bar\beta}H^\alpha \bar H^{\bar\beta}\sqrt{g}R(g). \tag{122}$$

The dilatation symmetry is generated by

$$K = \tfrac{1}{2}\left(H^\alpha(z)\partial_\alpha + \bar H^{\bar\alpha}(\bar z)\partial_{\bar\alpha}\right), \tag{123}$$

but this model has rigid symmetry generated by JK, which was not present in the real case. This leads to the infinitesimal transformations

$$\delta_\epsilon X^\alpha = \tfrac{1}{2}H^\alpha\epsilon, \quad \delta_\varphi X^\alpha = \tfrac{1}{2}\mathrm{i}H^\alpha\varphi, \quad \delta_\epsilon g_{\mu\nu} = -2\epsilon g_{\mu\nu}. \tag{124}$$

Assuming (115), the action is invariant under these transformations where ϵ can be local, but φ is still a global transformation parameter.

A relevant example is the finite transformation

$$\begin{aligned} &X^\alpha \longmapsto \lambda^{c/2}X^\alpha, \\ &g_{\mu\nu} \longmapsto |\lambda|^{-2}g_{\mu\nu}, \quad \lambda = |\lambda|\mathrm{e}^{\mathrm{i}\varphi} = 1 + \epsilon + \mathrm{i}\varphi + \cdots \in \mathbb{C}, \end{aligned} \tag{125}$$

for which

$$H^\alpha = cX^\alpha, \quad X^\gamma\partial_\gamma G_{\alpha\bar\beta} = 0. \tag{126}$$

In that case, the transformations (124) can be integrated to a finite transformation. In order to implement the local invariance under φ, we introduce a U(1) connection A_μ, which transforms as

$$A_\mu \longmapsto A_\mu + \partial_\mu\varphi,$$

and we couple it minimally to the scalar fields defining

$$\begin{aligned} \mathcal{L}_{\mathbb{C}} &= -\sqrt{g}g^{\mu\nu}G_{\alpha\bar\beta}D_\mu X^\alpha D_\nu \bar X^{\bar\beta} - \tfrac{1}{4}aG_{\alpha\bar\beta}H^\alpha \bar H^{\bar\beta}\sqrt{g}R(g), \\ D_\mu X^\alpha &= \partial_\mu X^\alpha - \tfrac{1}{2}\mathrm{i}A_\mu H^\alpha. \end{aligned} \tag{127}$$

For shortness, we denote

$$N = \frac{1}{c^2} G_{\alpha\bar\beta} H^\alpha \bar H^{\bar\beta}.$$

Using (77) for the Levi-Civita connection, we have also that

$$\partial_\alpha N = \frac{1}{c} G_{\alpha\bar\beta} \bar H^{\bar\beta}, \quad \partial_\alpha \partial_{\bar\beta} N = G_{\alpha\bar\beta}.$$

Hence, N is the Kähler potential of the manifold described by the X^α.

The field equation for the auxiliary field A_μ is algebraic and it allows us to solve for A_μ:

$$A_\mu = \frac{\mathrm{i}}{c^2 N} G_{\alpha\bar\beta} \big(H^\alpha \partial_\mu \bar X^{\bar\beta} - \partial_\mu X^\alpha \bar H^{\bar\beta} \big) = \frac{\mathrm{i}}{c\, N} \big(\partial_\mu \bar X^{\bar\beta} \partial_{\bar\beta} N - \partial_\mu X^\alpha \partial_\alpha N \big).$$

The first term of (127), the scalar kinetic term $\mathcal{L}_{\text{scalar}}$, is then as follows:

$$\begin{aligned} \frac{\mathcal{L}_{\text{scalar}}}{\sqrt{g}} &= -G_{\alpha\bar\beta} \partial_\mu X^\alpha \partial^\mu \bar X^{\bar\beta} - \frac{1}{4N} \big[\partial_\mu \bar X^{\bar\beta} \partial_{\bar\beta} N - \partial_\mu X^\alpha \partial_\alpha N \big]^2 \\ &= -\partial_\mu X^\alpha \partial^\mu \bar X^{\bar\beta} \left[\partial_\alpha \partial_{\bar\beta} N - \frac{1}{N} (\partial_\alpha N)(\partial_{\bar\beta} N) \right] - \frac{1}{4N} \big[\partial_\mu N\, \partial^\mu N \big] \\ &= -N \partial_\mu X^\alpha \partial^\mu \bar X^{\bar\beta} \partial_\alpha \bar\partial_{\bar\beta} \log |N| - \frac{1}{4N} \big[\partial_\mu N\, \partial^\mu N \big]. \end{aligned} \tag{128}$$

Notice that H^α is a zero mode of the quantity in square brackets in the second line.

We can fix the dilation gauge freedom (125) by taking as before

$$N = \frac{1}{c^2} G_{\alpha\bar\beta} H^\alpha \bar H^{\bar\beta} = -\frac{2}{a c^2 \kappa^2}. \tag{129}$$

The second term of (127) is then the Einstein–Hilbert action.

As N gets a fixed value, a function of N is not convenient as a Kähler potential for the restricted manifold. We will show now how to construct a Kähler potential, restricting to the case (126).

In that case, we rescale the coordinates X^α, introducing η^α by

$$\begin{aligned} \Phi_Y : \widetilde{\mathcal{M}} &\longrightarrow \widetilde{\mathcal{M}}, \\ \eta^\alpha &\longrightarrow X^\alpha = \eta^\alpha Y(\eta, \bar\eta), \end{aligned}$$

for an arbitrary function $Y(\eta, \bar\eta)$. Notice that this map is not holomorphic with respect to J. However, it induces a new complex structure on $\widetilde{\mathcal{M}}$, denoted as J', by the commutativity of the diagram

$$\begin{array}{ccc} T\widetilde{\mathcal{M}} & \xrightarrow{T\Phi_Y} & T\widetilde{\mathcal{M}} \\ {\scriptstyle J'}\downarrow & & \downarrow{\scriptstyle J} \\ T\widetilde{\mathcal{M}} & \xrightarrow[T\Phi_Y > T]{} & \widetilde{\mathcal{M}}. \end{array}$$

The map $T\Phi_Y$ then sends holomorphic vectors with respect to J' to holomorphic vectors with respect to J. In this sense, it is a holomorphic map.

Defining

$$\mathcal{K} = \frac{N}{Y\bar{Y}},$$

then $\mathcal{K}$ is a function of $(\eta, \bar{\eta})$. The homogeneity properties of N imply

$$\mathcal{K}(\lambda\eta, \bar{\lambda}\bar{\eta}) = \lambda\bar{\lambda}\mathcal{K}(\eta, \bar{\eta}),$$

and therefore also

$$\frac{\partial}{\partial X^\alpha} N = \bar{Y}\frac{\partial}{\partial \eta^\alpha}\mathcal{K}, \quad \frac{\partial^2}{\partial X^\alpha \partial \bar{X}^{\bar{\beta}}} N = \frac{\partial^2}{\partial \eta^\alpha \partial \bar{\eta}^{\bar{\beta}}}\mathcal{K}.$$

Hence, $c\eta^\alpha \frac{\partial}{\partial \eta^\alpha}$ is a holomorphic homothetic Killing vector with respect to J', and $\mathcal{K}$ defines the Kähler potential of a projective Kähler manifold, see Definition 4.2.

The action reduces to

$$\frac{\mathcal{L}_{\mathbb{C}}}{\sqrt{g}} = -N\,\partial_\mu \eta^\alpha \partial^\mu \bar{\eta}^{\bar{\beta}} \frac{\partial^2}{\partial \eta^\alpha \partial \bar{\eta}^{\bar{\beta}}} \log|\mathcal{K}| - \frac{1}{4N}\left[\partial_\mu N\, \partial^\mu N\right] - \frac{1}{4}ac^2 NR(g). \quad (130)$$

The first term in (130) is proportional to $-\tilde{\rho}_{\alpha\bar{\beta}}$ in (95), which is the pull back of a 2-form on the quotient manifold $\tilde{\rho} = \mathrm{i}\pi^*\rho$ as in (96). If z^i, $i = 1, \ldots, n$ are coordinates on the quotient, then similarly as in (97), a metric is defined. The appropriate normalization for the Kähler potential is

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} \left[-\frac{2}{ac^2\kappa^2} \log -\frac{ac^2}{2}\mathcal{K} \right].$$

On the quotient N is constant and thus $\partial_\mu N = 0$, so the action reduces to a sigma model in dimension n coupled to gravity in the standard way,

$$\mathcal{L}_{\mathbb{C}} = -\sqrt{g} g_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu \bar{z}^j + \frac{1}{\kappa^2}\sqrt{g}R(g).$$

Note that the $D = 4$ values of (115) and (120) lead to $ac^2/2 = 1/3$. That is also the value that one finds in $N = 1$ supergravity. For $N = 2$ supergravity, one has two scalar manifolds, the one of the vector multiplets, and the one of hypermultiplets. There is another auxiliary field, whose origin is beyond our discussion here, such that when one eliminates the hypermultiplets, the effective value of a is $1/2$, i.e. $ac^2/2 = 1$.

We remark that we need here the lower signs in (97) in order to get the positive kinetic energy for gravity, and the other signatures should all be $+$ in order to have positive kinetic terms of the sigma model.

6 Projective special Kähler manifolds

6.1 Definition of projective special Kähler manifolds

A projective special Kähler manifold $\mathcal{M}$ is a Kähler–Hodge manifold such that the manifold $\widetilde{\mathcal{M}}$ is rigid special Kähler. The flat connection on $\widetilde{\mathcal{M}}$ is an extra structure that also projects to $\mathcal{M}$. Here we have the precise definition.

Definition 6.1. Let $\widetilde{\mathcal{M}}$ be a rigid special pseudo-Kähler manifold with complex structure $\tilde{J}$, metric $\tilde{g}$, Kähler form $\widetilde{\Phi}$ and flat symplectic connection $\widetilde{\nabla}$. We assume that on $\widetilde{\mathcal{M}}$ there is a free holomorphic action of $\mathbb{C}^\times$ such that the fundamental vector field H is a non null, holomorphic homothetic Killing vector for the flat connection,

$$\widetilde{\nabla}_Y H = cY \quad \text{for all } Y \in T^{1,0}\widetilde{\mathcal{M}} \quad \text{and} \quad \tilde{g}(H, \bar{H}) \neq 0. \tag{131}$$

Then we say that on $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{C}^\times$ there is a projective special Kähler structure.

In fact, we will prove that (131) implies that H is also an holomorphic homothetic Killing vector for the Levi-Civita connection, that is,

$$\widetilde{D}_Y H = cY \quad \text{which is equivalent to} \quad \mathcal{L}_Y \tilde{g} = c\tilde{g} \quad \text{for all } Y \in T^{1,0}\widetilde{\mathcal{M}}, \tag{132}$$

so it is enough to require it for the flat connection. Let us look at (131) in special coordinates. From (48) we have

$$\widetilde{\nabla} H = \frac{\partial H^\beta}{\partial \eta^\alpha} \mathrm{d}\eta^\alpha \otimes \frac{\partial}{\partial \eta^\beta} - \frac{1}{2} \frac{\partial \tau_{\beta\gamma}}{\partial \eta^\alpha} H^\gamma \mathrm{d}\eta^\alpha \otimes \frac{\partial}{\partial y_\beta} = c\, \mathrm{d}\eta^\alpha \otimes \frac{\partial}{\partial \eta^\alpha}.$$

The first term is holomorphic, while in the second there is an holomorphic and an antiholomorphic part, since y_β is real. To cancel the antiholomorphic part necessarily

$$\frac{\partial \tau_{\beta\gamma}}{\partial \eta^\alpha} H^\gamma = 0, \tag{133}$$

and then

$$\frac{\partial H^\beta}{\partial \eta^\alpha} = c\delta^\beta_\alpha \iff H^\beta = c\eta^\beta \text{ (up to a constant)}. \tag{134}$$

We can always shift η^α by a constant, it is still a special coordinate. In particular, (134) implies that special coordinates are homogeneous coordinates as defined in (84). (133) and (134) imply that $\tau_{\alpha\beta}$ are homogeneous functions of η of degree 0,

$$\tau_{\alpha\beta}(\lambda\eta) = \tau_{\alpha\beta}(\eta), \quad \lambda \in \mathbb{C}^\times,$$

so they depend on the prepotential $\mathcal{F}$ as in (45):

$$\tau_{\alpha\beta} = -8 \frac{\partial^2 \mathcal{F}}{\partial \eta^\alpha \partial \eta^\beta}$$

must be a homogeneous function of η of degree 2,

$$\mathcal{F}(\lambda\eta) = \lambda^2\mathcal{F}(\eta), \quad \lambda \in \mathbb{C}^\times. \tag{135}$$

For the Levi-Civita connection, we have

$$DH = \frac{\partial H^\beta}{\partial\eta^\alpha}\mathrm{d}\eta^\alpha \otimes \frac{\partial}{\partial\eta^\beta} + \frac{1}{8}\mathrm{i}g^{\beta\bar\delta}\frac{\partial\tau_{\delta\gamma}}{\partial\eta^\alpha}H^\gamma \mathrm{d}\eta^\alpha \otimes \frac{\partial}{\partial\eta_\beta},$$

and using (133) and (134) we get

$$DH = c\,\mathrm{d}\eta^\alpha \otimes \frac{\partial}{\partial\eta^\alpha},$$

which proves (132).

Using (134) we can compute the integral surfaces (80) of H in special coordinates,

$$H(\rho_\lambda(x)) = \frac{\mathrm{d}\rho_\lambda}{\mathrm{d}\lambda} \iff c\rho^\alpha_\lambda(x) = \frac{\mathrm{d}\rho^\alpha_\lambda}{\mathrm{d}\lambda} \iff \rho^\alpha_\lambda(x) = \mathrm{e}^{c\lambda}\eta^\alpha(x),$$

since $\rho_0(x) = x$ and $\rho^\alpha_0(x) = \eta^\alpha(x)$.

We will denote also by $\Re_b(x) = \rho_\lambda(x)$ with $b = \mathrm{e}^{c\lambda} \in \mathbb{C}^\times$.

As we saw in general in (94), $\theta = c^2\mathcal{K}$.

Since ∇ is flat, it descends to $\mathcal{H}$ as a flat connection, and then it defines a connection on $L \otimes T\mathcal{M}$ as in (109). This connection is not necessarily flat.

Next we will see that also the holomorphic cubic form descends to an appropriate bundle over $\mathcal{M}$, and we will compute the curvature tensor in terms of it.

The holomorphic cubic form. We consider the holomorphic cubic form Ξ defined in Section 3.7. We want to see how it descends to the manifold $\mathcal{M}$. If X is a vector field on $\mathcal{M}$, its horizontal lift is (106)

$$\widetilde{X} = X^i\partial_i - y\kappa_i X^i\partial_y = X^i(\partial_i\eta^\alpha - \kappa_i\eta^\alpha)\partial_\alpha, \tag{136}$$

so

$$\frac{\partial\mathcal{F}}{\partial\eta^\alpha}\mathrm{d}\eta^\alpha(\widetilde{X}) = X^i(\partial_i\mathcal{F} - 2\kappa_i\mathcal{F}) = X^i D^L_i(\mathcal{F}) = D^L_X\mathcal{F},$$

where we have used the fact that $\mathcal{F}$ is homogeneous of degree 2. It is in fact a section of $(L^*)^{\otimes 2}$, and D^L denotes the covariant derivative with respect to the Hermitian connection. We can also write

$$D^L_i\mathcal{F} = \mathrm{e}^{2\log|\mathcal{K}|}\partial_i(\mathrm{e}^{-2\log|\mathcal{K}|}\mathcal{F}) = \mathcal{K}^2\partial_i\left(\frac{1}{\mathcal{K}^2}\mathcal{F}\right).$$

If Y, Z are also vector fields on $\mathcal{M}$ and $\widetilde{Y}, \widetilde{Z}$ are their horizontal lifts respectively, we have

$$\frac{\partial^3\mathcal{F}}{\partial\eta^\alpha\partial\eta^\beta\partial\eta^\gamma}\mathrm{d}\eta^\alpha\mathrm{d}\eta^\beta\mathrm{d}\eta^\gamma(\widetilde{X},\widetilde{Y},\widetilde{Z}) = D^L_X D^L_Y D^L_Z(\mathcal{F}).$$

This shows that the holomorphic cubic form on $\widetilde{\mathcal{M}}$, $\widetilde{\Xi}$ descends to a section of $(L^*)^{\otimes 2}\otimes (T^*\mathcal{M})^{\otimes 3}$,

$$\Xi(X,Y,Z) \equiv \widetilde{\Xi}(\widetilde{X},\widetilde{Y},\widetilde{Z}). \tag{137}$$

This procedure of lifting the vector fields on $\mathcal{M}$ to $\widetilde{\mathcal{M}}$ gives us also another way of computing the metric on $\mathcal{M}$.

The metric. As before, let $\widetilde{X}$ and $\widetilde{Y}$ be the horizontal lifts of X and Y, vector fields on $\mathcal{M}$. Then we have, using (136), (93) and (97),

$$\begin{aligned}\tilde{g}(\widetilde{X},\overline{\widetilde{Y}}) &= \partial_\alpha\partial_{\bar\beta}\mathcal{K}(\partial_i\eta^\alpha - \kappa_i\eta^\alpha)(\partial_{\bar\jmath}\bar\eta^{\bar\beta} - \bar\kappa_{\bar\jmath}\bar\eta^{\bar\beta})X^i\overline{Y}^{\bar\jmath}\\ &= \mathcal{K}\partial_i\partial_{\bar\jmath}(\log|\mathcal{K}|)X^i\overline{Y}^{\bar\jmath} = |\mathcal{K}|g(X,\overline{Y}).\end{aligned} \tag{138}$$

The Riemannian curvature on $\mathcal{M}$. From (109) we can compute the curvature tensor of the Levi-Civita connection on $\mathcal{M}$. Since $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ are Kähler manifolds we can use (11). We have

$$R^j{}_{ik\bar\ell} = -\partial_{\bar\ell}\Gamma^j_{ik} = \widetilde{R}^j{}_{ik\bar\ell} + \partial_{\bar\ell}(\delta^j_i\partial_k\log|\mathcal{K}| + \delta^j_k\partial_i\log|\mathcal{K}|).$$

These are the components of the curvature tensor in the coordinates z^i. To avoid confusion, we will split the coordinates η^α as (η^0,η^a), $a=1,\dots,n$. In this way the indices i,j,k will always refer to the coordinates z.

We can use (62) to express $\widetilde{R}^j{}_{ik\bar\ell}$ in terms of the cubic form $\widetilde{\Xi}$. We first have to write $\widetilde{\Xi}$ in terms of (y,z^i). We have

$$\mathrm{d}\eta^0 = \mathrm{d}y, \quad \mathrm{d}\eta^i = z^i\mathrm{d}y + y\mathrm{d}z^i. \tag{139}$$

Due to the homogeneity condition (135) and $\widetilde{\Xi}$ being a third derivative of $\mathcal{F}$ as in (57), we have

$$\eta^\alpha\widetilde{\Xi}_{\alpha\beta\gamma} = 0, \tag{140}$$

and therefore the $\mathrm{d}y$ terms in (139) do not contribute if we rewrite

$$\widetilde{\Xi} = \widetilde{\Xi}_{\alpha\beta\gamma}\mathrm{d}\eta^\alpha\mathrm{d}\eta^\beta\mathrm{d}\eta^\gamma = y^3\widetilde{\Xi}_{abc}\delta^a_i\delta^b_j\delta^c_k\mathrm{d}z^i\mathrm{d}z^j\mathrm{d}z^k = \Xi_{ijk}\mathrm{d}z^i\mathrm{d}z^j\mathrm{d}z^k,$$

where Ξ_{ijk} has been defined in (137). This leads to

$$\Xi_{ijk} = 2y^2\frac{\partial^3\mathcal{F}(1,z^i)}{\partial z^i\partial z^j\partial z^k},$$

where $\mathcal{F}(1,z^i)$ is $\mathcal{F}(\eta)$ with η^0 replaced by 1, and η^i by z^i.

On the other hand, using (110), we find

$$R^j{}_{ik\bar\ell} = \frac{1}{4\mathcal{K}^2}g^{j\bar\jmath'}g^{p'\bar p}\Xi_{p'ki}\,\overline{\Xi}_{\bar p\bar\jmath'\bar\ell} \pm \delta^j_i g_{k\bar\ell} \pm \delta^j_k g_{i\bar\ell}. \tag{141}$$

Notice that in (141) all the dependence in $y,\bar y$ cancels as $\mathcal{K}$ is proportional to $y\bar y$.

6.2 Examples of projective special Kähler manifolds

Example 6.2 (Projective space and unit ball as special Kähler manifolds). We consider the complex projective space $\mathbb{C}\mathrm{P}^m$ of lines in the complex space $\mathbb{C}^{m+1}$. It is a special case of Example 4.6, with $p = 1, q = m$.

We have a covering of $\mathbb{C}\mathrm{P}^m$ by open sets

$$U_i = \{\text{lines } S \text{ in } \mathbb{C}^{m+1} \text{ with } z^i|_S \neq 0\},$$

(these are the lines that do not lie in the hyperplane $z^i = 0$). Let us take a fixed index $i = 0$, then we have that

$$z^j|_S = t^j z^0|_S, \quad j \neq 0,$$

so $(t^1, \dots, t^m)$ is a set of coordinates on U_0.

The tautological bundle is already a line bundle, $L \subset \mathbb{C}\mathrm{P}^m \times \mathbb{C}^{m+1}$ so there is no need of taking the determinant. On $\mathbb{C}\mathrm{P}^m \times \mathbb{C}^{m+1}$ we have the fiber metric

$$\langle \zeta, \zeta' \rangle = \zeta^0 \bar{\zeta}^0 + \cdots + \zeta^m \bar{\zeta}^m, \tag{142}$$

which we will restrict to L. On L the fiber metric and the Hermitian connection are

$$h(t, \bar{t}) = 1 + \sum_{j=1}^{m} t^j \bar{t}^j, \quad \Gamma_i = \left(1 + \sum_{j=1}^{m} t^j \bar{t}^j\right)^{-1} \bar{t}^i.$$

The Ricci form (114) becomes

$$\rho_{i\bar{\jmath}} = -\mathrm{i} g_{i\bar{\jmath}} = -\mathrm{i}\partial_{\bar{\jmath}}\partial_i \log h = -\mathrm{i}\partial_{\bar{\jmath}}\partial_i \log(1 + t\bar{t}) = \frac{\mathrm{i}}{h^2}\bar{t}^i t^j - \frac{\mathrm{i}}{h}\delta^{ij}.$$

We can define the prepotential as

$$\mathcal{F} = \frac{1}{4}\mathrm{i}(\eta^0\eta^0 + \cdots + \eta^m\eta^m), \quad \eta^i = t^i\eta^0, \ i = 1, \dots, m.$$

Then the Kähler potential in $\widetilde{\mathcal{M}} = \mathbb{C}^{m+1}$ and in $\mathbb{C}\mathrm{P}^m$ is

$$\mathcal{K} = 2\Im\left(\frac{\partial \mathcal{F}}{\partial \eta}\bar{\eta}\right) = \eta\bar{\eta}; \quad \log|\mathcal{K}| = \log(1 + t\bar{t}) + \log \eta^0\bar{\eta}^0.$$

As the third derivative of the prepotential vanishes, the curvature is given by the last two terms of (141), where we have to use the + signs.

If we change the fiber metric (142) to

$$\langle \zeta, \zeta' \rangle = \zeta^0 \bar{\zeta}^0 - \cdots - \zeta^m \bar{\zeta}^m,$$

we obtain that h is positive on the unit ball

$$h(t, \bar{t}) = 1 - t\bar{t} > 0 \quad \text{for } t\bar{t} < 1,$$

which is the symmetric space

$$\frac{\mathrm{SU}(1, m)}{\mathrm{SU}(m) \times \mathrm{U}(1)}.$$

Notice that the metric is

$$g_{i\bar{\jmath}} = \frac{1}{h^2} t^i \bar{t}^j + \frac{1}{h} \delta^{ij} = -\partial_i \bar{\partial}_j \log h.$$

This means that we have to start with the negative Kähler potential $\mathcal{K}$. This sign is important in physical applications (supergravity), as we saw in Section 5. In this case, we have to use the $-$ signs in the last two terms of (141).

Example 6.3 (A pseudo-Riemannian special Kähler manifold). We want to describe now the pseudo-Riemannian symmetric space

$$\frac{\mathrm{SU}(1,2)}{\mathrm{SU}(1,1) \times \mathrm{U}(1)}. \tag{143}$$

We start the construction as for the projective space, on which we try to define a pseudo-Riemannian metric. As we did for the passage to the unit-ball, we will have to restrict to those points where this metric is non degenerate.

As a fiber metric on the trivial bundle $\mathbb{C}\mathrm{P}^2 \times \mathbb{C}^3$ we take

$$\langle \zeta, \zeta \rangle = \bar{\zeta}^1 \zeta^1 - \bar{\zeta}^2 \zeta^2 + \bar{\zeta}^3 \zeta^3. \tag{144}$$

The space is covered by the three open sets

$$U_i = \left\{ \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \end{pmatrix} \mid \zeta^i \neq 0 \right\}, \quad i = 1, 2, 3$$

as before. In the patch U_1, we have

$$\begin{pmatrix} \zeta^2 \\ \zeta^3 \end{pmatrix} = T^1 \zeta^1, \quad \text{with } T^1 = \begin{pmatrix} t^{21} \\ t^{31} \end{pmatrix},$$

and a local section of the tautological bundle is given by a function $\zeta^1(T^1)$,

$$\zeta(T_1) = \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \end{pmatrix} = \begin{pmatrix} 1 \\ T^1 \end{pmatrix} \zeta^1(T^1).$$

The inner product becomes

$$\langle \zeta(T^1), \zeta'(T^1) \rangle = \bar{\zeta}^1 (1, T^{1\dagger}) g \begin{pmatrix} 1 \\ T^1 \end{pmatrix} \zeta'^1 = \bar{\zeta}^1 (1 - \bar{t}^{21} t^{21} + \bar{t}^{31} t^{31}) \zeta'^1,$$

with

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Doing the same computation for U_2, U_3 we obtain

$$\begin{aligned}
\langle \zeta(T^1), \zeta'(T^1)\rangle &= \bar{\zeta}^1(1 - \bar{t}^{21}t^{21} + \bar{t}^{31}t^{31})\zeta'^1 \quad \text{for } U_1,\\
\langle \zeta(T^2), \zeta'(T^2)\rangle &= \bar{\zeta}^2(\bar{t}^{12}t^{12} - 1 + \bar{t}^{32}t^{32})\zeta'^2 \quad \text{for } U_2,\\
\langle \zeta(T^3), \zeta'(T^3)\rangle &= \bar{\zeta}^3(\bar{t}^{13}t^{13} - \bar{t}^{23}t^{23} + 1)\zeta'^3 \quad \text{for } U_3.
\end{aligned} \tag{145}$$

Equations (145) give us the fiber metric on the tautological (line) bundle. If in each U_i,

$$\langle \zeta, \zeta'\rangle = h_i \zeta^i \bar{\zeta}'^i \quad \text{(no sum over } i\text{)}, \tag{146}$$

then

$$\begin{aligned}
h_1(T^1) &= (1 - \bar{t}^{21}t^{21} + \bar{t}^{31}t^{31}),\\
h_2(T^2) &= (\bar{t}^{12}t^{12} - 1 + \bar{t}^{32}t^{32}),\\
h_3(T^3) &= (\bar{t}^{13}t^{13} - \bar{t}^{23}t^{23} + 1).
\end{aligned}$$

In the intersections, the change of coordinates

$$t^{ij} = \frac{1}{t^{ji}}, \quad \zeta^i = t^{ij}\zeta^j$$

leaves (146) invariant.

We have to restrict ourselves to the space where the fiber metric is positive definite. Let

$$\mathcal{U}_i = \{T^i \in U_i \mid h_i(T^i) > 0\}.$$

$\mathcal{U}_1$ and $\mathcal{U}_3$ are homeomorphic to $\mathbb{C}^2$, but $\mathcal{U}_2$ is U_2 minus a ball of radius 1 centered at $T^2 = 0$. The point $T^2 = 0$ is the only point in U_2 that is not contained in U_1 or U_3. So we can safely ignore $\mathcal{U}_2$, since $\{\mathcal{U}_1, \mathcal{U}_3\}$ form a covering of the space of points where the fiber metric is definite positive.

Notice that in U_1, $\xi^2 = t^{21}\xi^1$ serves as a coordinate and the same in U_3, $\xi^2 = t^{23}\xi^3$, so ξ^2 is a global coordinate and describes $\mathbb{C}$. The other coordinate, t^{21} or t^{12} respectively in U_1 and U_3 describe a sphere S^2, so we have that the topology of (143) is $S^2 \times \mathbb{C}$.

Let us compute the Ricci form in U_1. For simplicity we will denote $t^{21} = t^2$, $t^{31} = t^3$.

$$\rho_{i\bar{j}} = -\mathrm{i}\partial_i\partial_{\bar{j}} \log h = \frac{-\mathrm{i}}{h^2}\begin{pmatrix} -1 - t^3\bar{t}^3 & t^3\bar{t}^2 \\ t^2\bar{t}^3 & 1 - t^2\bar{t}^2 \end{pmatrix}.$$

The metric is

$$g_{i\bar{j}} = \mathrm{i}\rho_{i\bar{j}},$$

and it is easy to see that it has one positive and one negative eigenvalue.

The prepotential is

$$\mathcal{F} = \mathrm{i}(\eta^1\eta^1 - \eta^2\eta^2 + \eta^3\eta^3),$$

with

$$t^2 = \frac{\eta^2}{\eta^1}, \quad t^3 = \frac{\eta^3}{\eta^1},$$

and then the Kähler potential is

$$\mathcal{K} = 4(\eta^1\bar{\eta}^1 - \eta^2\bar{\eta}^2 + \eta^3\bar{\eta}^3) = 4\eta_1\bar{\eta}_1 h.$$

We can also see that $\Xi = 0$. Then (141) gives $R_{i\bar{\jmath}} = R^k{}_{ik\bar{\jmath}} = 3g_{i\bar{\jmath}}$ and $R = 6$. □

7 Conclusions

In this exposition we have extended the definition of special Kähler geometry to the case of arbitrary signature of the Kähler metric. For the rigid case, we have extended the definition given in [8], while for the projective case we have given a definition inspired in the conformal calculus framework.

We have seen that the non-existence of prepotential in some symplectic coordinates, which was known for projective special geometry [12] (the special geometry that occurs in supergravity), is in fact a characteristic of pseudo Riemannian manifolds, and applies also to the rigid case. This was masked by the fact that in physical applications of rigid special geometry one is only interested in the Euclidean signature, which gives positive definite kinetic energy for the scalar fields.

Projective (or 'local', referring to the local supersymmetry invariance of supergravity) special geometry is obtained from a rigid special manifold that has a closed homothetic Killing vector K. If K is such vector and J is the complex structure JK is a Killing vector, so the metric has an extra U(1) symmetry. The result is that the existence of a closed homothetic Killing vector is equivalent to the existence of a holomorphic homothetic Killing vector, which we define in (77). This means essentially that there is an action of the group $\mathbb{C}^\times$, and the procedure to obtain the projective special manifold is to take quotient of the rigid 'mother' manifold by this action (and from here, the name of 'projective' geometry). The positive signatures of the kinetic terms of scalars and gravity in supergravity theories require that the rigid manifold has signature $(2, 2n)$. We extend, however, projective special geometry to arbitrary signatures. If the signature of the projective manifold is (s, t) (s positive eigenvalues, t negative eigenvalues), then the signature of the 'mother' rigid manifold is either $(s + 1, t)$ or $(s, t + 1)$. It is the later case that occurs in supergravity. The standard formula for the curvature is generalized to (141), the lower choice in $\pm$ being the standard supergravity case. The other possibility allows us also to discuss special geometries with a compact isometry group.

In fact, this projectivization can be discussed for general Kähler manifolds, not necessarily special. We develop the formalism in this more general case and, for example, we prove that the projective Kähler manifold is automatically Kähler–Hodge. As this is the method that is used in conformal calculus, it implies that all the Kähler

manifolds that are constructed in this way for $N = 1$ or $N = 2$ supergravity satisfy the Kähler–Hodge condition that was introduced in reference [22]. We also give an interpretation of the Levi-Civita connection in these projective Kähler manifolds as induced from the connection of the 'mother' manifold in a particular way, making use of the line bundle and the Hermitian connection.

Appendices

A Some technical results

Lemma A.1 (See Lemma A1 in [7]). *Let*

$$V = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

be a $2n \times n$ *matrix of rank* n (α *and* β *are* $n \times n$ *matrices*). *Then there is a matrix* $S \in \mathrm{Sp}(2n, \mathbb{R})$ *such that the transformed matrix*

$$V' = SV = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$$

has the property that α' *itself has rank* n.

Proof. We give an outline of the proof. Let us denote

$$\alpha = \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix},$$

and let $r - 1 \leq n$ be the rank of α. If $r - 1 = n$ we have already the result, so we will take $r - 1 < n$. Without loosing generality, we can assume that $\alpha^1, \ldots, \alpha^{r-1}$ are linearly independent. Then

$$\alpha^r = \sum_{i=1}^{r-1} \lambda_i \alpha^i. \tag{147}$$

Let β_k be such that $\alpha^1, \ldots, \alpha^{r-1}, \beta_k$ are also linearly independent. For the particular case $k = r$ the symplectic matrix

$$S = \begin{pmatrix} \mathbb{1} - E_{r,r} & E_{r,r} \\ -E_{r,r} & \mathbb{1} - E_{r,r} \end{pmatrix}$$

gives an α' with rank r.

(We have used the standard notation $(E_{i,j})^l{}_k = \delta_i^l \delta_{jk}$.)

In the generic case $k \neq r$, we consider the symplectic matrix

$$S = \begin{pmatrix} \mathbb{1} - E_{r,r} - \frac{1}{\sigma} E_{k,r} & E_{r,k} + \sigma E_{r,r} \\ -\frac{1}{\sigma} E_{r,r} & \mathbb{1} - E_{r,r} \end{pmatrix},$$

where σ is a parameter $\sigma \neq 0$. It follows that it is always possible to choose σ such that the vectors ${\alpha'}^1, \ldots, {\alpha'}^r$ are independent. In fact, the conditions on σ are that σ must be different from some fixed quantities.

By iterating this procedure we see that transforming V with a finite number of symplectic matrices it is possible to construct a matrix V' such that $\operatorname{rank}(\alpha') = n$. □

Remark A.2. When passing from a constant matrix V to a point dependent matrix $V(z)$, one has first to restrict to a neighbourhood where the same components of V are independent (not only in number). Otherwise the theorem could not be applied. So we may have to enlarge the number of open sets of our covering.

Next, we want to consider a constant symplectic transformation in order to have flat Darboux coordinates in each open set. The constraints for σ (which must be constant) become now point dependent, namely σ must be different from certain functions of z and $\bar{z}$. This is always possible, but perhaps in an open subset of the original open set. For each point there is a neighbourhood contained in a compact set where the constraints can be satisfied. One can cover the manifold with such neighbourhoods and, assuming that the space is locally compact, one can pick up a subcovering which is locally finite.

Lemma A.3. *The subgroup of* $\mathrm{Sp}(2n, \mathbb{R})$ *formed by the matrices of the form*

$$\left\{ \begin{pmatrix} A & 0 \\ C & (A^{\top})^{-1} \end{pmatrix} \right\}$$

is a maximal subgroup.

Before going to the proof let us explain a way of seeing maximality. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}$ a Lie subalgebra with respective groups G and S with $S \subset G$. We are interested in deciding when $\mathfrak{s}$ is maximal in $\mathfrak{g}$. Note that the adjoint action of S on $\mathfrak{g}$ leaves $\mathfrak{s}$ stable and so S acts on $\mathfrak{a} := \mathfrak{g}/\mathfrak{s}$.

Theorem A.4. *If there is a subgroup* $T \subset S$ *such that the action of* T *on* $\mathfrak{a}$ *is irreducible, then* $\mathfrak{s}$ *is maximal in* $\mathfrak{g}$

(We can operate over the complex numbers as maximality over the complexes is stronger than maximality over the reals.)

Proof. Let $\mathfrak{h}$ be a subalgebra such that $\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{g}$ with $\mathfrak{h} \neq \mathfrak{s}$. We must show that $\mathfrak{h} = \mathfrak{g}$. Since $\mathfrak{h}$ is invariant under T, the image $\mathfrak{b}$ of $\mathfrak{h}$ in $\mathfrak{a}$ is stable under T. Since $\mathfrak{h}$ is strictly larger than $\mathfrak{s}$, the space $\mathfrak{b}$ is not 0 and is stable under T. By the irreducibility of the action of T we must have $\mathfrak{b} = \mathfrak{a}$ so that $\mathfrak{h} = \mathfrak{g}$. □

Let us go back to the proof of Lemma A.3.

Proof. In our case S is the lower triangular block group and so we can take $\mathfrak{a}$ to be the space of matrices

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad b = b^t.$$

We take T to be the subgroup

$$\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}, \quad A \in \mathrm{GL}(n).$$

Then the action of T on $\mathfrak{a}$ works out to be

$$(A, b) \longmapsto AbA^t$$

which is the representation of $\mathrm{GL}(n)$ on the symmetric tensors of the n-space, which is known to be irreducible. Applying Theorem A.4, we complete our proof. □

B Connection on a principal bundle and covariant derivative

We will relate now the definition 4.3 of connection on a principal bundle to the covariant derivative in associated bundles.

Let E be an associated vector bundle to P, with standard fiber F, and let $R\colon G \to \mathrm{End}(F)$ be the representation of G on F. For simplicity we will consider $G \approx R(G)$, although this is not necessary. We want to define the covariant derivative of a section of E in terms of the connection 1-form. Let $\{e_1, \ldots, e_k\}$ be a basis on F. A *local frame* of E is a set of $k = \mathrm{rank}(E)$ independent local sections of E. We will denote it by $\epsilon(m) = \{\epsilon_1(m), \ldots, \epsilon_k(m)\}$, with $m \in \mathcal{M}$. It can be interpreted as an invertible map $\epsilon(m)\colon F \to E_m$ such that

$$\epsilon(m)(e_a) = \epsilon_a(m),$$

so it provides with an identification of the fiber E_m with F. The set of frames is a principal bundle with structure group GL_n. P is a subbundle of the bundle of frames, so a local section $s\colon \mathcal{M} \to P$ is a local frame of E.

The pull-back $\Gamma = \sigma^*\omega$ defines a local $\mathfrak{g}$-valued 1-form on $\mathcal{M}$. The covariant derivative of a local section of E, $\sigma = \sigma^a\epsilon_a$, is given by

$$\nabla_i\sigma = \left(\partial_i\sigma^a + \Gamma_i{}^a{}_b\sigma^b\right)\epsilon_a, \quad \text{so} \quad \Gamma_i{}^a{}_b = (\nabla_i\epsilon_b)^a. \tag{148}$$

(148) relates the definition of connection as a $\mathfrak{g}$-valued 1-form on P with the notion of covariant derivative that we have been using through the text. (148) is given in terms of a local section σ on P, but changing the section gives the usual gauge transformation of the local connection 1-form on $\mathcal{M}$. The coordinate independent description of the covariant derivative can be found for example in reference [23].

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Chapter 5

Special geometry, black holes and Euclidean supersymmetry

Thomas Mohaupt

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1 Introduction

Special geometry was discovered more than 20 years ago [1]. While the term special geometry originally referred to the geometry of vector multiplet scalars in four-dimensional $N = 2$ supergravity, today it is used more generally for the geometries encoding the scalar couplings of vector and hypermultiplets in theories with 8 real supercharges. It applies to rigidly and locally supersymmetric theories in ≤ 6 space-time dimensions, both in Lorentzian and in Euclidean signature. The scalar geometries occurring in these cases are indeed closely related. In particular, they are all much more restricted than the Kähler geometry of scalars in theories with 4 supercharges, while still depending on arbitrary functions. In contrast, the scalar geometries of theories with 16 or more supercharges are completely fixed by their matter content. Theories with 8 supercharges have a rich dynamics, which is still constrained enough to allow one to answer many questions exactly. Special geometry lies at the heart of

the Seiberg–Witten solution of $N = 2$ gauge theories [2] and of the non-perturbative dualities between $N = 2$ string compactifications [3], [4].

While the subject has now been studied for more than twenty years, there are still new aspects to be discovered. One, which will be the topic of this chapter, is the role of real coordinates. Many special geometries, in particular the special Kähler manifolds of four-dimensional vector multiplets and the hyper-Kähler geometries of rigid hypermultiplets are complex geometries. Nevertheless, they also possess distinguished real parametrizations, which are natural to use for certain physical problems. Our first example illustrates this in the context of special geometries in theories with Euclidean supersymmetry. This part reviews the results of [5], [6], and gives us the opportunity to explore another less studied aspect of special geometry, namely the scalar geometries of $N = 2$ supersymmetric theories in Euclidean space-time. It turns out that the relation between the scalar geometries of theories with Lorentzian and Euclidean space-time geometry is (roughly) given by replacing complex structures by para-complex structures. One technique for deriving the scalar geometry of a Euclidean theory in D dimensions is to start with a Lorentzian theory in $D + 1$ dimensions and to perform a dimensional reduction along the time-like direction. The specific example we will review is to start with vector multiplets in four Lorentzian dimensions, which gives, by reduction over time, hypermultiplets in three Euclidean dimensions. This provides us with a Euclidean version of the so-called c-map. The original c-map [7], [8] maps any scalar manifold of four-dimensional vector multiplet scalars to a scalar manifold of hypermultiplets. For rigid supersymmetry, this relates affine special Kähler manifolds to hyper-Kähler manifolds, while for local supersymmetry this relates projective special Kähler manifolds to quaternion-Kähler manifolds. By using dimensional reduction with respect to time rather than space, we will derive the scalar geometry of Euclidean hypermultiplets. As we will see, the underlying geometry is particularly transparent when using real scalar fields rather than complex ones. The geometries of Euclidean supermultiplets are relevant for the study of instantons, and, by 'dimensional oxidation over time' also for solitons, as outlined in [5]. In this chapter we will restrict ourselves to the geometrical aspects.

Our second example is taken from a different context, namely BPS black hole solutions of matter-coupled $N = 2$ supergravity. The laws of black hole mechanics suggest to assign an entropy to black holes, which is, at least to leading order, proportional to the area of the event horizon. Since (super-)gravity presumably is the low-energy effective theory of an underlying quantum theory of gravity, the black hole entropy is analogous to the macroscopic or thermodynamic entropy in thermodynamics. A quantum theory of gravity should provide the fundamental or microscopic level of description of a black hole and, in particular, should allow one to identify the microstates of a black hole and to compute the corresponding microscopic or statistical entropy. The microscopic entropy is the information missing if one only knows the macrostate but not the microstate of the black hole. In other words, if a black hole with given mass, charge(s) and angular momentum (which characterise the macrostate) can be in d different microstates, then the microscopic entropy is $S_{\text{micro}} = \log d$. If the

area of the event horizon really is the corresponding macroscopic entropy, then these two quantities must be equal, at least to leading order in the semi-classical limit. In string theory it has been shown that the two entropies are indeed equal in this limit [9], at least for BPS states (also called supersymmetric states). These are states which sit in special representations of the supersymmetry algebra, where part of the generators act trivially. These BPS (also called short) representations saturate the lower bound set for the mass by the supersymmetry algebra, and, as a consequence, the mass is exactly equal to a central charge of the algebra.[1] In this chapter we will be interested in the macroscopic part of the story, which is the construction of BPS black hole solutions and the computation of their entropy. The near horizon limit of such solutions, which is all one needs to know in order to compute the entropy, is determined by the so-called black hole attractor equations [11], whose derivation is based on the special geometry of vector multiplets. The attractor equations are another example where real coordinates on the scalar manifold appear in a natural way. In the second part of the exposition we review how the attractor equations and the entropy can be obtained from a variational principle. When expressed in terms of real coordinates, the variational principle states that the black hole entropy is the Legendre transform of the Hesse potential of the scalar manifold. We also discuss how the black hole free energy introduced by Ooguri, Strominger and Vafa [12] fits into the picture, and indicate how higher curvature and non-holomorphic corrections to the effective action can be incorporated naturally. This part is based on [13] and on older work including [14], [15], [16].

Let us now explain how our two subjects are connected to the second topic of this volume, pseudo-Riemannian geometry. In both parts of the chapter we have two relevant geometries, the geometry of space-time and the geometry of the target manifold of the scalar fields. In the first case, space-time is Euclidean, but, as we will see, the scalar manifold is pseudo-Riemannian with split signature. In the second case the scalar geometry is positive definite, but space-time is pseudo-Riemannian with Lorentz signature. In fact, our two subjects, the c-map and black holes, can be related in a rather direct fashion, as follows: for a static black hole one can perform a dimensional reduction along the time-like direction in complete analogy to the dimensional reduction of flat Minkowski space. Then one can dualize the vector multiplets into hypermultiplets, which gives rise to a 'local' version of the c-map.[2] This construction can be used to study time-independent four-dimensional geometries from a three dimensional perspective, which has the advantage that all bosonic degrees of freedom (metric, gauge fields and scalars) become scalars in the reduced theory and can then be combined into a non-linear sigma model. This method has been used in Einstein–Maxwell theory already a long time [18], and has been elaborated on both for black holes [19] and brane-type solutions [20]. We refer to [21] for a review. More recently, dimensional reduction to three Euclidean dimensions and the corresponding version of the c-map have been used by [22] to elaborate on the

[1] See [10] Chapter 2.

[2] Here local means that supersymmetry is realized as a local, i.e., space-time dependent symmetry.

ideas of Ooguri, Strominger and Vafa [12] by quantizing static, spherically BPS black hole solutions. The three-dimensional solutions obtained by dimensional reduction for four-dimensional static black hole solutions can also be lifted to four Euclidean dimensions, where they describe wormhole solutions, which generalize the D-instanton of type-IIB string theory [17].

Let us finally mention two contributions to this volume which are closely related to our topics. The article [23] (which is based on [24]) discusses new insights into the geometry of the c-map, which have been obtained by relating vector and hypermultiplets to tensor multiplets. The contribution of [25] discusses new developments in para-quaternionic geometry. While we only discuss the 'rigid' version of the Euclidean c-map here, its 'local' (supergravity) version maps projective special Kähler manifolds to para-quaternionic manifolds.

2 Euclidean special geometry

2.1 Vector multiplets

We start by reviewing the geometry of vector multiplets in rigid four-dimensional $N = 2$ supersymmetry.[3] A vector multiplet consists of a gauge field A_m ($m = 0, \ldots, 3$ is the Lorentz index), two Majorana spinors λ^i ($i = 1, 2$) and one complex scalar X. We consider n such multiplets, labeled by an index $I = 1, \ldots, n$. The field equations for the gauge fields are invariant under $\mathrm{Sp}(2n, \mathbb{R})$ rotations which act linearly on the field strength F^I_{mn} and the dual field strength $G_{I|mn} = \frac{\delta L}{\delta F^I_{mn}}$, where L denotes the Lagrangian. These symplectic rotations generalize the electric-magnetic duality rotations of Maxwell theory and are in fact invariances of the full field equations. A thorough analysis shows that this has the important consequence that all vector multiplet couplings are encoded in a single holomorphic function of the scalars, $F(X^I)$, which is called the prepotential [1]. In superspace language the general action for vector multiplets can be written as a chiral superspace integral of the prepotential F, considered as a superspace function of n so-called restricted chiral multiplets $(X^I, \lambda^{I+}, F^{I-}_{mn})$, which encode the gauge invariant quantities of the n vector multiplets. Here λ^{I+} are the positive chirality projections of the spinors and F^{I-}_{mn} are the antiselfdual projections of the field strength. To be precise, the Lagrangian is the sum of a chiral and an antichiral superspace integral, the latter depending on the complex conjugated multiplets $(\bar{X}^I, \lambda^{I-}, F^{I+}_{mn})$. When working out the Lagrangian in components, all couplings can be expressed in terms of $F(X^I)$, its derivatives, which we denote $F_I, F_{IJ}, \ldots$ and their complex conjugates $\bar{F}_I, \bar{F}_{IJ}, \ldots$. For later use we specify the bosonic part of the Lagrangian:

$$L^{\text{4d VM}}_{\text{bos}} = -\tfrac{1}{2} N_{IJ} \partial_m X^I \partial^m \bar{X}^J - \tfrac{i}{2} (F_{IJ} F^{I-}_{mn} F^{J-mn} - \text{c.c.}), \tag{2.1}$$

[3] Some more background material and references on vector multiplets can be found in [26].

where

$$N_{IJ} = \partial_I \partial_{\bar{J}} \left(-i(X^I \bar{F}_I - F_I \bar{X}^I)\right) \tag{2.2}$$

can be interpreted as a Riemannian metric on the target space M_{VM} of the scalars X^I.[4] $N = 1$ supersymmetry requires this metric to be a Kähler metric, which is obviously the case, the Kähler potential being $K = -i(X^I \bar{F}_I - F_I \bar{X}^I)$. As a consequence of $N = 2$ supersymmetry the metric is not a generic Kähler metric, since the Kähler potential can be expressed in terms of the holomorphic prepotential $F(X^I)$. The resulting geometry is known as affine (also: rigid) special Kähler geometry. The intrinsic characterization of this geometry is the existence of a flat, torsionfree, symplectic connection ∇, called the special connection, such that

$$(\nabla_U I)V = (\nabla_V I)U, \tag{2.3}$$

where I is the complex structure and U, V are arbitrary vector fields [27]. It has been shown that all such manifolds can be constructed locally as holomorphic Langrangian immersions into the complex symplectic vector space $T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$ [29]. In this context X^I, F_I are flat complex symplectic coordinates on $T^*\mathbb{C}^n$ and the prepotential is the generating function of the immersion $\Phi\colon M_{VM} \to T^*\mathbb{C}^n$, i.e., $\Phi = dF$. For generic choice of Φ, the X^I provide coordinates on the immersed M_{VM}, while $F_I = \partial_I F = F_I(X)$ along M_{VM} . The X^I are non-generic coordinates, physically, because they are the lowest components of vector multiplets, mathematically, because they are adapted to the immersion. They are called special coordinates.

So far we have considered vector multiplets in a four-dimensional Minkowski space-time. In four-dimensional Euclidean space the theory has the same form, except that the complex structure I, $I^2 = -\mathbb{1}$ is replaced by a para-complex structure J. This is an endomorphism of TM_{VM} such that $J^2 = \mathbb{1}$, with the eigendistributions corresponding to the eigenvalues ± 1 having equal rank. Many notions of complex geometry, including Kähler and special Kähler geometry can be adapted to the para-complex realm. We refer to [5], [6] for the details. In particular, it can be shown that the target space geometry of rigid Euclidean vector multiplets is affine special para-Kähler. Such manifolds are the para-complex analogues of affine special Kähler manifolds. When using an appropriate notation, the expressions for the Lagrangian, the equations of motion and the supersymmetry transformation rules take the same form as for Lorentzian supersymmetry, except that complex quantities have to be re-interpreted as para-complex ones. For example, the analogue of complex coordinates $X^I = x^I + i u^I$, where x^I, u^I are real and i is the imaginary unit, are para-complex coordinates $X^I = x^I + e u^I$, where e is the para-complex unit characterized by $e^2 = 1$ and $\bar{e} = -e$, where the 'bar' denotes para-complex conjugation.[5] While in Lorentzian signature the selfdual and antiselfdual projections of the field strength are related by complex conjugation, in the Euclidean theory one can re-define the selfdual

[4] The scalar fields X^I might only provide local coordinates. We will work in a single coordinate patch throughout.

[5] It has been known for quite a while that the Euclidean version of a supersymmetric theory can sometimes be obtained by replacing $i \to e$ [30].

and antiselfdual projections by appropriate factors of e such that they are related by para-complex conjugation. One can also define para-complex spinor fields such that the fermionic terms of the Euclidean theory take the same form as in the Lorentzian one. The Euclidean bosonic Lagrangian takes the same form (2.1) as the Lorentzian one, with (2.2) replaced by

$$N_{IJ} = \partial_I \partial_{\bar{J}} \left(-e(X^I \bar{F}_I - F_I \bar{X}^I)\right). \tag{2.4}$$

Note that the Euclidean Lagrangian is real-valued, although the fields X^I and F^{I-}_{mn} are para-complex. We also remark that a para-Kähler metric always has split signature. The full Lagrangian, including fermionic terms, and the supersymmetry transformation rules can be found in [5]. There we also verified that it is related to the rigid limit of the general Lorentzian signature vector multiplet Lagrangian [31], [32] by replacing $i \to e$ (together with additional field redefinitions, which account for different normalizations and conventions).

2.2 Hypermultiplets

Our next step is to derive the geometry of Euclidean hypermultiplets. This can be done by either reducing the Lorentzian vector multiplet Lagrangian with respect to time or the Euclidean vector multiplet Lagrangian with respect to space [6]. Here we start from the Lorentzian Lagrangian and perform the reduction over space and over time in parallel. This is instructive, because the reduction over space corresponds to the standard c-map and gives us hypermultiplets in three-dimensional Minkowski spacetime, while the reduction over time is the new para-c-map and gives us hypermultiplets in three-dimensional Euclidean space.

Before performing the reduction, we rewrite the Lorentzian vector multiplet Lagrangian in terms of real fields. Above we noted that the intrinsic characteristic of an affine special Kähler manifold is the existence of the special connection ∇, which is, in particular, flat, torsionfree and symplectic [27]. The corresponding flat symplectic coordinates are

$$x^I = \mathrm{Re}\, X^I, \quad y_I = \mathrm{Re}\, F_I. \tag{2.5}$$

Note that since F is an arbitrary holomorphic function, these real coordinates are related in a complicated way to the special coordinates X^I. The real coordinates x^I, y_I are flat (or affine) coordinates with respect to ∇, i.e., $\nabla dx^I = 0 = \nabla dy_I$, and they are symplectic (or Darboux coordinates), because the symplectic form on M_{VM} is $\omega = 2dx^I \wedge dy_I$. While in special coordinates the metric of M_{VM} can be expressed in terms of the prepotential by (2.2), the metric has a Hesse potential when using the real coordinates $q^a = (x^I, y_I)$, where $a = 1, \ldots, 2n$ [27], [28]:

$$g_{ab} = \frac{\partial^2 H}{\partial q^a \partial q^b}. \tag{2.6}$$

The Hesse potential is related to the imaginary part of the prepotential by a Legendre transform [44]:

$$H(x,y) = 2\,\mathrm{Im}\,F(x+iu) - 2u^I y_I. \tag{2.7}$$

The two parametrizations of the metric on M_{VM} are related by

$$ds^2 = -\tfrac{1}{2}N_{IJ}dX^I d\bar{X}^J = -g_{ab}dq^a dq^b. \tag{2.8}$$

In order to rewrite the Lagrangian (2.1) completely in terms of real fields, we express the (anti)selfdual field strength $F^{I\pm}_{mn}$ in terms of the field strength $F^I_{mn} = F^{I+}_{mn} + F^{I-}_{mn}$ and their Hodge-duals $\tilde{F}^I_{mn} = i(F^{I+}_{mn} - F^{I-}_{mn})$. The result is

$$L^{\text{4d VM}}_{\text{bos}} = -g_{ab}\partial_m q^a \partial^m q^b - \tfrac{1}{4}N_{IJ}F^I_{mn}F^{Jmn} + \tfrac{1}{4}R_{IJ}F^I_{mn}\tilde{F}^{Jmn}, \tag{2.9}$$

where

$$\begin{aligned} R_{IJ} &= F_{IJ} + \bar{F}_{IJ}, \\ N_{IJ} &= i(F_{IJ} - \bar{F}_{IJ}) = \partial_I \partial_{\bar{J}} \left(-i(X^I \bar{F}_I - F_I \bar{X}^I)\right). \end{aligned} \tag{2.10}$$

We now perform the reduction of the Lagrangian (2.9) from four to three dimensions. We treat the reduction over space and over time in parallel. In the following formulae, $\epsilon = 1$ refers to reduction over time, which gives a Euclidean three-dimensional theory, while $\epsilon = -1$ refers to reduction over space. By reduction, one component of each gauge field becomes a scalar. We define

$$p^I = A^{I|0} \text{ for } \epsilon = 1, \quad p^I = A^{I|3} \text{ for } \epsilon = -1. \tag{2.11}$$

Moreover, the n three-dimensional gauge fields $A^{I|\hat{m}}$ obtained from dimensional reduction[6] can be dualized into n further real scalars s_I. Denoting the new scalars by

$$(\hat{q}_a) = (s_I, 2p^I), \tag{2.12}$$

the reduced bosonic Lagrangian takes the following, remarkably simple form:

$$L_{HM} = -g_{ab}(q)\partial_i q^a \partial^i q_b + \epsilon g^{ab}(q)\partial_i \hat{q}_a \partial^i \hat{q}_b, \tag{2.13}$$

where $g^{ab}(q)$ is the inverse of $g_{ab}(q)$. In this parametrization it is manifest that the hypermultiplet target space with metric $(g_{ab}(q)) \oplus (-\epsilon g^{ab}(q))$ is $N = M_{HM} = T^*M_{VM}$. The geometry underlying this Lagrangian was presented in detail in [6] for $\epsilon = 1$, and works analogously for $\epsilon = -1$. Here we give a brief summary. The special connection ∇ on $M = M_{VM}$, can be used to define a decomposition

$$T_\xi N = \mathcal{H}^\nabla_\xi \oplus T^v_\xi N \simeq T_q M \oplus T^*_q M, \tag{2.14}$$

where $\xi \in N$ is a point on N (with local coordinates $(q^a, \hat{q}_a)$), $q = \pi(\xi) \in M$ is its projection onto M, $\mathcal{H}^\nabla_\xi$ is the horizontal subspace with respect to the connection ∇ and $T^v_\xi N$ is the vertical subspace. The identification with $T_q M \oplus T^*_q M$ is canonical, and the scalar fields q^a, $\hat{q}_a$ obtained by dimensional reduction are adapted to the

[6] The three-dimensional vector index takes values $\hat{m} = 0,1,2$ for $\epsilon = -1$ and $\hat{m} = 1,2,3$ for $\epsilon = 1$.

decomposition. One can then define a complex structure J_1 on N, which acts on $T_\xi N \simeq T_q M \oplus T_q^* M$ by multiplication with

$$J_1 := J_1^\nabla = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \tag{2.15}$$

where J, J^* denote the action of the complex structure J of M on TM and T^*M, respectively. Let us now consider the Euclidean case $\epsilon = 1$ for definiteness. Using the Kähler form ω on M, one can further define

$$J_2 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{2.16}$$

where ω is interpreted as a map $T_q M \to T_q^* M$. This is a para-complex structure, $J_2^2 = \mathbb{1}$. Moreover, $J_3 = J_1 J_2$ is a second para-complex structure, and J_1, J_2, J_3 satisfy a modified version of the quaternionic algebra known as the para-quaternionic algebra. Thus, (J_1, J_2, J_3) is a para-hyper-complex structure on N. When defining, as in (2.13), the metric on N by

$$g_N = \begin{pmatrix} g & 0 \\ 0 & -g^{-1} \end{pmatrix}, \tag{2.17}$$

where g is the metric on M, then J_1 is an isometry, while J_2, J_3 are anti-isometries. This means that (J_1, J_2, J_3, g_N) is a para-hyper Hermitian structure.[7] Moreover, the structures J_α, $\alpha = 1, 2, 3$ are parallel with respect to the Levi-Civita connection on N. Thus the metric g_N is para-hyper Kähler, meaning that it is Kähler with respect to J_1 and para-Kähler with respect to J_2, J_3. The case $\epsilon = -1$ works analogously. Here one finds three complex structures satisfying the quaternionic algebra, and the metric defined by (2.13) is hyper-Kähler.

One can introduce (para-)complex fields such that one of the complex or (para-) complex structures becomes manifest in the three-dimensional Lagrangian [7], [6]. In these coordinates the Lagrangian is more complicated, and the geometrical structure reviewed above is less clear. Moreover one has singled out one of the three (para-) complex structures. Thus working in real coordinates has advantages, which should be exploited further in the future. Note in particular that for the c-map in local supersymmetry, the target space of hypermultiplets is quaternion-Kähler for Lorentzian space-time, while it is expected to be para-quaternion-Kähler for Euclidean space-time. In general, the almost (para-)complex structures of a (para-)quaternion-Kähler manifold need not be integrable. Then combining real scalar fields into (para-)complex fields is not natural, as these fields do not define local (para-)complex coordinates.

[7] Also note that J_1, J_2, J_3 are integrable, which follows from the integrability of J.

3 Black holes

In order to prepare for our second application of special geometry, we now give a brief self-contained introduction to certain aspects of black holes.[8] Somewhat surprisingly, one can associate thermodynamic properties to black holes. The so called laws of black hole mechanics, which have been derived in the framework of classical, matter-coupled Einstein gravity, formally have the same structure as the laws of thermodynamics [36]. While this was originally suspected to be a coincidence, the (theoretical) discovery of the Hawking effect [38] strongly suggested to take this observation seriously. More recently, developments in string theory have provided additional insights. Let us now briefly review this, starting with the laws of black hole mechanics in classical gravity.

3.1 The laws of black hole mechanics

The zeroth laws of black hole mechanics states that the surface gravity κ_S of a black hole is constant over the event horizon Δ.[9] The surface gravity can be defined if the event horizon is a Killing horizon, which is the case for all stationary black hole solutions of matter-coupled Einstein gravity. A Killing horizon is a hypersurface in space-time where a Killing vector field ξ becomes null: $\xi^\nu \xi_\nu = 0$. One can show that a Killing horizon is generated by the integral lines of the Killing vector fields, which are null geodesics. There are two natural normal vector fields: the Killing vector field ξ itself and the gradient of its norm-squared, $\nabla(\xi^\nu \xi_\nu)$. Both vector fields must be proportional, and the factor of proportionality is defined to be the surface gravity:

$$\nabla^\mu(\xi^\nu \xi_\nu) = -2\kappa_S \xi^\mu. \tag{3.1}$$

While this implies that κ_S is a function on the horizon, the zeroth law states that this function is constant. The physical interpretation of the surface gravity is that it measures the force which an observer outside the black hole must apply in order to keep a unit test mass fixed at the horizon. Thus it measures the strength of gravity at the horizon. Since the zeroth law of thermodynamics is that temperature is constant in thermodynamical equilibrium, this suggests to interpret surface gravity as temperature and stationary black holes as equilibrium states. At the classical level this interpretation cannot be defended against the obvious problem that a black hole does not emit radiation, a fact which is explicitly alluded to in the term 'black' hole. As we will review below this changes once quantum effects are taken into account. For the time being we focus on the assumptions needed to prove the zeroth law. The classical proof uses the explicit form of the Einstein equations, while the effects of matter are controlled by imposing a suitable condition on the energy-momentum tensor. Moreover

[8] See [33], [34], [35] for a detailed discussion.

[9] The term 'horizon' is unfortunately used for two different but closely related concepts. We will use Δ to denote the null hypersurface which is the boundary between the exterior and interior of the black hole in space-time, and H for the space-like surface which is the boundary in space, at given time. Thus H is a spatial section of Δ while Δ is the 'worldline' of H.

the solution must be stationary. It then follows from the field equations that the event horizon is a Killing horizon. However, it was realized later that the zeroth law does not depend on the details of the gravitational field equations. Instead, it can be proved for any covariant (diffeomorphism invariant) action, including actions which contain higher derivative and in particular higher curvature terms [37]. The relevance of such actions will be discussed below. The prize for not specifying the field equations is that one needs to make the following assumptions: (i) the field equations admit stationary black hole solutions with a Cauchy hypersurface, (ii) the event horizon is a Killing horizon, (iii) if the black hole is stationary but not static, then certain symmetry properties, which in Einstein geometry are consequences of the field equations, need to be imposed.[10]

Before proceeding, let us explain why it is desirable to admit actions containing higher derivative terms. The reason is that we would like to include so-called effective actions which incorporate quantum effects. In quantum field theory the effective action is defined to be the generating functional of the correlation functions. Since the classical action generates the classical contribution to the correlation functions (the leading part in an expansion in $\hbar$) the effective action might be considered to be its quantum version. Unfortunately the exact effective action is usually a rather formal and inaccessible object. However, certain approximations can be computed, and string theory provides a framework where quantum corrections to the gravitational action can and have been computed.[11] As expected on general grounds, quantum gravity manifests itself in the form of higher derivative terms in the effective action, in particular terms which contain higher powers of the Riemann tensor and its contractions. We will discuss a particular class of such terms in the next section.

Let us next turn to the first law of black hole mechanics, which states that for a stationary black hole an infinitesimal change of the mass M is related to infinitesimal changes of the horizon area A, of the angular momentum J and of the electric charge Q by

$$\delta M = \frac{\kappa_S}{8\pi}\delta A + \omega\delta J + \phi\delta Q, \tag{3.2}$$

where ω is the angular velocity and ϕ the electrostatic potential. This should be compared to the first law of thermodynamics (for a grand canonical ensemble),

$$\delta E = T\delta\mathcal{S} - p\delta V + \mu\delta N, \tag{3.3}$$

where E is energy, T is temperature, $\mathcal{S}$ is entropy, p is pressure, V is volume, μ is chemical potential and N is particle number. Given the relation between surface gravity and temperature, this suggests to interprets the area of the event horizon as the entropy of the black hole. This is surprising, since the entropy of normal thermody-

[10] A space-time is stationary if it admits a time-like Killing vector field. In a static space-time this Killing vector field is in addition required to be the normal vector field of a family of hypersurfaces. The additional requirements needed if the space-time is stationary but not static are that the black hole is axisymmetric and invariant under simultaneous reflection of time and the angle around the symmetry axis.

[11] We refer to [26] for more details.

namical systems is an extensive property, i.e., proportional to the volume rather than the surface area.

Like the zeroth law the first law can be derived for general covariant actions, under the same assumptions as for the zeroth law. Moreover, the mass, entropy, angular momentum and charge of the black hole are defined as surface charges, which are obtained by integrating a so-called Noether two-form over a closed spatial surface [69]. The Noether two-form is constructed out of Killing vector fields according to a certain algorithm. For the special case of Einstein gravity this definition reduces to the usual ones (i.e., the Komar or ADM constructions of mass and angular momentum, and the proportionality of entropy and horizon area).

Finally, let us turn to the second law of black hole mechanics, the Hawking area law. In contrast to the zeroth and first law, one does not assume the space-time to be stationary. Rather it can be time-dependent, and include processes such as the formation and fusion of black holes, as long as the time evolution is 'asymptotically predictable'.[12] The second law then states that the total area of event horizons is non-decreasing,

$$\delta A \geq 0, \tag{3.4}$$

which is obviously analogous to the second law of thermodynamics, which states the same for the entropy,

$$\delta \mathcal{S} \geq 0. \tag{3.5}$$

This reinforces the identification of area and entropy suggested by the first law. The second law has been derived using Einstein's field equation together with conditions on the energy-momentum tensor of matter (plus assuming 'predictability' of space-time). So far there is no general proof for the case of general covariant actions. However, examples have been studied, and the integrated Noether form is a good candidate for entropy in non-stationary space-times [40]. One interesting question is whether one should expect that the second law holds for all covariant actions. Since dynamical processes such as collision of black holes are admitted, the contents of the second law appears to be more sensitive to the details of the dynamics as the zeroth and first law. It is not clear whether all possible higher derivative actions give rise to 'sensible' physics which respects the second law. But one would certainly expect this to be true for string-effective actions, though this does not seem to have studied so far. Anyway, already the zeroth and first law provide compelling evidence for relating relating the surface gravity to the temperature and the area (integrated Noether two-form) to the entropy.

3.2 Quantum aspects of black holes

Let us now review the role of the Hawking effect [38] in making plausible the re-interpretation of geometrical as thermodynamic properties. This effect is derived

[12] We refer to [33] for a precise definition and more details.

by treating space-time geometry as a classical background, while matter is described by quantum field theory. In this framework it has been shown that black holes emit thermal radiation, even if there is no ingoing radiation or matter. Moreover, the so-called Hawking temperature of this radiation is indeed proportional to the surface gravity:

$$T_{\text{Hawking}} = \frac{\kappa_S}{2\pi}. \tag{3.6}$$

In Einstein gravity the factor of proportionality between area and entropy is then fixed by the first law:

$$\mathcal{S} = \frac{A}{4} \tag{3.7}$$

(Newton's and Planck's constant and the speed of light serve as natural units, $G_N = \hbar = c = 1$). When using a covariant higher derivative action, the entropy is given by integrating the Noether two form Q over the horizon H:

$$\mathcal{S} = 2\pi \oint_H Q. \tag{3.8}$$

It has been shown that the entropy can be expressed in terms of variational derivatives of the Lagrangian with respect to the Riemann tensor [39], [40]:

$$\mathcal{S} = 2\pi \oint_H \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} \sqrt{h}\, d^2\Omega, \tag{3.9}$$

where $\epsilon^{\mu\nu}$ is the normal bivector of H (with a certain normalization), and $\sqrt{h}\, d^2\Omega$ is the induced volume element. If $\mathcal{L}$ is the Einstein–Hilbert Lagrangian, this formula reproduces the area law. If further terms containing the Riemann tensor are present in $\mathcal{L}$ they induce explicit modifications of the area law.

Once the Hawking effect is taken into account, black holes can emit radiation, which implies that they loose mass and shrink, thus violating the second law of black hole mechanics. However, as soon as one takes the idea seriously that black holes carry entropy, one should consider the total entropy obtained by adding black hole entropy and the thermodynamical entropy of the exterior region. The generalized second law of thermodynamics, which states that the total entropy is non-decreasing, is expected to be valid in quantum gravity [41].

So far we have considered black hole entropy from what one might call the macroscopic or thermodynamical perspective. When dealing with many-constituent systems one distinguishes two levels of description. The fundamental or microscopic level of description requires knowledge of the precise state of the system. For a classical gas this would require to specify the positions and momenta (and other quantities if internal excitations exist) of all atoms or molecules. At the thermodynamical or macroscopic level of description one only considers collective properties of the system, such as temperature, volume and pressure. Statistical mechanics asserts that these macroscopic quantities arise by 'coarse graining' microscopic quantities. E.g., temperature is the average energy per degree of freedom. Obviously many microstates

will give rise to the same macrostate, where the latter is characterised by fixing only the macroscopic quantities. The so-called statistical or microscopic entropy measures how many different microstates give rise to the same macrosate. If $d(E,\dots)$ denotes the number of microstates corresponding to the macrostate with energy E, etc., then the corresponding microscopic entropy is

$$\mathcal{S}_{\text{micro}} = \log d(E,\dots). \tag{3.10}$$

In contrast the so-called macroscopic or thermodynamical entropy $\mathcal{S}_{\text{macro}}$ is a purely macroscopic quantity, which can be characterized by its relation to other macroscopic quantities, such as temperature, free energy, etc. Both entropies are expected to be equal in the thermodynamcial limit, i.e., when the number of constituents goes to infinity.

The geometrical black hole entropy is analogous to the macroscopic entropy, because it has been defined through relations which only involve collective properties of the black hole, such as mass, charge and angular momentum. Any theory of quantum gravity is expected to provide a corresponding microscopic description of black holes, which in particular allows one to identify its microstates. In particular the microscopic entropy should be equal to the macroscopic one, at least in the limit of large mass, which is analogous to the thermodynamical limit. This is widely regarded as a benchmark test for theories of quantum gravity.

3.3 Black holes and strings

In string theory four-dimensional black holes can be interpreted as arising from states in the full ten-dimensional string theory. These states might be string states, or winding states of higher-dimensional membranes (in particular D-branes) [42]. One can test the expected relation between macroscopic and microscopic entropy by counting the ten-dimensional states which give rise to the same four-dimensional black hole. This comparison generically involves the variation of parameters such as the string coupling, and it is not a priori clear whether the number of states is preserved under this interpolation. But for a special subclass of states, the so-called supersymmetric states or BPS states, which we will review below, the interpolation is at least highly plausible. Moreover, both the macroscopic and the microscopic entropy can often be computed to high precision and it has been found that they match [9], even when subleading corrections are included [15]. In particular these tests are sensitive to the distinction between the area law and the generalized formula (3.9), and clearly show that string theory 'knows' about the modifications of the area law. In performing these precision tests, special geometry plays a central role. It is the indispensable tool for constructing black hole solutions and extracting the macroscopic entropy from them. This will be the subject of the next section. We will not be able to cover the microscopic side of the story, i.e., the counting of microstates, in this exposition.

3.4 Black holes and supersymmetry

Before turning to the details, let us review the concepts of BPS states and BPS solitons.[13] Recall that the supercharges which generate supersymmetry transformations are spinors. If there is more than one such spinor, then the supersymmetry algebra admits central operators, which can be organised into a complex antisymmetric matrix. The skew eigenvalues $Z_{(i)}$ of this matrix are called the central charges. It can be shown that on any irreducible representation the mass is bounded from below by the absolute values of the charges:

$$M \geq |Z_{(1)}| \geq |Z_{(2)}| \geq \cdots . \tag{3.11}$$

Moreover, when the mass saturates one or several of these bounds, part of the supercharges operate trivially, and the corresponding multiplet is shorter than a generic massive multiplet. Such multiplets are called supersymmetric multiplets or BPS multiplets. When all inequalities are saturated, the resulting BPS multiplet is invariant under half of the supertransformations and has as many states as a massless multiplet. In the case of $N = 2$ supersymmetry considered here, the algebra has one single complex supercharge Z. Consequently, there are generic massive supermultiplets $M > |Z|$ and '$\frac{1}{2}$-BPS multiplets' with $M = |Z|$.

The concept of BPS state can be applied to solitons. By solitons we refer to solutions of the field equations which can be interpreted as particle-like objects. In particular, these solutions are required to have finite energy, and therefore must approach the ground state asymptotically. Since the energy is localized in a small part of space, such 'lumps' can be thought of as 'extended particles'. One also requires that the solution is static (describing 'a massive particle in its rest frame') and free of naked singularities (we admit singularities covered by event horizons in order to include black holes). A soliton is then called supersymmetric or BPS, if it is invariant under part of the supersymmetry transformations. Let us denote the fields of the underlying action collectively by Φ, the spinorial supersymmetry transformation parameters by ϵ, the corresponding supersymmetry transformation by δ_ϵ and the soliton solution by Φ_0. Then a solution is BPS if there exists a choice of ϵ such that

$$(\delta_\epsilon \Phi)_{|\Phi_0} = 0. \tag{3.12}$$

Particular examples of BPS solitons are provided by supersymmetric black hole solutions of supergravity actions. In supergravity the supersymmetry transformation parameters depend on space-time, $\epsilon = \epsilon(x)$. Therefore the BPS condition implies the existence of a spinor field which generates a supertransformation under which the black hole solution is invariant. This is analogous to a Killing vector field, which generates a diffeomorphism under which the metric (and possibly other fields) are invariant. Therefore such spinor fields $\epsilon(x)$ are called Killing spinors (more accurately

[13] In the following we use basic facts about supersymmetry algebras and their representations. See for example [10], Chapter II.

Killing spinor fields). The interested reader is referred to the monograph [43] for a detailed discussion of supersymmetric solutions.

4 Special geometry and black holes

4.1 Vector multiplets coupled to gravity

We are now in position to discuss BPS black hole solutions in $N = 2$ supergravity coupled to n vector multiplets. This is the relevant part of the effective action for string compactifications preserving $N = 2$ supersymmetry. The general $N = 2$ vector multiplet action was constructed using the superconformal calculus [31].[14] The idea of this method is to start with a theory of $n + 1$ rigidly supersymmetric vector multiplets and to impose that the theory is invariant under superconformal transformations. This implies that the prepotential has to be homogenous of degree 2 in addition to being holomorphic:

$$F(\lambda X^I) = \lambda^2 F(X^I), \quad \lambda \in \mathbb{C}^*, \tag{4.1}$$

where now $I = 0, 1, \ldots, n$. Next one 'gauges' the superconformal transformation, that is one makes the Lagrangian locally superconformally invariant by introducing suitable connections. The new fields entering through this process are encoded in the so-called Weyl multiplet.[15] Finally, one imposes gauge conditions which reduce the local superconformal invariance to a local invariance under standard (Poincaré) supersymmetry. Through the gauge conditions some of the fields become functions of the others. In particular, only n out of the $n + 1$ complex scalars are independent. A convenient choice for the independent scalars is

$$z^A = \frac{X^A}{X^0}, \tag{4.2}$$

where $A = 1, \ldots, n$. This provides a set of special coordinates for the scalar manifold M_{VM}. In contrast, all $n + 1$ gauge fields remain independent. While one particular linear combination, the so-called graviphoton, belongs to the Poincaré supergravity multiplet, the other n gauge fields sit in vector multiplets, together with the scalars z^A. The Weyl multiplet also provides physical degrees of freedom, namely the graviton and two gravitini.

From the underlying rigidly superconformal theory the supergravity theory inherits the invariance under symplectic rotations. For the gauge fields this is manifest, as $(F^I_{mn}, G_{I|mn})$ transforms as a vector under $\mathrm{Sp}(2(n + 1), \mathbb{R})$.[16] In the scalar sector

[14] Further references on $N = 2$ vector multiplet Lagrangians and the superconformal calculus include [45], [46], [1], [47].

[15] One also needs to add a further 'compensating multiplet', which can be taken to be a hypermultiplet. We won't need to discuss this technical detail here. See for example [26] for more background material and references.

[16] The dual gauge fields $G_{I|mn}$ were introduced at the beginning of Section 2.

(X^I, F_I), where $F_I = \partial_I F$, also transforms as a vector, while the gravitational degrees of freedom are invariant. To maintain manifest symplectic invariance, it is advantageous to work with (X^I, F_I) instead of z^A.

The underlying geometry can be described as follows [27], [28], [29]: the fields X^I provide coordinates on the scalar manifold of the associated rigidly superconformal theory. This manifold has complex dimension $n+1$, and can be immersed into $T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2(n+1)}$ just as described in the previous section. The additional feature imposed by insisting on superconformal invariance is that the prepotential is homogenous of degree 2. Geometrically this implies that the resulting affine special Kähler manifold is a complex cone. The scalar manifold of the supergravity theory is parametrized by the scalars z^A and has complex dimension n. It is obtained from the manifold of the rigidly superconformal theory by gauge-fixing the dilatation and $U(1)$ symmetry contained in the superconformal algebra. This amounts to taking the quotient of the complex cone with respect to the $\mathbb{C}^*$-action $X^I \to \lambda X^I$. Thus the scalar manifold M_{VM} is the basis of the conical affine special Kähler manifold $C(M_{VM})$ of the rigid theory. For many purposes, including the study of black hole solutions, it is advantageous to work on $C(M_{VM})$ instead of M_{VM}. In particular, this allows to maintain manifest symplectic covariance, as we already noted. In physical terms this means that one can postpone the gauge-fixing of the dilatation and $U(1)$ transformations. The manifolds which can be obtained from conical affine special Kähler manifolds by a $\mathbb{C}^*$-quotient are called projective special Kähler manifolds. These are the target spaces of vector multiplets coupled to supergravity. All couplings in the Lagrangian and all relevant geometrical data of M_{VM} are encoded in the prepotential. In particular, the affine special Kähler metric on $C(M_{VM})$ has Kähler potential

$$K_C(X^I, \bar{X}^I) = -i(X^I \bar{F}_I - F_I \bar{X}^I), \tag{4.3}$$

while the projective special Kähler metric on M_{VM} has Kähler potential

$$K(z^A, \bar{z}^{\bar{B}}) = -\log\left(-i(X^I \bar{F}_I - F_I \bar{X}^I)\right), \tag{4.4}$$

with corresponding metric

$$g_{a\bar{b}} = \frac{\partial^2 K(z^A, \bar{z}^{\bar{B}})}{\partial z^a \partial \bar{z}^b}. \tag{4.5}$$

In string theory the four-dimensional supergravity Lagrangians considered here are obtained by dimensional reduction of the ten-dimensional string theory on a compact six-dimensional manifold X and restriction to the massless modes. Then the scalar manifold M_{VM} is the moduli space of X. It turns out that the moduli spaces of Calabi–Yau threefolds provide natural realizations of special Kähler geometry [58]. Consider for instance the Calabi–Yau compactification of type-IIB string theory. In this case M_{VM} is the moduli space of complex structures of X, the cone M_{VM} is the moduli space of complex structures together with a choice of the holomorphic top-form, and $T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2(n+1)}$ is $H^3(X, \mathbb{C})$, see [59].

4.2 BPS black holes and the attractor mechanism

Let us then discuss BPS black hole solutions of $N = 2$ supergravity with n vector multiplets. These are static, spherically symmetric solutions of the field equations, which are asymptotically flat, have regular event horizons, and possess 4 Killing spinors. Since the $N = 2$ superalgebra has 8 real supercharges, these are $\frac{1}{2}$-BPS solutions. Let us first have a look at pure four-dimensional $N = 2$ supergravity, i.e., we drop the vector multiplets, $n = 0$. The bosonic part of this theory is precisely the Einstein–Maxwell theory. In pure $N = 2$ supergravity, BPS solutions have been classified [60], [61], [62]. The number of linearly independent Killing spinor fields can be 8, 4 or 0. This can be seen, for example, by investigating the integrability conditions of the Killing spinor equation.[17] Solutions with 8 Killing spinors are maximally supersymmetric and therefore considered as supersymmetric ground states. Examples are Minkowski space and $\mathrm{AdS}_2 \times S^2$. Solutions with 4 Killing spinors are called $\frac{1}{2}$-BPS, because they are invariant under half as many supersymmetries as the ground state. They are solitonic realisations of states sitting in BPS representations. For static $\frac{1}{2}$-BPS solutions the space-time metric takes the form [60], [61]

$$ds^2 = -e^{-2f(\vec{x})}dt^2 + e^{2f(\vec{x})}d\vec{x}^2, \tag{4.6}$$

where $\vec{x} = (x_1, x_2, x_3)$ are space-like coordinates and the function $f(\vec{x})$ must be such that $e^{f(\vec{x})}$ is a harmonic function with respect to $\vec{x}$. The solutions also have a non-trivial gauge field, which likewise can be expressed in terms of $e^{f(\vec{x})}$. This class of solutions of Einstein–Maxwell theory is known as the Majumdar–Papapetrou solutions [64], [65]. The only Majumdar–Papapetrou solutions without naked singularities are the multi-centered extremal Reissner–Nordstrom solutions, which describe static configurations of extremal black holes, see for example [66]. If one imposes in addition spherical symmetry, one arrives at the extremal Reissner–Nordstrom solution describing a single charged black hole. In this case the metric takes the form

$$ds^2 = -e^{-2f(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2), \tag{4.7}$$

where r is a radial coordinate and $d\Omega^2$ is the line element on the unit two-sphere. The harmonic function takes the form

$$e^{f(r)} = 1 + \frac{q^2 + p^2}{r}, \tag{4.8}$$

where q, p are the electric and magnetic charge with respect to the graviphoton. The solution has two asymptotic regimes. In one limit, $r \to \infty$, it becomes asymptotically flat: $e^f \to 1$. In the other limit, $r \to 0$, which is the near-horizon limit, it takes the

[17] The classification of supersymmetric solutions has recently moved to the focus of interest. Readers who want to get an idea how the classification of supersymmetric solutions of four-dimensional $N = 2$ supergravity would work with 'modern', systematic methods can consult [63], where all supersymmetric solutions of minimal five-dimensional supergravity were constructed.

form

$$ds^2 = -\frac{r^2}{q^2+p^2}dt^2 + \frac{q^2+p^2}{r^2}dr^2 + (q^2+p^2)d\Omega^2. \tag{4.9}$$

This is a standard form for the metric of $AdS_2 \times S^2$. The area of the two-sphere, which is the area of the event horizon of the black hole, is given by $A = 4\pi(q^2+p^2)$. The two limiting solutions, flat Minkowski space-time and $AdS_2 \times S^2$ are among the fully supersymmetric solutions with 8 Killing spinors that we mentioned before. Thus, the extremal Reissner–Nordstrom black hole interpolates between two supersymmetric vacua [48]. This is a property familiar from two-dimensional kink solutions, and motivates the interpretation of supersymmetric black hole solutions as solitons, i.e., as particle-like collective excitations.

Let us now return to $N = 2$ supergravity with an arbitrary number n of vector fields. We are interested in solutions which generalize the extremal Reissner–Nordstrom solution. Therefore we impose that the solution should be $\frac{1}{2}$-BPS, static, spherically symmetric, asymptotically flat, and that it should have a regular event horizon.[18] More general $\frac{1}{2}$-BPS solutions have been studied extensively in the literature, in particular in [49] and [16]. Recently, the classification of all $\frac{1}{2}$-BPS solutions was achieved in [50].

BPS black holes in theories with n vector multiplets depend on $n+1$ gauge fields and on n scalar fields. For any $\frac{1}{2}$-BPS solution, which is static and spherically symmetric, the metric can be brought to the form (4.8) [16]. The condition that the solution is static and spherically symmetric is understood in the strong sense, i.e., it also applies to the gauge fields and scalars. Thus gauge fields and scalars are functions of the radial coordinate r, only. Moreover the electric and magnetic fields are spherically symmetric, which implies that each field strength $F^I_{mn}(r)$ has only two independent components (see for example Appendix A of [26] for more details).

The electric and magnetic charges carried by the solution are defined through flux integrals of the field strength over asymptotic two-spheres:

$$(p^I, q_I) = \frac{1}{4\pi}\left(\oint F^I, \oint G_I\right), \tag{4.10}$$

where F^I, G_I are the two-forms associated with the field strength F^I_{mn} and their duals G_{Imn}. As a consequence, the charges transform as a vector under symplectic transformations. By contracting the charges with the scalars one obtains the symplectic function

$$Z = p^I F_I - q_I X^I. \tag{4.11}$$

This field is often called the central charge, which is a bit misleading because Z is a function of the fields X^I and F_I and therefore a function of the scalar fields z^A, which are space-time dependent.[19] Hence, in the class of backgrounds we consider, Z is a function of the radial coordinate r. However, when evaluating this field in the

[18] This excludes both naked singularities and null singularities, where the horizon coincides with the singularity and has vanishing area.

asymptotically flat limit $r \to \infty$, it computes the electric and magnetic charge carried by the graviphoton, which combine into the complex central charge of the $N = 2$ algebra [67].

In particular, the mass of the black hole is given by

$$M = |Z|_\infty = M(p^I, q_I, z^A(\infty)). \tag{4.12}$$

Thus BPS black holes saturate the mass bound implied by the supersymmetry algebra. Note that the mass does not only depend on the charges, but also on the values of the scalars at infinity, which can be changed continuously.

The other asymptotic regime is the event horizon. If the horizon is regular, then the solution must be fully supersymmetric in this limit [11]. Thus, while the bulk solution has 4 Killing spinors, both asymptotic limits have 8. In the near horizon limit, the metric (4.8) takes the form

$$ds^2 = -\frac{r^2}{|Z|^2_{\text{hor}}} dt^2 + \frac{|Z|^2_{\text{hor}}}{r^2} dr^2 + |Z|^2_{\text{hor}} d\Omega^2, \tag{4.13}$$

where $|Z|^2_{\text{hor}}$ is the value of $|Z|^2$ at the horizon. As in the extremal Reissner–Nordstrom solution, this is $\text{AdS}_2 \times S^2$. The area of the two-sphere, which is the area of the event horizon, is given by $A = 4\pi |Z|^2_{\text{hor}}$. Hence the Bekenstein Hawking entropy is

$$\mathcal{S}_{\text{macro}} = \frac{A}{4} = \pi |Z|^2_{\text{hor}}. \tag{4.14}$$

A priori, $\mathcal{S}_{\text{macro}}$ depends on both the charges and the values of the scalars at the horizon, and one might expect that one can change the latter continuously. This would be incompatible with relating $\mathcal{S}_{\text{macro}}$ to a statistical entropy $\mathcal{S}_{\text{micro}}$ which counts states. But it turns out that the values of the scalar fields at the horizon are themselves determined in terms of the charges. Here, it is convenient to define $Y^I = \bar{Z} X^I$ and $F_I = F_I(Y) = \bar{Z} F_I(X)$.[20] In terms of these variables, the black hole attractor equations [11], which express the horizon values of the scalar fields in terms of the charges, take the following, symplectically covariant form:

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I - \bar{F}_I \end{pmatrix}_{\text{hor}} = i \begin{pmatrix} p^I \\ q_I \end{pmatrix}. \tag{4.15}$$

The name attractor equations refers to the behaviour of the scalar fields as functions of the space-time radial coordinate r. While the scalars can take arbitrary values at $r \to \infty$, they flow to fixed points, which are determined by the charges, for $r \to 0$. This fixed point behaviour follows when imposing that the event horizon is regular. Alternatively, one can show that to obtain a fully supersymmetric solution with geo-

[19] One can analyse BPS solutions without imposing the gauge conditions which fix the superconformal symmetry, and in fact it is advantageous to do so [15], [16]. Then the scalars are encoded in the fields $X^I(r)$, which are subject to gauge transformations. Once gauge conditions are imposed, one can express $Z(r)$ in terms of the physical scalar fields $z^A(r)$. See [26] for more details.

[20] Note that F_I is homogenous of degree 1.

metry $AdS_2 \times S^2$ the scalars need to take the specific values dictated by the attractor equations [16]. This is due to the presence of non-vanishing gauge fields. The gauge fields in $AdS_2 \times S^2$ are covariantly constant, so that this can be viewed as an example of a flux compactification. In contrast, Minkowski space is also maximally supersymmetric, but the scalars can take arbitrary constant values, because the gauge fields vanish. In type-II Calabi–Yau compactifications, the radial dependence of the scalar fields defines a flow on the moduli space, which starts at an arbitrary point and terminates at a fixed point corresponding to an 'attractor Calabi–Yau.' Since the electric and magnetic charges (p^I, q_I), which determine the fixed point, take discrete values, such attractor threefolds sit at very special points in the moduli space. This has been studied in detail in [51].

Using the fields Y^I instead of X^I to parametrize the scalars simplifies formulae and has the advantage that the Y^I are invariant under the $U(1)$ transformations of the superconformal algebra. Note that

$$|Z|^2 = p^I F_I - q_I Y^I, \tag{4.16}$$

which is easily seen using the homogeneity properties of the prepotential. The diffeomorphism $X^I \to Y^I$ acts non-holomorphically on the cone $C(M_{VM})$, but operates trivially on its basis M_{VM}. Note in particular that

$$z^A = \frac{X^A}{X^0} = \frac{Y^A}{Y^0}. \tag{4.17}$$

4.3 The black hole variational principle

We now turn to the black hole variational principle, which was found in [14] and generalized in [13], motivated by the observations of [12]. First, we define two symplectic functions, the entropy function

$$\Sigma(Y^I, \bar{Y}^I, p^I, q_I) = \mathcal{F}(Y^I, \bar{Y}^I) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I) \tag{4.18}$$

and the black hole free energy

$$\mathcal{F}(Y^I, \bar{Y}^I) = -i\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right). \tag{4.19}$$

The reason for our choice of terminology will become clear later. Now we impose that the entropy function is stationary, $\delta\Sigma = 0$, under variations of the scalar fields $Y^I \to Y^I + \delta Y^I$. Using that the prepotential is homogenous of degree two, it is easy to see that the conditions for Σ being stationary are precisely the black hole attractor equations (4.15). Furthermore, at the attractor point we find that[21]

$$\begin{aligned} \mathcal{F}_{\text{attr}} &= -i\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right)_{\text{attr}} = \left(q_I Y^I - p^I F_I\right)_{\text{attr}} \\ &= \left(q_I \bar{Y}^I - p^I \bar{F}_I\right)_{\text{attr}} = -|Z|^2_{\text{attr}} \end{aligned} \tag{4.20}$$

[21] The relation $-i\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right) = \left(q_I Y^I - p^I F_I\right)$ follows from the definitions of Z and Y^I together with the homogeneity of the prepotential (once the dilatational symmetry of the fields X^I has been gauge fixed). Therefore it holds irrespective of whether the scalar fields take their attractor values or not.

and therefore

$$\Sigma_{\text{attr}} = |Z|^2_{\text{attr}} = \tfrac{1}{\pi}\mathcal{S}_{\text{macro}}(p^I, q_I). \tag{4.21}$$

Here and in the following we use the label 'attr' (instead of 'hor' used previously) to indicate that quantities are evaluated at the attractor point determined by the electric and magnetic charges.

Thus, up to a constant factor, the entropy is obtained by evaluating the entropy function at its critical point. Moreover, a closer look at the variational principle shows us that, again up to a factor, the black hole entropy $\mathcal{S}_{\text{macro}}(p^I, q_I)$ is the Legendre transform of the free energy $\mathcal{F}(Y^I, \bar{Y}^I)$, where the latter is considered as a function of $x^I = \text{Re}(Y^I)$ and $y_I = \text{Re}(F_I)$. At this point the real variables discussed in the previous section become important again. Note that the change of variables $(Y^I, \bar{Y}^I) \to (x^I, y_I)$ is well defined provided that $\text{Im}(F_{IJ})$ is non-degenerate. This assumption will be satisfied in general, but breaks down in certain string theory applications, where one reaches the boundary of the moduli space.[22]

We are therefore led to rewrite the variational principle in terms of real variables. First, recall that the Hesse potential $H(x^I, y_I)$ is the Legendre transform of (two times) the imaginary part of the prepotential, see (2.7).[23] This Legendre transform replaces the independent variables $(x^I, u^I) = (\text{Re}(Y^I), \text{Im}(Y^I))$ by the independent variables $(x^I, y_I) = (\text{Re}(Y^I), \text{Re}(F_I))$ and therefore implements the change of variables $(Y^I, \bar{Y}^I) \to (x^I, y_I)$. Using (2.7) we find

$$H(x^I, y_I) = -\tfrac{i}{2}(\bar{Y}^I F_I - \bar{F}_I Y^I) = \tfrac{1}{2}\mathcal{F}(Y^I, \bar{Y}^I). \tag{4.22}$$

Thus, up to a factor, the Hesse potential is the black hole free energy. We can now express the entropy function in terms of the real variables:

$$\Sigma(x^I, y_I, p^I, q_I) = 2H(x^I, y_I) - 2q_I x^I + 2p^I y_I. \tag{4.23}$$

If we impose that Σ is stationary with respect to variations of x^I and y_I, we get the black hole attractor equations in real variables:

$$\frac{\partial H}{\partial x^I} = q_I, \quad \frac{\partial H}{\partial y_I} = -p^I. \tag{4.24}$$

Plugging this back into the entropy function we obtain

$$\mathcal{S}_{\text{macro}} = 2\pi\left(H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I}\right)_{\text{attr}}. \tag{4.25}$$

Thus, up to a factor, the black hole entropy is the Legendre transform of the Hesse potential. This is an intriguing observation, because it relates the black hole entropy, which is a space-time quantity, directly to the special geometry encoding the scalar dynamics. In string theory compactifications this relates the geometry of four-dimensional space-time to the geometry of the compact internal space X. The Hesse

[22] See for example [13] for a discussion of some of the implications.

[23] Note that this is the Hesse potential of the affine special Kähler metric on $C(M_{VM})$. The projective special Kähler metric on M_{VM} is obtained by the $\mathbb{C}^*$-quotient.

potential appears to be closely related to the action functional underlying the geometry of X in Hitchin's approach to manifolds with special holonomy [52], [53], [54].

We can also relate the black hole free energy to another quantity of special geometry. In terms of complex variables we observe that

$$\mathcal{F}(Y^I, \bar{Y}^I) = K_C(Y^I, \bar{Y}^I) := i(\bar{Y}^I F_I - \bar{F}_I Y^I). \tag{4.26}$$

Comparing to (4.3) it appears that we should interpret $K_C(Y^I, \bar{Y}^I)$ as the Kähler potential of an affine special Kähler metric on $C(M_{VM})$. Since the diffeomorphism $X^I \to Y^I$ is non-holomorphic, this is not the same special Kähler structure as with (4.3). However, we already noted that the diffeomorphism acts trivially on M_{VM}, see (4.17). Moreover it is easy to see that when taking the quotient with respect to the $\mathbb{C}^*$-action $Y^I \to \lambda Y^I$, then the resulting projective special Kähler metric with Kähler potential $K(Y^I, \bar{Y}^I) = -\log K_C(Y^I, \bar{Y}^I)$ is the same as the one derived from (4.4), because the two Kähler potentials differ only by a Kähler transformation. It appears that in the context of black hole solutions the affine special Kähler metric associated with the rescaled scalars Y^I is of more direct importance than the one based on the X^I. The same remark applies to the Hesse potential, which depends on the real coordinates associated to Y^I. Note that the scalars Y^I do not only encode the values of the Calabi–Yau moduli z^A via (4.17) but also, via (4.16) the size of the two-sphere in the black-hole space-time.[24] While variations of the moduli correspond to variations along the basis of the cone $C(M)$, variations of the radius of the two-sphere correspond to motions along the radial direction of the cone.

Note that it is more natural to identify the free energy with the Hesse potential than the Kähler potential. The first reason is that the various Legendre transforms involve the real and not the complex coordinates. The second reason is that, as we will discuss below, we need to generalize the supergravity Lagrangian in order to take into account certain corrections appearing in string theory. We will see that this works naturally by introducing a generalized Hesse potential.

Before turning to this subject, we also remark that the terms in the entropy function (4.18) which are linear in the charges, and which induce the Legendre transform, have yet another interpretation in terms of supersymmetric field theory. Namely, the symplectic function

$$W = q_I Y^I - p^I F_I \tag{4.27}$$

has the form of an $N = 2$ superpotential. The four-dimensional supergravity Lagrangian we are studying does not have a superpotential. However, the near-horizon solution has the form $\mathrm{AdS}_2 \times S^2$ and carries non-vanishing, covariantly constant gauge fields. The dimensional reduction on S^2 is a flux compactification, with fluxes parametrized by (p^I, q_I), and the resulting two-dimensional theory will possess a superpotential. This also provides an alternative interpretation of the attractor mechanism, as the resulting scalar potential will lift the degeneracy of the moduli.

[24] This is not only true at the horizon but throughout the whole black hole solution.

4.4 Quantum corrections to black holes solutions and entropy

So far we only considered supergravity Lagrangians which contain terms with at most two derivatives. The effective Lagrangians derived from string theory also contain higher derivative terms, which modify the dynamics at short distances. These terms describe interactions between the massless states which are mediated by massive string states. While the effective Lagrangian does not contain the massive string states explicitly, it is still possible to describe their impact on the dynamics of the massless states.

In $N = 2$ supergravity a particular class of higher derivative terms can be taken into account by giving the prepotential an explicit dependence on an additional complex variable Υ, which is proportional to the lowest component of the Weyl multiplet [57], [68]. The resulting function $F(Y^I, \Upsilon)$ is required to be holomorphic in all its variables, and to be (graded) homogenous of degree two:[25]

$$F(\lambda Y^I, \lambda^2 \Upsilon) = \lambda^2 F(Y^I, \Upsilon). \tag{4.28}$$

Assuming that it is analytic at $\Upsilon = 0$ one can expand it as

$$F(Y^I, \Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y^I) \Upsilon^g. \tag{4.29}$$

Then $F^{(0)}(Y^I)$ is the prepotential, while the functions $F^{(g)}(Y^I)$ with $g > 0$ appear in the Lagrangian as the coefficients of various higher-derivative terms. These include in particular terms quadratic in the space-time curvature, and therefore one often loosely refers to the higher derivative terms as R^2-terms.

In type-II Calabi–Yau compactifications the functions $F^{(g)}(Y^I)$ can be computed using (one of) the topologically twisted version(s) of the theory [56]. They are related to the partition functions $Z^{(g)}_{\text{top}}$ of the topologically twisted string on a world sheet with genus g by $F^{(g)} = \log Z^{(g)}_{\text{top}}$. Therefore they are called the (genus-g) topological free energies.

It was shown in [15], [16] that the black hole attractor mechanism can be generalized to the case of Lagrangians based on a general function $F(Y^I, \Upsilon)$. The attractor equations still take the form (4.15), but the prepotential is replaced by the full function $F(Y^I, \Upsilon)$. The additional variable Υ takes the value $\Upsilon = -64$ at the horizon. The evaluation of the generalized entropy formula (3.9) for $N = 2$ supergravity gives [15]:

$$\mathcal{S}_{\text{macro}}(q^I, p_I) = \pi \left(|Z|^2 + 4\,\text{Im}(\Upsilon F_\Upsilon) \right)_{\text{attr}}, \tag{4.30}$$

where $F_\Upsilon = \partial_\Upsilon F$.[26] Note that symplectic covariance is manifest, as the entropy is the sum of two symplectic functions. While the first term corresponds to the area law, the second term is an explicit modification which depends on the coefficients $F^{(g)}$, $g > 0$, of the higher derivative terms.

[25] Since we are interested in black hole solutions, we use rescaled fields Y^I, Υ.

[26] At the attractor point, Υ takes the value $\Upsilon = -64$.

It was shown in [13] that the variational principle generalizes to the case with R^2-terms. The black hole free energy $\mathcal{F}$ is now proportional to a generalized Hesse potential $H(x^I, y_I, \Upsilon, \overline{\Upsilon})$, which in turn is proportional to the Legendre transform of the imaginary part of the function $F(Y^I, \Upsilon)$:

$$H(x^I, y_I, \Upsilon, \overline{\Upsilon}) = 2\,\mathrm{Im}\, F(x^I + iu^I, \Upsilon) - 2y_I u^I .$$

In terms of complex fields Y^I this becomes

$$\begin{aligned} H(x^I, y_I, \Upsilon, \overline{\Upsilon}) &= -\tfrac{i}{2}(\overline{Y}^I F_I - \overline{F}_I Y^I) - i(\Upsilon F_\Upsilon - \overline{\Upsilon}\,\overline{F}_{\overline{\Upsilon}}) \\ &= \tfrac{1}{2}\mathcal{F}(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon}). \end{aligned} \tag{4.31}$$

The entropy function (4.23), the attractor equations (4.24) and the formula for the entropy (4.25), which now includes correction terms to the area law, remain the same, except that one uses the generalized Hesse potential. From (4.31) it is obvious that the black hole free energy naturally corresponds to a generalized Hesse potential (defined by the Legendre transform of the prepotential) and not to a 'generalized Kähler potential', which would only give rise to the first term on the right hand side of (4.31).

There is a second class of correction terms in string-effective supergravity Lagrangians. Quantum corrections involving the massless fields lead to modifications which correspond to adding non-holomorphic terms to the function $F(Y^I, \Upsilon)$. The necessity of such non-holomorphic terms can be seen by observing that otherwise the invariance of the full string theory under T-duality and S-duality is not captured by the effective field theory. In particular, one can show that the black hole entropy can only be T- and S-duality invariant if non-holomorphic corrections are taken into account [55].[27] From the point of view of string theory the presence of these terms is related to a holomorphic anomaly [56], [57].

As the holomorphic R^2-corrections, the non-holomorphic corrections can be incorporated into the black hole attractor equations and the black hole variational principle [55], [13]. The non-holomorphic terms are encoded in a function $\Omega(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon})$, which is real valued and homogenous of degree two. To incorporate non-holomorphic terms into the variational principle one has to define the generalized Hesse potential as the Legendre transform of $2\,\mathrm{Im}\, F + 2\Omega$:

$$H(x^I, \hat{y}_I, \Upsilon, \overline{\Upsilon}) = 2\,\mathrm{Im}\, F(x^I + iu^I, \Upsilon, \overline{\Upsilon}) + 2\Omega(x^I, u^I, \Upsilon, \overline{\Upsilon}) - 2\hat{y}_I u^I, \tag{4.32}$$

where $\hat{y}_I = y_I + i(\Omega_I - \Omega_{\bar{I}})$ and $\Omega_I = \frac{\partial \Omega}{\partial Y^I}$ and $\Omega_{\bar{I}} = \frac{\partial \Omega}{\partial \overline{Y}^I}$. Up to these modifications, the attractor equations, the entropy function, and the entropy remain as in (4.24), (4.23) and (4.25). Also note from (4.32) that if Ω is harmonic, it can be absorbed into $\mathrm{Im}\, F$, because it then is the imaginary part of holomorphic function. Thus, the non-holomorphic modifications of the prepotential correspond to non-harmonic functions Ω.

[27] We are referring to compactifications with exact T- and S-duality symmetry. These are mostly compactifications with $N = 4$ supersymmetry, which, however, can be studied in the $N = 2$ framework. We refer to [55], [70], [13] for details.

In terms of the complex variables the attractor equation are

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I + 2i\,\Omega_I - F_I + 2i\,\Omega_{\bar{I}} \end{pmatrix} = i \begin{pmatrix} p^I \\ q_I \end{pmatrix}. \tag{4.33}$$

The modified expressions for the free energy and the entropy function can be found in [13].

At this point it is not quite clear what the R^2-corrections and the non-holomorphic corrections mean in terms of special geometry. Since they correspond to higher derivative terms in the Lagrangian, they do not give rise to modifications of the metric on the scalar manifold, which, by definition, is the coefficient of the scalar two-derivative term.[28] It would be very interesting to extend the framework of special geometry such that the functions $F^{(g)}$ get an intrinsic geometrical meaning.

4.5 Black hole partition functions and the topological string

Let us now discuss how the black hole variational principle is related to black hole partition functions and the topological string. We start by relating the variational principle described in the last sections to the variational principle used in [12]. One can start from the generalized Hesse potential and perform partial Legendre transforms by imposing only part of the attractor equations. If this subset of fields is properly chosen one obtains a reduced variational principle, which yields the remaining attractor equations, and, by further extremisation, the black hole entropy. Specifically, one can solve the magnetic attractor equations $Y^I - \bar{Y}^I = ip^I$ by setting[29]

$$Y^I = \tfrac{1}{2}(\phi^I + ip^I). \tag{4.34}$$

Plugging this back, the new, reduced entropy function is

$$\Sigma(p^I, \phi^I, q_I) = \mathcal{F}_E(p^I, \phi^I, \Upsilon, \bar{\Upsilon}) - q_I \phi^I, \tag{4.35}$$

where[30]

$$\mathcal{F}_E(p^I, \phi^I, \Upsilon, \bar{\Upsilon}) = 4\left(\mathrm{Im}\, F(Y^I, \Upsilon) + \Omega(Y^I, \bar{Y}^I, \Upsilon, \bar{\Upsilon})\right)_{\mathrm{mgn}} \tag{4.36}$$

Here the label 'mgn' indicates that the magnetic attractor equations have been imposed, i.e., $Y^I = \frac{1}{2}(\phi^I + ip^I)$. Both $\mathcal{F}(Y^I, \bar{Y}^I, \Upsilon, \bar{\Upsilon}) = 2H(x^I, \hat{y}_I, \Upsilon, \bar{\Upsilon})$ and $\mathcal{F}_E(p^I, \phi^I, \Upsilon, \bar{\Upsilon})$ are interpreted as free energies, which, however, refer to different statistical ensembles. In the microcanonical ensemble the electric and magnetic charges are kept fixed, while they fluctuate around a mean value in the canonical ensemble. The transition between these two ensemble is made by changing the independent variables, i.e., one eliminates the electric and magnetic charges q_I, p^I in

[28] See however [72], where such an interpretation was proposed.

[29] Obviously, $\phi^I = 2x^I$. We use ϕ^I to be consistent with the notation used in [13]. The conventions of [12] are slightly different.

[30] We suppressed the dependence of Σ on Υ, but indicated it for $\mathcal{F}_E$ in order to make explicit that we included the higher derivative corrections.

favour of the corresponding chemical potentials, which are the electrostatic and magnetostatic potentials ϕ^I, χ_I.[31] By virtue of the equations of motion the potentials coincide, up to a factor, with the real coordinates on $C(M_{VM})$: $\phi^I = 2x^I$, $\chi_I = 2\hat{y}_I$. In black hole thermodynamics the electrostatic and magnetostatic potentials are evaluated at the horizon. Note that both sets of thermodynamical variables correspond to different real symplectic coordinates on $C(M_{VM})$: the charges to the imaginary part, the potentials to the real part of the symplectic vector (Y^I, F_I).

As an intermediate step, one can go to the mixed ensemble, where the magnetic charges are kept fixed, while the electric charges fluctuate. Then the independent variables are p^I and ϕ^I. This indicates that $\mathcal{F}$ is the free energy with respect to the canonical ensemble, while $\mathcal{F}_E$ is the free energy with respect to the mixed ensemble.

If one imposes that $\Sigma(p^I, \phi^I, q_I)$ is stationary with respect to variations of ϕ^I, then one obtains the electric attractor equations $(F_I - 2i\,\Omega_I) - (\bar{F}_I + 2i\,\Omega_{\bar{I}}) = i q_I$ (4.33). Plugging these back one sees that at the stationary point $\Sigma_{\text{attr}} = \frac{1}{\pi}\mathcal{S}_{\text{macro}}(p^I, q_I)$ and that the macroscopic entropy is the partial Legendre transform of the free energy $\mathcal{F}_E(p^I, \phi^I, \Upsilon, \bar{\Upsilon})$.

Actually, the black hole free energy introduced in [12] includes the contribution from holomorphic higher derivative terms, but not the non-holomorphic corrections. Let us denote this quantity by $\mathcal{F}_{\text{OSV}}(p^I, \phi^I, \Upsilon)$. It is proportional to the imaginary part of the generalized holomorphic prepotential $F(Y^I, \Upsilon)$. If the model under consideration has been obtained by compactification of type-II string theory on a Calabi–Yau threefold, then the prepotential is in turn proportional to the so-called topological free energy F_{top}, which is the logarithm of the all-genus partition function of the topological type-II string, $Z_{\text{top}} = e^{F_{\text{top}}}$. In our conventions the precise relation between the free energies is

$$\pi\,\mathcal{F}_{\text{OSV}} = 4\pi\ \text{Im}\ F\ = 2\,\text{Re}\ F_{\text{top}}. \tag{4.37}$$

Therefore the free energy $\mathcal{F}_{\text{OSV}}$ is related to the topological partition function by [12]

$$e^{\pi\mathcal{F}_{\text{OSV}}(p,\phi,\Upsilon)} = |Z_{\text{top}}|^2. \tag{4.38}$$

This supports the idea to take the interpretation of $\mathcal{F}_{\text{OSV}}(p, \phi, \Upsilon)$ as the free energy of the black hole seriously. Then it should be related to the partition function of the black hole with respect to the mixed ensemble, which is defined by

$$Z_{\text{mixed}}(p, \phi) = \sum_q d(p, q) e^{q\phi}, \tag{4.39}$$

where $d(p, q)$ is the number of BPS microstates with charges p^I, q_J, and $q\phi := q_I \phi^I$. This relation is a formal discrete Laplace transform which relates the microscopic partition function, i.e., the state degeneracy, to the mixed partition function. The standard relation between free energy and partition function would imply that $Z_{\text{mixed}} = e^{\pi\mathcal{F}_{\text{OSV}}}$. However, from our discussion of the black hole variational principle and of the

[31] Since the charges play the roles of particle number in non-relativistic thermodynamics, it might appear more logical to call the 'microcanonical' ensemble canonical, and the 'canonical' ensemble grand canonical. However, we follow the terminology established in the recent literature on the OSV conjecture.

role of non-holomorphic corrections it appears to be natural to contemplate including non-holomorphic terms, thus replacing $\mathcal{F}_{\rm OSV}$ by $\mathcal{F}_E$.[32] Thus we should leave open the option that there are subleading corrections to the relation between the black hole partition function and the topological string partition function. The weak version of the OSV conjecture [12] is

$$Z_{\rm mixed}(p,\phi) \approx e^{\pi \mathcal{F}_{\rm OSV}(p,\phi)} = |Z_{\rm top}(p,\phi)|^2, \tag{4.40}$$

where $\approx$ means equality in the limit of large charges, which is the semiclassical limit. Evidence for this form of the conjecture will be given below. We will also see that the conjecture needs to be modified as soon as subleading corrections are included.

By a formal Laplace transform we can equivalently formulate this conjecture as a prediction of the state degeneracy in terms of the free energy, by

$$d(p,q) \approx \int d\phi e^{\pi[\mathcal{F}_{\rm OSV} - q\phi]}. \tag{4.41}$$

Here $d\phi = \prod_I d\phi^I$, and the ϕ^I are taken to be complex and integrated along a contour encircling the origin. The relation (4.41) is intriguing, as it relates the black hole microstates directly to the topological string partition function. Note that a saddle point evaluation of the integral gives

$$d(p,q) \approx e^{\mathcal{S}_{\rm macro}(p,q)}, \tag{4.42}$$

because at the critical point of the integrand we have $\pi[\mathcal{F}_E - q_I\phi^I]_{\rm attr} = \mathcal{S}_{\rm macro}(p,q)$. Thus the microscopic entropy $\mathcal{S}_{\rm micro}(p,q) = \log d(p,q)$ and the macroscopic entropy $\mathcal{S}_{\rm macro}(p,q)$ agree to leading order in the semiclassical limit.[33]

There are several problems which indicate that the proposal (4.41) must be modified. The number of states $d(p,q)$ should certainly be invariant under stringy symmetries such as S-duality and T-duality. In the context of compactifications with $N \geq 2$ supersymmetry, where duality symmetries are realized as symplectic transformation, this also means that $d(p,q)$ should be a symplectic function. However, in the approach of [12] the electric and magnetic charges are treated differently, so that there is no manifest symplectic covariance. A related issue is how to take into account non-holomorphic corrections. While [12] is based on the holomorphic function $F(Y^I,\Upsilon)$, it is clear that non-holomorphic terms have to enter one way or another, because they are needed to make $d(p,q)$ duality invariant. A concrete proposal for modifying (4.41) was made in [13]. It is based on the free energy $\mathcal{F} = 2H$, i.e., on the generalized Hesse potential, instead of $\mathcal{F}_{\rm OSV}$. This allows one to treat electric and magnetic charges on equal footing and to keep symplectic covariance manifest.

[32] This makes sense microscopically, because the non-holomorphic corrections to the supergravity effective action are related to the holomorphic anomaly of the topological string [56], [57]. The role of the holomorphic anomaly for the OSV conjecture has also been investigated in [71].

[33] $\mathcal{S}_{\rm macro}$ and $\mathcal{S}_{\rm micro}$ are expected to be different, once subleading terms are taken into account, because they refer to different statistical ensembles.

The covariant version of (4.40) is

$$e^{\pi \mathcal{F}(\phi,\chi)} = e^{2\pi H(x,\hat{y})} \approx Z_{\text{can}}(\phi,\chi) = \sum_{p,q} d(p,q) e^{\pi(q\phi - p\chi)}, \tag{4.43}$$

where $\phi^I = 2x^I$ and $\chi_I = 2\hat{y}_I$ are the electrostatic and the magnetostatic potentials, respectively, and $Z_{\text{can}}(\phi,\chi)$ is the partition function of the black hole with respect to the canonical ensemble. By a formal Laplace transform we can reformulate the conjecture (4.43) as a prediction of the state degeneracy:

$$d(p,q) \approx \int dx d\hat{y} e^{\pi \Sigma(x,\hat{y},p,q)}. \tag{4.44}$$

In lack of R^2- and non-holomorphic corrections, the measure $dxdy = \prod_{I,J} dx^I dy_J$ is proportional to the top power of the symplectic form $dx^I \wedge dy_I$ on $C(M_{VM})$ and therefore is symplectically invariant. In the presence of R^2- and non-holomorphic corrections, $dx d\hat{y}$ is the appropriate generalization. Since Σ is a symplectic function, we have found a manifestly symplectically covariant version of (4.41).

As before, the variational principle ensures that in saddle point approximation we have $d(p,q) \approx \exp(\mathcal{S}_{\text{macro}})$, as $\mathcal{S}_{\text{macro}}$ is the Legendre transform of the Hesse potential and hence the saddle point value of $\pi\Sigma$. In order to compare (4.44) to (4.41), we can rewrite (4.44) in terms of the complex variables and perform the integral over $\text{Im}\, Y^I$ in saddle point approximation, i.e., we perform a Gaussian integration with respect to the subspace where the magnetic attractor equations are satisfied. The result is [13]

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p,\phi)} e^{\pi[\mathcal{F}_E - q\phi]} \tag{4.45}$$

and modifies (4.41) in two ways: first, in contrast to [12] we have included non-holomorphic terms into the free energy $\mathcal{F}_E$; second, the integral contains a measure factor $\Delta^-(p,\phi)$, whose explicit form can be found in [13]. The measure factor is needed in order to be consistent with symplectic covariance.

The proposals (4.41) and (4.44) can be tested by comparing the black hole entropy to the microscopic state degeneracy. There are some cases where these are either known exactly, or where at least subleading contributions are accessible. While this chapter is far from being closed, there seems to be agreement by now that (4.41) needs to be modified by a measure factor [73], [72], [13]. In particular, the measure factors extracted from the evaluation of exact dyonic state degeneracies in $N = 4$ compactifications [74] are consistent, at the semiclassical level, with the proposal (4.44) [13]. Detailed investigations of microscopical $N = 2$ partition functions have clarified the origin of the asymptotic holomorphic factorization of the black hole partition function, $Z_{\text{mixed}} \approx |Z_{\text{top}}|^2$: it results from simultaneous contributions of branes and anti-branes to the state degeneracy [75], [76], [77], [78]. Recently, the refined analysis of [79] has identified a microscopic measure factor, which agrees with the one found in [72], [13] in the semiclassical limit.

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Part B

Generalized geometry

Chapter 6

Generalized geometry – an introduction

Nigel Hitchin

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1 Introduction

"Generalized geometry" is an approach to differential geometric structures which seems remarkably well-adapted to some of the concepts in String Theory and Supergravity, for example: 3-form flux, gauged sigma-models, D-branes, connections with skew torsion. It also incorporates in a natural way the role of the B-field as a symmetry.

It began life in [11] as an attempt to understand the meaning of critical points of a certain invariant functional but it has since developed considerably, partially at the hands of the author's former students M. Gualtieri, G. Cavalcanti and F. Witt. Here we shall offer an introduction to this geometry and a few of its applications, showing how in particular some relatively old results in the physics literature acquire a natural meaning within this new setting.

2 The basic scenario

The fundamental idea is to take a manifold M of dimension n and replace its tangent bundle by the direct sum $T \oplus T^*$ of the tangent bundle and its dual. This has a natural inner product of signature (n, n) defined by

$$(X + \xi, X + \xi) = i_X \xi$$

(there is a difference in sign here with [11] but this is immaterial and accords with what is now the common usage). The bundle of skew adjoint transformations from $T \oplus T^*$ to itself splits as

$$\text{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

and in particular a 2-form B, a section of $\Lambda^2 T^*$, acts. If we exponentiate it to an orthogonal action on $X + \xi \in T \oplus T^*$, then it is simply

$$X + \xi \mapsto X + \xi + i_X B.$$

The general idea now is to take geometric structures on $T \oplus T^*$ which are analogues of the usual objects one studies in differential geometry. We should define them only using the fact that $T \oplus T^*$ has an inner product, and then we will be able to transport them not only by the natural action of the diffeomorphism group on $T \oplus T^*$ but also by the action of a two-form – the B-field.

As an example, let us consider how to encode an ordinary Riemannian metric g into this picture. We think of g as an isomorphism $g : T \to T^*$ defined by $X \mapsto g(X, -)$. Its graph $V \subset T \oplus T^*$ is defined as the vectors $X + g(X, -)$ and the inner product restricted to this is $i_X g(X, -) = g(X, X)$.

Now let's try and define this purely in terms of $T \oplus T^*$ geometry. We take a subbundle $V \subset T \oplus T^*$ of rank n such that the inner product restricted to V is positive definite. Since the inner product on T^* is identically zero, $V \cap T^* = 0$ which means that V is the graph of a map from T to T^*. This has a symmetric part g and a skew symmetric part B, and a vector in V looks like

$$X + gX + i_X B$$

where we write gX for $g(X, -)$. This is simply the transform of a metric, as encoded above, by the 2-form B. So in this generalization, a metric comes along with a skew-symmetric part B – a B-field.

Note that the orthogonal complement $V^\perp$ has induced inner product which is negative definite, and is the graph of $X \mapsto -gX + i_X B$.

3 The Courant bracket

Thus far we have been discussing linear algebra, now comes differentiation. There is an analogue of the Lie bracket $[X, Y]$ of two vector fields for two sections $u =$

$X + \xi, v = Y + \eta$ of $T \oplus T^*$. It is called the Courant bracket [6] and is defined by

$$[u, v] = [X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi).$$

Without the last term, this would have a simple interpretation as the bracket on the Lie algebra of the group which is the semi-direct product of $\mathrm{Diff}(M)$ with the additive group Ω^1 of 1-forms on M. But the last term is indeed there and as a consequence this is not the Lie bracket for any Lie algebra, for in general it does not satisfy the Jacobi identity. Nevertheless, it is natural within the context here. Its characteristic properties are, for u, v, w sections of $T \oplus T^*$ and f a function, as follows:

$$[u, fv] = f[u, v] + (\pi(u) f)v - (u, v)df, \tag{1}$$

$$\pi(u)(v, w) = ([u, v] + d(u, v), w) + (v, [u, w] + d(u, w)). \tag{2}$$

Here $\pi(X + \xi) = X$. A key feature of the Courant bracket is that it is preserved by the automorphism $X + \xi \mapsto X + \xi + i_X B$ if B is *closed*.

4 Affine connections

We shall now put these two pieces of data together to define a natural connection associated to a metric with B-field. When the B-field vanishes we shall get the Levi-Civita connection.

We start with $V \subset T \oplus T^*$ a subbundle on which the inner product is positive definite. Given a vector field X we can lift it to a section X^+ of V by $X^+ = X + gX + i_X B$ and to a section $X^- = X - gX + i_X B$ of $V^\perp$.

Proposition 1. *Let v be a section of V and X a vector field. Let $[X^-, v]$ be the Courant bracket and $[X^-, v]^+$ the orthogonal projection onto V. Then*

$$\nabla_X v = [X^-, v]^+$$

defines a connection on V which preserves the inner product.

Proof. From (1)

$$[fX^-, v] = f[X^-, v] - (\pi(v) f)X^- + (X^-, v)df$$

but $X^- \in V^\perp$ is orthogonal to V so the projection onto V is $f[X^-, v]^+$. Thus

$$\nabla_{fX} v = f \nabla_X v.$$

From (1) again

$$[X^-, fv] = f[X^-, v] + (Xf)v$$

and so, projecting to V, we have

$$\nabla_X (fv) = f \nabla_X v + (Xf)v$$

and hence a connection.

Let v, w be sections of V and take $u = X^-$ in (2). Then we obtain

$$X(v, w) = ([X^-, v], w) + (v, [X^-, w])$$

since X^- is orthogonal to v and w. Since $v, w \in V$ the right hand side is

$$([X^-, v]^+, w) + (v, [X^-, w]^+) = (\nabla_X v, w) + (v, \nabla_X w)$$

which means the connection preserves the metric. □

The projection π identifies V with T so we have a metric connection on T. We write $v = Y^+$ and then $[X^-, Y^+]^+ = (\nabla_X Y)^+$. But $\pi([X^-, Y^+]) = [X, Y]$, so $[X^-, Y^+] - [X, Y]^-$ is a 1-form which we write as $2gZ$ for some vector field Z. The orthogonal decomposition of this is

$$2gZ = (Z + gZ + i_Z B) - (Z - gZ + i_Z B) = Z^+ - Z^- \tag{3}$$

and hence for a vector field Z, Z^+ is the V-component of $2gZ$. Since

$$([X^-, Y^+] - [X, Y]^-)^+ = [X^-, Y^+]^+ = (\nabla_X Y)^+,$$

this means that the connection on T is given by

$$2g\nabla_X Y = [X^-, Y^+] - [X, Y]^-. \tag{4}$$

Theorem 2. *The connection* (4) *has skew torsion* $-dB$.

Remark. Interchanging the roles of V and $V^\perp$, we get a connection with torsion $-dB$.

Proof. Consider

$$2g(\nabla_X Y - \nabla_Y X - [X, Y]) = [X^-, Y^+] - [Y^-, X^+] - 2[X, Y]^- - 2g[X, Y]. \tag{5}$$

One-forms Courant-commute so

$$[X^+ - X^-, Y^+ - Y^-] = 0. \tag{6}$$

Consider now the Courant bracket of $X + i_X B, Y + i_Y B$. If the 2-form B were closed, we would get $[X, Y] + i_{[X,Y]}B$ by invariance of the bracket. In general there is an extra term and we obtain

$$[X + i_X B, Y + i_Y B] = [X, Y] + i_{[X,Y]}B - i_X i_Y dB.$$

But

$$X^+ + X^- = 2(X + i_X B)$$

and so

$$\frac{1}{4}[X^+ + X^-, Y^+ + Y^-] = \frac{1}{2}([X, Y]^+ + [X, Y]^-) - i_X i_Y dB.$$

Using (6) this gives

$$[X^+, Y^-] + [X^-, Y^+] = [X, Y]^+ + [X, Y]^- - 2i_X i_Y dB.$$

Substitute this in the right hand side of (5), using the skew-symmetry of the Courant bracket, to get

$$[X, Y]^+ - [X, Y]^- - 2g[X, Y] - 2i_X i_Y dB$$

which from (3) is $-2i_X i_Y dB$. We therefore get that the torsion is $-dB$. □

Example. If we take $B = 0$ we get the familiar formula (using the summation convention) for the Levi-Civita connection:

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k\right] - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]^+ = \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right)dx_k$$
$$= 2g_{\ell k}\Gamma^\ell_{ij}dx_k.$$

Note that the extra term in the Courant bracket which causes the Jacobi identity to fail yields the third term in this standard formula.

With non-zero B-field we get the formula

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k + B_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k + B_{jk}dx_k\right]$$
$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right)dx_k + \left(\frac{\partial B_{jk}}{\partial x_i} - \frac{\partial B_{ik}}{\partial x_j}\right)dx_k.$$

Thus if

$$H_{ijk} = \frac{\partial B_{jk}}{\partial x_i} - \frac{\partial B_{ik}}{\partial x_j}$$

this connection is related to the Levi-Civita connection ∇ by

$$\nabla_i + \frac{1}{2}H_{ijk}g^{k\ell}. \tag{7}$$

It follows that we have two affine connections: one associated with V and the other with $V^\perp$. They are

$$\nabla^+_i = \nabla_i + \frac{1}{2}H_{ijk}g^{k\ell}, \quad \nabla^- = \nabla_i - \frac{1}{2}H_{ijk}g^{k\ell}.$$

5 Gerbes

For some reason, the Courant bracket is naturally associated with the differential geometry of gerbes (see [3] and [10] for more information on gerbes and their differential geometry). We adopt the simple-minded point of view on gerbes: we have an open covering $\{U_\alpha\}$ where all open sets and intersections are contractible and define

a gerbe by smooth circle-valued functions: $g_{\alpha\beta\gamma}\colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}(1)$ such that $g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = \cdots$ and $g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$.

We define a trivialization of the gerbe by functions $h_{\alpha\beta}$ such that $g_{\alpha\beta\gamma} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}$ on threefold intersections of open sets. Given two trivializations h and $\tilde{h}$, the expression $\tilde{h}_{\alpha\beta}h^{-1}_{\alpha\beta}$ is the transition function for a unitary line bundle so we may say that "the ratio of two trivializations is a line bundle".

In [3] a connection on a gerbe is introduced in two stages. The first stage – the "connective structure" is important for the geometry here. This consists of a collection of 1-forms $A_{\alpha\beta}$ on twofold intersections such that $A_{\alpha\beta} = -A_{\beta\alpha}$ and which satisfy the relation

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma}dg_{\alpha\beta\gamma}$$

on threefold intersections. A *flat* trivialization (relative to a connective structure) is defined by the condition

$$A_{\alpha\beta} = h^{-1}_{\alpha\beta}dh_{\alpha\beta}.$$

If $\tilde{h}_{\alpha\beta}$ and $h_{\alpha\beta}$ are two flat trivializations then $\tilde{h}_{\alpha\beta}h^{-1}_{\alpha\beta}$ is a constant so the line bundle this transition function defines is a flat one.

Remark. A circle has a good covering by three open intervals, and furthermore the triple intersections are empty. It follows that any connective structure on a gerbe over the circle has a flat trivialization. Any two flat trivializations differ by a flat line bundle and the holonomy of a flat line bundle takes values in $H^1(S^1, \mathrm{U}(1))$ which is $\mathrm{U}(1)$. Thus if we say that two flat trivializations are equivalent if they differ by a flat connection with trivial holonomy, the set of equivalence classes is acted on freely and transitively by $\mathrm{U}(1)$. This way a connective structure on a gerbe defines a principal $\mathrm{U}(1)$-bundle on the loop space of M – the fibre over a single loop is just this set of equivalence classes.

Given a connective structure on a gerbe we can produce a twisted version of $T \oplus T^*$. Taking the exterior derivative of the 1-forms $A_{\alpha\beta}$ gives

$$dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = d[g^{-1}_{\alpha\beta\gamma}dg_{\alpha\beta\gamma}] = 0. \tag{8}$$

On each twofold intersection we can patch $T \oplus T^*$ over U_α with $T \oplus T^*$ over U_β with the automorphism

$$X + \xi \mapsto X + \xi + i_X dA_{\alpha\beta}.$$

The relation (8) is the consistency condition to define a vector bundle E, an extension

$$0 \to T^* \to E \xrightarrow{\pi} T \to 0.$$

A global section of E consists of a vector field X and locally defined one-forms ξ_α such that on $U_\alpha \cap U_\beta$

$$\xi_\beta = \xi_\alpha + i_X dA_{\alpha\beta}.$$

Mathematically speaking (8) is a cocycle with values in $C^\infty(\mathrm{Hom}(T, T^*))$ which will define an extension. But it is more than that since it is skew and then the patching preserves the inner product on the local $T \oplus T^*$. It is also closed and so preserves the Courant bracket. Thus E inherits all the structure of $T \oplus T^*$ that we are using – the inner product and the Courant bracket, where by the latter we mean a bracket on sections of E which satisfies the conditions (1) and (2). We also retain an action of B-fields: $u \mapsto u + i_{\pi u} B$.

Given this set-up we make a general definition:

Definition 1. A generalized metric is a subbundle $V \subset E$ of rank n on which the inner product is positive definite.

Locally we have the previous situation – V is the graph of a map $X \mapsto X + g_\alpha X + i_X B_\alpha$ but now on the twofold intersections we have

$$g_\beta X + i_X B_\beta = g_\alpha X + i_X B_\alpha + i_X dA_{\alpha\beta}.$$

But B and dA are skew, so

$$g_\beta(X, X) = g_\alpha(X, X)$$

and we have a well-defined metric g. The remaining relation is

$$B_\beta = B_\alpha + dA_{\alpha\beta}.$$

Locally defined 2-forms with this property form a *curving* of the connective structure in the language of [3], so a generalized metric gives us an ordinary metric and a curving of the gerbe. The 3-form $H = dB_\alpha = dB_\beta$ is globally defined and is the curvature of the gerbe.

With this generalized viewpoint Theorem 2 defines a connection with skew torsion $-H$: the three-form H need not be globally exact.

6 Generalized complex structures

One of the most fertile developments of the generalized point of view is the notion of a generalized complex manifold, the basic properties of which can be found in [9]. We retain our twisted setting of replacing $T \oplus T^*$ by the extension E in what follows.

Definition 2. A *generalized complex structure* on a manifold M of dimension $2m$ with bundle E is an automorphism $J : E \to E$ such that

- $J^2 = -1$
- $(Ju, v) + (u, Jv) = 0$
- if $Ju = iu$, $Jv = iv$ then $J[u, v] = i[u, v]$ using the Courant bracket.

Here we have imitated the definition of a Kähler metric, but replaced T by E, a metric by the natural inner product, and the Lie bracket by the Courant bracket. The linear algebra data consists of a reduction of structure group of the bundle E to $\mathrm{U}(m,m) \subset \mathrm{SO}(2m,2m)$. The third condition can be replaced by the vanishing of a Nijenhuis-type of tensor: $[Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v] = 0$.

The interesting feature about this notion is that it includes both symplectic and complex manifolds and indeed the possible link with mirror symmetry for Calabi–Yau threefolds was one motivating force behind the author's paper [11].

For a complex manifold with $E = T \oplus T^*$ we take

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

In this case the $+i$ eigenspace of J is $[\ldots \partial/\partial z_i \ldots, \ldots d\bar{z}_i \ldots]$.

For a symplectic manifold

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

and the $+i$ eigenspace of J is $[\ldots, \partial/\partial x_j + i \sum_{i,j} \omega_{jk} dx_k, \ldots]$.

Another, more interesting, class of examples consists of holomorphic Poisson manifolds. Let M be a complex manifold with a holomorphic bivector field

$$\sigma = \sum_{i,j} \sigma^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

which satisfies $[\sigma, \sigma] = 0$ using the Schouten bracket. Then the $+i$ eigenspace

$$\Big[\ldots, \frac{\partial}{\partial z_j}, \ldots, d\bar{z}_k + \sum_{\ell} \bar{\sigma}^{k\ell} \frac{\partial}{\partial \bar{z}_\ell}, \ldots\Big]$$

defines a generalized complex structure. A simple case is to take a complex surface with a holomorphic bivector field σ – this is a section of the dual K^* of the canonical bundle (for example take $\mathbb{C}\mathrm{P}^2$ with a section of $\mathcal{O}(3)$). This is always a holomorphic Poisson manifold, since $[\sigma, \sigma]$ is a $(3,0)$ vector and these are all zero in dimension two. Although it is a complex manifold, from the generalized complex point of view the picture is rather different. The section σ vanishes on a curve C (an elliptic curve). Outside C, σ^{-1} is a well-defined holomorphic section of K and so is a holomorphic 2-form $\varphi = \omega_1 + i\omega_2$. The generalized complex structure there consists of the symplectic structure ω_2 transformed by the B-field ω_1. On C, the generalized complex structure is just the complex structure.

This change of type of the generalized complex structure – from symplectic to complex on the same manifold – is a characteristic feature. In four dimensions it takes place over a 2-torus, if the locus is smooth. In fact, as shown in [5], by doing surgery along an embedded torus in a certain symplectic manifold, it can be shown that the four-manifold $3\mathbb{C}\mathrm{P}^2 \# 19\overline{\mathbb{C}\mathrm{P}^2}$ has a generalized complex structure which is

complex on the torus and symplectic outside. But by Seiberg–Witten theory, there is no ordinary complex or symplectic structure on it.

7 Generalized Kähler structures

One step beyond a generalized complex structure is the idea of a generalized Kähler structure, introduced by Gualtieri. The surprising feature here is that, in a different language, this was a differential geometric structure which appeared in the physics literature over 20 years ago.

The starting point is that an ordinary Kähler manifold is a manifold with both a complex structure and a symplectic structure and a compatibility condition between the two. Both of these structures can be encoded as generalized complex structures, and it turns out that compatibility means they commute. Thus our generalized geometry definition is:

Definition 3. A *generalized Kähler structure* on a manifold M with bundle E consists of two commuting generalized complex structures J_1, J_2 such that the quadratic form $(J_1 J_2 u, u)$ is positive definite.

The main theorem of Gualtieri (in our formulation using the bundle E) is:

Theorem 3. *A generalized Kähler structure on a manifold M defines*

- *a generalized metric,*
- *two integrable complex structures I^+, I^- on M such that the metric g is Hermitian with respect to both,*
- *the connections ∇^+, ∇^- of the generalized metric preserve I^+, I^- respectively.*

Conversely, up to the action of a closed B-field, this data determines a generalized Kähler structure on M.

Such structures form the target spaces for the nonlinear sigma model with $(2,2)$ supersymmetry as discussed in [7]. We prove the first part of this theorem below.

Proof. Firstly, since J_1 and J_2 commute, we have $(J_1 J_2)^2 = 1$ and so E splits into ± 1 eigenspace bundles $V, V^\perp$. Since $(J_1 J_2 u, u)$ is positive definite, if $u = u^+ + u^-$ is the eigenspace decomposition of $u \in E$, then $(u^+, u^+) - (u^-, u^-)$ is positive definite. So V is a subbundle such that the inner product is positive definite and hence defines a generalized metric.

Now J_1 and J_2 preserve the eigenspaces of $J_1 J_2$ so $J_1 = -J_2$ acts on V. It defines the almost complex structure I^+ on M by

$$J_1 X^+ = (I^+ X)^+$$

and similarly for I^-. We first need to show that I^+ is integrable, so suppose $I^+X = iX$, then $J_1X^+ = iX^+$. But from the definition of a generalized complex structure if $J_1X^+ = iX^+$ and $J_1Y^+ = iY^+$, then $J_1[X^+, Y^+] = i[X^+, Y^+]$. Since $\pi[u, v] = [\pi u, \pi v]$ relates the Courant bracket to the Lie bracket, we have $I_+[X, Y] = i[X, Y]$ and the complex structure is integrable by Newlander–Nirenberg.

Next we need to determine the compatibility of I^+ with the connection with skew torsion coming from the generalized metric. We require $\nabla^+ I^+ = 0$, or equivalently $g(\nabla^+_X I^+Y, Z) = -g(\nabla^+_X Y, I^+Z)$. But from the definition of ∇^+,

$$2g(\nabla^+_X Y, Z) = ([X^-, Y^+] - [X, Y]^-, Z^+) = ([X^-, Y^+], Z^+),$$

so we need to show that

$$([X^-, J_1Y^+], Z^+) = -([X^-, Y^+], J_1Z^+).$$

We use the Nijenhuis condition $[Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v] = 0$ for integrability of a generalized complex structure. For J_1 this gives

$$[J_1X^-, J_1Y^+] - J_1[J_1X^-, Y^+] - J_1[X^-, J_1Y^+] - [X^-, Y^+] = 0$$

and for J_2 (recalling that $J_1 = -J_2$ on V and $J_1 = J_2$ on $V^\perp$) we have

$$-[J_1X^-, J_1Y^+] - J_2[J_1X^-, Y^+] + J_2[X^-, J_1Y^+] - [X^-, Y^+] = 0.$$

Adding these two gives

$$-(J_1 + J_2)[X^-, J_1Y^+] - (J_1 - J_2)[X^-, J_1Y^+] - 2[X^-, Y^+] = 0.$$

Now take the inner product with J_1Z^+:

$$\begin{aligned}&((J_1 + J_2)[X^-, J_1Y^+], J_1Z^+)\\ &\quad + ((J_1 - J_2)[X^-, J_1Y^+], J_1Z^+) + 2([X^-, Y^+], J_1Z^+) = 0.\end{aligned}$$

But $J_1 = -J_2$ on V, so $([X^-, J_1Y^+], Z^+) + ([X^-, Y^+], J_1Z^+) = 0$ which is the required identity. □

8 Spinors

One of the key features of generalized geometry is the way in which differential forms are regarded as spinors for $T \oplus T^*$. We define Clifford multiplication by a vector $u = X + \xi \in T \oplus T^*$ on a form α by

$$u \cdot \alpha = i_X\alpha + \xi \wedge \alpha$$

and then $u^2 = (u, u)1$.

There is an invariant bilinear form with values in $\Lambda^n T^*$ – the so-called Mukai pairing – defined by

$$\langle \alpha, \beta \rangle = [\alpha \wedge \sigma(\beta)]_n$$

where $\sigma(\beta) = 1$ if the degree of β is 0 or 3 mod 4 and -1 otherwise. (In cohomology this is a familiar operator – the Chern character of a bundle V is related to that of its dual by $\mathrm{ch}(V^*) = \sigma(\mathrm{ch}(V))$.)

The B-field action of a two-form B in the spin representation is the exponentiation of the exterior product: $\alpha \mapsto e^B\alpha$.

The exterior derivative still has a natural role when we regard forms as spinors. In fact it is in some respects dual to the Courant bracket. Recall the expression below which represents the usual formula relating the Lie bracket of vector fields to the exterior derivative:

$$2i_{[X,Y]}\alpha = d([i_X, i_Y]\alpha) + 2i_X d(i_Y\alpha) - 2i_Y d(i_X\alpha) + [i_X, i_Y]d\alpha.$$

Here the algebraic action of vector fields on forms is through the interior product i_X. Replacing this by the Clifford action of sections of $T \oplus T^*$ gives

$$2[u, v]\cdot\alpha = d((u\cdot v - v\cdot u)\cdot\alpha) + 2u\cdot d(v\cdot\alpha) - 2v\cdot d(u\cdot\alpha) + (u\cdot v - v\cdot u)\cdot d\alpha \quad (9)$$

where $[u, v]$ is now the Courant bracket.

What happens when we twist $T \oplus T^*$ to get the bundle E? The bundle E was constructed by patching together $T \oplus T^*$ with the cocycle $dA_{\alpha\beta}$. Now

$$e^{-B}(X + \xi)\cdot e^B\alpha = (X + \xi + i_X B)\alpha.$$

We construct the spinor bundle $S(E)$ by taking the exterior forms Λ^*T^* and identifying over twofold intersections via

$$\varphi \mapsto e^{dA_{\alpha\beta}}\varphi.$$

Clearly this does not preserve degrees, only the parity. But equally, since $dA_{\alpha\beta}$ is closed, it preserves the exterior derivative. We thus have a well-defined differential operator

$$d : C^\infty(S(E)) \to C^\infty(S(E)).$$

The cohomology of this, with the $\mathbb{Z}_2$-grading given by odd and even forms, is the *twisted cohomology*.

To relate this to the more normal description, we can choose a curving of the gerbe so that we have local one-forms B_α with $B_\beta - B_\alpha = dA_{\alpha\beta}$. Then a section of S given by local forms φ_α satisfying

$$\varphi_\beta = e^{dA_{\alpha\beta}}\varphi_\alpha = e^{B_\beta - B_\alpha}\varphi_\alpha$$

defines a global form

$$\psi = e^{-B_\alpha}\varphi_\alpha = e^{-B_\beta}\varphi_\beta.$$

It is no longer closed but instead

$$d\psi + H \wedge \psi = 0$$

where $dB_\alpha = H$ is the curvature.

We thus have a choice: use only a connective structure and think of spinors with the operator d; or choose a curving and think of forms but with operator $d + H$.

9 Structures defined by forms

Generalized geometric structures of interest are often defined by reducing the structure group of E from $\mathrm{SO}(n,n)$ to the stabilizer of a spinor. When we interpret spinors as forms then it sometimes happens that when the form is closed we get an integrability condition for the structure. The first example is that of a generalized Calabi–Yau manifold [11].

If φ is a spinor for $T \oplus T^*$, then its annihilator A is the vector space of $u \in (T \oplus T^*) \otimes \mathbb{C}$ such that $u \cdot \varphi = 0$. The restriction of the inner product to A is always zero, since

$$0 = u \cdot u \cdot \varphi = (u,u)\varphi.$$

A pure spinor is one for which the annihilator has maximal dimension n.

Definition 4. A generalized Calabi–Yau structure is a complex closed form ρ which is a pure spinor for $T \oplus T^*$ and satisfies $\langle \rho, \bar{\rho} \rangle \neq 0$.

The annihilator of ρ defines a rank n subbundle A of $(T \oplus T^*) \otimes \mathbb{C}$: the condition $\langle \rho, \bar{\rho} \rangle \neq 0$ means that $A \cap \bar{A} = 0$. We define a generalized complex structure by making A the $+i$ eigenspace of J. It satisfies the Courant integrability condition because ρ is closed, using equation (9). Clearly we can replace $T \oplus T^*$ by its twisted version E.

This is perhaps the easiest way to find generalized complex manifolds. A symplectic manifold (M,ω) for example is given by $\rho = e^{i\omega}$ and a B-field transform of it by

$$\rho = e^{B+i\omega}.$$

In this situation (which is generic when both forms are even) we can see how to solve the equations for a generalized Kähler structure. First (following [13]) consider how to achieve two commuting generalized complex structures:

Lemma 4. *Let $\rho_1 = e^{\beta_1}, \rho_2 = e^{\beta_2}$ be closed forms defining generalized complex structures J_1, J_2 on a manifold of dimension $4k$. Suppose that*

$$(\beta_1 - \beta_2)^{k+1} = 0 = (\beta_1 - \bar{\beta}_2)^{k+1}$$

and $(\beta_1 - \beta_2)^k$ and $(\beta_1 - \bar{\beta}_2)^k$ are non-vanishing. Then J_1 and J_2 commute.

Proof. Suppose that $(\beta_1 - \beta_2)^{k+1} = 0$ and $(\beta_1 - \beta_2)^k$ is non-zero. Then the 2-form $\beta_1 - \beta_2$ has rank $2k$, i.e. the dimension of the space of vectors X satisfying

$i_X(\beta_1 - \beta_2) = 0$ is $2k$. Since $i_X 1 + \xi \wedge 1 = 0$ if and only if $\xi = 0$, this means that the space of solutions $u = X + \xi$ to

$$u \cdot \exp(\beta_1 - \beta_2) = 0 = u \cdot 1$$

is $2k$-dimensional. Applying the invertible map $\exp \beta_2$, the same is true of solutions to

$$u \cdot \exp \beta_1 = 0 = u \cdot \exp \beta_2.$$

This is the intersection $A_1 \cap A_2$ of the two $+i$ eigenspaces. Repeating for $\beta_1 - \bar{\beta}_2$ we get $A_1 \cap \bar{A}_2$ to be $2k$-dimensional. These two bundles are common eigenspaces of (J_1, J_2) corresponding to the eigenvalues (i, i) and $(i, -i)$ respectively. Together with their conjugates they decompose $(T \oplus T^*) \otimes \mathbb{C}$ into a direct sum of common eigenspaces of J_1, J_2, thus $J_1 J_2 = J_2 J_1$ on every element. □

Now let us reformulate what we have without going through Theorem 3: the $2k$-dimensional space of vectors satisfying $i_X(\beta_1 - \beta_2) = 0$ form the $(0, 1)$ vectors of an almost complex structure I^+. Since $\beta_1 - \beta_2$ is closed, I^+ is integrable and moreover $\omega^+ = \beta_1 - \beta_2$ is a holomorphic symplectic form. Similarly $\omega^- = \beta_1 - \bar{\beta}_2$ is a holomorphic symplectic form for a complex structure I^-. Locally any two complex structures are equivalent by a diffeomorphism, and together with the holomorphic version of the Darboux theorem we have a local diffeomorphism F such that $F^*\omega^+ = \omega^-$. But

$$\omega^+ + \bar{\omega}^+ = \beta_1 + \bar{\beta}_1 - \beta_2 - \bar{\beta}_2 = \omega^- + \bar{\omega}^-.$$

So if $\omega^+ = \omega + i\omega'$, then $\omega^- = \omega + i\omega''$ and hence

$$F^*\omega = \omega, \quad F^*\omega' = \omega''.$$

Apart from the positivity condition, a generalized Kähler structure is thus determined by a local symplectic diffeomorphism F. We can define one by taking an arbitrary smooth function f and integrating its Hamiltonian vector field X_f – roughly the same freedom as for a Kähler structure.

So now let us address the definiteness of $(J_1 J_2 u, u)$ in Definition 3. The -1 eigenspace of $J_1 J_2$ is

$$V^\perp = A_1 \cap A_2 \oplus \bar{A}_1 \cap \bar{A}_2.$$

If X is a $(0, 1)$-vector it lies in the $2k$-dimensional space defined by $i_X(\beta_1 - \beta_2) = 0$ and then $u = X - i_X \beta_2$ satisfies $u \cdot \exp \beta_1 = 0 = u \cdot \exp \beta_2$, i.e. $u \in A_1 \cap A_2$. But then

$$(u + \bar{u}, u + \bar{u}) = -i_X \beta_2(\bar{X}) - i_{\bar{X}} \bar{\beta}_2(X) = -(\beta_2 - \bar{\beta}_2)(X, \bar{X}) \tag{10}$$

so we need $(\beta_2 - \bar{\beta}_2)(X, \bar{X})$ to be positive definite, which is $i(F^*\omega' - \omega')(X, \bar{X}) \geq 0$. In fact ω' is of type $(2, 0) + (0, 2)$ with respect to I^+ so $\omega'(X, \bar{X}) = 0$ and the condition is $iF^*\omega'(X, \bar{X}) \geq 0$. Now if F_t is the one-parameter group of symplectic

diffeomorphisms obtained from the Hamiltonian f, we have

$$\frac{\partial}{\partial t}F_t^*\omega'|_{t=0} = \mathcal{L}_X\omega' = d(i_{X_f}\omega').$$

But $(\omega + i\omega')(X, Y) = i(\omega + i\omega')(X, I^+Y)$ and $i_{X_f}\omega = df$ so that $i_{X_f}\omega' = I^+df$ and hence

$$\frac{\partial}{\partial t}F_t^*\omega'|_{t=0} = dI^+df.$$

Thus if dI^+df (this is $i\partial\bar{\partial}f$ for the complex structure I^+) defines a local Kähler form, for t small enough $iF_t^*\omega'(X, \bar{X})$ will be positive definite.

Generalized Kähler structures of this generic type thus seem to be very like Kähler metrics, but more nonlinear – given two Kähler potentials f, $\tilde{f}$ the *difference* $\tilde{f} - f$ is the real part of a holomorphic function, but two symplectic diffeomorphisms F, $\tilde{F}$ must have the *composition* $F^{-1}\tilde{F}$ holomorphic to describe the same generalized Kähler structure.

Note that the construction above has its origins in one of D. Joyce (see [2]) and also lies behind the potentials in [16].

10 Double structures

A generalized metric decomposes E into $V \oplus V^\perp$. This (with an orientation) is a reduction of the structure group of E from $\mathrm{SO}(n, n)$ to $\mathrm{SO}(n) \times \mathrm{SO}(n)$. A generalized Kähler structure gave us independent complex structures I^+, I^- on M which were more naturally described by J_1 or J_2 on V and $V^\perp$. So the picture is: a complex structure on V and another on $V^\perp$, or equivalently a reduction to $\mathrm{U}(m)\times\mathrm{U}(m)$. There is a further property in Gualtieri's theorem, that the holonomy of the natural connections on V and $V^\perp$ reduces to $\mathrm{U}(m)$. This is an example of what we shall call a double structure: a reduction to $G \times G \subset \mathrm{SO}(n) \times \mathrm{SO}(n)$ with corresponding holonomy reduction – the case when G preserves a spinor is important for supersymmetric considerations, as in [8]. We shall consider here how generalized geometry is a suitable setting for this (see [17] for a detailed account.)

Let w denote the orthogonal automorphism of the bundle E given by reflection in V. It lifts to ω in the Pin group which acts as an automorphism of the spinor bundle $S(E)$. We shall consider the equations

$$d\rho = 0 = d(\omega \cdot \rho)$$

for a section ρ of $S(E)$ satisfying certain algebraic conditions.

Since any double structure involves a generalized metric, having decomposed $E = V \oplus V^\perp$, it is useful to use the curving of the gerbe that it defines. Recall that this is given by local 1-forms B_α such that

$$B_\beta = B_\alpha + dA_{\alpha\beta}.$$

We interpret this as a splitting of the sequence

$$0 \to T^* \to E \xrightarrow{\pi} T \to 0$$

– the cocycle $dA_{\alpha\beta}$ is written as a specific coboundary $B_\beta - B_\alpha$.

Since there are various splittings involved, let us consider these more carefully. A splitting is by definition a homomorphism $S\colon T \to E$ such that $\pi S(X) = X$. If S, S' are two splittings then with this definition so is $(S + S')/2$, the average. More geometrically, the image of S is a subbundle isomorphic to T and since $S(T) \cap T^* = 0$ this is equivalent to an isomorphism $E \cong T \oplus T^*$.

The subbundle V of a generalized metric defines one splitting ($V = S(T)$), and its orthogonal complement $V^\perp$ defines another splitting S'. In the first case $S(T)$ is positive definite, in the second $S'(T)$ is negative definite. For the averaged splitting, $S'' = (S + S')/2$, $S''(T)$ is isotropic: this is the splitting given by the B_α.

Since the B_α are skew the isomorphism $E \cong T \oplus T^*$ defined by this splitting takes the inner product to the standard one, but the B_α are not in general closed so the Courant bracket transforms to a twisted version

$$[X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) - i_X i_Y H$$

where $H = dB_\alpha$. The spinor bundle now becomes identified with the standard bundle of exterior forms but the d-operator is replaced by $\alpha \mapsto d\alpha + H \wedge \alpha$, and $\omega = \pm\sigma *$ where $*$ is the Hodge star operator for the metric.

Let ∇ be the Levi-Civita connection and $X_1, \ldots, X_n$ a local orthonormal basis of vector fields. Then because ∇ is torsion-free and $2gX = X^+ - X^-$, we can write the exterior derivative of an arbitrary form α as

$$d\alpha = \sum_i gX_i \wedge \nabla_i \alpha = \frac{1}{2} \sum_i (X_i^+ - X_i^-) \cdot \nabla_i \alpha \tag{11}$$

and

$$H \wedge \alpha = \frac{1}{48} \sum_{i,j,k} H_{ijk} (X_i^+ - X_i^-) \cdot (X_j^+ - X_j^-) \cdot (X_k^+ - X_k^-) \cdot \alpha. \tag{12}$$

Now, in terms of the orthonormal basis of V, the element $\omega = X_1^+ \cdot X_2^+ \cdot \cdots \cdot X_n^+$. Since the X_i^+ and X_j^- are orthogonal they anticommute except with themselves, so

$$\omega \cdot X_i^+ = (-1)^{n-1} X_i^+ \cdot \omega, \quad \omega \cdot X_i^- = (-1)^n X_i^- \cdot \omega.$$

Thus, since ω is covariant constant with respect to ∇,

$$d(\omega \cdot \alpha) = (-1)^n \omega \cdot \frac{1}{2} \sum_i (X_i^+ + X_i^-) \cdot \nabla_i \alpha \tag{13}$$

and

$$H \wedge (\omega \cdot \alpha) = (-1)^n \omega \cdot \frac{1}{48} \sum_{i,j,k} H_{ijk} (X_i^+ + X_i^-) \cdot (X_j^+ + X_j^-) \cdot (X_k^+ + X_k^-) \cdot \alpha. \tag{14}$$

hence, using (11) and (13), and (12) and (14), the two equations

$$(d + H)\alpha = 0 = (d + H)(\omega \cdot \alpha)$$

are equivalent to

$$\sum_i X_i^+ \cdot \nabla_i \alpha + \frac{1}{24} \sum_{i,j,k} H_{ijk}(X_i^+ \cdot X_j^+ \cdot X_k^+ + 3X_i^+ \cdot X_j^- \cdot X_k^-) \cdot \alpha = 0,$$

$$\sum_i X_i^- \cdot \nabla_i \alpha + \frac{1}{24} \sum_{i,j,k} H_{ijk}(X_i^- \cdot X_j^- \cdot X_k^- + 3X_i^- \cdot X_j^+ \cdot X_k^+) \cdot \alpha = 0.$$

Now suppose that $G \subset \mathrm{SO}(n)$ is the stabilizer of a unit length spinor φ (here, as in [17] we shall consider only a real spinor representation for $\mathrm{Spin}(n)$ so that $G = G_2$ or $\mathrm{Spin}(7)$). Then $G \times G$ is the stabilizer of φ^+, a spinor for V and φ^-, a spinor for $V^\perp$. But our decomposition of $T \oplus T^* = V \oplus V^\perp$ gives us a tensor product decomposition of spinors for the two orthogonal spaces, so that $\varphi^+ \otimes \varphi^-$ can be considered as a spinor for $\mathrm{Spin}(n, n)$, or equivalently an exterior form. It, or any non-zero multiple, is stabilized by $G \times G$.

From this viewpoint a vector $X^+ \in V$ acts by the Euclidean Clifford action of X on the left hand factor and X^-, coming from $V^\perp$ action, on the right – though we must remember that the inner product on $V^\perp$ is negative definite and V positive. This difference appears in the spin representation: if $a \in \mathfrak{so}(n)$ is given relative to an orthonormal basis X_i by the skew-symmetric matrix a_{ij}, then if $X_i \cdot X_i = -1$, the action on spinors is

$$-\frac{1}{4} \sum_{i,j} a_{ij} X_i \cdot X_j$$

and if $X_i \cdot X_i = +1$, the opposite.

Suppose then that $\rho = e^f \varphi^+ \otimes \varphi^-$ satisfies the equations

$$(d + H)\rho = 0 = (d + H)(\omega \cdot \rho),$$

then setting $\alpha = \rho$ in the first equation above gives (using the summation convention)

$$\begin{aligned} X_i \cdot \nabla_i \varphi^+ \otimes \varphi^- + X_i \cdot \varphi^+ \otimes \nabla_i \varphi^- + \frac{1}{3} X_i \cdot H_i^+ \varphi^+ \otimes \varphi^- + X_i \cdot \varphi^+ \otimes H_i^- \varphi^- \\ = -df \cdot \varphi^+ \otimes \varphi^-. \end{aligned} \tag{15}$$

Here

$$H_i^+ = \frac{1}{8} \sum_{j,k} H_{ijk} X_j \cdot X_k = \frac{1}{2} \delta(H_i)$$

where δ is the spin representation on the Lie algebra (recall that $X_i^+ \cdot X_i^+ = 1$). Similarly $H_i^- = -\delta(H_i)/2$.

Considering the right hand spinor factor in (15) we get

$$\nabla_i \varphi^- - \frac{1}{2}\delta(H_i)\varphi^- = \lambda_i \varphi^-,$$

but φ^- has unit length so $\lambda_i = 0$. We deduce (see (7)) that

$$\nabla_i^- \varphi^- = \left(\nabla_i - \frac{1}{2}\delta(H_i)\right)\varphi^- = 0$$

so that the holonomy of the connection ∇^- reduces to the stabilizer G of φ^-. Applying the same argument to the second equation (and taking account of the change of sign of the metric) we get

$$\nabla_i^+ \varphi^+ = \left(\nabla_i + \frac{1}{2}\delta(H_i)\right)\varphi^+ = 0.$$

Now put this into the equation (15) and consider the left hand spinor factor: we get the equation

$$\left(df + \frac{1}{12}\sum_{ijk} H_{ijk} X_i \cdot X_j \cdot X_k\right) \cdot \varphi^+ = 0.$$

Together with a similar one for φ^- we obtain the Type II supersymmetric geometries described in [8]. This last relation says that the spinor $e^f \varphi^+$ is annihilated by the so-called *cubic Dirac operator* [1].

Remark. In 7 and 8 dimensions, the equations $(d + H)\rho = 0 = (d + H)(\omega \cdot \rho)$ arise in a very natural way. First, in 7 dimensions there is an open set of (even or odd) forms for which the stabilizer in $\mathrm{Spin}(7, 7)$ is $G_2 \times G_2$. This means that the form itself defines, in a complicated nonlinear fashion, the generalized metric. The equations then appear as variational ones for a natural invariant functional, as in the 6-dimensional case of [11]. This was the starting point for Witt's work. In 8 dimensions there is a form ρ whose stabilizer is $\mathrm{Spin}(7) \times \mathrm{Spin}(7)$ and this satisfies $\rho = \pm\omega \cdot \rho$. Thus we only need it to be closed to satisfy the equations – it is however algebraically special. In fact here there are two cases – when the form is odd or even (see [8] and [17]).

11 Group actions

Suppose we have a smooth action of a Lie group G (with Lie algebra $\mathfrak{g}$) on M. There is a natural Lie algebra homomorphism from $\mathfrak{g}$ to the space of vector fields:

$$\mathfrak{g} \to C^\infty(T).$$

In generalized geometry we replace T, with its Lie bracket, by E, with the Courant bracket, so we can ask what the natural generalization should be.

The most obvious request is a lift of this homomorphism to sections of E

$$e\colon \mathfrak{g} \to C^\infty(E)$$

such that

$$[e(a), e(b)] = e([a, b])$$

using the Courant bracket on sections of E.

Some things need to be adapted, however. First recall that the Courant bracket does not satisfy the Jacobi identity. In fact

$$3[[u, v], w] + [[w, u], v] + [[v, w], u] = d(([u, v], w) + ([w, u], v) + ([v, w], u)).$$

So if $u, v, w = e(a), e(b), e(c)$ the right hand side must vanish but then $(([u, v], w) + ([w, u], v) + ([v, w], u))$ is a constant element of $\Lambda^3\mathfrak{g}^*$ which we need to incorporate into the picture. Because this is expressed as inner products, it will vanish if we insist that for each $a \in \mathfrak{g}$, $(e(a), e(a)) = 0$.

Secondly, note that in the normal case of a *free* group action, the image of $\mathfrak{g}$ in T spans a subbundle which is at each point the tangent space to the G-orbit through that point. In particular it is an integrable distribution: its sections are closed under the Lie bracket. In the generalized case, let F be the subbundle of E spanned by $e(\mathfrak{g})$. Since e is a homomorphism, F has a basis of sections which are closed under Courant bracket. But from (1), it is not always true that this will hold for a linear combination with C^∞ coefficients. However if $(e(a), e(a)) = 0$ (i.e. F is isotropic), (1) shows that all sections of F are closed under the Courant bracket.

It is natural therefore to demand the isotropy condition for e. In more generality one ought to extend the notion of Lie algebra to incorporate an analogue of the Courant bracket and the authors of [4] go some way towards doing that, but we shall stick here to the simplest, most geometrical, situation.

Explicitly, $e(a)$ is a section of E which maps under $\pi\colon E \to T$ to X (which we sometimes write as X_a), the vector field generated by a: locally $e(a) = X + \xi_\alpha$ where

$$\xi_\beta - \xi_\alpha = i_X dA_{\alpha\beta}. \tag{16}$$

We also sometimes write $e(a) = e_X$, to think of a specific lift of X to E.

We now want to extend the natural Lie derivative action of X on $T \oplus T^*$ to a Lie algebra action of $\mathfrak{g}$ on E. Now

$$\begin{aligned}\mathcal{L}_X(Y + \eta_\alpha) &= [X, Y] + \mathcal{L}_X(\eta_\beta - i_Y dA_{\alpha\beta})\\ &= [X, Y] + \mathcal{L}_X\eta_\beta - i_{[X,Y]}dA_{\alpha\beta} - i_Y d(\mathcal{L}_X A_{\alpha\beta}),\end{aligned}$$

so the usual action does not make global sense because of the last term. However, from (16)

$$d\xi_\beta = d\xi_\alpha + d(i_X dA_{\alpha\beta}) = d\xi_\alpha + d(\mathcal{L}_X A_{\alpha\beta}),$$

so we can define a new Lie derivative L_X by

$$L_X(Y + \eta_\alpha) = \mathcal{L}_X(Y + \eta_\alpha) - i_Y d\xi_\alpha$$

which *is* globally defined. This is sometimes called the Dorfman bracket of $X + \xi_\alpha$ and $Y + \eta_\alpha$: a useful formula is

$$L_X u = [e_X, u] + d(e_X, u). \tag{17}$$

Is L_X really an action? We need to show that

$$[L_X, L_Y](Z + \zeta_\alpha) = L_{[X,Y]}(Z + \zeta_\alpha)$$

for any $Z + \zeta_\alpha$.

The vector field part is clear. The one-form part of $[L_X, L_Y](Z + \zeta_\alpha)$ is

$$\begin{aligned} &\mathcal{L}_X(\mathcal{L}_Y\zeta_\alpha - i_Z d\eta_\alpha) - i_{[Y,Z]}d\xi_\alpha - \mathcal{L}_Y(\mathcal{L}_X\zeta_\alpha - i_Z d\xi_\alpha) + i_{[X,Z]}d\eta_\alpha \\ &\quad = \mathcal{L}_{[X,Y]}\zeta_\alpha - i_Z d(\mathcal{L}_X\eta_\alpha - \mathcal{L}_Y\xi_\alpha). \end{aligned} \tag{18}$$

But $e_{[X,Y]} = [e_X, e_Y]$ and

$$[e_X, e_Y] = [X, Y] + \mathcal{L}_X\eta_\alpha - \mathcal{L}_Y\xi_\alpha - d(i_X\eta_\alpha - i_Y\xi_\alpha)/2 = [X, Y] + \omega_\alpha,$$

and so

$$L_{[X,Y]}(Z+\zeta_\alpha) = \mathcal{L}_{[X,Y]}(Z+\zeta_\alpha) - i_Z d\omega_\alpha = \mathcal{L}_{[X,Y]}(Z+\zeta_\alpha) - i_Z d(\mathcal{L}_X\eta_\alpha - \mathcal{L}_Y\xi_\alpha),$$

so this checks with (18).

Note that the inner product is preserved by the Lie derivative:

$$X(u, v) = (L_X u, v) + (u, L_X v).$$

This follows directly from formula (17) and (2).

The extended Lie derivative defines a Lie algebra action on E, and given suitable global conditions on M and G, we can integrate this to a normal G-action on the vector bundle E.

Remark. We can also define a Lie derivative on spinors by imitating the Cartan formula for forms: $\mathcal{L}_X\alpha = d\,i_X\alpha + i_X d\alpha$. We define

$$L_X\alpha = d(e_X \cdot \alpha) + e_X \cdot d\alpha.$$

This clearly commutes with d and one may check, using the definition of $L_X u$ above, that

$$L_X(u \cdot \alpha) = L_X u \cdot \alpha + u \cdot L_X\alpha.$$

Having seen how to define an action on E, we can now tackle the question of invariant generalized geometrical structures. Note that the inner product is invariant and also the Courant bracket (since d is invariant and the bracket is defined by this and the inner product), so any structure defined by these is transformed by the G-action to another.

First consider a *curving* of a gerbe. As we saw earlier, this is defined by an isotropic splitting of E – a subbundle $S(T)$ such that $\pi\colon S(T) \to T$ is an isomorphism. Invariance of a curving under the action of G is just the invariance of $S(T)$. A local

section is of the form $Y + i_Y B_\alpha$, so

$$\begin{aligned} L_X(Y + i_Y B_\alpha) &= [X, Y] + \mathcal{L}_X i_Y B_\alpha - i_Y d\xi_\alpha \\ &= [X, Y] + i_{[X,Y]} B_\alpha + i_Y(\mathcal{L}_X B_\alpha - d\xi_\alpha) \end{aligned}$$

and for invariance we require

$$\mathcal{L}_X B_\alpha = d\xi_\alpha. \tag{19}$$

Note that

$$\xi_\beta - \xi_\alpha = i_X dA_{\alpha\beta}, \quad B_\beta - B_\alpha = dA_{\alpha\beta}$$

so that $\xi_\alpha - i_X B_\alpha$ is a globally defined 1-form c. Moreover

$$dc = d\xi_\alpha - d i_X B_\alpha = i_X dB_\alpha = i_X H \tag{20}$$

from (19). There is one such form c for each generator of $\mathfrak{g}$ so we have a 1-form κ with values in the dual of the Lie algebra:

$$\kappa \in \Omega^1(M, \mathfrak{g}^*)$$

analogous to the moment map in symplectic geometry.

For a generalized metric, invariance is just the invariance of $V \subset E$. The generalized metric was given by two quantities – a metric and a curving. For the metric we have just the usual invariance $\mathcal{L}_X g = 0$ and we considered the curving above.

12 Quotients

One of the simplest constructions in Riemannian geometry is the quotient metric – a free isometric action of G on a manifold defines a natural metric on the quotient space. For example, the quotient of the standard metric on the unit sphere in $\mathbb{C}^{n+1}$ by the S^1 action $z \mapsto e^{i\theta} z$ gives the Fubini–Study metric. We now ask how to adapt this process for a generalized metric. We shall follow the formalism developed here but in fact it gives what the physicists term the "gauged WZW model" and has been known for some time.

Recall first the standard quotient. We have a free action of G with quotient map $p\colon M \to M/G$. The image of $\mathfrak{g}$ spans a subbundle $T_F \subset TM$ (the tangent bundle along the fibres) and the tangent space of the quotient is identified with TM/T_F by p. Given a G-invariant metric g on M, the orthogonal complement $T_F^\perp$ maps isomorphically to $T(M/G)$ by p. Now if $\bar{X}$ is a tangent vector at a point $\bar{x}$ on M/G we lift it to a tangent vector $X \in T_F^\perp$ to M at some point x in the orbit over $\bar{x}$ and define $\bar{g}_{\bar{x}}(\bar{X}, \bar{X}) = g_x(X, X)$. By G-invariance the definition is independent of choices.

Now consider the generalized version. We let F denote the subbundle of E spanned by $e(\mathfrak{g})$, so $\pi : E \to T$ projects F to T_F. Since F is isotropic by assumption, $F \subset F^\perp$ and so $F^\perp/F$ is a bundle of rank $(2n - \dim G - \dim G) = 2(n - \dim G)$, and with an induced nondegenerate inner product. Since E projects to TM and πF is the

tangent space of the G-orbits, we have a surjective map from $F^\perp/F$ to $p^*T(M/G)$. Its kernel consists of the 1-forms which annihilate πF, and this is $p^*T^*(M/G)$. In particular this is an isotropic subspace of half the dimension, so the signature of the quadratic form is $(n-\dim G, n-\dim G)$. Everything is G-invariant and so descends to a bundle $\bar{E}$ on M/G.

More importantly, there is an induced Courant bracket. To see this, note that the sections of $\bar{E}$ on M/G pull back precisely to the G-invariant sections of $F^\perp/F$. We saw above that since F is isotropic, all its sections are closed under Courant bracket.

Now let u, v be invariant sections of $F^\perp$, so that $L_X u = 0 = L_X v$ and for all $e_X = e(a)$, $(e_X, u) = 0$. From (17) we have

$$L_X u = [e_X, u] + d(e_X, u) = 0$$

so that $[e_X, u] = 0$. Now use (2):

$$\pi(u)(v, w) = ([u, v] + d(u, v), w) + (v, [u, w] + d(u, w))$$

with $w = e_X$. Since $(u, e_X) = 0 = (v, e_X)$ and $[e_X, u] = 0$ we get

$$0 = ([u, v], e_X) + (d(u, v), e_X) = ([u, v], e_X) + X(u, v)/2$$

But since $L_X u = L_X v = 0$, $X(u, v) = (L_X u, v) + (u, L_X v) = 0$ so $([u, v], e_X) = 0$ and $[u, v]$ is also a section of $F^\perp$. Hence the sections of $\bar{E}$ have a Courant bracket.

The twisting bundle E descends to a bundle $\bar{E}$ with the same structure. The issue about whether a gerbe with connective structure descends is more complicated – here we content ourselves with just the bundle $\bar{E}$ – an *exact Courant algebroid* in other language.

Now for the generalized metric: if $V \subset E$ is a positive-definite invariant subbundle, then take the image of $V \cap F^\perp$ to define $\bar{V} \subset F^\perp/F$. Since $V^\perp$ is negative definite and F is null, $V^\perp + F$ has dimension $n + \dim G$. Thus its orthogonal subspace $V \cap F^\perp$ has dimension $2n - n - \dim G = n - \dim G$. It has a positive definite inner product and injects into $F^\perp/F$, and intersects the kernel of the projection to T^*M/G in zero, again by the definiteness of the metric. It thus defines a generalized metric on the quotient.

This procedure has here been obtained from the viewpoint of generalized geometry, but in fact in a different language it appeared in the physics literature [14], [15] many years ago. What we have done here may be considered quite abstract, so let us see what happens to the two concrete fields defined by a generalized metric – the actual metric g and the three-form H.

The quotient metric is induced from $V \cap F^\perp$, which is the subspace of E defined by

$$(Y + gY + i_Y B_\alpha, X + \xi_\alpha) = 0$$

for all $e(a) = X + \xi_\alpha$. But

$$2(Y + gY + i_Y B_\alpha, X + \xi_\alpha) = g(Y, X) + i_X i_Y B_\alpha + i_Y \xi_\alpha$$

and we defined above for each $a \in \mathfrak{g}$ a global one-form c_a where $c_a = \xi_\alpha - i_X B_\alpha$ if $e(a) = e_X$. So

$$V \cap F^\perp = \{Y^+ \in E : g(Y, X_a) + c_a(Y) = 0\} \tag{21}$$

for all $a \in \mathfrak{g}$. This is a modified form of the horizontality condition $g(Y, X_a) = 0$ which gives the usual quotient metric. Let $\mathcal{H} \subset TM$ be the G-invariant subbundle defined by this condition, then the procedure is the same: we identify tangent vectors on M/G with vectors in $\mathcal{H}$ and take the induced inner product.

To find the three-form on the quotient, we consider the pull-back of $\bar{E} = F^\perp/F$ as

$$p^*\bar{E} = V \cap F^\perp \oplus V^\perp \cap F^\perp,$$

this decomposition defining the generalized metric on the quotient. So a vector in $V \cap F^\perp$ is locally given by

$$Y + gY + i_Y B_\alpha$$

where Y satisfies $g(Y, X_a) + c_a(Y) = 0$, and a vector in $V^\perp \cap F^\perp$ is

$$Y' - gY' + i_{Y'} B_\alpha$$

where Y' satisfies the similar linear constraint $g(Y', X_a) - c_a(Y') = 0$.

If $p(Y') = p(Y)$, then

$$Y' = Y + \sum_i b_i(Y) X_i \tag{22}$$

where $X_1, \ldots, X_m$ are the vector fields generated by a basis a_i of $\mathfrak{g}$. From

$$g(Y, X_i) + c_i(Y) = 0 = g(Y', X_i) - c_i(Y')$$

we get

$$2c_i(Y) = \sum_i b_j(Y)(g(X_i, X_j) - c_i(X_j)). \tag{23}$$

Note here that

$$c(X) = \xi_\alpha(X) - i_X i_X B_\alpha = (X + \xi_\alpha, X + \xi_\alpha) = 0$$

by the isotropy condition, hence

$$c_i(X_j) = -c_j(X_i).$$

In particular, it follows that $g(X_i, X_j) - c_i(X_j)$ is always invertible and (23) can be uniquely solved for $b_i(Y)$ for any Y – it defines 1-forms b_i by

$$b_i = 2\sum_j (G - C)^{-1}_{ij} c_j \tag{24}$$

where $C_{ij} = c_i(X_j), G_{ij} = g(X_i, X_j)$.

We now adopt the point of view mentioned earlier that the average of the positive definite and negative definite splitting is the curving, whose curvature we are aiming

to find. This is

$$\frac{1}{2}(Y + gY + i_Y B_\alpha + Y' - gY' + i_{Y'} B_\alpha)$$

which from (22) we can write as

$$Y + i_Y B_\alpha + \frac{1}{2}\sum_i b_i(Y)(X_i + i_{X_i} B_\alpha - gX_i). \tag{25}$$

But $c = \xi_\alpha - i_X B_\alpha$ and we are working in $\bar{E} = F^\perp/F$ where F is spanned by the $X + \xi_\alpha$, so in $F^\perp/F$, we have

$$c_i = -X_i - i_{X_i} B_\alpha$$

which means we write the splitting (25) as

$$Y + i_Y B_\alpha - \frac{1}{2}\sum_i b_i(Y)(c_i + gX_i).$$

From (21), this is

$$Y + i_Y\Big(B_\alpha - \frac{1}{2}\sum_i b_i \wedge (c_i + gX_i)\Big)$$

and from (24) we have

$$\frac{1}{2}\sum_i b_i \wedge (c_i + gX_i) = \sum_{i,j}(G-C)^{-1}_{ij} c_j \wedge (c_i + gX_i) = \sum_{i,j} c_j \wedge (G+C)^{-1}_{ji}(c_i + gX_i)$$

using the symmetry of G and skew-symmetry of C. Now

$$\sum_{i,j}(G + C)^{-1}_{ji}(c_i + gX_i)a_j$$

is a Lie-algebra valued 1-form which annihilates $\mathcal{H} \subset TM$ and on X_j takes the value a_j – it is therefore the connection form A for the connection on M, considered as a principal G-bundle over M/G, defined by the horizontal distribution $\mathcal{H}$. Thus we have $A \in \Omega^1(M, \mathfrak{g})$ and $\kappa \in \Omega^1(M, \mathfrak{g}^*)$ so using the duality pairing we write the splitting as

$$Y + i_Y\left(B_\alpha - \langle \kappa, A\rangle\right).$$

Taking the exterior derivative, this means that the three-form on M/G is

$$H - d\langle \kappa, A\rangle.$$

Example. The simplest situation is where $G = S^1$ and $H = 0$. Then since $dc = i_X H = 0$ from (20), the three-form on the quotient is

$$-d(c \wedge A) = c \wedge F$$

where $F = dA$ is the curvature of M considered as a principal circle bundle. Here we see that non-trivial three-form fluxes can be created from trivial ones by a quotient.

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Chapter 7

Generalizing geometry – algebroids and sigma models

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1 Introduction

In this exposition we want to summarize some ideas in the overlap of differential geometry and mathematical physics. In particular we focus on the interplay of so-called sigma models with geometrical structures being related to algebroids in one way or another. Several traditional geometrical notions received various kinds of generalizations in recent years. Some of them give rise to novel sigma models, while also some sigma models bring attention to possible new geometries.

Sigma models are action functionals (variational problems) where the underlying space of fields (maps) has a target space equipped with some geometry; in the most standard case one regards maps $\mathcal{X}$ from one given Riemannian manifold (Σ, h) to another one (M, g) and considers the functional

$$S[\mathcal{X}] = \frac{1}{2}\int_\Sigma \|\mathrm{d}\mathcal{X}\|^2, \tag{1.1}$$

where $\|\alpha\|^2 = (\mathcal{X}^* g)(\alpha \,\hat{\wedge}\, *\alpha)$ for any $\alpha \in \Omega^p(\Sigma, \mathcal{X}^*TM)$ with $*$ denoting the h-induced Hodge duality operation on Σ,[1] the critical points of which are precisely

[1] In local coordinates σ^μ on Σ and x^i on M this reads more explicitly as $\alpha_{i\mu_1\ldots\mu_p}\alpha^{i\mu_1\cdots\mu_p}\mathrm{dvol}_\Sigma$ where indices μ_i and i are raised and lowered by means of the metric h and (the pullback by $\mathcal{X}$ of) g, respectively, and $\mathrm{dvol}_\Sigma = \sqrt{\det(h)}\mathrm{d}^d\sigma$, d denoting the dimension of Σ. In this chapter we use the Einstein sum convention, i.e., a sum over repeated indices is always understood.

the maps $\mathcal{X}$ which are harmonic. For the special case that M is just $\mathbb{R}^n$ equipped with the standard flat metric this functional reduces to

$$S[\phi^i] = \frac{1}{2}\int_\Sigma \mathrm{d}\phi^i \wedge *\mathrm{d}\phi^i, \tag{1.2}$$

the action of n "free scalar fields" on Σ (here ϕ^i denotes the function on Σ obtained by restricting $\mathcal{X}$ to the i-th coordinate in $\mathbb{R}^n$ – it is understood that the index i on the l.h.s. is an "abstract index", i.e., S depends on all the scalar fields, ϕ^1 to ϕ^n).

Another example of a sigma model is the Poisson sigma model [36], [21], where the source manifold Σ is necessarily 2-dimensional and the target manifold M carries a Poisson structure instead of a Riemannian one. In fact, one considers a functional on the space of vector bundle morphisms from $T\Sigma$ to T^*M in this case. It is "topological", which we want to interpret as saying that the space of classical solutions (stationary points of the functional) modulo gauge transformations (invariances or symmetries of the functional) is finite dimensional – besides the fact that it does not depend on geometrical structures of the source like a Riemannian metric h (which in this case is even absent in the definition of the functional). The tangent bundle of any manifold as well as the cotangent bundle of a Poisson manifold give rise to what are called Lie algebroids (whose definition is properly recalled in the body of the chapter below – cf. in particular Examples 3.2 and 2.3 below) and the above functional becomes stationary precisely on the morphisms of these Lie algebroid structures [4].

But this also works the other way around: given a functional of such a form defined by an a priori arbitrary bivector field Π (and, in some particular extension of the functional, also a closed 3-form H) the respective functional becomes topological, *iff* [35], [23] Π defines a (for non-zero H twisted) Poisson structure. In fact, twisted Poisson structures (c.f. also [34]), i.e., bivectors Π together with closed 3-forms H defined over a manifold M satisfying

$$[\Pi, \Pi] = (\Pi^\natural)^{\otimes 3} H \tag{1.3}$$

where $[\cdot,\cdot]$ denotes the Schouten–Nijenhuis bracket of multi-vector fields and $\Pi^\natural\colon T^*M \to TM$ is the natural operator induced by the bivector Π as follows: $\Pi^\natural(\alpha) = \Pi(\alpha,\cdot)$, were even *found* first in such a manner [23], [30]. This is typical for the interplay of geometrical notions and sigma models: the former are needed to define the latter, but sigma models sometimes also give indications about (focus on) particularly interesting geometrical notions. In this example, in the space of bivectors on a manifold the ones which are Poisson are singled out by the sigma model, or in the space of pairs (Π, H) those satisfying equation (1.3) (which can be seen to define a particular Dirac structure [34] – we introduce to Dirac structures in the main text in detail). Another example for such an interplay are supersymmetric sigma models and bihermitian geometry [14]. The latter geometry received renewed and revived interest recently by its elegant reformulation in terms of so-called generalized complex structures [20].

What is, on the other hand, meant more specifically by generalizing traditional geometrical notions? In fact, also the generalized notions can usually be expressed in terms of ordinary differential geometrical ones, in which case it just boils down to a different way of thinking about them. In any such case one has usually some particular kind of a so-called *algebroid* in the game.

There are several kinds of an algebroid considered in the literature. All of them have the following data in common, which we thus want to use as a definition of the general term:

Definition 1.1. We call an algebroid $E = (E, \rho, [\cdot, \cdot])$, a vector bundle $E \to M$ together with a homomorphism of vector bundles $\rho\colon E \to TM$, called the anchor of E, and a product or bracket on the sections of E satisfying the Leibniz rule ($\psi, \psi' \in \Gamma(E)$, $f \in C^\infty(M)$):

$$[\psi, f\psi'] = f[\psi, \psi'] + (\rho(\psi) f)\ \psi'. \tag{1.4}$$

Note that the map on sections induced by ρ is denoted by the same letter conventionally. Depending on further conditions placed on the bracket (like its symmetry properties or its Jacobiator) and further structures defined on E, one has different kinds of algebroids: Lie algebroids, Courant algebroids, strong homotopy Lie (or L_∞-) algebroids, and so on. Lie algebroids are obtained e.g. by requiring in addition that the bracket is a Lie bracket, i.e., antisymmetric and with vanishing Jacobiator. The definition of a Courant algebroid requires a fiber metric on E controlling the symmetric part of the bracket as well as its Jacobiator. We will come back to all these various kinds of algebroids in more detail in the text below.

The philosophy now is that we can do differential geometry by replacing TM or, more generally, also the tensor bundle $\tau^p_q(M) = TM^{\otimes p} \otimes T^*M^{\otimes q}$ by E and $E^{\otimes p} \otimes E^{*\otimes q}$, respectively. Let us call a section t of $\bigoplus_{p,q} \left(E^{\otimes p} \otimes E^{*\otimes q}\right)$ an E-tensor field. The Leibniz property permits us to define a (Lie) "derivative" of t along any section ψ of E: Indeed, set ${}^E L_\psi(f) := \rho(\psi) f$, ${}^E L_\psi(\psi') := [\psi, \psi']$, and extend this to powers of E by the Leibniz property w.r.t. tensor multiplication and to E^* by means of compatibility with contraction: ${}^E L_\psi\ (\langle \psi', \omega \rangle) = \langle {}^E L_\psi(\psi'), \omega \rangle + \langle \psi', {}^E L_\psi(\omega) \rangle$, defining the Lie derivative for any $\omega \in \Gamma(E^*)$. This implies that given an algebroid E (not necessarily Lie), i.e., data that at least include those of the definition above, we can define a "Lie derivative" of any E-tensor along sections of E. By construction, this generalizes the notion of an ordinary Lie derivative of ordinary tensor fields: indeed, the usual formulas are reproduced in the case where the algebroid is chosen to be a so-called standard Lie algebroid, i.e., where E is the tangent bundle of the base manifold M, the anchor ρ is the identity map and the bracket is the standard Lie bracket of vector fields. Such type of geometrical notions have properties in common with their prototypes in traditional geometry (like, in this case e.g., by construction, the Leibniz property of the generalized Lie derivatives), but they in general also have pronounced differences, possibly depending on the type of algebroid, i.e., on the additional structures. In the above example one can ask e.g. if the commutator of such

Lie derivatives is the Lie derivative of the bracket of the underlying sections. In general this will not always be the case. However, for an important subclass, containing Lie and Courant algebroids, it will (cf. Lemma 2.6 below).

Another important example of such generalized structures is "generalized geometry" or *generalized complex geometry* in the sense of Hitchin (cf. also the contribution of N. Hitchin to this volume). It is a particular case of the above viewpoint where E is taken to be $TM \oplus T^*M$ equipped with projection to the first factor as anchor as well as the so-called Courant or Dorfman bracket (in fact, slightly more general and conceptually preferable, one takes E to be what is called an exact Courant algebroid – cf. Definition 2.14 below). Now, by definition a generalized complex structure is what usually would be a complex structure, just replacing the standard Lie algebroid TM by an exact Courant algebroid. In particular, it is an endomorphism of $E \cong TM \oplus T^*M$ squaring to minus one and satisfying an integrability condition using the bracket on E (cf., e.g., Proposition 4.3 below). It is not difficult to see that this notion generalizes simultaneously ordinary complex structures as well as symplectic ones. In fact, the situation is closely related to (real) Dirac structures (particular Lie subalgebroids of exact Courant algebroids, cf. Definition 3.1 below), mentioned already previously above: these generalize simultaneously Poisson and presymplectic structures on manifolds. In fact, generalized complex structures did not only find at least part of their inspiration from real Dirac structures, but they can be even defined equivalently as imaginary Dirac structures – which is the perspective we want to emphasize in the present note.[2]

An elegant and extremely useful viewpoint on some algebroids arises within the language of differential graded manifolds, sometimes also called Q-manifolds (Q denoting a homological degree-one vector field on the graded manifold, i.e., its differential). We devote a section to explaining this relation, after having introduced the reader to the above mentioned notions of algebroids, Dirac structures, and generalized complex structures, in the three sections to follow. Together these four sections provide our exposition on algebroids and this kind of generalized geometry.

Some of the following sections then deal with the respective sigma models: Given a Q-manifold with a compatible graded symplectic structure on it, one can always associate a topological sigma model to it [1]. We review this construction in some detail and specialize it to lowest dimensions (of the source manifold Σ), reproducing topological models corresponding to Poisson manifolds ($\dim \Sigma = 2$, this is the above mentioned Poisson sigma model), to Courant algebroids ($\dim \Sigma = 3$, such models were considered in [31]). Some space is devoted to describing these models, somewhat complementary to what is found in the literature, since they can be used to introduce part of the formalism that is needed for the last section to this contribution.

There are also topological models that, at least up to now, have not yet been related to the AKSZ formalism and corresponding e.g. to Dirac structures. We recall these models, called Dirac sigma models [25] and generalizing the Poisson sigma models essentially such as (real) Dirac structures generalize Poisson manifolds, in a separate

[2]This point of view was maybe less known at the time when we this note was started, while in the mean time it has received some attention also elsewhere.

section. These as well as the AKSZ models share the property that they are topological and that the solutions to their field equations generalize (only) *flat* connections to the algebroid setting.

The final section is devoted to sigma models in arbitrary spacetime dimension (dimension of Σ which then is taken to be pseudo-Riemannian) with a relation to algebroids but which are nontopological and which generalize connections that are not necessarily flat but instead satisfy the Yang–Mills field equations.[3] This deserves some further motivating explanation already in the introduction:

Consider replacing in equation (1.2) the functions on Σ by 1-forms A^a, $a = 1, \dots, r$, yielding

$$S[A^a] = \frac{1}{2}\int_\Sigma \mathrm{d}A^a \wedge *\mathrm{d}A^a. \tag{1.5}$$

For $r = 1$, i.e., there is just one 1-form field A, this is a famous action functional, describing the electromagnetic interactions (the electric and the magnetic fields can be identified with the components of the "field strength" 2-form $\mathrm{d}A$). Having several such 1-forms in the game, $r > 1$, one obtains the functional describing r free (i.e., mutually independent) 1-form fields[4]. The most standard way of making scalar fields interact is to add some "potential term" to the functional (1.2), i.e., to add the integral over $V(\phi^i)$ multiplied by the h-induced volume form on Σ (where V is some appropriately smooth function on $\mathbb{R}^n$, mostly even only a low degree polynomial so as to not spoil "renormalizability").[5] Turning (1.5) into an interacting theory (without introducing further fields and not spoiling its gauge invariance, at most "deforming" the latter one appropriately) is not so simple. In fact, the result is rather restricted (cf., e.g., uniqueness theorems in the context of the deformation theory of gauge theories [3]) and one is lead to only replace $\mathrm{d}A^a$ by the expression

$$F = \mathrm{d}A + \tfrac{1}{2}[A \overset{\wedge}{,} A] \tag{1.6}$$

for the curvature of the Lie algebra valued connection 1-form A of a (trivialized) principal bundle for some r-dimensional, quadratic Lie group G. "Quadratic" means that its Lie algebra $\mathfrak{g}$ admits an ad-invariant inner product κ which, when starting by deforming (1.5), needs to have definite signature such that a sum over the index a

[3] Being nontopological is required for most physical applications so as to host the degrees of freedom necessary to describe realistic interactions. – There are also another type of (nontopological) sigma models than those explained in the last section that are related to algebroids and in particular generalized and bihermitian geometry. These are supersymmetric 2-dimensional sigma models, i.e., *string theories*. Although also highly interesting, we will not touch this issue here but refer e.g. to the review article [42] and references therein.

[4] In the physics language they are often called "vector fields" (as opposed to "scalar fields" used in (1.2)). We avoid this somewhat misleading/ambiguous nomenclature, but we will, however, from time to time refer to them as "gauge fields", despite the fact that in the mathematical setting they correspond to connections in a principal G-bundle (here trivialized with $G = \mathrm{U}(1)^r$).

[5] An alternative way of having scalar fields interact is coupling them to 1-form gauge fields so that they start being correlated (i.e., interacting) via these 1-form fields. In fact, both ways of interactions are realized in the standard model of elementary particle physics, where the gauge fields describe interaction particles like the photon and the scalar fields describe "matter", essentially like electrons (or the – not yet discovered – Higgs particles).

results from a term of the form

$$S_{\mathrm{YM}}[A] = \int_{\Sigma} \kappa(F \wedge *F) \tag{1.7}$$

after choosing a κ-orthonormal basis in $\mathfrak{g}$. The resulting theory is called a Yang–Mills (gauge) theory and was found to govern all the strong, the weak, and the electromagnetic interactions.

If one considers an algebroid E as defined above over a 0-dimensional base manifold M, M degenerating to a point, one is left only with a vector space. For E being a Lie algebroid this vector space becomes a Lie algebra, for E being a Courant algebroid it becomes a quadratic Lie algebra. In fact, this is a second, algebraic part that is incorporated in algebroids: general Lie algebroids can be thought of as a common generalization of the important notions of, on the one hand, a Lie algebra $\mathfrak{g}$ and, on the other hand, standard geometry (i.e., geometry defined for TM or $\tau_q^p(M)$), as we partially explained already above. (A similar statement is true for general Courant algebroids, TM then being replaced by the "standard Courant algebroid" $TM \oplus T^*M$ and $\mathfrak{g}$ by a quadratic Lie algebra together with its invariant scalar product). From this perspective it is thus tempting to consider the question if one can define e.g. a theory of principal bundles with connections in such a way that the structural Lie algebra is replaced by (or better generalized to) appropriately specified structural algebroids. Likewise, from the more physical side, can one generalize a functional such as the Yang–Mills functional (1.7) to a kind of sigma model, replacing the in some sense flat Lie algebra $\mathfrak{g}$ by nontrivial geometry described via some appropriate Lie or Courant algebroid? These questions will be addressed and, at least part of them, answered to the positive in the final section to this chapter.

Between the part on sigma models and the one on algebroids it would have been nice to also include a section on current algebras (cf. [2] as a first step), as another link between the two; lack of spacetime, however, made us decide to drop this part in the present contribution.

2 Lie and Courant algebroids

In the present section, we recall the notions of Lie and Courant algebroids in a rigorous manner and study some of their properties. We also brielfly introduce some of their higher analogues, like Lie 2-algebroids and vector bundle twisted Courant algebroids. In a later section we will provide another, alternative viewpoint of all these objects by means of graded manifolds, which permits an elegant and concise reformulation.

Definition 2.1. A *Loday algebroid* is an algebroid $(E, \rho, [\cdot,\cdot])$ as defined in Definition 1.1 where the bracket defines a Loday algebra on $\Gamma(E)$, i.e., it satisfies the Loday (or left-Leibniz) property,

$$[\psi_1, [\psi_2, \psi_3]] = [[\psi_1, \psi_2], \psi_3] + [\psi_2, [\psi_1, \psi_3]]. \tag{2.1}$$

An *almost Lie algebroid* is an algebroid E where the bracket is antisymmetric. E becomes a *Lie algebroid*, if $[\cdot,\cdot]$ defines a Lie algebra structure on $\Gamma(E)$, i.e., if E is simultaneously Loday and almost Lie.

Here we adapted to the nomenclature of Kosmann-Schwarzbach, who prefers to use the name of Loday in the context of (2.1) so as to reserve the terminus Leibniz for compatibility w.r.t. multiplication of sections by functions, (1.4), for which "Leibniz rule" has become standard.

Example 2.2. Obviously, $(TM, \rho = \mathrm{Id})$ together with the Lie bracket of vector fields is a Lie algebroid; it is called the *standard* Lie algebroid.

If M is a point, on the other hand, a Lie algebroid reduces to a Lie algebra. More generally, if the anchor of a Lie algebroid map vanishes (i.e., maps E to the image of the zero section in TM), E is a bundle of Lie algebras; in general not a Lie algebra bundle, since Lie algebras of different fibers need not be isomorphic.

Example 2.3. A less trivial Lie algebroid is the cotangent bundle T^*M of a Poisson manifold.[6] The anchor is provided by contraction with the Poisson bivector field Π and the bracket of exact 1-forms $[\mathrm{d}f, \mathrm{d}g] := \mathrm{d}\{f, g\}$ is extended to all 1-forms by means of the Leibniz rule (1.4).

Example 2.4. An example of a Loday algebroid with a non-antisymmetric bracket is the following one: $E = TM \oplus T^*M$ with ρ being projection to the first factor and the bracket being given by

$$[\xi + \alpha, \xi' + \alpha'] = [\xi, \xi'] + L_\xi \alpha' - \imath_{\xi'} \mathrm{d}\alpha; \tag{2.2}$$

here ξ, ξ' and α, α' denote vector fields and 1-forms on M, respectively, and the bracket on the r.h.s. is the usual Lie bracket of vector fields. This is the so-called Dorfman bracket. Note that if one takes the antisymmetrization of this bracket (the original bracket Courant has introduced [10]), E does *not* become an algebroid in our sense, since equation (1.4) will not be valid any more (as a consequence of the non-standard behavior of the above bracket under multiplication by a function of the section in the first entry).

Lemma 2.5. *The anchor map of a Loday algebroid is a morphism of brackets.*

Proof. First, from (2.1) we find $[[\psi_1, \psi_2], f\,\psi_3] = [\psi_1, [\psi_2, f\,\psi_3]] - (\psi_1 \leftrightarrow \psi_2)$. Using the Leibniz rule for the l.h.s., we obtain $\rho([\psi_1, \psi_2])\, f\ \psi_3 + f\,[[\psi_1, \psi_2], \psi_3]$. Applying it twice to the first term on the r.h.s., we get $\rho(\psi_1)\rho(\psi_2)\, f\ \psi_3 + f\,[\psi_1, [\psi_2, \psi_3]] +$

[6] By definition, one obtains a Poisson structure on a smooth manifold M, if the space of functions is equipped with a Lie bracket $\{\cdot,\cdot\}$ satisfying $\{f, gh\} = g\{f, h\} + h\{f, g\}$ for all $f, g, h \in C^\infty(M)$. The latter condition, together with the antisymmetry of the bracket, is equivalent to the existence of a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ such that $\{f, g\} = \langle \Pi, \mathrm{d}f \otimes \mathrm{d}g \rangle$.

$\rho(\psi_1) f\,[\psi_2,\psi_3] + \rho(\psi_2) f\,[\psi_1,\psi_3]$. The last two terms drop out upon antisymmetrizing in ψ_1 and ψ_2. In the remaining equation the terms proportional to f cancel by means of (2.1), and one is left with

$$\rho([\psi_1,\psi_2])f\ \psi_3 = [\rho(\psi_1),\rho(\psi_2)]f\ \psi_3, \tag{2.3}$$

valid for all sections ψ_i, $i = 1,2,3$, and functions f. This completes the proof. □

Lemma 2.6. *The E-Lie derivative provides a representation of the bracket of a Loday algebroid on E-tensors,* $[{}^E L_{\psi_1}, {}^E L_{\psi_2}] = {}^E L_{[\psi_1,\psi_2]}$.

Proof. This follows from (2.1), the previous lemma, and the extension of the E-Lie derivative to tensor powers and the dual by means of a Leibniz rule (using that commutators of operators satisfying a Leibniz rule for some algebra – here that of the tensor product as well as that of contractions – are of Leibniz type again). □

A Lie algebroid permits to go further in extrapolating usual geometry on manifolds to the setting of more general vector bundles (algebroids). In particular, there is a straightforward generalization of the de Rham differential in precisely this case:[7] In any almost Lie algebroid we may define the following degree-one map ${}^E\mathrm{d}$ on E-differential forms $\Omega_E(M) \equiv \Gamma(\Lambda^\cdot E^*)$. For any function f and E-1-form ω we set

$$\begin{aligned} \langle {}^E\mathrm{d}f, \psi\rangle &:= \rho(\psi) f, \\ \langle {}^E\mathrm{d}\omega, \psi\otimes\psi'\rangle &:= \rho(\psi)\langle\omega,\psi'\rangle - \rho(\psi')\langle\omega,\psi\rangle - \langle\omega,[\psi,\psi']\rangle \end{aligned} \tag{2.4}$$

and extend this by means of a graded Leibniz rule to all of $\Omega_E(M)$. Clearly, for the standard Lie algebroid (TM, Id) this reduces to the ordinary de Rham differential. As one proves by induction, with this one finds in generalization of the Cartan–Koszul formula:

$$\begin{aligned} {}^E\mathrm{d}\omega(\psi_1,\dots,\psi_{p+1}) := &\sum_{i=1}^{p+1}(-1)^{i+1}\rho(\psi_i)\omega(\dots,\hat{\psi}_i,\dots) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([\psi_i,\psi_j]\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots), \end{aligned} \tag{2.5}$$

valid for any $\omega \in \Omega^p_E(M)$ and $\psi_i \in \Gamma(M,E)$. This text is in part supposed to contain explicit proofs:

Proof of equation (2.5). The property holds for $p = 1$ by definition. Suppose we proved (2.5) for all forms of order at most $p-1$. It suffices to see that (2.5) holds for each $\omega = \alpha\wedge\tau$, where $\alpha \in \Omega^1_E(M)$, $\tau \in \Omega^{p-1}_E(M)$. ${}^E\mathrm{d}(\omega) = {}^E\mathrm{d}(\alpha)\wedge\tau - \alpha\wedge{}^E\mathrm{d}(\tau)$,

[7] We remark in parenthesis that there is also an option to generalize the de Rham differential different from formula (2.4) below, using the language of graded manifolds; *that* generalization can be used also for Courant algebroids (cf. e.g. [32]) or even arbitrary L_∞ algebroids, cf. Section 5 below.

therefore

$$
\begin{aligned}
{}^{E}\mathrm{d}(\omega)(\psi_1,\dots,\psi_{p+1}) &= \sum_{i<j}(-1)^{i+j+1}\ {}^{E}\mathrm{d}\alpha(\psi_i,\psi_j)\tau(\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)\\
&\quad+\sum_i(-1)^i\alpha(\psi_i)\ {}^{E}\mathrm{d}\tau(\dots,\hat{\psi}_i,\dots)\\
&=\sum_{i<j}(-1)^{i+j+1}\ \big(\rho(\psi_i)\alpha(\psi_j)-\rho(\psi_j)\alpha(\psi_i)\\
&\qquad-\alpha([\psi_i,\psi_j])\big)\tau(\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)\\
&\quad+\sum_i(-1)^i\alpha(\psi_i)\Big(\sum_{j<i}(-1)^{j+1}\rho(\psi_j)\tau(\dots,\hat{\psi}_j,\dots,\hat{\psi}_i,\dots)\\
&\quad+\sum_{i<j}(-1)^j\rho(\psi_j)\tau(\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)\\
&\quad+\sum_{\substack{k<l<i\\ i<k<l}}(-1)^{k+l}\tau([\psi_k,\psi_l],\dots,\widehat{\psi_{i,k,l}},\dots)\\
&\quad+\sum_{k<i<l}(-1)^{k+l+1}\tau([\psi_k,\psi_l],\dots,\widehat{\psi_{i,k,l}},\dots)\Big),
\end{aligned}
$$

Collecting all terms proportional to ρ, we find

$$
\begin{aligned}
&\sum_{i<j}(-1)^{i+j+1}\ \big(\rho(\psi_i)\alpha(\psi_j)-\rho(\psi_j)\alpha(\psi_i)\big)\,\tau(\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)\\
&\qquad+\sum_i(-1)^i\alpha(\psi_i)\Big(\sum_{j<i}(-1)^{j+1}\rho(\psi_j)\tau(\dots,\hat{\psi}_j,\dots,\hat{\psi}_i,\dots)\\
&\qquad+\sum_{i<j}(-1)^j\rho(\psi_j)\tau(\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)\Big)\\
&\quad=\sum_{i=1}^{p+1}(-1)^{i+1}\rho(\psi_i)\omega(\dots,\hat{\psi}_i,\dots),
\end{aligned}
$$

while the remaining terms yield

$$
\begin{aligned}
&\sum_{i<j}(-1)^{i+j}\alpha([\psi_i,\psi_j])\tau(\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)\\
&\qquad+\sum_i(-1)^i\alpha(\psi_i)\Big(\sum_{\substack{k<l<i\\ i<k<l}}(-1)^{k+l}\tau([\psi_k,\psi_l],\dots,\widehat{\psi_{i,k,l}},\dots)\\
&\qquad+\sum_{k<i<l}(-1)^{k+l+1}\tau([\psi_k,\psi_l],\dots,\widehat{\psi_{i,k,l}},\dots)\Big)\\
&\quad=\sum_{i<j}(-1)^{i+j}\omega([\psi_i,\psi_j]\dots,\hat{\psi}_i,\dots,\hat{\psi}_j,\dots)
\end{aligned}
$$

which completes the proof. □

One may now verify that the above operator squares to zero, ${}^E\mathrm{d}^2 = 0$, *iff* (2.1) is satisfied (turning E into a Lie algebroid). In fact, *all* the information of a Lie algebroid is captured by ${}^E\mathrm{d}$ as seen from the following

Proposition 2.7. *The structure of a Lie algebroid on a vector bundle $E \to M$ is in one-to-one correspondence with the structure of a differential complex on $(\Omega_E(M), \wedge)$.*

Proof. As obvious from (2.4), the skew-symmetric bracket on sections of E and the anchor map are uniquely determined by the canonical differential ${}^E\mathrm{d}$. In particular, (2.4) can be rewritten as

$$\rho(\psi) f = [\imath_\psi, {}^E\mathrm{d}] f, \tag{2.6}$$

$$\imath_{[\psi,\psi']}\omega = \big[\imath_\psi, [\imath_{\psi'}, {}^E\mathrm{d}]\big]\,\omega, \tag{2.7}$$

where $\imath_\psi$ denotes the contraction with $\psi \in \Gamma(M, E)$ and $[\ ,\ ]$ the super bracket of super derivations of the graded commutative algebra $\Omega_E(M)$. Note that (2.7) has been verified for E-1-forms ω (and it is trivially satisfied if ω is a function); due to the required Leibniz property of ${}^E\mathrm{d}$, however, it then necessarily holds for arbitrary $\omega \in \Omega_E(M)$ (which is generated by E-1-forms and functions). With these formulas we now find

$$(\rho([\psi_1, \psi_2]) - [\rho(\psi_1), \rho(\psi_2)]) f = \imath_{\psi_1}\imath_{\psi_2}\,{}^E\mathrm{d}^2 f,$$

$$(\imath_{[\psi_1,[\psi_2,\psi_3]]} + \text{cyc. perm.})\omega = \imath_{\psi_1}\imath_{\psi_2}\imath_{\psi_3}\,{}^E\mathrm{d}^2\omega - (\imath_{\psi_1}\imath_{\psi_2}\,{}^E\mathrm{d}^2(\imath_{\psi_3}\omega) + \text{cyc. perm.}),$$

valid for any $f \in C^\infty(M)$, $\omega \in \Omega^1_E(M)$, and $\psi_i \in \Gamma(M, E)$. Thus we see that the anchor map is a morphism of brackets if and only if the canonical differential squares to zero on functions, and the Jacobi condition for the almost Lie bracket holds if and only if ${}^E\mathrm{d}^2 = 0$ on functions and E-1-forms simultaneously. This extends to all of $\Omega_E(M)$ since ${}^E\mathrm{d}^2 = \frac{1}{2}[{}^E\mathrm{d}, {}^E\mathrm{d}]$ is a super derivative of the algebra $\Omega_E(M)$. Noting that the Leibniz rule follows from the derivative property of ${}^E\mathrm{d}$ as well and that any Lie algebroid gives rise to such a differential finalizes the proof. □

Equation (2.7) shows that the bracket of a Lie algebroid is a so-called *derived bracket* [24], generalizing the well-known formulas of Cartan: equation (2.6) generalizes the fact that on differential forms one has $L_\xi = [\imath_\xi, \mathrm{d}]$ (while on a function the Lie derivative reduces to application of the vector field). Equation (2.7) reduces to the evident identity $-\imath_{L_{\xi'}(\xi)} = [\imath_\xi, L_{\xi'}]$ for the standard Lie algebroid (using $L_{\xi'}(\xi) = -[\xi, \xi']$). We remark in parenthesis that in the case of E being a Lie algebroid the E-Lie derivative, defined for a general Loday algebroid, satisfies also the usual formula

$${}^E L_\psi = \imath_\psi\,{}^E\mathrm{d} + {}^E\mathrm{d}\,\imath_\psi \tag{2.8}$$

on E-differential forms. Later in this chapter we also show how the Courant (or Dorfman) bracket can be put into the form of a derived bracket – for the case of

equation (2.2) already at the end of this section, while for the more general setting, which we are going to define in what follows, in Section 5 below.

Definition 2.8. A *Courant algebroid* is a Loday algebroid $(E, \rho, [\cdot, \cdot])$ together with an invariant E-metric ${}^E\mathrm{g}$ such that ${}^E\mathrm{g}([\psi, \psi], \psi') = \frac{1}{2}\rho(\psi'){}^E\mathrm{g}(\psi, \psi)$.

We will mostly denote ${}^E\mathrm{g}(\psi, \psi')$ simply as (ψ, ψ'), except if the appearance of the metric shall be stressed. The E-metric is used to control the symmetric part of the bracket. It permits to deduce the behavior of the bracket under multiplication of the first section w.r.t. a function for example:

$$[f\psi_1, \psi_2] = f[\psi_1, \psi_2] - (\rho(\psi_2)f)\,\psi_1 + (\psi_1, \psi_2)\,\rho^*(\mathrm{d}f). \tag{2.9}$$

In a general Loday algebroid there is *no* restriction to the bracket like this at all.

Invariance of ${}^E\mathrm{g}$ means that for any section $\psi \in \Gamma(M, E)$ one has ${}^E L_\psi\, {}^E\mathrm{g} = 0$; this is the same as requiring

$$([\psi_1, \psi_2], \psi_3) + (\psi_2, [\psi_1, \psi_3]) = \rho(\psi_1)(\psi_2, \psi_3) \tag{2.10}$$

for any $\psi_i \in \Gamma(M, E)$. When ρ, considered as a section of $\mathrm{Hom}(E, TM)$, is nonzero at any point of M, remarkably the invariance can be also concluded from the remaining three axioms of a Courant algebroid cf. [18]. Also, it is easy to see that just equation (2.10) by itself permits to conclude the Leibniz identity (1.4): Replace ψ_2 by $f\psi_2$ to obtain $([\psi_1, f\psi_2], \psi_3) - f([\psi_1, \psi_2], \psi_3) = (\rho(\psi_1)f)(\psi_2, \psi_3)$, which yields the claimed equation by bilinearity and non-degeneracy of the inner product. So, any anchored vector bundle equipped with some bracket on its sections and an invariant fiber metric in the sense of (2.10) is an algebroid as defined in the introduction.

Sometimes in the literature an antisymmetrization of the above bracket $[\cdot, \cdot]$ is used in the definition, cf. e.g. the contribution of Hitchin to this volume;[8] this antisymmetrized bracket, however, has *neither* of the two nice properties (1.4) and (2.1), for which reason we preferred the present version of axiomatization.

For M a point, the definition of a Courant algebroid is easily seen to reduce to a quadratic Lie algebra, i.e., to a Lie algebra endowed with a nondegenerate, ad-invariant inner product: Indeed, since then the r.h.s. of the last equation in Definition 2.8 vanishes, the bracket becomes antisymmetric and (2.10) apparently reduces to the notion of ad-invariance of the metric. (Evidently in this context, or whenever ρ has zeros, the invariance needs to be required separately). The fact that Courant algebroids provide a generalization of quadratic Lie algebras may be one motivation for introducing them, since in particular there is also the following simple observation:

[8] At least in the case of exact Courant algebroids with $\boldsymbol{H} = \mathbf{0}$, introduced below, it is often the antisymmetrized bracket that is called the Courant bracket in the more recent literature while the bracket used above then is refered to as the Dorfman bracket. Despite this tendency, we will mostly stick to the terminus "Courant bracket" for the non-antisymmetric bracket in this article. In any case, in the axiomatization of a general Courant algebroid, both types of brackets can be used, and, following [34], we prefer the non-antisymmetric bracket for the reason to follow.

Proposition 2.9. *Let E be a Lie algebroid. Then it admits an invariant fiber metric ${}^E\mathrm{g}$ only in the (very restrictive) case of $\rho \equiv 0$.*

Proof. The statement follows at once from the formula (deduced from the Leibniz property of the E-Lie derivative and formula (2.8) applied to E-1-forms)

$$ {}^E L_{f\psi}\, {}^E\mathrm{g} = f\, {}^E L_\psi\, {}^E\mathrm{g} + 2\, {}^E\mathrm{d}f \vee \iota_\psi\, {}^E\mathrm{g}, \tag{2.11} $$

valid for any $f \in C^\infty(M)$, $\psi \in \Gamma(M, E)$, where $\vee$ denotes the symmetrized tensor product: according to the first formula (2.4), ${}^E\mathrm{d}f$ vanishes for all functions f only if $\rho \equiv 0$. □

Nevertheless, the option (2.10) is not the only possibility to generalize quadratic Lie algebras to the realm of algebroids. There are at least two more, which will also play a role in an attempt to generalize Yang–Mills theories to the context of structural algebroids/groupoids, as explained in the last section. We thus introduce these two, which are also interesting in their own right.

Using the notion of E-Lie derivatives and inspired by the philosophy to treat E as the tangent bundle of a manifold with the ensuing geometrical intuition, we consider

Definition 2.10. Let E be a Loday algebroid. We call it a *maximally symmetric E-Riemannian space* if E is equipped with a definite fiber metric ${}^E\mathrm{g}$ which permits a (possibly overcomplete) basis of sections $\psi_\alpha \in \Gamma(E)$, $\langle \psi_\alpha(x) \rangle = E_x$ for all $x \in M$, $\alpha = 1, \ldots, s \geq \operatorname{rank} E$, such that ${}^E L_{\psi_\alpha}\, {}^E\mathrm{g} = 0$.

Likewise we can define a maximally symmetric E-pseudo-Riemannian space by dropping the condition of the definiteness of the fiber metric. For $E = TM$ this reduces to the standard notion of a maximally symmetric pseudo-Riemannian space (manifold), whereas for a Lie algebra (a Lie algebroid over a point) this reproduces the notion of quadratic Lie algebras.

So, in contrast to (2.10), *this* notion is compatible with a Lie algebroid with non-vanishing anchor, although it also poses restrictions (like it is already the case in the example of Lie algebras). Another option for generalizing quadratic Lie algebras to Lie algebroids is the following one using an algebroid type covariant derivative. For this we define

Definition 2.11. Let $E \to M$ be an algebroid and $V \to M$ another vector bundle over the same base manifold. Then an *E-covariant derivative* ${}^E\nabla$ on V is a map from $\Gamma(E) \otimes \Gamma(V) \to \Gamma(V)$, $(\psi, v) \mapsto {}^E\nabla_\psi v$ that is $C^\infty(M)$ linear in the first entry and satisfies the Leibniz rule

$$ {}^E\nabla_\psi(fv) = f\, {}^E\nabla_\psi v + \rho(\psi) f\, v \quad \text{for all } f \in C^\infty(M). \tag{2.12} $$

Clearly, this notion reduces to an ordinary covariant derivative on V for the case of the standard Lie algebroid $E = TM$. Moreover, any ordinary connection ∇ on V gives rise to an E-covariant derivative by means of ${}^E\nabla_\psi := \nabla_{\rho(\psi)}$, while certainly

not every E-connection is of this form (for $\rho \equiv 0$, e.g., this expression vanishes identically, while in this case *any* section of $E^* \otimes \mathrm{End}(V)$ defines an E-connection). In the particular case of a Lie algebroid (and only there) one can generalize the notions of curvature and torsion to the algebroid setting, with ${}^E R \in \Omega^2_E \otimes \Gamma(\mathrm{End}\, V)$ and ${}^E T \in \Omega^2_E \otimes \Gamma(E)$. Note that a "flat" E-connection, i.e., one with ${}^E R = 0$, is what one calls a *Lie algebroid representation* on V (generalizing the ordinary notion of a Lie algebra representation on a vector space, to which this reduces for the case of M being a point).

Given a fiber-metric on V it is called compatible with an E-connection on V, if it is annihilated by E-covariant derivatives. In generalization of the well-known fact of the uniqueness of a metric-compatible, torsion-free connection on a manifold ($E = TM$), also for a general Lie algebroid E one can prove that there is a unique E-torsion-free E-connection compatible with a (pseudo-)Riemannian metric ${}^E g$. On the other hand, given an ordinary connection ∇ on an algebroid E, we can also regard the following canonically induced E-connection

$$\widetilde{{}^E\nabla}_{\psi_1}\psi_2 = \nabla_{\rho(\psi_2)}\psi_1 + [\psi_1, \psi_2], \tag{2.13}$$

which, for E a Lie algebroid, differs from ${}^E\nabla_\psi := \nabla_{\rho(\psi)}$ by subtraction of its own E-torsion. It is easy to see that a Lie algebroid E with fiber metric ${}^E g$ compatible with $\widetilde{{}^E\nabla}$, induced by some ordinary connection on E, reduces to a quadratic Lie algebra; here one may or may not want to impose that this E-connection is flat, while only in the first case one would call it an E-metric invariant under the adjoint *representation* induced by the connection ∇.

Before discussing the second motivation of introducing Courant algebroids, which will lead us into the world of Dirac and generalized complex structures, we use the opportunity of having introduced E-covariant derivatives for a natural generalization of Courant algebroids:[9]

Definition 2.12. A vector bundle twisted or *V-twisted Courant algebroid* is given by a Loday algebroid structure $(E, \rho, [\cdot,\cdot])$ on $E \to M$ together with a second vector bundle $V \to M$, an E-covariant derivative ${}^E\nabla$ on V, and a surjective, non-degenerate bilinear map $(\cdot,\cdot)\colon E \times_M E \to V$ such that

$$([\psi,\psi],\psi') = \tfrac{1}{2}{}^E\nabla_{\psi'}(\psi,\psi) = ([\psi',\psi],\psi). \tag{2.14}$$

It is not difficult to see that for the case of V being a trivial $\mathbb{R}$-bundle over M, admitting an E-covariantly constant basis section, the above definition reduces to the one of an ordinary Courant algebroid. A V-twisted Courant algebroid defined over a point does no more reduce to Lie algebras, but rather gives a Leibniz algebra E only, with V being an E-module (cf. [17] for further details on these statements). V-twisted Courant algebroids play a role in the context of higher gauge theories, in particular nonabelian gerbes.

[9] This notion was introduced independently and with different, but complementary motivations in [17], [9].

For later use we also recall the definition of a (strict) Lie 2-algebroid from [17]:

Definition 2.13. A *strict Lie 2-algebroid* are two Lie algebroids over M, $(E \to M, \rho, [\cdot,\cdot])$ as well as $(V \to M, 0, [\cdot,\cdot]_V)$ together with a morphism $t : V \to E$ and a representation ${}^E\nabla$ of E on V such that

$$ {}^E\nabla_{t(v)} w = [v, w]_V, \quad t({}^E\nabla_\psi v) = [\psi, t(v)] \quad \text{for all } v, w \in \Gamma(V) \text{ and } \psi \in \Gamma(E). \tag{2.15} $$

Note that the second Lie algebroid V is a bundle of Lie algebras only, $\rho_V \equiv 0$. The general notion of a morphism for Lie algebroids is somewhat involved but most easily defined in terms of the Q-language developed later (and thus provided in Section 5 below); over the same base manifold as here, however, it simply implies $t([v, w]) = [t(v), t(w)]$ as well as $\rho \circ t = \rho_V$. For M being a point, the above definition reduces to the one of a strict Lie 2-algebra or, equivalently, a differential crossed module. The above definition may be twisted by an E-3-form H taking values in V, cf. Theorem 3.1 in [17], in which case one obtains what one might call an H-twisted or semistrict Lie 2-algebroid or simply a Lie 2-algebroid. (As one of the names suggests, it reduces to a semistrict Lie 2-algebra when M is a point. The last name is most natural in view of the relation to so-called Q-manifolds, cf. Section 5 below).

We now return to ordinary Courant algebroids. The other motivation for considering them comes from the quest for generalizing the notions of (pre-) symplectic, Poisson, and complex manifolds to so-called (real and/or complex) Dirac structures, our subject in the next section. For their definition one restricts to particular kinds of Courant algebroids, so called exact Courant algebroids: For a general Courant algebroid one obviously has the following sequence of vector bundles

$$ 0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0, \tag{2.16} $$

where ρ^* is the fiberwise transpose of ρ combined with the isomorphism induced by ${}^E\mathrm{g}$.[10]

Definition 2.14. An exact Courant algebroid is a Courant algebroid such that the sequence (2.16) is exact.

For the rest of this section E will always denote an exact Courant algebroid.

Proposition 2.15. *The image of T^*M in E is a maximally isotropic subbundle w.r.t.* ${}^E\mathrm{g}$.

Proof. By the definition of ρ^*, $(\rho^*\omega, \psi) = \langle \omega, \rho(\psi)\rangle$ therefore $(\omega_1, \omega_2) = 0$ for all $\omega_i \in \Omega^1(M)$. Thus $\rho^*T^*M \subset (\rho^*T^*M)^\perp$; by a dimensional argument, using that the E-metric is nondegenerate, we may conclude equality. □

Definition 2.16. A splitting (also sometimes called a connection) of an exact Courant algebroid is a map $j : TM \to E$, such that $\rho \circ j = \mathrm{id}$ and $j(TM)$ is isotropic.

[10] In fact, this even defines a complex of sheaves, since for a general Courant algebroid $\rho \circ \rho^* \equiv 0$, cf. e.g. [10].

Evidently $j(TM)$ defines a maximally isotropic subbundle of E complementary to T^*M. Hereafter (within this and the following two sections) we identify a 1-form with its image in the exact Courant algebroid and, given a splitting, we can likewise do so for the image of vector fields w.r.t. the splitting. A general element of E can thus be written as the sum of an element of T^*M and TM, provided a splitting is given. Existence of the latter one is guaranteed by the following

Lemma 2.17. *There always exists a splitting in E. The set of splittings is a torsor over $\Omega^2(M)$.*

Proof. Let us take any splitting of the exact sequence $0 \to T^*M \to E \to TM \to 0$ in the category of vector bundles, $j : TM \to E$. Since T^*M is maximally isotropic, the pairing between $j(TM)$ and T^*M has to be nondegenerate. Therefore for any vector field ξ there exists a unique 1-form $\beta(\xi)$ such that $(j(\xi), j(\xi')) = \langle\beta(\xi), \xi'\rangle$. The new splitting given by $j(\xi) - \frac{1}{2}\beta(\xi)$ is maximally isotropic (note that for any 1-form $\alpha = \beta(\xi)$ one has by definition of ρ^* and j: $(\rho^*(\alpha), j(\xi')) = \langle\alpha, (\rho \circ j)(\xi')\rangle = \langle\alpha, \xi'\rangle$). Any other splitting differs from the chosen one by a section B of $T^*M \otimes T^*M$, i.e., by sending

$$j_B(\xi) = j(\xi) + B(\xi, \cdot). \tag{2.17}$$

The tensor field B is necessarily skew-symmetric because the image of TM is required to be maximally isotropic. □

Proposition 2.18. *For each $\omega \in \Omega^1(M)$ and $\psi \in \Gamma(E)$ one has*

$$[\psi, \rho^*\omega] = \rho^* L_{\rho(\psi)}\, \omega \quad \textit{and} \quad [\rho^*\omega, \psi] = -\rho^* \imath_{\rho(\psi)} \mathrm{d}\omega. \tag{2.18}$$

Proof. Invariance of the E-metric implies

$$\rho(\psi_1)(\rho^*\omega, \psi_2) = ([\psi_1, \rho^*\omega], \psi_2) + (\rho^*\omega, [\psi_1, \psi_2]),$$

and therefore $([\psi_1, \rho^*\omega], \psi_2) = \langle L_{\rho(\psi_1)}\, \omega, \rho(\psi_2)\rangle$, where we used $\rho\,([\psi_1, \psi_2]) = [\rho(\psi_1), \rho(\psi_2)]$. The E-metric being nondegenerate, we may conclude the first rule of multiplication. The second one follows from this by the symmetry property of the bracket in a Courant algebroid: $[\psi, \psi] = \rho^*\mathrm{d}(\psi, \psi)$. □

A direct consequence is:

Corollary 2.19. *The image of T^*M is an abelian ideal of E.*

A result in the classification of exact Courant algebroids, ascribed to P. Ševera [33], is the following one:

Proposition 2.20. *Up to isomorphism there is a one-to-one correspondence between exact Courant algebroids and elements of $H^3(M, \mathbb{R})$. In particular, with the choice of a splitting of (2.16) it takes the form*

$$[\xi + \alpha, \xi' + \alpha'] = [\xi, \xi'] + H(\xi, \xi', \cdot) + L_\xi \alpha' - \imath_{\xi'} \mathrm{d}\alpha, \tag{2.19}$$

where $H \in \Omega^3(M)$ *and* $\mathrm{d}H = 0$. *For a change of splitting parameterized by a two-form* B *as in* (2.17)*, the* 3*-form transforms as* $H \mapsto H + \mathrm{d}B$.

Proof. Given a splitting of (2.16), one has $[j(\xi), j(\xi')] = j([\xi, \xi']) + H(\xi, \xi')$, with H the "curvature" of the splitting, taking values in 1-forms. The Leibnitz property of the Courant bracket applied to the formula above implies that H is $C^\infty(M)$-linear with respect to its first two arguments, hence H can be identified with a section of $(T^*M)^{\otimes 3}$. The image of TM by j is an isotropic subbundle of E. Therefore the Courant bracket restricted to $j(TM)$ is skew-symmetric and therefore also the curvature H, i.e., $H(\cdot, \cdot)$. Moreover, using that the Courant metric is invariant w.r.t. the Courant bracket and that the image of j is isotropic, we obtain

$$\begin{aligned}\langle H(\xi_1, \xi_2), \xi_3\rangle &= ([j(\xi_1), j(\xi_2)], j(\xi_3)) \\ &= -(j(\xi_2), [j(\xi_1), j(\xi_3)]) \\ &= -\langle H(\xi_3, \xi_2), \xi_1\rangle,\end{aligned}$$

thus H is totally skew-symmetric and can be regarded as a 3-form. By the formula (2.18) the bracket of two sections of E with the splitting j is indeed found to take the form of (2.19), the Courant (or Dorfman, cf. also footnote 8) bracket.

Let us calculate the Jacobiator of sections $J(\psi_1, \psi_2, \psi_3) = [\psi_1, [\psi_2, \psi_3]] - [[\psi_1, \psi_2], \psi_3]] - [\psi_2, [\psi_1, \psi_3]]$ for $\psi_i \in \Gamma(E)$, which has to vanish by the property of a Courant algebroid. Again we want to be explicit here. Using $\psi_i = \xi_i + \alpha_i$, where ξ_i are vector fields and α_i 1-forms, we obtain

$$\begin{aligned}[\psi_1, [\psi_2, \psi_3]] &= \big[\xi_1 + \alpha_1, [\xi_2, \xi_3] + H(\xi_2, \xi_3, \cdot) + L_{\xi_2}\alpha_3 - \imath_{\xi_3}\mathrm{d}\alpha_2\big] \\ &= [\xi_1, [\xi_2, \xi_3]] + H(\xi_1, [\xi_2, \xi_3], \cdot) + L_{\xi_1}H(\xi_2, \xi_3, \cdot) \\ &\quad + L_{\xi_1}L_{\xi_2}\alpha_3 - L_{\xi_1}\imath_{\xi_3}\mathrm{d}\alpha_2 - \imath_{[\xi_2,\xi_3]}\mathrm{d}\alpha_1,\end{aligned}$$

$$\begin{aligned}[[\psi_1, \psi_2], \psi_3]] &= \big[[\xi_1, \xi_2] + H(\xi_1, \xi_2, \cdot) + L_{\xi_1}\alpha_2 - \imath_{\xi_2}\mathrm{d}\alpha_1, \xi_3 + \alpha_3\big] \\ &= [[\xi_1, \xi_2], \xi_3] + H([\xi_1, \xi_2], \xi_3, \cdot) + L_{[\xi_1,\xi_2]}\alpha_3 - \imath_{\xi_3}\mathrm{d}H(\xi_1, \xi_2, \cdot) \\ &\quad - \imath_{\xi_3}L_{\xi_1}\mathrm{d}\alpha_2 + \imath_{\xi_3}L_{\xi_2}\mathrm{d}\alpha_1,\end{aligned}$$

$$\begin{aligned}[\psi_2, [\psi_1, \psi_3]] &= \big[\xi_2 + \alpha_2, [\xi_1, \xi_3] + H(\xi_1, \xi_3, \cdot) + L_{\xi_1}\alpha_3 - \imath_{\xi_3}\mathrm{d}\alpha_1\big] \\ &= [\xi_2, [\xi_1, \xi_3]] + H(\xi_2, [\xi_1, \xi_3], \cdot) + L_{\xi_2}H(\xi_1, \xi_3, \cdot) \\ &\quad + L_{\xi_2}L_{\xi_1}\alpha_3 - L_{\xi_2}\imath_{\xi_3}\mathrm{d}\alpha_1 - \imath_{[\xi_1,\xi_3]}\mathrm{d}\alpha_2,\end{aligned}$$

so one can rewrite the Jacobiator as

$$\begin{aligned}J(\psi_1, \psi_2, \psi_3) = {} & [\xi_1, [\xi_2, \xi_3]] - [[\xi_1, \xi_2], \xi_3] - [\xi_2, [\xi_1, \xi_3]] \\ & + L_{\xi_1}L_{\xi_2}\alpha_3 - L_{[\xi_1,\xi_2]}\alpha_3 - L_{\xi_2}L_{\xi_1}\alpha_3 \\ & - L_{\xi_1}\imath_{\xi_3}\mathrm{d}\alpha_2 + \imath_{\xi_3}L_{\xi_1}\mathrm{d}\alpha_2 + \imath_{[\xi_1,\xi_3]}\mathrm{d}\alpha_2 \\ & - \imath_{[\xi_2,\xi_3]}\mathrm{d}\alpha_1 - \imath_{\xi_3}L_{\xi_2}\mathrm{d}\alpha_1 + L_{\xi_2}\imath_{\xi_3}\mathrm{d}\alpha_1 \\ & + (\mathrm{d}H)(\xi_1, \xi_2, \xi_3, \cdot).\end{aligned}$$

Apparently, the first four lines vanish and therefore the 3-form H entering into the product formula has to be closed. Finally, by a simple calculation one obtains

$$[\,\psi_1 + B(\rho(\psi_1),\cdot), \psi_2 + B(\rho(\psi_2),\cdot)\,] = [\psi_1,\psi_2] + \mathrm{d}B(\rho(\psi_1),\rho(\psi_2),\cdot),$$

which finalizes the proof of the Proposition 2.20. □

Suppose a splitting of (2.16) is chosen. Then, each section of E, $\psi = \xi + \alpha$, can be thought of as an operator acting on differential forms by the formula

$$\mathsf{c}(\psi)\,\omega = \iota_\xi\,\omega + \alpha\wedge\omega. \tag{2.20}$$

It is useful to consider the space of forms as a spinor module over the Clifford algebra of E with the quadratic form given by the Courant metric. The next simple proposition, the proof of which we leave for readers, shows that the notion of derived brackets can be exploited also in the case of Courant algebroids.

Proposition 2.21. *The following identity holds true:*

$$\mathsf{c}([\psi_1,\psi_2]) = [[\mathsf{c}(\psi_1),\mathrm{d}+H],\mathsf{c}(\psi_2)]\,. \tag{2.21}$$

3 Dirac structures

Definition 3.1. Suppose E is an exact Courant algebroid over M and D a maximal totally isotropic (maximally isotropic) subbundle of E with respect to the Courant scalar product. Then D is called a Dirac structure, if the Courant bracket of two sections of D is again a section of D.

Example 3.2. T^*M is a Dirac structure, see Proposition 2.15 and Corollary 2.19. Given a connection in E (see Definition 2.16) TM is a maximally isotropic subbundle of E; TM is a Dirac structure, iff the curvature H is zero.

Suppose a splitting of a Courant algebroid E is chosen, $E = T^*M \oplus TM$, the curvature H of which is zero. We then have the following two examples showing that Poisson and presymplectic geometry give rise to particular Dirac structures.

Example 3.3. Let D be a graph of a tensor field $\Pi \in \Gamma(\otimes^2 TM)$, considered as a map from T^*M to TM. Then D is a Lagrangian subbundle iff Π is skew-symmetric, i.e., iff it is a bivector field, $\Pi \in \Gamma(\Lambda^2 TM)$. The projection of D to T^*M is non-degenerate. Vice versa, any Lagrangian subbundle of E which has a non-degenerate projection to T^*M is a graph of a bivector field. The graph of a bivector field D is a Dirac structure if and only if Π is a Poisson bivector, $[\Pi,\Pi] = 0$.

Example 3.4. Let D be a graph of a 2-form $\omega \in \Omega^2(M)$ considered as a skew-symmetric map from TM to T^*M, then D is a Lagrangian subbundle and the projection

of D to TM is non-degenerate. Any Lagrangian subbundle of E which has a non-degenerate projection to TM is a graph of a 2-form. D is a Dirac structure if and only if ω is closed.

In the second example we could also have started with a general $(2,0)$-tensor field certainly, completely parallel to the first example. For the case that H is non-zero, one obtains twisted versions of both examples, cf. equation (1.3) in the case of Example 3.3.

The complexification of E, $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$, is a real bundle, i.e., a bundle over $\mathbb{C}$ endowed with a $\mathbb{C}$-anti-linear operator σ acting on E^c, such that $\sigma^2 = 1$ and $E = \ker(\sigma - 1)$. We next turn to a description of the algebraic set of complex subbundles of E^c, which are maximally isotropic with respect to the complexified Courant scalar product. In order to do so, we introduce an additional structure, namely a particular kind of fiber metric ${}^E\mathrm{g}' =: g_\tau$ (cf. also [25]), different from the canonical one ${}^E\mathrm{g}$, called sometimes a generalized Riemannian metric. Note that according to the philosophy presented here, a generalized Riemannian metric on a Courant algebroid would be any positive definite fiber metric; we will still adopt this partially established terminology now and then show that, given a splitting, it corresponds to an ordinary Riemannian metric g on M together with a 2-form B (cf. [25] as well as the contribution of N. Hitchin within this volume).

Definition 3.5. A positive E-metric g_τ which can be expressed via an operator $\tau \in \Gamma(\mathrm{End}(E))$ squaring to the identity such that for any $\psi_i \in \Gamma(E)$

$$g_\tau(\psi_1, \psi_2) = (\tau\psi_1, \psi_2)$$

is called a generalized Riemannian metric.

It follows from the definition, that τ has to be self-adjoint with respect to the Courant metric ${}^E\mathrm{g}(\cdot,\cdot) \equiv (\cdot,\cdot)$; because of $\tau^2 = 1$, it is orthogonal moreover. From $g_\tau(\psi,\psi) \geq 0$ we conclude that the $+1$ and -1 eigen-subspaces of τ are positive and negative definite, respectively, again with respect to the canonical metric in E, while $(\tau\psi_1, \tau\psi_2) = (\psi_1, \psi_2)$ shows that they are orthogonal to one another. As a corollary, ${}^E\mathrm{g}$ having signature (n,n), the dimensions of the two eigenvalue subspaces are equal. Vice versa, let us take any positive definite subbundle V of maximal rank, then there exists a unique τ which is postulated to be 1 on V and -1 on the orthogonal subbundle $V^\perp$ (with respect to the canonical metric). The operator τ satisfies the properties of the definition as above.

Given a splitting of the exact sequence (2.16), there is a one-to-one correspondence between generalized Riemannian metrics in the sense of the Definition 3.5 and the one of Hitchin within this volume. Since V is positive, it has zero intersection with any Lagrangian subspace of E and in particular with TM and T^*M (i.e., with $j(TM) \cong TM$ and $\rho^*(T^*M) \cong T^*M$). Hence V is a graph of an invertible bundle map $TM \to T^*M$ which can be identified with a non-degenerate tensor in $\Gamma(T^*M \otimes T^*M)$ such that its symmetric component g in the decomposition into the symmetric and antisymmetric parts, $g + B$, is a Riemannian metric on TM.

Define a new real form by using the complexification of τ: $\tau_\sigma := \sigma\tau$. Since τ is a real operator, its complexification commutes with the complex conjugation σ, therefore $\tau_\sigma^2 = 1$. Moreover, τ_σ is an anti-linear operator as a composition of linear and anti-linear operators, thus it defines a new real structure in E^c (different from the old one σ). Hereafter we shall distinguish the σ- and τ_σ-real forms by calling them "real" and "τ-real", respectively.

Proposition 3.6. *The real and imaginary subbundles of E^c with respect to the real structure τ_σ are the subbundles (over $\mathbb{R}$) $E^+ := V \oplus iV^\perp$ and $E^- := V^\perp \oplus iV$, respectively. Any complex Dirac structure in E^c is a totally complex subbundle w.r.t. τ_σ, i.e., its intersections with the τ_σ-real and totally complex subbundles of E^c are trivial.*

Proof. The first statement follows trivially from the definition of τ_σ: one needs to take into account that τ_σ is anti-linear. To check the second statement, it suffices to notice that the (real) eigen-subspaces of the new real structure τ_σ, E^+ and E^-, are positive (negative) definite with respect to the restriction of the (complexified) Courant metric in E^c, therefore $E^\pm$ do not contain any nonzero isotropic vectors. □

Let D be a maximally isotropic complex subbundle of E^c; by the proposition above, D is a totally complex subbundle as well as $\tau_\sigma(D)$. Define a linear complex structure J in E by requiring $J(D) = i$ and $J(\tau_\sigma(D)) = -i$. By construction, J is a τ-real operator. Moreover, the restriction of J to E^+ is an orthogonal operator with respect to the induced positive metric in E^+.

Proposition 3.7. *There is a one-to-one correspondence between complex maximally isotropic subbundles in E^c and $\mathbb{R}$-linear orthogonal operators J in E^+ which satisfy $J^2 = -1$. In particular, if D is a real maximally isotropic subbundle, then J is uniquely represented by a real orthogonal operator $S : V \to iV^\perp$ as follows:*

$$J_\sigma : \begin{pmatrix} 0 & -S^{-1} \\ S & 0 \end{pmatrix}. \tag{3.1}$$

Proof. We need to check only the last statement. Suppose D is a complexification of some real maximally isotropic subbundle of E, then D is preserved by the real structure σ and $\tau_\sigma(D) = \tau(D)$. Now, by construction, J anti-commutes with σ. Let us identify E^+ with $V \oplus V^\perp$ such that σ acts as $+1$ on the first and as -1 on the second factors. This identification is an isometry, if we supply $V^\perp$ with an opposite metric (the original one on $V^\perp$ is negatively definite). Since J anti-commutes with σ, it has only off-diagonal entries. Now, taking into account that J is orthogonal and squares to -1, we immediately get the required form (3.1). □

Proposition 3.8. *Let D be a maximally isotropic real subbundle of E provided with the metric induced by g_τ. Then the projector of D to V and $V^\perp$ is an isometry up to a factor $\frac{1}{2}$ and $-\frac{1}{2}$, respectively.*

Proof. Taking into account that the orthogonal projector to V and $V^\perp$ can be expressed as $P_V = \frac{1}{2}(1+\tau)$ and $P_V^\perp = \frac{1}{2}(1-\tau)$, respectively, we obtain:

$$(P_V\psi_1, P_V\psi_2) = \tfrac{1}{2}((\tau\psi_1,\psi_2)+(\psi_1,\psi_2)),$$
$$(P_V^\perp\psi_1, P_V^\perp\psi_2) = \tfrac{1}{2}(-(\tau\psi_1,\psi_2)+(\psi_1,\psi_2))$$

for all $\psi_1, \psi_2 \in \Gamma(E)$. Imposing $\psi_i \in \Gamma(D)$, we get the required property. □

Let us take $D = T^*M$ supplied with the positive metric, induced by g_τ in E as above.

Corollary 3.9. *There is a one-to-one correspondence between real maximally isotropic subbundles and orthogonal operators acting point-wisely in TM.*

Proof. The composition of maps $P_V^\perp \circ (P_{V|D})^{-1}\colon V \to V^\perp$ is an isometry up to a factor -1 (an anti-isometry). Combining this map with the natural anti-isometry of real spaces, $V^\perp \to iV^\perp$, given by the multiplication with i, we get an isometric identification of V and $iV^\perp$, which allows to identify S with an orthogonal operator acting in V. Using the conjugation by P_V, we identify S with an orthogonal operator acting in T^*M or, by the duality, in TM. □

The above operator S is a section of $\mathrm{O}(TM)$, the associated bundle $P \times_{\mathrm{O}(n)} \mathrm{O}(n)$ where P is the bundle of orthogonal frames and $\mathrm{O}(n)$ acts on itself by conjugation. As it was constructed, the homotopy class of the section S depends only on the homotopy class of the corresponding maximally isotropic subbundle. One knows that, for a Lie group G, any principal G-bundle over M is the pull-back by the canonical one over the universal classifying space, $EG \to BG$ with respect to some map $\varphi_0\colon M \to BG$. Then $P \times_G Y$ is the pull-back of $EG \times_G Y$ for each G-space Y. Applying this to $G = Y = \mathrm{O}(n)$, one has the following commutative diagram:

$$\begin{array}{ccc} \mathrm{O}(TM) & \xrightarrow{\varphi} & E\mathrm{O}(n) \times_{\mathrm{O}(n)} \mathrm{O}(n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi_0} & B\mathrm{O}(n). \end{array} \tag{3.2}$$

Given any section $S\colon M \to \mathrm{O}(TM)$, the pull-back of $\varphi \circ S$ defines a map[11] from the equivariant cohomology of $\mathrm{O}(n)$ with coefficients in some ring $\mathcal{R}$ to the cohomology of M :

$$(\varphi \circ S)^*\colon H^i(E\mathrm{O}(n) \times_{\mathrm{O}(n)} \mathrm{O}(n), \mathcal{R}) = H^i(\mathrm{O}(n), \mathcal{R})^{\mathrm{O}(n)} \to H^i(M, \mathcal{R}). \tag{3.3}$$

Since all the arrows are defined up to homotopy, we obtain a characteristic map from the product of the set of homotopy classes of maximally isotropic subbundles with $H^i(\mathrm{O}(n), \mathcal{R})$ to the cohomology of M. The explicit construction of this characteristic

[11] This construction was suggested by A. Alekseev during our joint work on Dirac structures.

map for $\mathcal{R} = \mathbb{R}$ can be done by use of the secondary characteristic calculus. Let ∇ be the Levi-Civita connection of the induced metric, Φ an ad-invariant polynomial on the Lie algebra $\mathfrak{o}(n)$, then define the corresponding characteristic class as follows:

$$\mathrm{c}_{\Phi}(S, g) = \int_0^1 \Phi(\mathrm{d}t \wedge A + F(t)), \tag{3.4}$$

where $A = S^{-1}\nabla(S)$, and $F(t)$ is the curvature of $\nabla + tA$. One can easily check that

$$\mathrm{dc}_{\Phi}(S, g) = \Phi\left((S^{-1} \circ \nabla \circ S)^2\right) - \Phi(R) = 0,$$

where R is the curvature of ∇ (the Riemann curvature), since Φ is ad-invariant. The cohomology class of $c_{\Phi}(S, g)$ does not change under the homotopy of S and g.

Proposition 3.10. *Let D be a maximally isotropic subbundle of E. Suppose that there is a connection, i.e., an isotropic splitting of $0 \to T^*M \to E \to TM \to 0$, such that the projection of D to TM or T^*M is non-degenerate. Then the section S, constructed as above, is homotopic to 1 and -1, respectively, thus all characteristic classes vanish.*

Proof. Obviously, whatever metric g is taken, T^*M corresponds to $-1 \in \Gamma(\mathrm{O}(TM))$. Assume that the projection of D to T^*M is non-degenerate, then, apparently, D is homotopic to T^*M and thus the section S, corresponding to D, is homotopic to 1. If the projection of L to TM is non-degenerate, then D is homotopic to image TM and thus to any maximally isotropic subbundle with zero intersection with T^*M. Now it suffices to take the orthogonal complement of T^*M in E with respect to the generalized Riemann metric g_τ: this is a maximally isotropic subbundle of the required type which corresponds to 1. □

Proposition 3.11. *Let D and D' be maximally isotropic subbundles corresponding to S and $S' \in \Gamma(\mathrm{O}(T^*M))$, respectively, then the intersection $D \cap D'$ is zero if and only if $S^{-1}S' - 1$ is non-degenerate.*

Proof. The proof follows from the explicit parameterization of D by T^*M:

$$D_x = \{(1 + \tau)\eta + (1 - \tau)S\eta \mid \eta \in T_x^*M\}. \tag{3.5}$$

□

Corollary 3.12. *D and T^*M admit a common complementary maximal isotropic subbundle, if and only if there exists $R \in \Gamma(\mathrm{O}(TM))$ such that $R - 1$ and $RS - 1$ are non-degenerate.*

Previously we proved that if a Lagrangian subbundle D and T^*M has a common complementary maximally isotropic subbundle, then the operator S which corresponds to D can be deformed to the minus identity section (we take the connection in E such

that the image of TM coincides with the chosen common complementary subbundle). In general, we can not reverse this statement, i.e., even if D is homotopic to $T^*M^\perp$, there may be no splitting in E such that the projection of D to T^*M becomes non-degenerate.

Example 3.13. Let M be a 2-dimensional surface, then $\det S = \det R = 1$. Therefore both operators S and R can be thought of as points in S^1.

Let us assume that M contains a loop such that S passes $1 \in S^1$ and covers S^1 many times in both directions. We only claim that the total degree of the map from the loop to S^1 equals to zero (hence S is homotopically equivalent to the identity map). For example, $M = T^2$. One can take a smooth function $\phi(t) = a\sin(2\pi i t)$. The quotient $t \mapsto \phi(t) \mod \mathbb{Z}$ defines a smooth map $S^1 \to S^1$ of degree zero, because the limit of $\phi(t)$ at $a \to 0$ is zero. Nevertheless, the image of ϕ wraps S^1 many times for sufficiently large a. Now we can trivially extend the map to T^2 which gives S. Since the function, which would correspond to R, is not permitted to coincide with ϕ and to pass through 1, we conclude that there is "no room" for R.

4 Generalized complex structures

Generalized complex structures (GCS) were invented by Nigel Hitchin [20]. It turned out that GCS provides a mathematical background of certain sigma models (for instance, those the target space of which is endowed with a bihermitian metric, cf. [14], [27]). Generalized complex structures interpolate naturally between symplectic and holomorphic Poisson geometry.

Definition 4.1. Let E be an exact Courant algebroid. A generalized complex structure is a maximally isotropic pure totally complex Dirac subbundle of the complexification of E with the complexified Courant scalar product, that is, a maximally isotropic subbundle D of $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$ such that D is closed w.r.t. the Courant bracket and $D \cap \bar{D} = \{0\}$.[12]

As in Section 3, we uniquely associate a maximally isotropic totally complex subbundle D of E^c with a real point-wisely acting operator J which squares to $-\mathrm{Id}_E$, such that, at any $x \in M$ the fibers D_x and $\bar{D}_x$ are the $+i$ and $-i$ eigen-subspaces of J_x, respectively. The isotropy condition of D implies that J is skew-symmetric with respect to the Courant scalar product $(,)$. From now on we shall consider only an "untwisted" version of an exact Courant algebroid together with an isomorphism $E \simeq TM \oplus T^*M$, where the latter direct sum is supplied with the canonical scalar product and the (Dorfman) bracket (2.2). We also identify bivector fields and 2-forms on M with sections of $\mathrm{Hom}(T^*M, TM)$ and $\mathrm{Hom}(TM, T^*M)$, respectively, by use

[12] We call such a subbundle D totally complex because its "real part" $D \cap \bar{D}$ is zero.

of the corresponding contractions. The next lemma gives a complete set of algebraic conditions for J.

Lemma 4.2. *J is an operator of the form*

$$J = \begin{pmatrix} I & \Pi \\ \Omega & -I^* \end{pmatrix}, \tag{4.1}$$

where $I \in \Gamma(\operatorname{End} TM)$, $\Pi \in \Gamma(\Lambda^2 TM)$, and $\Omega \in \Omega^2(M)$, which renders the condition $J^2 = -\mathrm{Id}$ equivalent to the following relations:

$$\begin{aligned} I^2 + \Pi\Omega &= -\mathrm{Id}_{TM}, \\ (I^*)^2 + \Omega\Pi &= -\mathrm{Id}_{T^*M}, \\ I^*\Omega - \Omega I &= 0, \\ I\Pi - \Pi I^* &= 0. \end{aligned} \tag{4.2}$$

Proof. Straightforward calculation. □

Similarly to the end of Section 2, we treat differential forms on M as sections of the spin module over the Clifford bundle $C(E)$, that is, the bundle of associative algebras generated by E subject to the relations $\varphi^2 = (\varphi, \varphi)$ for all $x \in M$ and $\varphi \in E_x$. The spinor action of $\Gamma(C(E))$ on $\Omega^*(M)$ is the extension of (2.20). Taking into account that the generalized complex structure operator J is skew-symmetric with respect to the scalar product, it can be thought of as a section of $C(E)$, which we denote by the same letter J, such that $J(\psi) = [J, \psi]$ for each $\psi \in \Gamma(E)$. Here $[\ ,\ ]$ is the (super-)commutator of sections of the Clifford bundle. It is easy to check that

$$\begin{aligned} &\mathsf{c}(J)(\alpha_1 \wedge \cdots \wedge \alpha_p) \\ &\quad = -\sum_{i=1}^{p} \alpha_1 \wedge \cdots \wedge I^*(\alpha_i) \wedge \cdots \wedge \alpha_p + (\imath_\Pi + \Omega\wedge)\alpha_1 \wedge \cdots \wedge \alpha_p, \end{aligned} \tag{4.3}$$

where $\alpha_i \in \Omega^1(M)$, $i = 1, \ldots, p$.

We now focus on the question when a maximally isotropic totally complex subbundle D is closed w.r.t. the Courant bracket or, in other words, when it is integrable.

Proposition 4.3. *D is a generalized complex structure if and only if for any $\psi, \psi' \in \Gamma(E)$ one has*

$$J\left([J\psi, \psi'] + [\psi, J\psi']\right) + [\psi, \psi'] - [J\psi, J\psi'] = 0. \tag{4.4}$$

Proof. The l.h.s. of the equation (4.4) is bilinear and real, thus it is sufficient to check the property for $\psi_1, \bar{\psi}_2$ and ψ_1, ψ_2, where $\psi_i \in \Gamma(D)$. While in the first case the expression vanishes identically, the second one gives

$$J\left([J\psi_1, \psi_2] + [\psi_1 J\psi_2]\right) + [\psi_1, \psi_2] - [J\psi_1, J\psi_2] = 2i\left(J[\psi_1, \psi_2] - i[\psi_2, \psi_2]\right),$$

which is identically zero if and only if $[\psi_1, \psi_2]$ is again a section of D. The latter is nothing else but the integrability condition for D. □

The meaning of equation (4.4) is the same as for an almost complex structure in the usual sense, that is, the vanishing of a certain tensor called the (generalized) Nijenhuis tensor of J. The next lemma provides an explicit construction of the Nijenhuis tensor in terms of the spin module.

Lemma 4.4. *The operator* $N_J := \frac{1}{2}\left([\mathsf{c}(J), [\mathsf{c}(J), \mathrm{d}]] + \mathrm{d}\right)$ *is a point-wisely acting map, such that for any* $\psi, \psi' \in \Gamma(E)$ *the following identity holds:*

$$[\mathsf{c}(\psi'), [\mathsf{c}(\psi), N_J]] = \mathsf{c}\left(J\left([J\psi, \psi'] + [\psi, J\psi']\right) + [\psi, \psi'] - [J\psi, J\psi']\right).$$

Proof. The first part of the lemma follows from the identity

$$2[N_J, f] = [[\mathsf{c}(J), [\mathsf{c}(J), \mathrm{d}]], f] + [\mathrm{d}, f] = \mathsf{c}(J^2(\mathrm{d}f) + \mathrm{d}f) = 0,$$

which holds for each $f \in C^\infty(M)$. The second part requires a simple computation which is essentially based upon the derived formula of the Courant bracket (2.21) when $H = 0$. Indeed, let us write the Nijenhuis tensor in the form $N_J = \frac{1}{2}(\mathrm{ad}_J^2(\mathrm{d}) + \mathrm{d})$ where $\mathrm{ad}_J(a) := [\gamma_J, a]$ for any a. It is clear that

$$[\mathsf{c}(\psi), \mathrm{ad}_J^2(a)] = -[\mathsf{c}(\psi), a] - 2\mathrm{ad}_J\left([\mathsf{c}(J\psi), a]\right) + \mathrm{ad}_J^2\left([\mathsf{c}(\psi), a]\right).$$

Thus, taking into account that $[\mathrm{ad}_J, \mathsf{c}(\psi)] = \mathsf{c}(J\psi)$ and $J^2\psi = -\psi$, we obtain

$$[\mathsf{c}(\psi), \mathrm{ad}_J^2(\mathrm{d}) + \mathrm{d}] = -2\mathrm{ad}_J\left(L_{J\psi}\right) + \mathrm{ad}_J^2\left(L_\psi\right),$$

where $L_\psi := [\mathrm{d}, \mathsf{c}(\psi)]$, and finally

$$\begin{aligned}
[\mathsf{c}(\psi'), [\mathsf{c}(\psi), \mathrm{ad}_J^2(\mathrm{d}) + \mathrm{d}]] &= -2[\mathsf{c}(\psi'), \mathrm{ad}_J\left(L_{J\psi}\right)] + [\mathsf{c}(\psi'), \mathrm{ad}_J^2\left(L_\psi\right)] \\
&= 2[\mathsf{c}(J\psi'), L_{J\psi}] - 2\mathrm{ad}_J\left([\mathsf{c}(\psi'), L_{J\psi}]\right) \\
&\quad - 2\mathrm{ad}_J\left([\mathsf{c}(J\psi'), L_\psi]\right) - [\mathsf{c}(\psi'), L_\psi] \\
&\quad + \mathrm{ad}_J^2\left([\mathsf{c}(\psi'), L_\psi]\right) \\
&= 2\mathsf{c}\big(J\left([J\psi, \psi'] + [\psi, J\psi']\right) \\
&\quad + [\psi, \psi'] - [J\psi, J\psi']\big),
\end{aligned}$$

which completes the proof. □

As a corollary, the integrability condition (4.4) admits the following equivalent form:

$$[\mathsf{c}(J), [\mathsf{c}(J), \mathrm{d}]] + \mathrm{d} = 0. \tag{4.5}$$

Let us remark that, if we decompose the l.h.s. of (4.5) into the sum of homogeneous components with respect to the natural grading in $\Omega^*(M)$, the lowest degree term will give us $[\imath_\Pi, [\imath_\Pi, \mathrm{d}] = 0$; this is equal to $\imath_{[\Pi,\Pi]} = 0$, where $[\Pi, \Pi]$ is the Schouten–Nijenhuis bracket of Π. Thus the vanishing of N_J implies, in particular, that Π is a

Poisson bivector. The explicit derivation of the remaining homogeneous terms gives a complete set of compatibility conditions for I, Π, and ω (cf. [12] for the details).

Proposition 4.5. (1) *If $\omega = 0$ then I is a complex structure and Π is a real part of a complex Poisson bivector which is holomorphic with respect I.*

(2) *If, on the other hand, $I = 0$ we reobtain an ordinary symplectic structure from a generalized complex one, such that Π then is the respective induced Poisson structure.*

Proof. (1) The algebraic conditions (4.2), imposed on J, will give us $I^2 = -\mathrm{Id}_{TM}$ and $I\Pi = \Pi I^*$. The first identity implies that I is an almost complex structure in the usual sense, therefore the complexified tangent and cotangent bundles admit the usual decomposition into the sum of $(1,0)$ and $(0,1)$ parts with respect to I:

$$T(M)^c = T^{(1,0)} \oplus T^{(0,1)}, \quad T^*(M)^c = T^{(1,0)^*} \oplus T^{(0,1)^*}.$$

The second commutation relation implies that the complexification of Π belongs to the direct sum of $\mathrm{Hom}(T^{(1,0)^*}, T^{(1,0)})$ and $\mathrm{Hom}(T^{(0,1)^*}, T^{(0,1)})$. Taking into account that Π is real, we get $\Pi = \Pi_h + \bar{\Pi}_h$, where $\Pi_h \in \Gamma(\Lambda^2 T^{(1,0)} M)$. The degree-0 homogeneous component of the integrability condition (4.5) asserts the vanishing of the Nijenhuis tensor of I, which means that I is a complex structure on M in the usual sense. The degree-1 homogeneous component of (4.5) gives us

$$[I^*, \delta_\Pi] + [\mathrm{d}_I, \iota_\Pi] = 0, \tag{4.6}$$

where $\delta_\Pi := [\mathrm{d}, \Pi]$ and $\mathrm{d}_I := [\mathrm{d}, I^*]$. The Hodge decomposition of differential forms with respect to I allows to decompose d into the sum of $(1,0)$ and $(0,1)$ parts, $\partial + \bar{\partial}$, and ι_Π into the sum of $(2,0)$ and $(0,2)$ parts (the contractions with $\bar{\Pi}_h$ and Π_h, correspondingly). Similarly, we have the decomposition of d_I: $\mathrm{d}_I = i(\partial - \bar{\partial})$. Let us look at the components of the l.h.s. of (4.6) which are homogeneous with respect to the Hodge decomposition. The identity (4.6) gives us only two (dependent) conditions, the first of which is conjugated to the second one: $[\bar{\partial}, \iota_{\Pi_h}] = 0$ and $[\partial, \iota_{\bar{\Pi}_h}] = 0$. This holds if and only if Π_h is a holomorphic Poisson bivector.

(2) Let $I = 0$, then the algebraic relations (4.2) gives us the only independent condition $\Pi\omega = -\mathrm{Id}_{TM}$, which means that ω is a non-degenerate 2-form and Π is the corresponding bivector field (up to a sign convention). As it was mentioned above, once J is integrable, Π is necessarily Poisson. Thus ω has to be a symplectic form. Conversely, provided ω is a symplectic form, we define Π such that $\Pi\omega = -\mathrm{Id}_{TM}$. Now we need to check the integrability condition (4.5). This gives us the only identity to verify: $[\delta_\Pi, \omega\wedge] + \mathrm{d} = 0$. Since M is a symplectic manifold, its dimension is even, say $\dim M = 2m$. It is easy to check that $[\iota_\Pi, \omega\wedge] = \bar{n} - m$, where $\bar{n}$ is an operator which counts the degree of a differential form and m is simply the multiplication on m. The identity $\mathrm{d}\omega = 0$ is obviously equivalent to the commutator relation $[\mathrm{d}, \omega\wedge] = 0$.

Therefore

$$[\delta_\Pi, \omega\wedge] + \mathrm{d} = [[\imath_\Pi, \mathrm{d}], \omega\wedge] = -[\mathrm{d}, [\imath_\Pi, \omega\wedge]] + \mathrm{d}$$
$$= -[\mathrm{d}, \bar{n} - m] + \mathrm{d} = -\mathrm{d} + \mathrm{d} = 0. \qquad \square$$

5 Algebroids as Q-manifolds

Any Lie algebra $\mathfrak{g}$ gives rise to a complex $(\Lambda^\cdot\mathfrak{g}^*, \mathrm{d}_{CE})$, where d_{CE} denotes the Chevalley–Eilenberg differential, $\mathrm{d}_{CE}\alpha(\xi, \xi') = -\alpha([\xi, \xi'])$ for $\alpha \in \mathfrak{g}^*$ (and $\xi, \xi' \in \mathfrak{g}$) etc. By definition an element $\alpha \in \mathfrak{g}^*$ is a linear function on $\mathfrak{g}$. Correspondingly, we may identify $\Lambda^\cdot\mathfrak{g}^*$ with the space of functions on $\mathfrak{g}$ if we declare the multiplication of two elements $\alpha, \alpha' \in \mathfrak{g}^*$, regarded as two functions, to be anticommuting, $\alpha\alpha' = -\alpha'\alpha$. This modified law of pointwise multiplication of functions is denoted by an additional Π (indicating parity reversion), $\Lambda^\cdot\mathfrak{g}^* \cong C^\infty(\Pi\mathfrak{g})$. In fact, in what follows it will be important to not only consider a $\mathbb{Z}_2$-grading, but a $\mathbb{Z}$-grading, inducing the $\mathbb{Z}_2$-grading. Thus we declare elements of $\mathfrak{g}^*$ to have degree 1, and then

$$\Lambda^\cdot\mathfrak{g}^* \cong C^\infty(\mathfrak{g}[1]),$$

the bracket indicating that the canonical degree of the before mentioned vector space (which in the case of $\mathfrak{g}$ is zero) is shifted by minus one. Given a basis e_a of elements in $\mathfrak{g}$, its dual basis θ^a becomes a set of coordinates on $\mathfrak{g}[1]$, any homogeneous element $\omega \in C^\infty(\mathfrak{g}[1])$ can be written as $\omega = \frac{1}{p!}\omega_{a_1\ldots a_p}\theta^{a_1}\ldots\theta^{a_p}$, and the Chevalley–Eilenberg differential becomes a vector field of degree $+1$ (it raises the degree of homogeneity of any element by one), which we will denote by Q,

$$Q = -\frac{1}{2}\theta^a\theta^b C^c_{ab}\frac{\partial}{\partial\theta^c}, \tag{5.1}$$

where C^c_{ab} denote the structure constants, $[e_a, e_b] = C^c_{ab}e_c$. By construction, $Q^2 = 0$, i.e., the vector field Q is *homological*.

This construction generalizes to the case where the Lie algebra acts on a manifold M by vector fields, $\xi \mapsto \rho(\xi) \in \Gamma(TM)$. Denoting $\rho(e_a)$ by ρ_a, the following vector field

$$Q = \theta^a\rho_a - \frac{1}{2}\theta^a\theta^b C^c_{ab}\frac{\partial}{\partial\theta^c}, \tag{5.2}$$

on $M \times \mathfrak{g}[1]$ is again homological. The corresponding cohomology in degree zero (functions containing no θs) is obviously isomorphic to the space of functions on M invariant under the flow generated by the Lie algebra $\mathfrak{g}$. (5.2) and its cohomology may be viewed as the "BRST description" of the space of "gauge invariant" functions on M.

In the above ρ_a was a vector field on M, so in some local coordinates x^i of M one has $\rho_a = \rho^i_a\partial_i$, where ρ^i_a are (local) functions on M (and $\partial_i \equiv \partial/\partial x^i$). We may now

consider to drop the restriction that C^c_{ab} is constant in (5.2) but instead also a function on M and pose the question under what conditions on ρ^i_a and C^c_{ab} the corresponding vector field (5.2) squares to zero. In this context Q is a vector field on a *graded manifold* $\mathcal{M}$[13] where the structure sheaf has local generators x^i and θ^a of degree zero and one, respectively; in this case we say that $\mathcal{M}$ is of degree one. Note that since transition functions are by definition required to be degree-preserving, a change of chart in $\mathcal{M}$ requires $\tilde{\theta}^a = M^a_b \theta^b$, with M^a_b a local function on the *body* M of $\mathcal{M}$; correspondingly, a graded manifold of degree one is isomorphic to a vector bundle $E \to M$, θ_a corresponding to a frame of local sections. Indicating that fiber-linear coordinates on E have degree one in the superlanguage, one writes $\mathcal{M} \cong E[1]$ in this case.

A general graded manifold is equipped with an Euler vector field ϵ, such that the grading of the space of functions $\mathcal{F}(\mathcal{M})$ corresponds to the eigenvalue-decomposition of ϵ:

$$\mathcal{F}^k(\mathcal{M}) = \{f \in \mathcal{F}(\mathcal{M}) \mid \epsilon f = k f\}. \tag{5.3}$$

One may consider $k \in \mathbb{Z}$, but we will restrict ourselves to non-negative integers k except if explicitly stated otherwise.

Apparently, in the degree-1 case the extension of the algebra of functions on the body, $C^\infty(M) \cong \mathcal{F}^0(\mathcal{M})$, to the algebra of functions $\mathcal{F}(\mathcal{M})$ is generated by $\mathcal{F}^1(\mathcal{M})$, which can be thought of as the space of sections of a vector bundle (since $\mathcal{F}^1(\mathcal{M})$ has to be a locally free module over $\mathcal{F}^0(\mathcal{M})$). If such a graded manifold is equipped with a degree-$(+1)$ homological vector field Q, the most general ansatz of which has the form of (5.2) with now C^a_{bc} being permitted to be functions on M, i.e., if one considers what is called a *Q-manifold of degree one* [37], the vector bundle $E \to M$ becomes equipped with the structure of a Lie algebroid: This may be regarded as a reformulation of Proposition 2.7 by the simple identification $C^\infty(\mathcal{M}) \cong \Omega_E(M)$. Alternatively, one may check directly that $Q^2 = 0$ implies in homogeneity degree two the morphism property of ρ, Lemma 2.5, and in degree three the Jacobi condition (2.1), both expressed for the local frame e_a. Note that in this picture the Leibniz rule (1.4) follows only from a change of coordinates on $\mathcal{M}$, requiring $Q^2 = 0$ to be valid in all possible frames; on the other hand, the bracket becomes automatically antisymmetric when defined by means of $[e_a, e_b] := C^c_{ab} e_c$, again to hold in any frame (which turns out to be consistent with the Leibniz rule (1.4)).[14]

If in a Lie algebroid there exists a frame such that the homological vector field takes the form with *constant* structure functions C^c_{ab}, then this algebroid is called an *action Lie algebroid*. The bundle E is then isomorphic to $M \times \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$ acting on M. Certainly, in general a Lie algebroid is not of this form and there does

[13] In this article graded manifolds will always signify $\mathbb{Z}$-graded manifolds or even their special case of $\mathbb{N}_0$-graded manifolds (cf. also below in the text). The $\mathbb{Z}$-grading induces naturally a $\mathbb{Z}_2$-grading, which governs the signs of the algebra of "functions" on the graded manifold. By the forgetful functor a $\mathbb{Z}$-graded manifold thus also becomes a particular supermanifold.

[14] For a detailed discussion of such type of arguments, which, at least in a slightly more general context, turn out to be more tricky than one may expect at first sight, cf. [18], [17].

not exist a frame, not even locally, such that the structure functions would become constants.

On the other hand one may consider a graded manifold $\mathcal{M}$ that carries a *homogeneous* symplectic form ω of degree n, i.e.,

$$L_\epsilon \omega = n\omega, \tag{5.4}$$

in which case $\mathcal{M}$ is called a *P-manifold* of the degree n. Note that the non-degeneracy of the symplectic form then requires that also $\mathcal{M}$ has degree n (as mentioned, we do not consider graded manifolds with generators of negative degrees here, in which case this statement would no more be true). In the case of $n = 1$, we already found that $\mathcal{M}$ is canonically isomorphic to $E[1]$, E a vector bundle over M. It is now easy to see that the P-structure restricts this further [32], $\mathcal{M} \cong T^*[1]M$, $(x^i, \theta^a) \cong (x^i, p_i)$, equipped with the canonical symplectic form $\omega = \mathrm{d}x^i \wedge \mathrm{d}p_i$. Indeed, since the symplectic form is of degree $n = 1$, equation (5.4) implies $\omega = \mathrm{d}\alpha$ with $\alpha = \imath_\epsilon \omega$. Suppose, (x^i, θ^a) are local coordinates of degree 0 and 1, respectively. Taking into account that ω is of degree 1, we immediately obtain that the expression of the symplectic form cannot contain $\mathrm{d}\theta^a \wedge \mathrm{d}\theta^b$, and since $\epsilon = \theta^a \partial/\partial\theta^a$, α it has to be of the form: $\alpha = \alpha_i^j(x)\theta_j \mathrm{d}x^i$. α provides a morphism $TM \to E^*$; nondegeneracy of ω requires that this is an isomorphism, and thus $(\mathcal{M}, \omega)$ is isomorphic to $T^*[1]M$ together with the canonical symplectic form. Note that functions on $T^*[1]M$ may be identified with multivector fields. The odd Poisson bracket induced by ω is then easily identified with the Schouten–Nijenhuis bracket.

A *PQ-manifold* of degree $n \in \mathbb{N}_0$ is then simultaneously a Q- and a P-manifold of degree n, such that Q preserves the symplectic form ω, in which case it turns out to be even Hamiltonian (cf. Lemma 2.2 in [32]): $Q = \{\mathcal{Q}, \cdot\}$ for some function $\mathcal{Q}$ of degree $n + 1$ (since the Poisson bracket decreases the degree by n), $Q^2 = 0$ reducing to $\{\mathcal{Q}, \mathcal{Q}\} = 0$.

In general, we use the following sign conventions: the algebra of differential forms on a graded manifold $\mathcal{M}$ is defined as $C^\infty(T[1]\mathcal{M})$. Then the degree of $\mathrm{d}h$ is $|h| + 1$, where $|h|$ is the degree of $h \in C^\infty(\mathcal{M})$. The Hamiltonian function of a Hamiltonian vector field is obtained from the relation $\imath_{X_h}\omega = (-1)^{|h|+1}\mathrm{d}h$. The advantage of such a sign convention is that, if we produce a Poisson bracket by the formula $\{f, h\} = X_f(h)$, then the Lie algebra morphism property will hold: $[X_f, X_h] = X_{\{f,h\}}$. For a symplectic structure ω of degree n written in local Darboux coordinates $\omega = \sum_\alpha \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha$ the local Poisson bracket follow to be $\{p_\alpha, q^\beta\} = (-1)^{n|q^\alpha|}\delta_\alpha^\beta$.

Now it is easy to see that a PQ-manifold of degree one is in one-to-one correspondence with a Poisson manifold M: $\mathcal{M} \cong T^*[1]M$ and a degree-two function has the form $\mathcal{Q} \equiv \Pi = \frac{1}{2}\Pi^{ij} p_i p_j$, corresponding to a bivector field on M. The condition that the Schouten–Nijenhuis bracket of Π with itself vanishes, $\{\mathcal{Q}, \mathcal{Q}\} = 0$, is just one way of expressing the Jacobi condition for the Poisson bracket of functions (cf. also Examples 2.3 and 3.3). The usual Poisson bracket $\{\cdot, \cdot\}_M$ between functions on M is reobtained here as a derived bracket: $\{f, g\}_M = \{\{f, -\Pi\}, g\}$ for any

$f, g \in C^\infty(M) \subset C^\infty(\mathcal{M})$, since the right hand side indeed yields $\Pi^{ij} f_{,i}\, g_{,j}$ in local coordinates.

We quote a likewise result for the degree-two case without proof, which is due to Roytenberg [32]:

Theorem 5.1. *A P-manifold of degree two is in one-to-one correspondence with a pseudo-Euclidean vector bundle $(E, {}^E\mathrm{g})$. A PQ-manifold of degree two is in one-to-one correspondence with a Courant algebroid $(E, {}^E\mathrm{g}, \rho, [\cdot,\cdot])$.*

We add some remarks on this theorem for illustration. In appropriate Darboux-like coordinates on $\mathcal{M}$, the symplectic form reads

$$\omega = \mathrm{d}x^i \wedge \mathrm{d}p_i + \tfrac{1}{2}\kappa_{ab}\mathrm{d}\theta^a \wedge \mathrm{d}\theta^b, \tag{5.5}$$

where (x^i, θ^a, p_i) are coordinates of degree zero, one, and two, respectively, the vector bundle E corresponds to the graded submanifold spanned by (x^i, θ^a), and the constants κ_{ab} correspond to the fiber metric ${}^E\mathrm{g}$ evaluated in some orthonormal frame θ_a.

The symplectic form is of degree two, so $\mathcal{Q}$ is necessarily of degree three and thus of the form

$$\mathcal{Q} = \rho^i_a \theta^a p_i - \tfrac{1}{6} C_{abc}\theta^a\theta^b\theta^c, \tag{5.6}$$

with coefficient functions ρ^i_a and C_{abc} depending on x^i only. Sections ψ of E may be identified with functions of degree one on $\mathcal{M}$, $\psi = \psi^a\theta_a$, $\theta_a \equiv \kappa_{ab}\theta^b$, functions f on M with functions of degree zero on $\mathcal{M}$. The anchor ρ and the Courant bracket $[\cdot,\cdot]$ now follow as derived brackets,

$$\rho(\psi) f = \{\{\psi, \mathcal{Q}\}, f\}, \tag{5.7}$$

$$[\psi, \psi'] = \{\{\psi, \mathcal{Q}\}, \psi'\}, \tag{5.8}$$

while the fiber metric comes from the normal Poisson bracket: ${}^E\mathrm{g}(\psi, \psi') = \{\psi, \psi'\}$. From (5.7) one obtains in particular $\rho(\theta_a)x^i = \rho^i_a$ and $[\theta_a, \theta_b] = C_{ab}{}^c\theta_c$, where the last index in C_{abc} has been raised by means of the fiber metric and θ_a is regarded as a local (orthonormal) frame in E.

Note that a derived bracket is in general not antisymmetric; from (5.8) one concludes $[\psi, \psi] = \frac{1}{2}\{\{\psi, \psi\}, \mathcal{Q}\} = \frac{1}{2}\theta^a\rho^i_a\partial_i\, {}^E\mathrm{g}(\psi, \psi)$, which reproduces the last axiom in Definition 2.8. On the other hand, evaluated in an orthonormal frame θ_a the bracket does become antisymmetric, $[\theta_a, \theta_b] = C_{ab}{}^c\theta_c$.[15] It is also obvious from (5.7) and (5.8) that the Courant bracket satisfies the Leibniz property in the second entry of the bracket (while it does not in the first one due to the non-antisymmetry).

The study of higher degree PQ-manifolds is certainly more involved. They, however, always give rise to Loday algebroids in the following way

[15] The difference to the situation with C^c_{ab} in the vector field Q of a Lie algebroid is that there such an equation holds in all frames and also that here a change in frame results in a different transformation property of C_{abc} in (5.6) by lifting this transformation to a canonical one on $\mathcal{M}$ (which prescribes a particular induced transformation property for p_i). Cf. also [18] for many more details on this issue.

Proposition 5.2. *Given a PQ-manifold $\mathcal{M}$ of degree $n > 1$ the functions $\psi, \psi', \ldots$ of degree $n-1$ can be identified with sections in a vector bundle E. The formulas (5.7) and (5.8), where $\mathcal{Q}$ denotes the Hamiltonian for the Q-structure, equip E with the structure of a Loday algebroid.*

Proof. First we note that the functions of any fixed degree d on an $\mathbb{N}_0$-graded manifold are a *locally free module* over the functions of degree zero, which in turn are isomorphic to $C^\infty(M)$. This implies the existence of vector bundles for any of those degrees d over M. For $d = n-1$ we call this bundle E.

As remarked above, the vector field Q compatible with the symplectic form is always Hamiltonian (even for $n \in \mathbb{N}$); since the (graded) canonical Poisson bracket has degree $-n$, the respective Hamiltonian $\mathcal{Q}$ has to have degree $n+1$. By an elementary computation adding up respective degrees, one then finds that $\rho(\psi)$ indeed maps functions $f \in C^\infty(M)$ to functions and the bracket $[\cdot,\cdot]$ takes again values in the sections of E.[16] It is also obvious that $\rho(\psi)$ is a vector field on M, since the r.h.s. of the defining expression (5.7) satisfies an (ungraded) Leibniz rule for f being a product of two functions. To have an algebroid (cf. our Definition 1.1), we need to verify two things: first, the Leibniz property (1.4), which follows at once from the two defining expressions on E above and the graded Leibniz property of the Poisson bracket $\{\cdot,\cdot\}$. Second, that $\rho(\psi)$ is indeed $C^\infty(M)$-linear in ψ so as to really give rise to a bundle map $\rho\colon E \to TM$. The only potentially dangerous term which may violate this condition may arise when θ has terms quadratic or higher in the momenta p_i conjugate to the coordinates x^i on M. However, since the coordinates p_i necessarily have degree n and $\mathcal{Q}$ has degree $n+1$, this is not possible for $n > 1$.

We are left with verifying the Loday property (2.1). It is only here where the condition $\{\mathcal{Q},\mathcal{Q}\} = 0$, resulting from $Q^2 = 0$, comes into the game. We leave the respective calculation, which also makes use of the graded Jacobi identity of the Poisson bracket, as an exercise to the reader. □

Some remarks: First of all, it is clear from the above proof that any symplectic graded manifold of degree $n \geq 2$ equipped with an arbitrary function $\mathcal{Q}$ of degree $n+1$ gives, by the above construction, rise to an algebroid structure (in our sense, cf. Definition 1.1). If in addition $\{\mathcal{Q},\mathcal{Q}\} = 0$, this algebroid becomes a Loday algebroid.

Certainly, the higher in degrees we go, the more additional structures arise. Already for the case of $n = 2$ the Loday algebroid had the additional structures making it into a Courant algebroid. For higher n, however, there will often be other algebroids out of which the Loday algebroid will be composed. Let us illustrate this for $n = 3$: In Darboux coordinates the symplectic structure will have the form

$$\omega = \mathrm{d}x^i \wedge \mathrm{d}p_i + \mathrm{d}\theta^a \wedge \mathrm{d}\xi_a, \tag{5.9}$$

[16] We denote the expressions on the graded manifold and those isomorphic to them on $E \to M$ by the same symbols.

where the coordinates x^i, θ^a, ξ_a, and p_i have the degrees 0, 1, 2, and 3, respectively. From this we learn that a PQ-manifold of degree 3 has to be symplectomorphic to $T^*[3](V[1])$ with its canonical symplectic form. Here $V \to M$ is a vector bundle and the brackets indicate shifts of degree in the respective fiber coordinates (note that without a shift the momenta ξ_a conjugate to the degree-1 fiber linear coordinates θ^a on $V[1]$ have to have degree -1 while they are now shifted so as to have degree 2). With $\psi = \varphi^a\xi_a + \frac{1}{2}\alpha_{ab}\theta^a\theta^a$, we see that the above vector bundle E in this case is isomorphic to $V \oplus \Lambda^2 V^*$. Now let us consider the Hamiltonian for the Q-structure, i.e., a function of degree 4; it thus have to have the form

$$\mathcal{Q} = \rho^i_a\theta^a - \tfrac{1}{2}C^c_{ab}\theta^a\theta^b\xi_c + \tfrac{1}{2}\beta^{ab}\xi_a\xi_b + \tfrac{1}{24}\gamma_{abcd}\theta^a\theta^b\theta^c\theta^d. \tag{5.10}$$

Obviously, for $\beta = \gamma = 0$ this is nothing but the canonical lift of the Q-structure (5.2) corresponding to a Lie algebroid, thus equipping V with the structure of a Lie algebroid. In the general case V still is an almost Lie algebroid and E can be considered as an appropriate extension into a Loday algebroid. (V itself is *not* a Loday algebroid, having a Jacobiator controlled by a contraction of β and γ. Adding a V-2-form α to the section φ, on which the anchor ρ acts trivially, one can restore the Loday property (2.1).)

We know that for $n = 1$ we also obtained an algebroid, even a Lie algebroid. However, this was defined on T^*M. The above construction leads to a trivial $\mathbb{R}$-bundle over M instead, the sections of which can be identified with functions on M, and $\rho\colon C^\infty(M) \to \Gamma(TM)$ is $\mathbb{R}$-linear but no more $C^\infty(M)$-linear, in fact it corresponds (up to a sign being subject of conventions) to the map from functions to their Hamiltonian vector fields. In fact, both equations (5.8) and (5.7) become equivalent in this degenerate case, equipping $C^\infty(M)$ with the Poisson bracket $\{\cdot,\cdot\}_M$, as already remarked above. (This bracket defines a Lie algebra structure on the sections of $M \times \mathbb{R}$, but not a Lie algebroid or even a general algebroid structure on this bundle since one does not have an anchor map for it).

To obtain the Lie algebroid structure on T^*M from the $(P)Q$-manifold $T^*[1]M$ we need to proceed differently. In fact, this provides a procedure that is applicable for any Lie algebroid $E \to M$, corresponding to a degree-(-1) Q-manifold $E[1]$. Sections of E are in 1-1 correspondence with vector fields $\psi, \psi', \ldots$ of degree -1 on $E[1]$, and their derived bracket $[[\psi, Q], \psi']$, where the brackets denote the (super)commutator of vector fields, is rather easily verified to reproduce the Lie algebroid bracket between the respective sections.

This procedure can in fact be considered for a Q-manifold of any degree n. While for general n this leads to what one may call Vinogradov algebroids (Loday algebroids with additional structures, cf. [17]), for $n = 2$ one obtains a V-twisted Courant algebroid as defined in Section 2 above. The degree-(-1) vector fields define the sections of the bundle E, the degree-(-2) vector fields the sections of the other bundle V; the derived bracket $[[\psi, Q], \psi']$ of the degree-(-1) vector fields defines the Loday bracket on E, the ordinary commutator bracket $[\psi, \psi']$, which apparently takes values in the degree-(-2) vector fields, gives the V-valued inner product on E. All the

defining properties of a V-twisted Courant algebroid are rather easy to verify in this case.

On the other hand, a Q-manifold $\mathcal{M}$ of degree 2 gives always rise to $V[2] \to \mathcal{M} \to W[1]$, where the first map is an embedding, setting the degree-one coordinates in $\mathcal{M}$ to zero and the second map a projection, forgetting about the degree-two coordinates. (Here V is the same bundle as the one in the previous paragraph and W another vector bundle over the same base manifold). A somewhat lengthy analyses (cf. [17]) shows that after one has chosen an embedding of $W[1]$ into $\mathcal{M}$, which composed with the projection giving the identity map, the degree-two Q-manifold is in bijection with a Lie 2-algebroid. The vector bundle E of the V-twisted Courant algebroid picture is then composed of W and V, similarly to the situation of the degree-three PQ-manifold discussed above; one finds easily that $E \cong W \oplus W^* \otimes V$, and, under only a few more assumptions (like that the rank of V is at least two) on a V-twisted Courant algebroid of this form, also vice versa, the latter arises always from a Lie 2-algebroid in such a way.

We finally remark that if $\mathcal{M}$ is a Q-manifold of degree n, then $T^*[n]\mathcal{M}$ is a PQ-manifold of the same degree.[17] E.g. $T^*[1]E[1]$, with $E[1]$ a Lie algebroid (and Q lifted canonically, certainly), is a degree-1 PQ-manifold, and thus isomorphic to the Lie algebroid of a Poisson manifold. Indeed, the dual bundle of a Lie algebroid is canonically a Poisson manifold,[18] in accordance with the easy-to-verify isomorphism $T^*[1]E[1] \cong T^*[1]E$. So, Poisson geometry on M can be viewed as a particular case of Lie algebroid geometry (considering T^*M), but also vice versa, a Lie algebroid structure on E as a particular (fiber-linear) Poisson structure (on E^*) – and likewise so in higher degrees: A Courant algebroid is a particular case of a V-twisted Courant algebroid and also of a Lie 2-algebroid, but, in an appropriate sense, also Lie 2-algebroids and their corresponding V-twisted Courant algebroids, can be viewed as particular Courant algebroids. However, as already the example of a Lie algebroid shows, this is not always the most convenient way of viewing them.

We conclude this section by returning to the question of morphisms of Lie algebroids and, more generally, of any algebroid described by a Q- or a PQ-manifold. A morphism of Q-manifolds is a degree-preserving map $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ such that its pullback $\varphi^*\colon C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ is a chain map, i.e., one has $Q_1 \circ \varphi^* = \varphi^* \circ Q_2$. It is a morphism of PQ-manifolds, if in addition it preserves the symplectic form, $\varphi^*\omega_2 = \omega_1$.

[17] T.S. is grateful to D. Roytenberg for this remark in the context of a talk on V-twisted Courant algebroids.

[18] The dual E^* of any Lie algebroid E becomes a Poisson manifold in the following manner: In order to define a Poisson bracket on E^*, it is obviously sufficient to do so on the fiber constant and fiber linear functions. The former are functions that arise as pullbacks from functions on the base manifold M, the latter are sections of E. It is then straightforward to verify that

$$\{f, f'\} = 0, \quad \{\psi, f\} = \rho(\psi)f, \quad \{\psi, \psi'\} = [\psi, \psi'], \tag{5.11}$$

valid for all $f, f' \in C^\infty(M)$, $\psi, \psi' \in \Gamma(E)$, defines a Poisson structure on E^*.

6 Sigma models in the AKSZ-scheme

In this section we want to discuss a particular class of topological sigma models that can be constructed in the context of algebroids. By topological we want to understand that the space of solutions to the classical field equations (the Euler–Lagrange equations of the functional) modulo gauge transformations does not depend on structures defined on the base manifold Σ in addition to its topology and that for "reasonable" topology (the fundamental group of Σ having finite rank etc) it is finite dimensional.

In the context of ordinary gauge theories, one such a well-known space is the moduli space of flat connections on Σ. A functional producing such a moduli space is the Chern–Simons theory

$$S_{\mathrm{CS}}[A] = \frac{1}{2}\int_{\Sigma} \kappa\left(A \overset{\wedge}{,} (\mathrm{d}A + \tfrac{1}{3}[A \overset{\wedge}{,} A])\right), \tag{6.1}$$

defined on the space of connections of a trivialized G-bundle over an orientable 3-dimensional base manifold Σ when specifying κ such that $(\mathfrak{g} = \mathrm{Lie}(G), [\cdot,\cdot], \kappa)$ gives a quadratic Lie algebra; such a connection is represented by a $\mathfrak{g}$-valued 1-form A on Σ, $A \in \Omega^1(\Sigma, \mathfrak{g})$.

To find an appropriate generalization of this theory to the present context, let us first reinterpret the fields A and the field equations $F = 0$ of this model within the present context such that it permits a straightforward generalization. First of all, a $\mathfrak{g}$-valued 1-form A on Σ is evidently equivalent to a degree-preserving map (a morphism)

$$a\colon T[1]\Sigma \to \mathfrak{g}[1]. \tag{6.2}$$

If e_a denotes a basis of $\mathfrak{g}$ and θ^a the linear odd coordinates on $\mathfrak{g}[1]$ corresponding to a dual basis, then the 1-forms $A^a \equiv A^a_\mu \vartheta^\mu$ in $A = A^a \otimes e_a$ are given by the pullback of θ^a with respect to the map a, $A^a = a^*(\theta^a)$; here $\vartheta^\mu = \mathrm{d}\sigma^\mu$ are the degree-one coordinates on $T[1]\Sigma$ induced by (local) coordinates σ^μ on Σ. This is easy to generalize, given the background of the previous sections: As our generalized gauge fields we will consider morphisms of graded manifolds

$$a\colon \mathcal{M}_1 \to \mathcal{M}_2 \tag{6.3}$$

keeping $\mathcal{M}_1 = T[1]\Sigma$ (so these gauge fields will be a collection of differential forms on Σ of various form degrees) and taking $\mathcal{M}_2$ as a more general $\mathbb{N}_0$-graded manifold than $\mathfrak{g}[1]$.

In the example, not only $\mathcal{M}_1 = T[1]\Sigma$ but also $\mathcal{M}_2 = \mathfrak{g}[1]$ is not any graded manifold, but even a Q-manifold, Q_1 is the de Rham differential and Q_2 the Chevalley–Eilenberg differential (5.1). Let us reconsider the field equations $F = 0$ with $F = F^a \otimes e_a$ the curvature (1.6) of A in this context. Obviously $F^a = 0$, iff $\mathrm{d}A^a = -C^a_{bc} A^b \wedge A^c$. With the identification of the vector fields Q_1 and Q_2 above, this in turn is now seen to be $Q_1 a^*(\theta^a) = a^* Q_2(\theta^a)$. In other words, the map $a^*\colon C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ needs to be a chain map, or, according to the definition of a Q-morphism, the field equations of the Chern–Simons gauge theory express that

(6.3) is not only a morphism of graded manifolds but even a Q-morphism (a morphism of differential graded manifolds).

We are thus searching a functional defined on morphisms (6.3) such that its Euler–Lagrange equations forces these to become morphisms

$$a\colon (\mathcal{M}_1, Q_1) \to (\mathcal{M}_2, Q_2). \tag{6.4}$$

Certainly, for a true generalization of the *gauge* theory defined by means of (6.1), we also need to reinterpret its gauge transformations appropriately, so that we can formulate also the desiderata for the gauge symmetries of the searched for action functional. It turns out that on the solutions of (6.4) the gauge symmetries receive the interpretation of Q-homotopy.[19] We are thus searching for a functional defined on (6.3) such that the moduli space of classical solutions modulo gauge transformations is the space of Q-morphisms from $(T[1]\Sigma, \mathrm{d})$ to the target Q-manifold $(\mathcal{M}_2, Q_2)$ modulo Q-homotopy.

A functional can be obtained by the so-called AKSZ-method [1] (cf. also [7] and [31]) for the case that the target carries also a compatible symplectic form, i.e., that the target is a PQ-manifold. However, this method yields in fact already the BV extension of the searched-for ("classical") functional. We thus briefly recall some basic ingredients of the BV formalism and we will do this at the example of a toy model, where we consider functions instead of functionals, as well as for the Chern–Simons theory (6.1) above.

The toy model is the following one: Consider a function S_{cl} on a manifold M invariant with respect to the action of some Lie algebra $\mathfrak{g}$: $v(S_{\mathrm{cl}})(x) = 0$ for all points $x \in M$ and all elements $v \in \Gamma(TM)$ corresponding to the action of an element of $\mathfrak{g}$. We learnt that $M \times \mathfrak{g}$ carries the structure of a Lie algebroid, the action Lie algebroid. Let us thus, more generally, consider a Lie algebroid $E \to M$ together with a function $S_{\mathrm{cl}} \in C^\infty(M)$ constant along the Lie algebroid orbits on M, i.e., such that $\rho(\psi)S_{\mathrm{cl}} = 0$ for all $\psi \in \Gamma(E)$.

$S_{\mathrm{cl}}(x)$ is supposed to mimic a functional invariant w.r.t. some gauge transformations (in the more general sense, cf., e.g., [19], also for general details on the BV formalism). Consider for simplicity first the case of the action Lie algebroid again. The gauge invariance here corresponds to $\delta_\epsilon S_{\mathrm{cl}} = 0$ where $\delta_\epsilon x^i = \epsilon^a \rho_a^i(x)$ with some arbitrary $\mathfrak{g}$-valued parameters $\epsilon = \epsilon^a e_a$ (and $\rho(e_a) = \rho_a^i(x)\frac{\partial}{\partial x^i}$). In the BRST–BV approach one first replaces the parameters ϵ^a by anticommuting variables θ^a so that the action of the BRST operator δ on the original, classical fields (coordinates here) becomes $\delta x^i = \theta^a \rho_a^i(x)$. This is completed by the action of δ on the "odd parameters" θ^a, $\delta\theta^a = -\frac{1}{2}C_{bc}^a \theta^b\theta^c$ rendering δ nilpotent, $\delta^2 = 0$. In fact, this BRST operator is evidently nothing but the operator Q defined on $E[1]$, $\delta \equiv Q$, cf. equation (5.2), and this works for a general Lie algebroid E with

$$\rho(e_a) = \rho_a^i(x)\frac{\partial}{\partial x^i}, \quad [e_a, e_b] = C_{ab}^c(x)e_c, \tag{6.5}$$

[19] We refer to [4] for the details.

where now e_a is a local frame of sections of E and C^c_{ab} thus became structure functions (instead of just structure constants).

To obtain the BV-form of the action, one now turns to the (graded) phase space version of this, i.e., one introduces momenta (shifted in degree, cf. below) for each of the fields (coordinates on $E[1]$ in our example), which, conventionally, are called the *antifields*. In our example x^*_i conjugate to x^i and θ^*_a conjugate to θ^a. Adding the Hamiltonian lift of δ to the classical action we obtain

$$S_{\mathrm{BV}} = S_{\mathrm{cl}}(x) + \theta^a \rho^i_a(x) x^*_i - \tfrac{1}{2} C^c_{ab}(x)\theta^a\theta^b\theta^*_c. \tag{6.6}$$

To have this to have a uniform total degree, we need to shift the momenta in degree by minus one, $\deg x^*_i = -1$, $\deg \theta^*_a = -2$. Thus the BV odd phase space we are looking at together with the BV-function are of the form

$$\mathcal{M}_{\mathrm{BV}} = T^*[-1]E[1], \quad \omega_{\mathrm{BV}} = \mathrm{d}x^*_i \wedge \mathrm{d}x^i + \mathrm{d}\theta^*_a \wedge \mathrm{d}\theta^a, \quad S_{\mathrm{BV}} = S_{\mathrm{cl}} + \mathcal{Q}, \tag{6.7}$$

where $\mathcal{Q}$ is the Hamiltonian of the canonical Hamiltonian lift of the vector field Q of the Lie algebroid $E[1]$, the odd Poisson bracket, the so-called BV-bracket $\{\cdot,\cdot\}_{\mathrm{BV}}$ has degree +1, so that $Q_{\mathrm{BV}} = \{S_{\mathrm{BV}}, \cdot\}_{\mathrm{BV}}$ has degree $+1$ as well. Q_{BV} is a differential, i.e., S_{BV} satisfies the so-called *classical master equation*

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\}_{\mathrm{BV}} = 0, \tag{6.8}$$

which is completely obvious from our perspective: $\mathcal{Q}$ Poisson commutes with itself since it is the Hamiltonian for the Lie algebroid differential Q (there are no "odd constants", $\{\mathcal{Q},\mathcal{Q}\}_{\mathrm{BV}}$ having degree 1), $S_{\mathrm{cl}}(x)$ Poisson commutes with itself since it depends on coordinates only, and $\{S_{\mathrm{cl}}(x), \mathcal{Q}\} = 0$ since it corresponds to the Lie algebroid action on S_{cl}, which is zero by assumption.

So, $(\mathcal{M}_{\mathrm{BV}}, \omega_{\mathrm{BV}}, Q_{\mathrm{BV}} = (S_{\mathrm{BV}}, \cdot)_{\mathrm{BV}})$ defines a PQ-manifold. In contrast to the previous $\mathbb{Z}$-graded Q-manifolds, this PQ-manifold also has negative degree generators. In fact, it is a cotangent bundle of an $\mathbb{N}_0$-graded Q-manifold with a shift in the cotangent coordinate degrees such that it is precisely the momenta (antifields) that have negative degrees. The total degree is called the *ghost number* conventionally; so the classical fields (coordinates x in the toy model) have ghost number zero, the "odd gauge parameters" θ^a, the ghosts, have ghost number one, and the antifields have negative ghost number. We can also just consider the number of momenta or antifields: denoting this number by a subscript, we see that $S_{\mathrm{BV}} = S_0 + S_1$ here, where $S_0 = S_{\mathrm{cl}}$ and $S_1 = \mathcal{Q}$.

In general, the BV formalism is more involved, there can be terms of higher subscript. Still, always S_0 is the classical action S_{cl}. Also we see that $\{x^*_i, S_{\mathrm{BV}}\}_{\mathrm{BV}}|_0 = \frac{\partial S_{\mathrm{cl}}}{\partial x^i}$. Applying Q_{BV} to the classical fields and setting the antifield-less part to zero yields the critical points of S_{cl}, i.e., the classical field equations.

We now turn to the BV formulation of the Chern–Simons gauge theory (6.1). As before in (6.6), we add to the classical action (6.1) a term linear in the classical antifields A^*_a with a coefficient that is the (infinitesimal) gauge transformations, replacing the gauge parameters by odd fields β^a (so, naturally, A^*_a should be a 2-form on Σ), and we

complete the expression by a term proportional to the odd (anti)fields β^*_a, 3-form on Σ, containing the structure constants such that the master equation (6.8) is satisfied;[20] this yields

$$S_{\text{CS-BV}}[A, A^*, \beta, \beta^*] = S_{\text{CS}}[A] + \int_\Sigma (\mathrm{d}\beta^a + C^a_{bc} A^b \beta^c) A^*_a + \tfrac{1}{2} C^c_{ab} \beta^a \beta^b \beta^*_c \quad (6.9)$$

which is in a striking similarity with our toy model (6.6). Remembering that we could rewrite the toy model in a much more elegant form using the Q-language, cf. (6.7), we strive for a similar simplification in the present context.

For this purpose we first recall that the quadratic Lie algebra used to define the Chern–Simons theory is a Courant algebroid over a point, which in turn is a degree-2 PQ-manifold over a point (cf. Theorem 5.1):

$$\mathcal{M}_2 = \mathfrak{g}[1], \quad \omega = \tfrac{1}{2}\kappa_{ab}\mathrm{d}\theta^a \wedge \mathrm{d}\theta^b, \quad \mathcal{Q} = -\tfrac{1}{6} C_{abc}\theta^a\theta^b\theta^c. \quad (6.10)$$

This is the target of the map (6.3), with the source, $\mathcal{M}_1 = T[1]\Sigma$, being a Q-manifold ($Q_1 = \mathrm{d}$). The map (6.3) corresponds to the classical fields A, which we amended with further fields β, β^*, A^* above.

It is tempting to collect all these fields together into a super field (indices were raised by means of κ)

$$\mathcal{A}^a = \beta^a + A^a + A^a_* + \beta^a_* \quad (6.11)$$

by adding them up with increasing form degrees. In fact, this corresponds to an extension of the morphism (6.3) to what is called a *map* $\tilde{a}$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. But before commenting on this extension on this more abstract level, we want to first convince ourselves that the concrete expression (6.11) is useful. Let us consider the action (6.1) simply replacing A by $\mathcal{A}$ – and clearly keeping only the top degree forms for the integration over Σ. Viewing differential forms on Σ as graded functions on $T[1]\Sigma$, we can also write this as a Berezin integral over that graded manifold and we will partially do so in what follows. Let us consider the first part of this action first:

$$\begin{aligned} S_{\text{source}}[\mathcal{A}] &= \frac{1}{2}\int_{T[1]\Sigma} \kappa(\mathcal{A}\,\hat{,}\,\mathrm{d}\mathcal{A}) \\ &= \frac{1}{2}\int_\Sigma \kappa(\beta, \mathrm{d}A^*) + \kappa(A\,\hat{,}\,\mathrm{d}A) + \kappa(A^*\,\hat{,}\,\mathrm{d}\beta). \end{aligned} \quad (6.12)$$

We see that taking Σ to have no boundary for simplicity, with appropriate sign rules (see 6.20, in general), the first and the third term become identical and they reproduce all the terms containing a d in the BV-action above. Similarly,

$$S_{\text{target}}[\mathcal{A}] = \frac{1}{6}\int_{T[1]\Sigma} \kappa(\mathcal{A}\,\hat{,}\,[\mathcal{A}\,\hat{,}\,\mathcal{A}])) \equiv \frac{1}{6}\int_{T[1]\Sigma} C_{abc}\,\mathcal{A}^a \wedge \mathcal{A}^b \wedge \mathcal{A}^c \quad (6.13)$$

[20] Again, also this example is a relatively simple one for what concerns the BV formalism and the simpler BRST approach would be sufficient to yield the same results. However, already the models generalizing the Chern–Simons theory that we will discuss below, like the Poisson sigma model or the more general AKSZ sigma model, have a more intricate ghost and antifield structure.

is easily seen to reproduce the remaining terms in (6.9). So we see that

$$S_{\text{CS-BV}}[A, A^*, \beta, \beta^*] = S_{\text{CS}}[\mathcal{A}] = S_{\text{source}}[\mathcal{A}] + S_{\text{target}}[\mathcal{A}] \tag{6.14}$$

and we will strive at understanding this action in analogy to the toy model situation (6.7).

For this purpose we need to identify the BV phase space of this situation. We thus first return to a further discussion of the superfields (6.11). The notion of a smooth map of graded manifolds is an extension of the notion of a morphism:[21]

$$\tilde{a} \in \text{Map}(\mathcal{M}_1, \mathcal{M}_2), \quad a \in \text{Mor}(\mathcal{M}_1, \mathcal{M}_2) = \text{Map}_0(\mathcal{M}_1, \mathcal{M}_2), \quad P_0\tilde{a} = a. \tag{6.15}$$

In the case of a flat target manifold like $\mathfrak{g}[1]$ the description of a smooth map is rather clear from the example (6.11): we (formally) allow the functional dependence of the target coordinates on arbitrary degrees on the source. $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$ is naturally graded: The coefficients in the expansion like (6.11) are coordinates on this map space. Since the coordinates on the target $\mathcal{M}_2 = \mathfrak{g}[1]$ have degree 1, each term in the expansion (6.11) has degree 1 as well. Correspondingly, the ghost β^a has degree 1, $A^a_\mu(\sigma)$, the coefficient in $A^a = A^a_\mu \vartheta^\mu$ has degree 0 (since $\vartheta^\mu = \mathrm{d}\sigma^\mu$, coordinates on the source $T[1]\Sigma$, have degree 1), $(A^a_*)_{\mu\nu}(\sigma)$, the coefficients of the 2-form field A^a_*, are fields of degree -1, etc. A field (or antifield) is the same as a coordinate on $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$. Its degree-zero part $\text{Map}_0(\mathcal{M}_1, \mathcal{M}_2)$ is the space of morphisms or the space of the original classical maps a, which correspond to the (classical) fields $A^a_\mu(\sigma)$. a results from $\tilde{a}$ by projection (formal operator P_0 in (6.15)) to its degree-zero part, which are those maps that are degree-preserving: in the example (6.11) this is keeping the second term.

Generally, the space of maps between graded manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, denoted as $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$, is uniquely determined by the functorial property (cf. for example [13] or also [32]): for any graded manifold Z and a morphism $\psi : Z \times \mathcal{M}_1 \to \mathcal{M}_2$ there exists a morphism $\tilde{\psi} : Z \to \text{Map}(\mathcal{M}_1, \mathcal{M}_2)$ such that $\psi = \widetilde{\text{ev}} \circ (\tilde{\psi} \times \text{Id})$, where the evaluation map $\widetilde{\text{ev}}$ is (formally) defined in the obvious way:

$$\widetilde{\text{ev}} : \text{Map}(\mathcal{M}_1, \mathcal{M}_2) \times \mathcal{M}_1 \to \mathcal{M}_2, \ (\tilde{a}, \tilde{\sigma}) \mapsto \tilde{a}(\tilde{\sigma}). \tag{6.16}$$

Though the map $\tilde{a}$ underlying (6.11) is not a morphism of graded manifolds (it is does not even induce an ungraded morphism of the associative algebras of functions!), the evaluation map $\widetilde{\text{ev}}$ is.

So the BV-phase space is $\mathcal{M}_{\text{CS-BV}} = \text{Map}(T[1]\Sigma, \mathfrak{g}[1])$, or, more generally, $\mathcal{M}_{\text{BV}} = \text{Map}(\mathcal{M}_1, \mathcal{M}_2)$. It is canonically an odd (infinite dimensional, weakly) symplectic manifold. The symplectic form ω_{BV} is induced by means of the one on the

[21] Strictly speaking, the first and consequently the third formula do not make sense: $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$ turns out to be an infinite dimensional graded manifold and, as any graded manifold, only its degree-zero part, the body, contains points; a graded manifold, like a supermanifold, is not even a set. Still, it is useful to *think* like this; like everything in supergeometry, the real definitions are to be given algebraically on the dual level.

target, equation (6.10):

$$\omega_{\mathrm{BV}} = \int_{T[1]\Sigma} \widetilde{\mathrm{ev}}^*\omega = \frac{1}{2}\int_{T[1]\Sigma} \kappa_{ab}\, \delta\mathcal{A}^a \wedge \delta\mathcal{A}^b, \tag{6.17}$$

where δ denotes the de Rahm differential on $\mathcal{M}_{\mathrm{BV}}$ (so as to clearly distinguish it from the *vector field* d on $\mathcal{M}_1$!). Note that the pullback by means of the evaluation map (6.16) of the 2-form ω produces a (highly degenerate) 2-form on $\mathcal{M}_{\mathrm{BV}} \times \mathcal{M}_1$, which, moreover, still has degree 2 since $\widetilde{\mathrm{ev}}$ is a morphism, as remarked above. The Berezin integration over $T[1]\Sigma$ then reduces the degree of the resulting differential form by 3 (since Σ is 3-dimensional), so that ω_{BV} has degree -1, such as in our toy model (6.7). We could write out the right hand side of (6.17) similarly to (6.12); this then makes it clear that, after the integration, the resulting 2-form is indeed (weakly) non-degenerate. Let us stress at this point that certainly this construction would *not* work when sticking to the purely classical fields A: A likewise expression $\int_\Sigma \kappa(\delta A \overset{\wedge}{,} \delta A)$ would be nonzero only for a surface Σ of dimension *two* – in fact, this then is the symplectic form of the classical phase space of the Chern–Simons theory. So, also at this point the extension from $a \in \mathrm{Mor}(\mathcal{M}_1, \mathcal{M}_2)$ to $\tilde{a} \in \mathrm{Map}(\mathcal{M}_1, \mathcal{M}_2) \equiv \mathcal{M}_{\mathrm{BV}}$ is essential, because only the latter space is naturally symplectic, and indeed the symplectic form has degree -1 so that the BV-bracket will have degree $+1$, as it should be.

From the above the first steps in the generalization of the Chern–Simons BV-action to a more general setting is clear: We will keep $(T[1]\Sigma, \mathrm{d})$ as our source Q-manifold $(\mathcal{M}_1, Q_1)$, with Σ having a dimension d different from three in general. As target we need to choose at least a symplectic graded manifold, but in fact, like in our guiding example (6.10), we will consider a PQ-manifold $(\mathcal{M}_2, \omega_2, Q_2)$, of degree n in general (for $n > 0$ we can also replace the symplectic vector field Q_2 by its generating Hamiltonian function $\mathcal{Q}_2$ of degree $n+1$). Now $\int_{T[1]\Sigma} \widetilde{\mathrm{ev}}^*\omega_2$ gives a degree-(-1) symplectic 2-form on $\mathcal{M}_{\mathrm{BV}} = \mathrm{Map}(\mathcal{M}_1, \mathcal{M}_2)$, iff $d = n+1$ (since the Berezin integration reduces the degree of the 2-form by d).

It remains to rewrite the action (6.9) or (6.14) in a form that will resemble somewhat the BV-function of the toymodel (6.7). In particular, according to our assumptions on source $\mathcal{M}_1 = T[1]\Sigma$ and target (6.10), we are having a vector field $Q_1 = \mathrm{d}$ and $Q_2 = \{\mathcal{Q}, \cdot\}$ on the source and the target, respectively. Both of them give rise to a vector field on $\mathcal{M}_{\mathrm{BV}} = \mathrm{Map}(\mathcal{M}_1, \mathcal{M}_2)$:[22] Identifying the tangent space at $\tilde{a} \in \mathcal{M}_{\mathrm{BV}}$ with $\Gamma(\mathcal{M}_1, \tilde{a}^* T\mathcal{M}_2)$, the two vector fields give rise to

$$\tilde{a}_* Q_1 \quad \text{and} \quad Q_2 \circ \tilde{a}, \tag{6.18}$$

respectively. These two vector fields on $\mathcal{M}_{\mathrm{BV}}$, both of degree 1, are (graded) commuting, as acting on the right and the left of the map $\tilde{a}$. Their difference

$$\tilde{f} := \tilde{a}_* Q_1 - Q_2 \circ \tilde{a} \tag{6.19}$$

is nothing but the BV Operator Q_{BV}. It turns out that both vector fields are Hamiltonian with respect to the symplectic form (6.17), with the Hamiltonians being given (6.12)

[22] Details for the remaining part of the paragraph and the following one can be found in [1], [7], and [31].

and (6.13), respectively. In the spirit of (6.7), we can now also bring the BV-action into the form

$$S_{\text{BV-AKSZ}} = \int\limits_{T[1]\Sigma} \iota_{\mathrm{d}} \widetilde{\mathrm{ev}}^*(\alpha) + (-1)^d \widetilde{\mathrm{ev}}^* \mathcal{Q}, \tag{6.20}$$

which defines the general AKSZ sigma model. Here d is the dimension of Σ, in the example of the Chern–Simons theory thus $d = 3$ and $\mathcal{Q}$ is the Hamiltonian (6.10). In the first term α is a primitive of ω, $\omega = \mathrm{d}\alpha$ and ι_{d} denotes the contraction with $Q_1 = \mathrm{d}$. With Σ having no boundary, the action (6.20) is independent of the choice of α. If q^α denote Darboux coordinates of the target PQ-manifold, which, as mentioned, has degree $d-1$, i.e., $\omega = \frac{1}{2}\omega_{\alpha\beta}\mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta$ with ω_{ab} being constants, we can choose $\alpha = \frac{1}{2}\omega_{\alpha\beta}\, q^\alpha \wedge \mathrm{d}q^\beta$. Let $\mathcal{A}^\alpha = \tilde{a}^*(q^\alpha)$. Then we can (somewhat formally) rewrite the AKSZ action "evaluated" at $\tilde{a}$ more explicitly as

$$S_{\text{BV-AKSZ}}[\mathcal{A}] = \int_{T[1]\Sigma} \tfrac{1}{2}\omega_{\alpha\beta}\mathcal{A}^\alpha \mathrm{d}\mathcal{A}^\beta + (-1)^d \tilde{a}^* \mathcal{Q}\,. \tag{6.21}$$

In this form it is very easy to see that we reproduce from this the Chern–Simons theory in its form (6.14) upon the choice (6.13) together with $d = 3$.

In general, the AKSZ sigma model is defined for a degree $d-1$ PQ-manifold on a d-dimensional base. The classical action results from the BV form of it simply by replacing $\tilde{a}$ by its degree-zero part a, i.e., with $A^\alpha = a^* q^\alpha$

$$S_{\text{AKSZ}}[A] = \int_{T[1]\Sigma} \tfrac{1}{2}\omega_{\alpha\beta} A^\alpha \mathrm{d}A^\beta + (-1)^d a^* \mathcal{Q}. \tag{6.22}$$

Again, in the Chern–Simons case we easily find the classical action (6.1) reproduced.

We can, however, now also parameterize this sigma model more explicitly by means of the considerations in Section 5 for the lowest dimensions of Σ: For $d = 2$, we need to regard degree 1 PQ-manifolds, which we had found to be always of the form

$$\mathcal{M}_2 = T^*[1]M, \quad \omega = \mathrm{d}p_i \wedge \mathrm{d}x^i, \quad \mathcal{Q} = \tfrac{1}{2}\Pi^{ij} p_i\, p_j \tag{6.23}$$

with Π a Poisson bivector (cf. Example 3.3); i.e., the target data are uniquely determined by a Poisson manifold (M, Π). The most general AKSZ sigma model for $d = 2$ is thus seen to be the Poisson sigma model [36], [21]:

$$S_{\text{PSM}}[X^i, A_i] = \int_\Sigma A_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\Pi^{ij}(X) A_i \wedge A_j \tag{6.24}$$

where $X^i = a^* x^i$ are 0-forms on Σ, $A_i = a^*(p_i)$ 1-forms, and we used a more standard notation of integration over the (orientable) base manifold Σ.

For $d = 3$ we see that we get a (topological) sigma model for any Courant algebroid, cf. Theorem 5.1, and the corresponding sigma model is easily specified by means of (5.5) and (5.6). We call it the Courant sigma model. It was obtained first by Ikeda in [22] and later by Roytenberg more elegantly by the present method [31].

For $d = 4$ the geometrical setting of the target, a degree-three PQ-manifold, has not yet been worked out in detail or given any name. But again we can write down the explicit sigma model in this case using (5.9) and (5.10).

How we presented the AKSZ sigma model, its main purpose is to find a topological action functional such that its classical field equations are precisely Q-morphisms, (6.4). While this was proven for the Poisson sigma model explicitly in [4], the present formalism permits a short general proof when using that (6.20) is the Hamiltonian for (6.17) (alternatively one may also directly perform a variation of (6.22)). As the comparison of (6.22) with (6.21) shows, the difference between the classical action and its BV extension is that we merely have to perform the replacement (6.4) to go from one to the other. This is a very specific feature of the present topological models, the BV extension is usually not that simple to obtain for a general gauge theory; here, however, it works like this as we saw above. The BV-AKSZ functional (6.21) becomes stationary precisely when (6.19) vanishes (since this is its Hamiltonian vector field and the symplectic form is non-degenerate). Correspondingly, the variation of (6.22) results into the same equation, but where $\tilde{a}$ is replaced by a, so the Euler–Lagrange equations are equivalent to the vanishing of

$$f : \mathcal{M}_1 \to T[1]\mathcal{M}_2, \quad f := a_* Q_1 - Q_2 \circ a, \tag{6.25}$$

where we shifted the degree of the tangent bundle to the target so as to have f being degree-preserving like a. This in turn is tantamount to the chain property of a^* (cf. also Lemma 8.1 below).

We finally remark that also Dirac structures and generalized complex structures can be formulated with profit into the language of super geometry (the former ones as particular Lagrangian Q-submanifolds in the degree-two PQ-manifold $T^*[2]T[1]M$), which in part can be used also to formulate particular sigma models for them within the present scheme not addressed in the present exposition (and different from those of the following section). We refer the reader for example to [16] and [8].

7 Sigma models related to Dirac structures

As we have mentioned before, a Poisson manifold gives a particular example of a Dirac structure, determined by the graph of the corresponding bivector in $TM \oplus T^*M$. Similar to the Poisson sigma model (PSM), the target space of which is a Poisson manifold, we now want to consider a topological sigma model associated to any Dirac subbundle of an exact Courant algebroid. This Dirac sigma model (DSM) [25] is supposed to be at least equivalent to the PSM for the special choice of a Dirac structure that is the graph of a Poisson bivector. Also, we want to continue pursuing our strategy that its classical field equations should be appropriate morphisms. In fact, in lack of a good notion of a morphism of a Dirac structure, we will content ourselves with a Q-morphism again, i.e., a Lie algebroid morphism in this case (since any Dirac structure is in particular a Lie algebroid structure). The model will be 2-dimensional.

As we saw in the previous section that the most general 2-dimensional model obtained by the AKSZ scheme is the PSM, the DSM does not result from this method, at least not by its direct application.

The target space of the Dirac sigma model is a manifold together with a Dirac structure D in an exact Courant algebroid twisted by a closed 3-form H. The space-time is a 2-dimensional surface Σ. We need also some auxiliary structures – a Riemannian metric g on M and a Lorentzian metric h on Σ. A classical field of the DSM is a bundle map $T\Sigma \to D$. First this corresponds to the base map $X\colon \Sigma \to M$, which is corresponds to a collection of "scalar fields". Taking into account that $D \subset TM \oplus T^*M$, we represent the remaining field content by a couple of sections $V \in \Omega^1(\Sigma, X^*TM)$, satisfying the constraint that this couple combines into a section of $T^*\Sigma \otimes X^*D$. Here $\Omega^p(N, E)$ is by definition $\Gamma(\Lambda^p T^*N \otimes E)$ for any smooth manifold N and vector bundle $E \to N$. The DSM action is the sum of two terms, one using the auxiliary geometrical structures and one that uses the topological structures only: $S_{\mathrm{DSM}}[X, V, A] = S_{\mathrm{geom}} + S_{\mathrm{top}}$, where

$$S_{\mathrm{geom}} := \frac{\alpha}{2} \int_\Sigma \|\mathrm{d}X - V\|^2, \tag{7.1}$$

$$S_{\mathrm{top}} := \int_\Sigma \left(\langle A \overset{\wedge}{,} \mathrm{d}X - \tfrac{1}{2}V\rangle\right) + \int_{N^3} H. \tag{7.2}$$

Here α is a real (non-vanishing, in general) constant[23], the absolute value in the first term corresponds to the canonical pairing induced by $h \otimes g$ on $T\Sigma \otimes TM$,[24] the brackets $\langle , \rangle$ denote the pairing between TM and T^*M, and the last term in the topological part of the action is the integral of H over an arbitrary map $N^3 \to M$ for $\partial N^3 = \Sigma$ which extends $X\colon \Sigma \to M$ (here we assumed for simplicity that X is homotopically trivial[25], which is e.g. always the case if the second homotopy group of M is zero).

This sigma model generalizes the G/G Wess–Zumino–Witten (GWZW) [41], [15] and (H-twisted) Poisson sigma model, simultaneously. We first comment on the relation to the PSM. Let us first choose $H = 0$ (cf. Example 3.3) and $\alpha = 0$ (yielding $S_{\mathrm{geom}} \equiv 0$); with D being the graph of a bivector Π, we have $V = X^*\Pi(A, \cdot)$ and comparison with (6.24) shows that we indeed have $S_{\mathrm{DSM}} = S_{\mathrm{PSM}}$ in that case. For H non-zero we get the twisted version of the Dirac structure of Example 3.3 and by means of equation (7.2) of the corresponding sigma model [23], respectively. (We will comment on non-zero α below). The GWZW model results from a special choice of the Dirac structure in an exact Courant algebroid on a quadratic Lie group $M = G$, H is the Cartan 3-form, g the biinvariant Riemannian metric on G, and $\alpha = 1$. In the description of a Dirac structure by an orthogonal operator $S \in \mathrm{End}(TM)$ once a

[23]The Lorentzian signature of h is chosen for simplicity. In the Riemannian version of the Dirac sigma model the coupling constant α has to be totally complex, that is, $\alpha \in i\mathbb{R}$.

[24]More explicitely this expression was defined after formula (1.1).

[25]For less a topologically less restrictive setting, one permits action functionals up to integer multiplies of $2\pi\hbar$ – here we refer to the literature on Wess–Zumino terms, cf. [25] and references therein.

metric g has been fixed on M, cf. Corollary 3.9, the Dirac structure on G is the one given by the adjoint action $S = \mathrm{Ad}_g$, where $g \in M$ is the respective base point.

The GWZW model is in so far an important special case as it is not only a well known model in string theory (cf., e.g., [15]), but it is also *known* to be topological, despite the appearance of auxiliary structures needed to define it. Also, its Dirac structure can be shown to *not* be a graph, e.g. by employing the characteristic classes described in Section 3.

Part of the topological nature for the general DSM can be already verified on the level of the classical field equations: Do they depend on the auxiliary structures like g or h? Here we cite the following result from [25]:

Theorem 7.1. *Let $\alpha \neq 0$, then a field (X, V, A) is a solution of the equations of motion, if and only if the corresponding bundle map $T\Sigma \to D$ induces a Lie algebroid morphism.*

Since the notion of a Lie algebroid morphism does not depend on auxiliary structures as those mentioned above, we see that at least this condition is satisfied for nonvanishing α.

The proof of this theorem is somewhat lengthy,[26] so that we do not want to reproduce it here; instead we want to prove it for the simplest possible Dirac structure, $D = TM$, Example 2.2 above, and refer for the general fact to [25].

We start by calculating the field equations of the sigma model; but for $D = TM$ and $H = 0$ (cf. Example 2.2) the topological part of the action is identically zero. So, it remains to look at the variation of (7.1). Since V is an independent (unconstrained) field in this case, the variation of the quadratic term w.r.t. V yields

$$\mathrm{d}X = V \tag{7.3}$$

while the X-variation vanishes on behalf of that equation. Mathematically, this equation is tantamount to saying that the vector bundle morphism $a\colon T\Sigma \to TM$ is the push forward of a map $X\colon \Sigma \to M$, $a = X_*$. We obtain the required statement in this special case by use of the following

Lemma 7.2. *A Lie algebroid morphism from the standard Lie algebroid over a manifold Σ to the standard Lie algebroid over a manifold M is the push forward of a smooth map $X\colon \Sigma \to M$.*

Proof. Recall from Section 5 that the Lie algebra morphism above can be defined best by a degree-preserving map $\bar{a}\colon T[1]\Sigma \to T[1]M$ such that $\bar{a}^*\colon C^\infty(T[1]M) \to C^\infty(T[1]\Sigma)$ commutes with the respective differentials characterizing the Lie algebroid, here being just the respective de Rahm differentials. Let us choose local coordinates $\sigma^\mu, \vartheta^\mu = \mathrm{d}\sigma^\mu$ and $x^i, \theta^i = \mathrm{d}x^i$, respectively. Then $X^i = \bar{a}^* x^i$ corresponds to

[26] One of the complications is that the fields A and V are not independent from one another in general and this has to be taken care of when performing the variations. One way of doing that is by using the orthogonal operator S mentioned in Corollary 3.9, which permits us to express these two fields by means of an independent $W \in \Omega^1(\Sigma, X^*TM)$ according to $A = (1 + S)W$ and $V = (1 - S)W$.

the base map X of the lemma. On the other hand $V^i := \bar{a}^*\theta^i = \mathrm{d}\bar{a}^* x^i = \mathrm{d}X^i$, where in the second equality we used that $\bar{a}^*$ commutes with d. $V^i = \mathrm{d}X^i$ or, equivalently, $V^i_\mu = X^i{}_{,\mu}$ is the searched-for equation. □

From the example we also see that $\alpha \neq 0$ is a necessary condition for the theorem to hold. Were $\alpha = 0$ in that special case, there were *no* field equations and the vector bundle morphism $a : T\Sigma \to TM$ were unrestricted and in general not a Lie algebroid morphism. On the other hand, it is not difficult to see (but it certainly also follows from the theorem) that for the (possibly H-twisted) PSM the field equations *do not change* when adding the term S_{geom} with some non-vanishing α. We thus propose to consider S_{DSM} for non-vanishing α in general.[27]

There are two more important issues which we need to at least mention in this context. First, if the DSM is to be associated to a Dirac structure on M and all other structures used in the definition of the functional are to be auxiliary, one needs to show that it is in the end the *cohomology class* of H only that enters the theory effectively. In [25] we proved

Proposition 7.3. *The DSM action transforms under a change of splitting* (2.17) *according to*

$$S_{\mathrm{DSM}} \mapsto S_{\mathrm{DSM}} + \int_\Sigma X^*(B)(\mathrm{d}X - V \,\hat{,}\, \mathrm{d}X - V). \tag{7.4}$$

Let $\alpha \neq 0$ and B be a "sufficiently small" 2-form, then there exists a change of variables $\bar{V} = V + \delta V$ and $\bar{A} = A + \delta A$, such that

$$S_{\mathrm{DSM}}[X, \bar{V}, \bar{A}] = S_{\mathrm{DSM}}[X, V, A] + \int_\Sigma X^*(B)(\mathrm{d}X - V \,\hat{,}\, \mathrm{d}X - V).$$

So, a change of the splitting can be compensated for by a (local) diffeomorphism on the space of fields.

Secondly, we did not yet touch the issue of the gauge symmetries, neither in the previous section on the AKSZ sigma models nor for the DSMs. Certainly the gauge symmetries are of utmost importance in topological field theories (without them we were never able to arrive at a finite dimensional moduli space of solutions for instance). While the BV formalism produces them by means of the BV operator for the AKSZ sigma models (although possibly in a coordinate dependent way, cf., e.g., [4] addressing this issue), they are less obvious to find for the DSM. In fact, here also all the *auxiliary structures do enter*, cf. [25] for the corresponding formulas. It is only *onshell*, i.e., on using the field equations (here only $\mathrm{d}X = V$), that the gauge

[27] In [25] it is conjectured that the theory with $\alpha = 0$ is (essentially) equivalent to the theory with $\alpha \neq 0$ in general. E.g. for $D = TM$, and $\alpha \neq 0$ the moduli space of solutions to the field equations (maps $X : \Sigma \to M$) *up to gauge symmetries* (which at least contain the homotopies of this map X) is 0-dimensional, like the moduli space for the vanishing action $\alpha = 0$. It is argued that the geometrical part serves as a kind of regulator for the general theory, which also should permit localization techniques on the quantum level.

transformations obtain a nice geometrical interpretation: they turn out to become "Lie algebroid homotopies" (cf. also [4]) in this case. We do, however, not want to go into further details on this here; in the present work we decided to focus more on the field equations, reassuring the reader in words that the more intricate gauge symmetries fit nicely into the picture as well, completing it in an essential way.

8 Yang–Mills type sigma models

In the Introduction we recapitulated the idea of sigma models: one wants to replace the flat target space $\mathbb{R}^n$ of usually a collection of n scalar fields (functions on spacetime Σ) by some geometrical object, like a Riemannian manifold. We posed the question, if, in the context of gauge fields (1-forms on space time), we can replace in a likewise fashion the "flat" Lie algebra $\mathbb{R}^n$ (or, more generally, $\mathfrak{g}$) by some nontrivially curved geometrical object. In fact, the Poisson sigma model (6.24), or more generally, the AKSZ sigma model (6.20), provides, in some sense, half a step into this direction: Let us consider the 3-dimensional case, where this is most evident. The quadratic Lie algebra $\mathfrak{g}$ needed for the definition of the Chern–Simons gauge theory can be generalized to a Courant algebroid, associated to which is the Courant sigma model [22], [31], which when specializing to the "flat case" $\mathfrak{g}$ reproduces the Chern–Simons theory. While these models realize in a geometrically nice way the right (classical) field content, the target algebroid being represented by a PQ-manifold and the Lie algebra valued 1-forms of a Yang–Mills theory being interpreted as and generalized to degree-preserving maps from $T[1]\Sigma$ to the respective target, and also the gauge transformations are generalized in a reasonably looking way, there is, from the *physical* point of view, a major drawback or "flaw" of these theories: What mathematically is usually considered an advantage of a field theory, namely to be topological, in the context of physics rules out a theory for being feasible to describe the degrees of freedom we see realized in the interacting world around us.

Indeed, the space of flat connections, which are the field equations of the Chern–Simons theory, modulo gauge transformations is (for, say, Σ without boundary and of finite genus) a finite dimensional space and this generalizes in a likewise fashion to the moduli space of solutions modulo gauge transformations for all the AKSZ models, where one considers the space of Q-morphisms modulo Q-homotopy. The moduli space needed to host a physical particle (like a photon, electron, Higgs, etc), on the other hand, is always infinite dimensional (like the space of harmonic functions on Σ for a Laplacian corresponding to a (d–1,1)–signature metric). There is also another way of seeing that one has gone half way only: Such as we want that when in a sigma model for scalar fields the choice of a "flat background" (i.e., in that case, that the target is a flat Riemannian manifold $\mathbb{R}^n$) the action reduces to (1.2), we want that when in a Yang–Mills type sigma model the target is chosen to be a Lie algebra $\mathbb{R}^n$ or, more generally, $\mathfrak{g}$, the gauge theory reduces to (1.5) and (1.7), respectively. So we will pose

this condition, maybe adding that there should be a "comparable number" of gauge symmetries in the general model as in the special case.

In a similar way we may generalize this condition by extending it to also higher form gauge fields B^b, namely that for an appropriate flat choice of the target geometry one obtains from the (higher) Yang–Mills type sigma model

$$S[B^b] = \frac{1}{2}\int_\Sigma \mathrm{d}B^b \wedge *\mathrm{d}B^b. \tag{8.1}$$

For the case that B's are 2-forms, this will yield an action functional for nonabelian gerbes.

Before continuing we bring the standard Yang–Mills action (1.7) into a form closer to a topological model first. We consider a trivial bundle in what follows, in which case the curvature or field strength is a $\mathfrak{g}$-valued 2-form. If d denotes the dimension of spacetime Σ, we introduce a $\mathfrak{g}^*$-valued $d-2$ form Λ in addition to the connection 1-forms A. With κ^{-1} denoting the scalar product on $\mathfrak{g}^*$ induced by κ on $\mathfrak{g}$, then

$$S'_{\mathrm{YM}}[A,\Lambda] := \int_\Sigma \langle \Lambda, F\rangle + \kappa^{-1}(\Lambda \mathbin{\hat{,}} *\Lambda) \tag{8.2}$$

is easy to be seen as equivalent to (1.7) on the classical level (Euler–Lagrange equations) after elimination of the auxiliary field Λ. (Equivalence on the quantum level follows from a Gaussian integration over the field Λ, as it enters the action quadratically only – up to an overall factor, which is irrelevant for the present considerations).

The first part of this action is topological, it is only the second term, breaking some of the symmetries of the topological one, that renders the theory physical. Let us first generalize the topological part, a so-called BF-theory, by an appropriate reformulation. In fact, we can obtain the BF-theory from the AKSZ-method by considering $\mathcal{M}_2 = T^*[d-1]\mathfrak{g}[1]$ as a target PQ-manifold (as before d is the dimension of spacetime Σ). Indeed, it is canonically a symplectic manifold with the symplectic form being of degree $d-1$, as it was required to be in Section 6, and the Q-vector field on $\mathfrak{g}[1]$, equation (5.1), can always be lifted canonically to the cotangent bundle with the following Hamiltonian [28]

$$\mathcal{Q} = (-1)^d \tfrac{1}{2} C^a_{bc}\theta^b\theta^c p_a. \tag{8.3}$$

The AKSZ action (6.22) now just produces

$$S_{\mathrm{BF}}[\Lambda, A] = \int_\Sigma \Lambda_a(\mathrm{d}A^a + \tfrac{1}{2}C^a_{bc}A^b \wedge A^c) \tag{8.4}$$

where we used $A^a = a^*(\theta^a)$ and $\Lambda_a = a^*(p_a)$; and this is just equal to the first term in (8.2). It is now easy to find a generalization of this part of the action for a general ($\mathbb{N}_0$-graded) Q-manifold $(\widetilde{\mathcal{M}}_2, \widetilde{Q}_2)$: Just consider $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ as a target

[28] We use the following sign convention: $\{p_a, q^b\} = (-1)^{m|q^b|}\delta^b_a$, where m is the degree of a symplectic form.

PQ-manifold with the canonical lift of $\widetilde{Q}_2$ to the Hamiltonian vector field Q_2 (with Hamiltonian $\mathcal{Q}_2 = \mathcal{Q}$).

Let us exemplify this at the example of a Lie algebroid, where $\widetilde{\mathcal{M}}_2 = E[1]$ and $\widetilde{Q}_2$ is given by means of formula (5.2) (with $\rho_a = \rho_a^i \partial/\partial x^i$) and $\rho_a^i(x)$, $C_{ab}^c(x)$ being the Lie algebroid structure functions in the local coordinates x^i, θ^a on $E[1]$). Now the same procedure yields the action functional for a "Lie algebroid BF-theory" [38], [39], [5]

$$S_{\mathrm{LABF}}[\Lambda_i, \Lambda_a, X^i, A^a] = \int_\Sigma (\Lambda_i \wedge F^i + \Lambda_a \wedge F^a) \tag{8.5}$$

where Λ_i and Λ_a are $(d-1)$-forms and $(d-2)$-forms, respectively, and

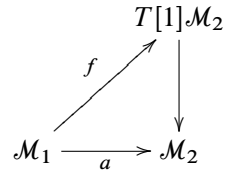
$$F^i = \mathrm{d}X^i - \rho_a^i(X) A^a \tag{8.6}$$

$$F^a = \mathrm{d}A^a + \tfrac{1}{2} C_{bc}^a(X) A^b \wedge A^c \tag{8.7}$$

are the generalizations of the YM-curvatures, which we prefer to call *field strengths* in the general case. We use the notation $A^\alpha = a^*(q^\alpha)$, where q^α are graded coordinates on $\widetilde{\mathcal{M}}_2$ and $\Lambda_\alpha = a^*(p_\alpha)$, the corresponding momenta (or anti-fields) on $\mathcal{M}_2 = T[1]\widetilde{\mathcal{M}}_2$. Note that the map $a\colon T[1]\Sigma \to \mathcal{M}_2$ is degree-preserving, so that $X^i :\equiv A^i = a^*(x^i)$ are 0-forms on Σ or scalar fields, $A^a = a^*(\theta^a)$ are 1-forms, and Λ_i and Λ_a are $(d-1)$- and $(d-2)$-forms, respectively.

We remark in parenthesis that although (8.5) is inherently coordinate and frame independent, the expression (8.7) is not. It is the splitting of the field strength(s) into two independent parts (8.6) and (8.7), which cannot be performed canonically. One way of curing this is by introducing an additional connection on the (target) Lie algebroid E, cf., e.g., [4], [39]. Better is to realize that the whole information is captured by the map (6.25), taking values in the *tangent* bundle over the target $\mathcal{M}_2 = T^*\widetilde{\mathcal{M}}_2$. We now turn to this perspective.

The map f defined in (6.25) covers the map $a\colon \mathcal{M}_1 \to \mathcal{M}_2$ (it is also this fact that makes the difference in (6.25) well-defined, being the difference between two elements in the fiber, which is a vector space, over the same point)



and measures the deviation of a to be a Q-morphism, in which case the following diagram would commute:

$$\begin{array}{ccc} T[1]\mathcal{M}_1 & \xrightarrow{a_*} & T[1]\mathcal{M}_2 \\ {\scriptstyle Q_1}\uparrow & & \uparrow{\scriptstyle Q_2} \\ \mathcal{M}_1 & \xrightarrow[a]{} & \mathcal{M}_2. \end{array}$$

Denote by $x^A = (q^\alpha, p_\alpha)$ the graded coordinates on $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ and by $\mathrm{d}x^A$ the induced fiber-linear coordinates on $T[1]\mathcal{M}_2$. To use the map f in practise, the following lemma is helpful:

Lemma 8.1. *Let $(x^A, \mathrm{d}x^A)$ be local coordinates on $T[1]\mathcal{M}_2$, $a \in \mathrm{Mor}(T[1]\Sigma, \mathcal{M}_2)$, and f the map defined in (6.25). Then*

$$f^*(x^A) = a^*(x^A), \quad f^*(\mathrm{d}x^A) = (\mathrm{d}a^* - a^*Q_2)x^A. \tag{8.8}$$

Proof. The first part is evident from f covering a as remarked above already. The second part is rather straightforward and we show only one first step in the calculation. The vector field Q_2 of degree 1 is a *degree-preserving* section of the tangent bundle over $\mathcal{M}_2$, if the latter is shifted by 1 in degree, i.e., as here, one considers $T[1]\mathcal{M}_2$. The section maps a point with coordinates x^A to the point $(x^A, Q_2^A(x))$. Thus we find for the pull-back of $\mathrm{d}x^A$ by this section: $(Q_2)^*\mathrm{d}x^A = Q_2^A(x)$, which, however, can also be rewritten as Q_2 applied to x^A. The remaining similar steps we leave as an exercise to the reader. We just remark that we had used $Q_1 = \mathrm{d}d$ and that certainly pull-backs go the reverse order as the respective map, cf the second diagram above. □

We remark also that $\mathrm{d}x^A$ is considered as a *function* on $T[1]\mathcal{M}_2$ and is pulled back as such, *not* as a differential form on $T[1]\mathcal{M}_2$, while a function on $T[1]\mathcal{M}_2$ is, on the other hand, equally well a differential form on $\mathcal{M}_2$. In fact, the so-understood pull back will *not* be a chain map w.r.t. the de Rahm differential (a vector field on $T[1]\mathcal{M}_2$), but only w.r.t. a modified vector field on $T[1]\mathcal{M}_2$, cf. Proposition 8.3 below.

We see that the field strengths F^i and F^a are nothing but the pull back of $\mathrm{d}x^i$ and $\mathrm{d}\theta^a$ by f and thus find [40]

Corollary 8.2. *The action functional (8.5) can be written more compactly as*

$$S_{\mathrm{LABF}}[a] = \int_\Sigma f^*(p_\alpha \mathrm{d}q^\alpha), \tag{8.9}$$

where $a\colon T[1]\Sigma \to \mathcal{M}_2 = T^*[d-1]E[1]$ and $p_\alpha \mathrm{d}q^\alpha$ is the canonical 1-form on $\mathcal{M}_2$.

Now we are in the position to generalize the Yang–Mills action functional (1.7) or better (8.2) to the Lie algebroid setting, i.e., to a situation where the structural Lie algebra $\mathfrak{g}$ is replaced by a Lie algebroid E (cf. Definition 2.1). A Lie algebroid Yang–Mills theory is defined as a functional on $a \in \mathrm{Mor}(T[1]\Sigma, \mathcal{M}_2 = T^*[d-1]E[1])$ and

is of the form [39]

$$S'_{\mathrm{LAYM}}[a] = S_{\mathrm{LABF}}[a] + \int_\Sigma {}^E\mathrm{g}^{-1}(\Lambda_{(d-2)} \hat{,} *\Lambda_{(d-2)}) \tag{8.10}$$

where $\Lambda_{(d-2)} \in \Omega^{d-2}(\Sigma, X^*E)$ are the Lagrange multiplier fields of the 2-form field strengths only. Equivalently, eliminating precisely those Lagrange multipliers like in the transition from (8.2) to (1.7), we can regard the somewhat more explicit action functional

$$S_{\mathrm{LAYM}}[A^a, X^i, \Lambda_i] = \int_\Sigma \Lambda_i \wedge F^i + \tfrac{1}{2}\, {}^E\mathrm{g}(X)_{ab} F^a \wedge *F^b, \tag{8.11}$$

where F^i and F^a are the field strengths (8.6) and (8.7), respectively, detailed above. In a standard YM theory, the metric on the Lie algebra needs to be ad-invariant. Here we can ask that $(E, {}^E\mathrm{g})$ define a maximally symmetric E-Riemannian space (cf. Definition 2.10) or that the fiber metric is invariant w.r.t. a representation (2.13) induced by any auxiliary connection on E. (For still less restrictive conditions cf. [29]).

This is now easy to generalize to higher form degrees of the gauge field. Let us consider a tower of gauge fields $X^i, A^a, \ldots, B^B$, which are 0-forms, 1-forms etc, respectively, up to a highest degree p, so B^B being p-forms. The role of the structural Lie algebra will now be played by a degree-p Q-manifold $(\widetilde{\mathcal{M}}_2, Q_2)$. The gauge fields are collected into a (degree-preserving) map $a\colon T[1]\Sigma \to \widetilde{\mathcal{M}}_2$. This is extended to $a \in \mathrm{Mor}(T[1]\Sigma, \mathcal{M}_2)$ with $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ the corresponding PQ-manifold with canonical 1-form $p_\alpha \mathrm{d}q^\alpha$. We then consider [40]

$$S'_{\text{higher YM}}[a] = \int_\Sigma f^*(p_\alpha \mathrm{d}q^\alpha) + \int_\Sigma {}^V\mathrm{g}^{-1}(\Lambda_{(d-p-1)} \hat{,} *\Lambda_{(d-p-1)}) \tag{8.12}$$

where ${}^V\mathrm{g}$ is a fiber metric on the vector bundle $V \to M$ corresponding to the degree-p variables on $\mathcal{M}_2$ (in fact, there is a canonical quotient of $\mathcal{M}_2$ yielding $V[p]$). Similarly to before we can also eliminate the lowest form degree Lagrange multipliers $\Lambda_{(d-p-1)}$) in $S'_{\text{higher YM}}$ to obtain $S_{\text{higher YM}}[a]$ in which the highest form degree field strength is squared by means of the fiber metric ${}^V\mathrm{g}$.

There is always a condition on this fiber metric generalizing the ad-invariance in the standard YM case. Let us specify this condition in the case $p = 2$. We were mentioning in Section 5 that a degree-two Q-manifold corresponds to a Lie 2-algebroid, cf. Definition 2.13 and the ensuing text. In this case the condition to be placed on ${}^V\mathrm{g}$ is its invariance w.r.t. the E-connection ${}^E\nabla$ on V [40]:

$${}^E\nabla\, {}^V\mathrm{g} = 0. \tag{8.13}$$

For $p = 2$ this is a possible definition of an action functional for nonabelian gerbes.

In this contribution we never discussed the gauge transformations in any detail. In a gauge theory they are certainly of utmost importance and much further motivation for the theories discussed here come from looking at their gauge invariance also.

We make here only two general remarks on the gauge transformations: First, for a general algebroid type gauge theory the gauge symmetries have a "generic part" that comes from the structural algebroid (possibly including some particularly important geometrical ingredients from the target like a compatible symplectic form), but there can be also contributions to them that show all possible structural ingredients used in the definition of the action functional. An (impressive) example for the latter scenario is provided by the Dirac sigma model, cf. Section 7 and [25]. Second, for the higher Yang–Mills action functionals proposed above, or also any other *physical* gauge theory proposed in this setting, the BV-formulation, which also captures elegantly the gauge symmetries of a theory, will *not* just consist in replacing the morphisms a by supermaps $\tilde{a}$ as it was the case with the AKSZ type sigma models. In fact, the space $\mathrm{Map}(T[1]\Sigma, \mathcal{M}_2)$ is too big: It contains ghosts and ghosts for ghosts etc for gauge symmetries of the BF-part of the action that are broken by the fiber metric part. It is tempting, however, to try to obtain the BV-formulation of the full physical model by a kind of supersymplectic reduction induced by the symmetry breaking terms.

In the present chapter we used the framework of the AKSZ sigma models to develop the field content of the higher gauge theories and, by the material at hand from the previous sections, it was natural to require the structural Lie algebra to be replaced by a general Q-manifold. This is in so far a physical approach as that the notion of Q-manifolds arose in a physical context, namely to describe the supergeometric framework underlying the BV-formulation [37]. In this way, the generalization of the connection 1-forms of a Yang–Mills theory to the field content considered sounds much less compulsory than it really is. As mentioned in the introduction, physical considerations often suggest geometrical notions and generalizations into very particular directions. This is also the case in the context of "higher gauge theories" – that is gauge theories where the field content is required to consist locally of a tower of differential forms up to some highest degree p. It is evident that then locally we can introduce (in a canonical way) a graded manifold $\mathcal{M}_2$ of degree p and interpret the field content as a morphism from $T[1]\Sigma$ to $\mathcal{M}_2$. Even without yet considering gauge transformations acting on these fields, there are sensible requirements on the properties of the theory related to a generalization of the Bianchi identities in ordinary gauge theories that *require* that the target $\mathcal{M}_2$ has to be a Q-manifold. This approach would have fitted the logic of the present chapter very well, too, but it would have extended its length further as well so that we decided to not reproduce these considerations, which were presented in [40] and can be found in [17].

Up to now we only considered the generalization of gauge theories in the context of trivial bundles. In principle, it is not at all clear that the local construction generalizes in a straightforward way to a global one. But fortunately here this is indeed the case. We briefly sketch how this works, since, from the *mathematical* point of view, much of the interest in such gauge theories would precisely stem from the global picture. For further details we have to refer the reader to [26].

We again start from ordinary Yang–Mills theories. There the space of fields are not Lie algebra valued 1-forms on our spacetime Σ, but rather connections in a principal

G-bundle P over Σ. Associated to each principal bundle, however, there is the so-called Atiyah algebroid, which is a very particular Lie algebroid over Σ. As a total space it is equal to TP/G, which is canonically a Lie algebroid by use of the de Rahm differential on TP. The image of its anchor map is $T\Sigma$. Shifting the fiber degrees by 1, this becomes a bundle in the category of Q-manifolds $T[1]P/G \to T[1]\Sigma$, the typical fiber of which is the Q-manifold $\mathfrak{g}[1]$ (as before, $\mathfrak{g}$ is the Lie algebra of the structure group G). We call this a Q-bundle. Moreover, a connection on P is in bijection to a section of this bundle *viewed* as a bundle of graded manifolds (it is the *flat* connections that are the sections of the Q-bundle).

The generalization is obvious now: As total spaces we consider Q-bundles $\pi\colon (\mathcal{M}, Q) \to (\mathcal{M}_1, Q_1)$, where as the base $(\mathcal{M}_1, Q_1)$ we keep $(T[1]\Sigma, \mathrm{d})$ for simplicity, but as typical fibers we permit any (non-negatively graded) Q-manifold $(\mathcal{M}_2, Q_2)$. Gauge fields are any degree-preserving maps $a\colon \mathcal{M}_1 \to \mathcal{M}$ such that $\pi \circ a = \mathrm{id}_{\mathcal{M}_1}$. Many constructions that were done for gauge fields $a\colon \mathcal{M}_1 \to \mathcal{M}_2$, like the definition of the field strength f, can now be repeated without modification by replacing merely $\mathcal{M}_2$ by the total space $\mathcal{M}$. The (generic part of the) gauge symmetries receive the nice interpretation of (possibly a graded subgroup of) vertical inner automorphisms of this bundle. And the gauge invariance of action functionals like (8.12) permits to glue local expressions for functionals together to global ones.

The global setting certainly also opens gates for studying characteristic classes for the bundles. In particular, there exists a generalization of the Chern–Weil formalism. In the classical setting, a connection on a principal bundle together with an ad-invariant polynomial on the Lie algebra gives a cohomology class on the base manifold, which does not depend on the connection and which vanishes when the bundle is trivial. In the context of the classical construction one considers the so-called Weil algebra $W(\mathfrak{g}) = S^\cdot(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ with a certain differential Q_W on it – the connection on P then induces a map from a subcomplex of so-called G-basic elements of W to the de Rahm complex on Σ.

To generalize this construction, one may first observe that W can be identified with differential forms on $\Omega(\mathfrak{g}[1])$, which we can identify with functions on $T[1]\mathfrak{g}[1]$. This reminds one of the target of the map f, namely when $\mathcal{M}_2 = \mathfrak{g}[1]$, cf. the first diagram above. Given a general Q-manifold $\mathcal{M}_2$ and a degree-preserving map $a\colon \mathcal{M}_1 \to \mathcal{M}_2$, can we equip $T[1]\mathcal{M}_2$ with a Q-structure $Q_{T\mathcal{M}_2}$, such that $f\colon (\mathcal{M}_1, Q_1) \to (T[1]\mathcal{M}_2, Q_{T\mathcal{M}_2})$ is *always* a Q-morphism? The answer is affirmative [26]:

Proposition 8.3. *The map* (6.25) *is a Q-morphism, if we equip $T[1]\mathcal{M}_2$ with the Q-structure $Q_{T\mathcal{M}_2} = \mathrm{d} + \mathcal{L}_{Q_2}$, where* d *is the de Rahm differential on $\mathcal{M}_2$ and $\mathcal{L}_v$ the Lie derivative w.r.t. a vector field v on $\mathcal{M}_2$, both viewed as degree-*1 *vector fields on $T[1]\mathcal{M}_2$:*

$$Q_1 f^* = f^* Q_{T\mathcal{M}_2}. \tag{8.14}$$

Indeed, the canonical lift $Q_{T\mathcal{M}_2}$ of the differential Q_2 to its tangent reproduces the Weil differential Q_W mentioned above for $\mathcal{M}_2 = \mathfrak{g}[1]$. One now generalizes also

the notion of being basic, where in the general setting a graded gauge or holonomy group $\mathcal{G}$ plays an important role. In the end one obtains [26] a generalization of the Chern–Weil map in the form

Theorem 8.4. *Let $\pi : \mathcal{M} \to T[1]\Sigma$ be a Q-bundle with a typical fiber $\mathcal{M}_2$, a holonomy group $\mathcal{G}$, and a a "gauge field", i.e., a section of π (in the graded sense). Then there is a well-defined map in cohomology*

$$H^p(\Omega(\mathcal{M}_2)_{\mathcal{G}}, Q_{T\mathcal{M}_2}) \to H^p_{\text{deRahm}}(\Sigma), \tag{8.15}$$

which does not depend on homotopies of a.

We want to end this section and thus this chapter by commenting on a rather intriguing link of this construction with the topological sigma models we were considering in a previous section, namely Section 6. There is a famous relation of a characteristic class on a principal bundle with a quadratic structure group, the second Chern class or the first Pontryagin class, with the integrand of the Chern–Simons gauge theory: locally "Second Chern form $=$ d Chern–Simons form". This class results from the Chern–Weil formalism by applying the map to the ad-invariant metric κ on $\mathfrak{g}$, viewed as quadratic polynomial. In Section 5 we learnt that κ can be viewed as a symplectic form ω of degree 2 on $\mathfrak{g}[1]$. Locally, the above generalized Chern–Weil map is nothing but f^*. On the other hand, the AKSZ sigma model resulted from choosing a PQ-manifold $(\mathcal{M}_2, Q_2, \omega)$ as a target – and for the Chern–Simons theory it is this ω that corresponds to κ.

We are thus lead to ask for a relation of the AKSZ sigma models with $f^*\omega$ for a general PQ-manifold (that would serve as a typical fiber in a Q-bundle). In Corollary 8.2 we found that the integrand of the AKSZ sigma model can be written as the pullback of the canonical 1-form, if $\mathcal{M}_2$ is the cotangent bundle $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ with $\widetilde{\mathcal{M}}_2$ a degree-1 Q-manifold.[29] We thus compute $\mathrm{d} f^*(p_\alpha \mathrm{d}q^\alpha) = f^*(\mathrm{d} + \mathcal{L}_{Q_2}) p_\alpha \mathrm{d}q^\alpha$, where we used the chain property (8.14). But the canonical 1-form is invariant w.r.t. the Hamiltonian lift Q_2 of a vector field $\widetilde{Q}_2$ on its base $\widetilde{\mathcal{M}}_2$; so only the first term remains on the r.h.s., which indeed has the form $f^*\omega$. In fact, this statement that we proved for $\mathcal{M}_2 = T^*[d-1]E[1]$, is true for any PQ-manifold; one has [26]

Theorem 8.5. *Consider a PQ-manifold $\mathcal{M}_2$ with symplectic form ω of positive degree $d-1$ and a $(d+1)$-dimensional manifold N with boundary $\partial N = \Sigma$, and $a \in \mathrm{Mor}(T[1]N, \mathcal{M}_2)$. Then*

$$\int_N f^*\omega = S_{\text{AKSZ}}[a], \tag{8.16}$$

the AKSZ sigma model (6.22) on Σ.

[29] In fact, this is true for a general Q-manifold, as one can prove directly in generalization of Corollary 8.2 or deduce from Theorem 8.5 below.

As mentioned, these observations generalize the relation of the Pontryagin class to the topological Chern–Simons theory to a much wider range of bundles and topological models.

Certainly, in some cases Q-manifolds and Q-bundles may be "integrable" (already not every Lie algebroid is integrable to a Lie groupoid – cf. [11] for the necessary and sufficient conditions for the integrability of Lie algebroids). Similar to the relation of Atiyah algebroids to principal bundels, integrable Q-bundles will correspond to rather intricate other type of bundles – in the case of the Lie algebroid Yang–Mills theories, e.g., these are bundles with typical fiber being a groupoid, in the case of the fibers being degree-2 Q-manifolds over a point these will be the nonabelian bundle gerbes of [6]. What we called gauge fields and field strengths and others might call "connections" and "curvatures", respectively, are often rather intricate to describe in the integrated pictures – in contrast to what happens in the Q-bundle setting.

It is our impression that the Q-bundle picture has several advantages in the description of higher gauge theories. Also, they exist and give rise to potentially interesting action functionals even if the underlying Q-manifolds are not integrable; a topological example of this kind is the Poisson sigma model, which exists and has its merits even if the target Poisson manifold M (the Lie algebroid T^*M) is not integrable.

In any case, we hope to have convinced the reader who was not yet familiar with all the concepts used in this chapter, that algebroids of various kinds are interesting as mathematical objects on the one hand, interpolating in a novel way between various established notions like tensor fields on manifolds and Lie algebras or symplectic and complex geometry, and, on the other hand, that they appear in many different facets in the context of sigma models and higher gauge theories, in fact in such a way that they have to be considered indispensable for an elegant and technically and conceptually convincing discussion of such theories.

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Chapter 8

A potential for generalized Kähler geometry

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1 Introduction

Generalized Kähler geometry [1] is a special and particularly interesting example of generalized geometry [2], [1]. It succinctly encodes the bihermitian geometry of Gates, Hull and Roček [3] in terms of a pair of commuting generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$. Here we use the equivalent bihermitian data to give a complete (local) description of generalized Kähler geometry in a neighbourhood of a regular point. We show that there exists a function K with the interpretation of a generating function for symplectomorphisms between two sets of coordinates in terms of which all geometric quantities may be described. In particular, the combination $E = \frac{1}{2}(g + B)$ of the metric and antisymmetric B-field is given as a nonlinear expression in terms of second derivatives of K. When further analyzed, this expression has the structure of a quotientquotient from some higher dimensional space. We find this auxiliary space, which we call an ALP space, and display the quotient structure. Finding a generalized Kähler geometry is thus reduced to a linear construction in terms of the corresponding K on the ALP.

The bihermitian geometry was first derived as the target space geometry of $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma models [3]. Correspondingly, all our constructions have a sigma model realization, and many of the proofs were derived in that setting. In particular, the "generalized Kähler potential" K has an interpretation as the

superspace sigma model Lagrangian[1]. Detailed descriptions of this approach may be found in the two articles [5], [6] on which the presentation is based.

2 Generalized complex geometry (GCG)

This section contains a brief recapitulation of the salient features of GCG. The references for the mathematical aspects are chiefly [2], [1], while a presentation more accessible to physicists (containing, *e.g.,* coordinate and matrix descriptions) may be found in [7], [8].

A generalized almost complex structure is an algebraic structure on the sum of the tangent and cotangent bundles of a manifold $\mathcal{M}$ defined via

$$\mathcal{J} \in \mathrm{End}(T \oplus T^*) \tag{2.1}$$

such that

$$\mathcal{J}^2 = -1 \tag{2.2}$$

and

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}, \tag{2.3}$$

where $\mathcal{I}$ is the metric corresponding to the natural pairing of elements $X + \eta, Y + \lambda \in T \oplus T^*$:

$$\langle X + \eta, Y + \lambda \rangle = \tfrac{1}{2}(\imath_X \lambda + \imath_Y \eta). \tag{2.4}$$

The generalized almost complex structure is a (twisted) generalized complex structure *iff* it is integrable with respect to the (twisted) Courant bracket

$$[X + \eta, Y + \lambda]_C := [X, Y]_L + \mathcal{L}_X \lambda - \mathcal{L}_Y \eta - \tfrac{1}{2} d(\imath_X \lambda - \imath_Y \eta) + \imath_X \imath_Y H, \tag{2.5}$$

where $[X, Y]_L$ is the Lie-bracket and "twisted" refers to inclusion of the last term involving the closed three-form H. Integrability is defined by the requirement that the distributions defined by the projection operators

$$\tfrac{1}{2}(1 \pm i\mathcal{J}) \tag{2.6}$$

are involutive with respect to the Courant-bracket. Of great importance to physical applications is the fact that the automorphisms of the Courant-bracket, in addition to diffeomorphisms, also contain the b-transform:

$$e^b(X + \eta) = X + \eta + \imath_X b \tag{2.7}$$

where b is a closed two-form: $db = 0$. When acting on a generalized complex structure $\mathcal{J}$ it produces an equivalent generalized complex structure $\mathcal{J}_b$. In a matrix

[1] A similar situation arises in the projective superspace description of hyper-Kähler sigma models [4].

representation this reads as follows:

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}. \tag{2.8}$$

We now restrict our attention to a subset of generalized complex geometries: generalized Kähler geometries (GKG). These are GCG's with two commuting generalized complex structures $\mathcal{J}_{1,2}$ whose product defines a positive definite metric $\mathcal{G}$ on $T \oplus T^*$ that squares to the identity [1];

$$\begin{aligned} [\mathcal{J}_1, \mathcal{J}_2] &= 0, \\ \mathcal{G} &= -\mathcal{J}_1 \mathcal{J}_2, \\ \mathcal{G}^2 &= 1. \end{aligned} \tag{2.9}$$

This is the proper setting for the bihermitian geometry of [3], which we now describe.

3 Bihermitian geometry

The data found in [3] to be necessary and sufficient to describe the target space geometry of a supersymmetric nonlinear sigma model with $(2, 2)$ supersymmetry consists of two (ordinary) complex structures J_+ and J_-, a metric g, hermitian with respect to both complex structures, and a closed three-form H with a (local) two-form potential B.

The structures $(J_\pm, g, B)$ satisfy

$$\begin{aligned} J_\pm^2 &= -1, \\ J_\pm^t g J_\pm &= g, \\ \nabla^{(\pm)} J_\pm &= 0, \\ H = d_+^c \omega_+ &= -d_-^c \omega_-, \end{aligned} \tag{3.1}$$

where $\omega_\pm$ are the two-forms $gJ_\pm$, $d_\pm^c$ is d^c with respect to $J_\pm$ respectively, and the covariant derivatives have (Bismut) torsion [9]

$$\nabla^{(\pm)} := \nabla \pm g^{-1} H, \quad dH = 0, \tag{3.2}$$

where ∇ is the Levi-Civita connection.

When $J_\pm$ commute, it was found in [3] that there exist coordinates z, $\bar{z}$ and z', $\bar{z}'$ that coordinatize $\ker(J_+ \mp J_-)$, respectively, and for which the metric and three-form are given as derivatives of a function $K(z, \bar{z}, z, \bar{z}')$. Explicitly:

$$\begin{aligned} \omega_\pm = (d_\mp^c d + d_\Pi^c d_\pm^c) K &\iff g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K, \, g_{z'\bar{z}'} = -\partial_{z'} \partial_{\bar{z}'} K, \\ H = d d_+^c d_-^c K &\iff H_{z,\bar{z},z'} = \partial_z \partial_{\bar{z}} \partial_{z'} K, \; H_{z,\bar{z},\bar{z}'} = -\partial_z \partial_{\bar{z}} \partial_{\bar{z}'} K, \\ &\quad\; H_{z',\bar{z}',z} = -\partial'_z \partial_{\bar{z}'} \partial_z K, \; H_{z',\bar{z}',\bar{z}} = \partial_{z'} \partial_{\bar{z}'} \partial_{\bar{z}} K \end{aligned} \tag{3.3}$$

where d^c_Π is the d^c operator defined with respect to the product $\Pi := J_+ J_-$ and the differentials d^c_+, d^c_-, d^c_Π anticommute.

A complete description for geometries with a nonvanishing coker$[J_+, J_-]$ was an open problem until our paper [5]. Before describing the solution to the problem in sections 5 and 6, we recall Gualtieri's map of the bihermitian data $(J_\pm, g, B)$ into generalized Kähler geometry [1]:

$$\mathcal{J}_{1,2} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \tag{3.4}$$

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{3.5}$$

Note that what looks like a b-transform is not, since $dB = H \neq 0$.

It is an interesting fact that if we introduce the usual GCS's corresponding to $J_\pm$,

$$\mathcal{J}_\pm = \begin{pmatrix} J_\pm & 0 \\ 0 & -J_\pm^t \end{pmatrix}, \tag{3.6}$$

then the map (3.4) may be summarized as[2]

$$\mathcal{J}_{1,2} = \Pi_+ \mathcal{J}_{+B} \pm \Pi_- \mathcal{J}_{-B}, \tag{3.7}$$

with

$$\Pi_\pm := \tfrac{1}{2}(1 \pm \mathcal{G}). \tag{3.8}$$

4 Poisson-structures

There are three Poisson-structures relevant to our discussion. We first consider two real ones:

$$\pi_\pm := (J_+ \pm J_-)g^{-1} = -g^{-1}(J_+ \pm J_-)^t. \tag{4.1}$$

These were introduced in [10], where they were used essentially as described below, simplifying an earlier derivation in [11]. They ensure the existence of coordinates adapted to $\ker(J_+ - J_-) \oplus \ker(J_+ + J_-)$.

In a neighborhood of a regular point x_0 of π_- we may choose coordinates x^A whose tangents lie in the kernel of π_-:

$$\pi_-^{A\mu} = 0 \implies J_{+\mu}^A = J_{-\mu}^A. \tag{4.2}$$

Similarly, in a neighborhood of a regular point of π_+ we have coordinates $x^{A'}$ such that

$$\pi_+^{A'\mu} = 0 \implies J_{+\mu}^{A'} = -J_{-\mu}^{A'}. \tag{4.3}$$

[2]This structure is related to how the map is derived in chapter 6 of [1]

It then follows from the nondegeneracy of $\pi_+ \pm \pi_-$ that the Poisson brackets defined by π_+ and by π_- cannot have common Casimir functions. In other words, the directions A and A' cannot coincide. The result is that we may write

$$J_\pm = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & \pm I_t \end{pmatrix}, \tag{4.4}$$

in coordinates adapted to

$$\ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \operatorname{coker}[J_+, J_-], \tag{4.5}$$

where we write a canonical complex structure as

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.6}$$

and c labels the A and t labels the A' directions.

A third Poisson structure σ, related to the real Poisson structures, was introduced in [12]:

$$\sigma := [J_+, J_-]g^{-1} = \pm(J_+ \mp J_-)\pi_\pm = \mp(J_+ \pm J_-)\pi_\mp. \tag{4.7}$$

The relation to $\pi_\pm$ implies that

$$\ker \sigma = \ker \pi_+ \oplus \ker \pi_-. \tag{4.8}$$

Tangents to the symplectic leaf for σ lie in $\operatorname{coker}[J_+, J_-]$. Using (4.8), we can use σ to investigate the remaining directions in (4.4). We need that

$$J_\pm \sigma J_\pm^t = -\sigma \implies \sigma = \sigma^{(2,0)} + \bar{\sigma}^{(0,2)}, \tag{4.9}$$

with the decomposition with respect to *both* J_+ and J_-. Further $\sigma^{(2,0)}$ is holomorphic [12]:

$$\bar{\partial}\sigma^{(2,0)} = 0, \tag{4.10}$$

We now investigate the structure of the cokernel.

5 The structure of coker$[J_+, J_-]$

It is convenient to first treat the case $\ker[J_+, J_-] = \emptyset$. Then we have at our disposal a symplectic form Ω:

$$\Omega := \sigma^{-1}, \quad d\Omega = 0, \quad J_\pm^t \Omega J_\pm = -\Omega. \tag{5.1}$$

Choosing complex coordinates with respect to J_+,

$$J_+ = \begin{pmatrix} I_s & 0 \\ 0 & I_s \end{pmatrix}, \tag{5.2}$$

with I as in (4.6), we have the decomposition

$$\Omega = \Omega_+^{(2,0)} + \bar{\Omega}_+^{(0,2)}, \tag{5.3}$$

and

$$\partial\Omega_+^{(2,0)} = 0, \quad \bar{\partial}\Omega_+^{(2,0)} = 0, \tag{5.4}$$

which means in particular that $\Omega_+^{(2,0)}$ is holomorphic. We may then choose Darboux coordinates such that

$$\Omega_+^{(2,0)} = dq^a \wedge dp^a, \quad \bar{\Omega}_+^{(0,2)} = d\bar{q}^{\bar{a}} \wedge d\bar{p}^{\bar{a}}. \tag{5.5}$$

Alternatively, we can choose complex coordinates with respect to J_- and similarly derive that

$$\Omega_-^{(2,0)} = dQ^{a'} \wedge dP^{a'}, \quad \bar{\Omega}_-^{(0,2)} = d\bar{Q}^{\bar{a}'} \wedge d\bar{P}^{\bar{a}'}. \tag{5.6}$$

The transformation $\{q, p\} \to \{Q, P\}$ is a canonical transformations (symplectomorphism) and can thus be specified by a generating function[3] $K(q, P)$. Thus in a neighborhood, the canonical transformation is given by the generating function $K(q, P)$

$$p = \frac{\partial K}{\partial q}, \quad Q = \frac{\partial K}{\partial P}. \tag{5.7}$$

We calculate all our geometric structures J_+, J_-, Ω and g in the "mixed" coordinates $\{q, P\}$, using the transformation matrices

$$\frac{\partial(q, p)}{\partial(q, P)} = \begin{pmatrix} 1 & 0 \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial P} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial^2 K}{\partial q \partial q} & \frac{\partial^2 K}{\partial P \partial q} \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ K_{LL} & K_{LR} \end{pmatrix}, \tag{5.8}$$

and

$$\frac{\partial(Q, P)}{\partial(q, P)} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial P} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 K}{\partial q \partial P} & \frac{\partial^2 K}{\partial P \partial P} \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} K_{RL} & K_{RR} \\ 0 & 1 \end{pmatrix}, \tag{5.9}$$

where the labels L and R are shorthand for the coordinates $\{q^a, \bar{q}^{\bar{a}}\}$ and $\{P^{a'}, \bar{P}^{\bar{a}'}\}$ in (5.5) and (5.6) above. The expressions for $J_\pm$ are nonlinear and can be read off from the formulae given below for the general case with $\ker[J_+, J_-] \neq \emptyset$. The symplectic structure Ω is the only linear function of K

$$\Omega = K_{AA'} dq^A \wedge dP^{A'}, \tag{5.10}$$

where we use the collective notation $A = \{a, \bar{a}\}$, $A' = \{a', \bar{a}'\}$. We write Ω as a matrix

$$\Omega = \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}. \tag{5.11}$$

[3]There always exists at least one polarization such that any symplectomorphism, can be written in terms of such a generating function [13].

and find the metric and B-field using this matrix as follows [14] (*cf.* 4.7)

$$g = \Omega[J_+, J_-], \quad B = \Omega\{J_+, J_-\}. \tag{5.12}$$

6 The general case

In the general case when both $\ker[J_+, J_-]$ and $\operatorname{coker}[J_+, J_-]$ are nonempty, we combine the discussion leading to (4.4) in section 4 with that of the previous section. The formulas that we compute are rather complicated, and we have not found a suitable coordinate free way to express them.

We assume that in a neighborhood of x_0, the ranks of $\pi_\pm$ are constant, and as result, the rank of σ is constant. We work in coordinates $\{q, p, z, z'\}$ adapted to the symplectic foliation of σ as well as to the description of $\ker[J_+, J_-]$ given in section 4. In such coordinates J_+ may be taken to have the canonical form[4]

$$J_+ = \begin{pmatrix} I_s & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & I_t \end{pmatrix}, \tag{6.1}$$

where $\{q, p\}$ are Darboux coordinates for a symplectic leaf of σ and $\{z, z'\}$ parametrize the kernels of $\pi_\mp$. Similarly, there are coordinates $\{Q, P, z, z'\}$ where J_- takes a diagonal form (identical to that of J_+ in (6.1) except for a change of sign in the last entry; $I_t \to -I_t$). For every leaf separately we may now apply the arguments of section 5. There thus exists a generating function $K(q, P, z, z')$ for the symplectomorphisms between the two sets of coordinates. The transformation matrices to the coordinates $\{q, P, z, z'\}$ are given by the obvious extensions of (5.8) and (5.9) to include z, z'. The expression for $J_\pm$ in the "mixed" coordinates are then[5]:

$$J_+ = \begin{pmatrix} I_s & 0 & 0 & 0 \\ K_{RL}^{-1} C_{LL} & K_{RL}^{-1} I_s K_{LR} & K_{RL}^{-1} C_{Lc} & K_{RL}^{-1} C_{Lt} \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & I_t \end{pmatrix}, \tag{6.2}$$

and

$$J_- = \begin{pmatrix} K_{LR}^{-1} I_s K_{RL} & K_{LR}^{-1} C_{RR} & -K_{LR}^{-1} C_{Rc} & K_{LR}^{-1} A_{Rt} \\ 0 & -I_s & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & -I_t \end{pmatrix}, \tag{6.3}$$

[4]For historical reasons related to the origin in sigma models involving chiral and twisted chiral superfields, we again use the labels c and t for the z and z' directions.

[5]The rows and columns of this matrix correspond to the directions along $\{q, P, z, z'\}$.

where

$$
\begin{aligned}
K_{LR}^{-1} &= (K_{RL})^{-1}, \\
C &= IK - KI = \begin{pmatrix} 0 & 2iK \\ -2iK & 0 \end{pmatrix}, \\
A &= IK + KI = \begin{pmatrix} 2iK & 0 \\ 0 & -2iK \end{pmatrix},
\end{aligned}
\tag{6.4}
$$

where we suppress the indices in the last two entries[6]. The relations (5.12) no-longer hold when $\ker[J_+, J_-] \neq \emptyset$. The definition (4.7) of the Poisson structure σ still determines the metric, except along the kernel. However, the additional relation $H = \pm d_{\pm}^c \omega_{\pm}$ from (3.1) provides us with an equation for the remaining components of g and allows us to find the B-field. From the sigma model we already know the solution; the sum $E = \frac{1}{2}(g + B)$ of the metric g and B-field takes on the explicit form:

$$
\begin{aligned}
E_{LL} &= C_{LL} K_{LR}^{-1} I_s K_{RL} \\
E_{LR} &= I_s K_{LR} I_s + C_{LL} K_{LR}^{-1} C_{RR} \\
E_{Lc} &= K_{Lc} + I_s K_{Lc} I_c + C_{LL} K_{LR}^{-1} C_{Rc} \\
E_{Lt} &= -K_{Lt} - I_s K_{Lt} I_t + C_{LL} K_{LR}^{-1} A_{Rt} \\
E_{RL} &= -K_{RL} I_s K_{LR}^{-1} I_s K_{RL} \\
E_{RR} &= -K_{RL} I_s K_{LR}^{-1} C_{RR} \\
E_{Rc} &= K_{Rc} - K_{RL} I_s K_{LR}^{-1} C_{Rc} \\
E_{Rt} &= -K_{Rt} - K_{RL} I_s K_{LR}^{-1} A_{Rt} \\
E_{cL} &= C_{cL} K_{LR}^{-1} I_s K_{RL} \\
E_{cR} &= I_c K_{cR} I_s + C_{cL} K_{LR}^{-1} C_{RR} \\
E_{cc} &= K_{cc} + I_c K_{cc} I_c + C_{cL} K_{LR}^{-1} C_{Rc} \\
E_{ct} &= -K_{ct} - I_c K_{ct} I_t + C_{cL} K_{LR}^{-1} A_{Rt} \\
E_{tL} &= C_{tL} K_{LR}^{-1} I_s K_{RL} \\
E_{tR} &= I_t K_{tR} I_s + C_{tL} K_{LR}^{-1} C_{RR} \\
E_{tc} &= K_{tc} + I_t K_{tc} I_c + C_{tL} K_{LR}^{-1} C_{Rc} \\
E_{tt} &= -K_{tt} - I_t K_{tt} I_t + C_{tL} K_{LR}^{-1} A_{Rt}
\end{aligned}
\tag{6.5}
$$

In view of this last relation (6.5) as well as (6.2), (6.3) and (5.11) it seems appropriate to call K the generalized Kähler potential.

[6] In (6.4), the rows and columns correspond to $\{A, \bar{A}\}$, where $\{A\}$ are any one of the $\{q, P, z, z'\}$ directions in (6.2), (6.3), (6.5).

7 Linearization of generalized Kähler geometry

As can be seen in (6.5), the expressions for the metric and the B-field in terms of the generalized Kähler potential are highly nonlinear in contrast to what is the case in ordinary Kähler geometry. The nonlinearities all stem from the cokernel of $[J_+, J_-]$: when $[J_+, J_-] = 0$, the metric and B-field can be expressed as linear functions of the Hessian of K. We note that the structure of the nonlinearities are such as one might expect from a quotient construction. One is therefore faced with a natural question: Is there a space in which all the geometric data is encoded linearly with respect to the generalized Kähler potential and which gives the nonlinear generalized Kähler geometry through a quotient construction?

Inspired by superspace sigma-models, we have found the following local prescription for generating such a space: In local coordinates, we simply make the substitution

$$q \to z_L + z'_L, \quad P \to z_R + \bar{z}'_R \tag{7.1}$$

in the generalized Kähler potential K and use the naive commuting complex structures $J_\pm$ in which the z-coordinates are biholomorphic and the z'-coordinates are J_+ holomorphic and J_- antiholomorphic. This introduces isometries of the metric (3.3) corresponding to shifts of z_L, z'_L that preserve the sum, and similarly for $z_R, \bar{z}_R$; taking a quotient with respect to these isometries leads to the original nonlinear model. We now attempt to make mathematical sense of this prescription. We begin with some definitions:

Definition 1 (BiLP). A generalized Kähler geometry with commuting complex structures will be called *a bihermitian local product space* (BiLP).

We also give a special name to a BiLP with the isometries sketched above:

Definition 2 (ALP). A BiLP with these additional isometries (further characterized below) we will call an *auxiliary product space* (ALP).

The isometries are generated by $2n$ complex Abelian Killing vectors that preserve all the BiLP geometric data. They can be separated into two groups of n Killing vectors $\{k_{LA}\}$ and $\{k_{RA}\}$, $A = 1, \ldots, n$, satisfying the further requirement

$$\imath_{k_{LA}} H = d\alpha_{LA}, \quad \imath_{k_{RA'}} H = -d\alpha_{RA'}, \tag{7.2}$$

where the one-form α is the dual of the killing vector k: $\alpha = g(k)$; this is equivalent to saying that the Killing vectors are covariantly constant with respect to the covariant derivative with torsion (3.2). Such Killing vectors are called Kac–Moody Killing vectors. We require that each group of Killing vectors k_L and k_R by themselves span maximally isotropic subspaces, *i.e.,* that

$$g(k_{LA}, k_{LB}) = 0, \quad g(k_{RA'}, k_{RB'}) = 0, \tag{7.3}$$

but that the inner product

$$h_{A'B} = g(k_{RA'}, k_{LB}) \tag{7.4}$$

is nondegenerate.

Theorem. *Locally, any generalized Kähler manifold M is a quotient of an ALP by its Kac–Moody isometries.*

The quotient is performed by going to the orbits of the action of the Killing vector and choosing a horizontal subspace by specifying a connection. In the particular case with left and right Kac–Moody Killing vectors there is a corresponding left and right connection given by

$$(\theta_R)_\mu^{A'} = g_{\mu\nu} k_{LB}^\nu h^{BA'}, \quad (\theta_L)_\mu^A = h^{AB'} k_{RB'}^\nu g_{\nu\mu}, \tag{7.5}$$

where $h^{AB'}$ is the inverse of the nondegenerate matrix (7.4). The connections (7.5) satisfy the following properties

$$\theta_L(k_L) = \mathbf{1}, \quad \theta_R(k_R) = \mathbf{1}, \quad \theta_L(k_R) = 0, \quad \theta_R(k_L) = 0. \tag{7.6}$$

Vectors in the horizontal subspaces are defined as the vectors lying in the kernel of *both* θ_L and θ_R.

All geometric structures defined on the ALP can now be defined on the quotient space. As usual, forms are projected using the connection so that the contraction with any vertical vector is zero, whereas vectors are pushed forward to the quotient using the bundle projection $\mathbf{p}$. When we project the complex structures

$$\tilde{J}_\pm(X) := \mathbf{p}_*\big(J_\pm(X) - \theta_L(X) \cdot J_\pm(k_L) - \theta_R(X) \cdot J_\pm(k_R)\big) \quad \text{for all } X, \tag{7.7}$$

it is clear that the projected complex structures $\tilde{J}_\pm$ do *not* commute even though $J_\pm$ do.

Projecting the metric and B-field, we find

$$\tilde{g} = g - h(\theta_L, \theta_R) - h(\theta_R, \theta_L), \tag{7.8}$$

$$\tilde{B} = B - h(\theta_L, \theta_R) + h(\theta_R, \theta_L). \tag{7.9}$$

It is straightforward to see that $\tilde{g}$ gives zero when contracted with any vertical vector while this is not so for $\tilde{B}$. This (apparent) mystery is resolved by showing that $\tilde{H} = d\tilde{B}$ can be written as

$$\tilde{H} = H - \theta_L \wedge \imath_{k_L} H - \theta_R \wedge \imath_{k_R} H - \theta_L \wedge \theta_R \wedge \imath_{k_L} \imath_{k_R} H \tag{7.10}$$

so that the geometrically meaningful object $\tilde{H}$ defined by $d\tilde{B}$ is indeed well defined on the quotient. The quotients $\tilde{g}$ and $\tilde{H}$ agree with the expressions for the original generalized Kähler manifold.

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Part C

Geometries with torsion

Chapter 9

Non-integrable geometries, torsion, and holonomy

Ilka Agricola

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1 Background and motivation

1.1 Introduction

Since the fifties, the French school founded by M. Berger had developed the idea that Riemannian manifolds should be subdivided into different classes according to their holonomy group, and the name *special* (*integrable*) *geometries* has become customary for those which are not of general type. At the beginning of the seventies, A. Gray generalized the classical holonomy concept by introducing a classification principle for *non-integrable special Riemannian geometries* discovered in this context nearly Kähler manifolds in dimension six and nearly parallel G_2-manifolds in dimension seven. Non-integrable geometries then entered mathematical physics when it became clear in the eighties that they are strongly related to twistor theory and the study of small eigenvalues of the Dirac operator.

The interest in non-integrable geometries was revived in the past years through developments of superstring theory. Firstly, integrable geometries (Calabi–Yau manifolds, Joyce manifolds etc.) are exact solutions of the Strominger model (1986), though with vanishing B-field. If one deforms these vacuum equations and looks for models with non-trivial B-field, a new mathematical approach going back a decade implies that solutions can be constructed geometrically from non integrable geometries with torsion. In this way, manifolds *not* belonging to the field of algebraic geometry (integrable geometries) become candidates for interesting models in theoretical physics.

Before discussing the deep mathematical and physical backgrounds, let us give a – very intuitive – explanation of why the traditional Yang-Mills approach needs modification in string theory and how torsion enters the scene. Point particles move along world-lines, and physical quantities are typically computed as line integrals of some potential that is, mathematically speaking, just a 1-form. The associated field strength is then its differential – a 2-form – and interpreted as the curvature of some connection. In contrast, excitations of extended 1-dimensional objects (the 'strings') are 'world-surfaces', and physical quantities have to be surface integrals of certain potential 2-forms. Their field strengths are thus 3-forms and cannot be interpreted as curvatures anymore. The key idea is to supply the (pseudo)-Riemannian manifold underlying the physical model with a non-integrable G-structure admitting a 'good' metric G-connection ∇ with torsion, which in turn will play the role of a B-field strength; and the art is to choose the G-structure so that the connection ∇ admits the desired parallel objects, in particular spinors, interpreted as supersymmetry transformations.

In this chapter, we shall establish connections with skew-symmetric torsion as one of the main tools to understand non-integrable geometries. To this aim a series of key examples is presented and successively dealt with using the notions of intrinsic torsion and characteristic connection of a G-structure as unifying principles. The General Holonomy Principle bridges over to parallel objects, thus motivating the discussion

of geometric stabilizers, with emphasis on spinors and differential forms. Several Weitzenböck formulas for Dirac operators associated with torsion connections enable us to discuss spinorial field equations, such as those governing the common sector of type II superstring theory. They also provide the link to Kostant's cubic Dirac operator.

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1.2 Mathematical motivation

While the classical symmetry approach in differential geometry was based on the *isometry group* of a manifold, it turned out by the mid-fifties that a second intrinsic group associated to a Riemannian manifold was deeply related to fundamental features like curvature and parallel objects. The so-called *holonomy group* determines how a vector can change under parallel transport along a closed loop inside the manifold. Berger's theorem (1955) classifies all possible restricted holonomy groups of a simply connected, irreducible and non-symmetric Riemannian manifold (M, g) (see [37], [217] for corrections and simplifications in the proof and [56] for a status report). The holonomy group can be either $\mathrm{SO}(n)$ in the generic case or one of the groups listed in Table 1 (here and in the sequel, ∇^g denotes the Levi-Civita connection).

Table 1. Possible Riemannian holonomy groups ('Berger's list').

$\dim M$	$\mathrm{Hol}(M)$	name	parallel object	curvature
$4n$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	quaternionic Kähler manifold	—	$\mathrm{Ric} = \lambda g$
$2n$	$\mathrm{U}(n)$	Kähler manifold	$\nabla^g J = 0$	—
$2n$	$\mathrm{SU}(n)$	Calabi–Yau manifold	$\nabla^g J = 0$	$\mathrm{Ric} = 0$
$4n$	$\mathrm{Sp}(n)$	hyper-Kähler manifold	$\nabla^g J = 0$	$\mathrm{Ric} = 0$
7	G_2	parallel G_2-manifold	$\nabla^g \omega^3 = 0$	$\mathrm{Ric} = 0$
8	Spin(7)	parallel Spin(7)-manifold	$\nabla^g \beta^4 = 0$	$\mathrm{Ric} = 0$
[16]	[Spin(9)]	[parallel Spin(9)-manifold]	—	—

Manifolds having one of these holonomy groups are called manifolds with special (integrable) holonomy, or *special* (*integrable*) *geometries* for short. We put the case $n = 16$ and $\mathrm{Hol}(M) = \mathrm{Spin}(9)$ into brackets, because Alekseevski and Brown/Gray showed independently that such a manifold is necessarily symmetric ([16], [54]). The point is that Berger proved that the groups on this list were the only possibilities, but he was not able to show whether they actually occurred as holonomy groups of compact manifolds. It took another thirty years to find out that – with the exception of $\mathrm{Spin}(9)$ – this is indeed the case: The existence of metrics with holonomy $\mathrm{SU}(m)$ or $\mathrm{Sp}(m)$ on compact manifolds followed from Yau's solution of the Calabi Conjecture posed in 1954 [247]. Explicit non-compact metrics with holonomy G_2 or $\mathrm{Spin}(7)$ are due to R. Bryant [55] and R. Bryant and S. Salamon [58], while compact manifolds with holonomy G_2 or $\mathrm{Spin}(7)$ were constructed by D. Joyce only in 1996 (see [156], [157] [158] and the book [159], which also contains a proof of the Calabi Conjecture). Later, compact exceptional holonomy manifolds have also been constructed by other methods by Kovalev ([175]).

As we will explain later, the General Holonomy Principle relates manifolds with $\mathrm{Hol}(M) = \mathrm{SU}(n), \mathrm{Sp}(n), G_2$ or $\mathrm{Spin}(7)$ with ∇^g-parallel spinors (see Section 3). Already in the sixties it had been observed that the existence of such a spinor implies in turn the vanishing of the Ricci curvature ([48] and Proposition 2.2) and restricts the holonomy group of the manifold ([139], [241]), but the difficulties in constructing explicit compact manifolds with special integrable Ricci-flat metrics inhibited further research on the deeper meaning of this result.

There was progress in this direction only in the homogeneous case. Symmetric spaces are the "integrable" geometries inside the much larger class of homogeneous reductive spaces. Given a non-compact semisimple Lie group G and a maximal compact subgroup K such that $\mathrm{rank}\, G = \mathrm{rank}\, K$, consider the associated symmetric space G/K. The Dirac operator can be twisted by a finite-dimensional irreducible unitary representation τ of K, and it was shown by Parthasarathy, Wolf, Atiyah and Schmid that for suitable τ most of the discrete series representations of G can be realized on the L^2-kernel of this twisted Dirac operator ([201], [246], [29]). The crucial step is to relate the square of the Dirac operator with the Casimir operator Ω_G of G; for trivial τ, the corresponding formula reads

$$D^2 = \Omega_G + \frac{1}{8}\,\mathrm{Scal}. \tag{1}$$

Meanwhile many people began looking for suitable generalizations of the classical holonomy concept. One motivation for this was that the notion of Riemannian holonomy is too restrictive for vast classes of interesting Riemannian manifolds; for example, contact or almost Hermitian manifolds cannot be distinguished merely by their holonomy properties (they have generic holonomy $\mathrm{SO}(n)$), and the Levi-Civita connection is not adapted to the underlying geometric structure (meaning that the defining objects are not parallel).

In 1971 A. Gray introduced the notion of *weak holonomy* ([128]), "one of his most original concepts" and "an idea much ahead of its time" in the words of N. Hitchin [141]. This concept turned out to yield interesting non-integrable geometries in dimensions $n \leq 8$ and $n = 16$. In particular, manifolds with weak holonomy $\mathrm{U}(n)$ and G_2 became known as *nearly Kähler* and *nearly parallel* G_2-manifolds, respectively. But whereas metrics of compact Ricci-flat integrable geometries have not been realized explicitly (so far), there are many well-known homogeneous reductive examples of non-integrable geometries ([127], [86], [33], [107], [49], [92] and many others). The relation to Dirac operators emerged shortly after Th. Friedrich proved in 1980 a seminal inequality for the first eigenvalue λ_1 of the Dirac operator on a compact Riemannian manifold M^n of non-negative curvature [95]:

$$(\lambda_1)^2 \geq \frac{n}{4(n-1)} \min_{M^n}(\mathrm{scal}), \tag{2}$$

In this estimate, equality occurs precisely if the corresponding eigenspinor ψ satisfies the Killing equation

$$\nabla^g_X \psi = \pm \frac{1}{2} \sqrt{\frac{\min(\mathrm{scal})}{n(n-1)}} X \cdot \psi =: \mu X \cdot \psi.$$

The first non-trivial compact examples of Riemannian manifolds with Killing spinors were found in dimensions 5 and 6 in 1980 and 1985, respectively ([95], [101]). The link to non-integrable geometry was established shortly after; for instance, a compact, connected and simply connected 6-dimensional Hermitian manifold is nearly Kähler if and only if it admits a Killing spinor with real Killing number μ [133]. Similar results hold for Einstein–Sasaki structures in dimension 5 and nearly parallel G_2-manifolds in dimension 7 ([105], [106]). Remarkably, the proof of inequality (2) relies on introducing a suitable spin connection – an idea much in line with recent developments. A. Lichnerowicz established the link to twistor theory by showing that on a compact manifold the space of twistor spinors coincides – up to a conformal change of the metric – with the space of Killing spinors [184].

1.3 Physical motivation – torsion in gravity

The first attempts to introduce torsion as an additional 'datum' for describing physics in general relativity goes back to Cartan himself [64]. Viewing torsion as some intrinsic angular momentum, he derived a set of gravitational field equations from a variational principle, but postulated that the energy-momentum tensor should still be divergence-free, a condition too restrictive for making this approach useful. The idea was taken up again in broader context in the late fifties. The variation of the scalar curvature (and an additional Lagrangian generating the energy-momentum tensor) on a space-time endowed with a metric connection with torsion yielded the two fundamental equations of *Einstein–Cartan theory*, first formulated by Kibble [163] and Sciama (see his article in [148]). The first equation is (formally) Einstein's classical field equation of general

relativity with an effective energy momentum tensor T_{eff} depending on torsion, the second one can be written in index-free notation as

$$Q(X,Y) + \sum_{i=1}^{n} \big(Q(Y,e_i) \lrcorner e_i\big) \cdot X - \big(Q(X,e_i) \lrcorner e_i\big) \cdot Y = 8\pi S(X,Y).$$

Here Q denotes the torsion of the new connection ∇, S the spin density and $e_1, \ldots, e_n$ any orthonormal frame. A. Trautman provided an elegant formulation of Einstein–Cartan theory in the language of principal fibre bundles [229]. The most striking predictions of Einstein–Cartan theory are in cosmology. In the presence of very dense spinning matter, nonsingular cosmological models may be constructed because the effective energy momentum tensor T_{eff} does not fulfill the conditions of the Penrose–Hawking singularity theorems anymore [230]. The first example of such a model was provided by W. Kopczyński [171], while J. Tafel found a large class of such models with homogeneous spacial sections [226]. For a general review of gravity with spin and torsion including extensive references, we refer to the article [136].

In the absence of spin, the torsion vanishes and the whole theory reduces to Einstein's original formulation of general relativity. In practice, torsion turned out to be hard to detect experimentally, since all tests of general relativity are based on experiments in empty space. Einstein–Cartan theory is pursued no longer, although some concepts that it inspired are still of relevance (see [137] for a generalization with additional currents and shear, [231] for optical aspects, [208] for the link to the classical theory of defects in elastic media). Yet, it may be possible that Einstein–Cartan theory will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion.

1.4 Physical motivation – torsion in superstring theory

Superstring theory (see for example [131], [185]) is a physical theory aiming at describing nature at small distances ($\simeq 10^{-25}$ m). The concept of point-like elementary particles is replaced by one-dimensional objects as building blocks of matter – the so-called strings. Particles are then understood as resonance states of strings and can be described together with their interactions up to very high energies (small distances) without internal contradictions. Besides gravitation, string theory incorporates many other gauge interactions and hence is an excellent candidate for a more profound description of matter than the standard model of elementary particles. Quantization of superstrings is only possible in the critical dimension 10, while M-theory is a non-perturbative description of superstrings with "geometrized" coupling, and lives in dimension 11. Mathematically speaking, a 10- or 11-dimensional configuration space Y (a priori not necessarily smooth) is assumed to be the product

$$Y^{10,11} = V^{3\text{-}5} \times M^{5\text{-}8}$$

of a low-dimensional spacetime V describing the 'external' part of the theory (typically, Minkowski space or a space motivated from general relativity like anti-de-Sitter space), and a higher-dimensional 'internal space' M with some special geometric structure. The metric is typically a direct or a warped product. On M, internal symmetries of particles are described by parallel spinor fields, the most important of which being the existing supersymmetries: a spinor field has spin $1/2$, so tensoring with it swaps bosons and fermions. By the General Holonomy Principle (see Theorem 2.3), the holonomy group has to be a subgroup of the stabilizer of the set of parallel spinors inside $\mathrm{Spin}(9,1)$. These are well known and summarized in Table 2. We shall explain how to derive this result and how to understand the occurring semidirect products in Section 3.4.

Table 2. Possible stabilizers of invariant spinors inside $\mathrm{Spin}(9,1)$.

# of invariant spinors	stabilizer groups
1	$\mathrm{Spin}(7) \ltimes \mathbb{R}^8$
2	G_2, $\mathrm{SU}(4) \ltimes \mathbb{R}^8$
3	$\mathrm{Sp}(2) \ltimes \mathbb{R}^8$
4	$\mathrm{SU}(3)$, $(\mathrm{SU}(2) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^8$
8	$\mathrm{SU}(2)$, $\mathbb{R}^8$
16	$\{e\}$

Since its early days, string theory has been intricately related with some branches of algebraic geometry. This is due to the fact that the integrable, Ricci-flat geometries with a parallel spinor field with respect to the Levi-Civita connection are exact solutions of the Strominger model for a string vacuum with vanishing B-field and constant dilaton. This rich and active area of mathematical research lead to interesting developments such as the discovery of mirror symmetry.

1.5 First developments since 1980

In the early eighties, several physicists independently tried to incorporate torsion into superstring and supergravity theories in order to get a more physically flexible model, possibly inspired by the developments in classical gravity ([238], [113], [142], [80], [207]). In fact, simple supergravity is equivalent to Einstein–Cartan theory with a massless, anticommuting Rarita–Schwinger field as source. But contrary to general relativity, one difficulty stems from the fact that there are several models in superstring

theory (type I, II, heterotic,…) that vary in the excitation spectrum and the possible interactions.

In his article "Superstrings with torsion" [222], A. Strominger describes the basic model in the common sector of type II superstring theory in form of a 6-tuple $(M^n, g, \nabla, T, \Phi, \Psi)$ consisting of a Riemannian spin manifold (M^n, g), a 3-form T, a dilaton function Φ and a spinor field Ψ. The field equations can be written in the following form (recall that ∇^g denotes the Levi-Civita connection):

$$\mathrm{Ric}_{ij} - \frac{1}{4} T_{imn} T_{jmn} + 2\nabla^g_i \partial_j \Phi = 0, \quad \delta(e^{-2\Phi} T) = 0,$$

$$\left(\nabla^g_X + \frac{1}{4} X \lrcorner T\right)\psi = 0, \quad (2d\Phi - T)\cdot\psi = 0.$$

If one introduces a new metric connection ∇ whose torsion is given by the 3-form T,

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2} T(X, Y, -),$$

one sees that the third equation is equivalent to $\nabla\Psi = 0$. The remaining equations can similarly be rewritten in terms of ∇. For constant dilaton Φ, they take the particularly simple form [151]

$$\mathrm{Ric}^{\nabla} = 0, \quad \delta^g(T) = 0, \quad \nabla\Psi = 0, \quad T\cdot\Psi = 0, \tag{3}$$

and the second equation ($\delta^g(T) = 0$) now follows from the first equation ($\mathrm{Ric}^{\nabla} = 0$). For M compact, it was shown in [4, Theorem 4.1] that a solution of all equations necessarily forces $T = 0$, i.e., an integrable Ricci-flat geometry with classical holonomy given by Berger's list. By a careful analysis of the integrability conditions, this result could later be extended to the non-compact case ([14], see also Section 5.4). Together with the well-understood Calabi–Yau manifolds, Joyce manifolds with Riemannian holonomy G_2 or Spin(7) thus became of interest in recent times (see [30], [75]). From a mathematical point of view, this result stresses the importance of tackling easier problems first, for example partial solutions. As first step in the investigation of metric connections with totally skew-symmetric torsion, Dirac operators, parallel spinors etc., Th. Friedrich and S. Ivanov proved that many non-integrable geometric structures (almost contact metric structures, almost Hermitian and weak G_2-structures) admit a unique invariant connection ∇ with totally skew-symmetric torsion [102], thus being a natural replacement for the Levi-Civita connection. Non-integrable geometries could then be studied by their holonomy properties.

In fact, in mathematics the times had been ripe for a new look at the intricate relationship between holonomy, special geometries, spinors and differential forms: in 1987, R. Bryant found the first explicit local examples of metrics with exceptional Riemannian holonomy (see [55] and [58]), Ch. Bär described their relation to Killing spinors via the cone construction [31]. Building on the insightful vision of Gray, S. Salamon realized the centrality of the concept of *intrinsic torsion* ([209] and, for recent results, [91], [69], [74]). Swann successfully tried *weakening holonomy* [225],

and N. Hitchin characterized non-integrable geometries as critical points of some linear functionals on differential forms [141]. In particular, he motivated a generalization of Calabi–Yau-manifolds [140] and of G_2-manifolds [244], and discovered a new, previously unknown special geometry in dimension 8 ("weak PSU(3)-structures", see also [245]). Friedrich reformulated the concepts of non-integrable geometries in terms of principle fiber bundles [99] and discussed the exceptional dimension 16 suggested by A. Gray years before ([97], [98]). Analytic problems – in particular, the investigation of the Dirac operator – on non-integrable Riemannian manifolds contributed to a further understanding of the underlying geometry ([42], [23], [115]). Finally, the Italian school and collaborators devoted over the past years a lot of effort to the explicit construction of homogeneous examples of non-integrable geometries with special properties in small dimensions (see for example [2], [93], [94], [210] and the literature cited therein), making it possible to test the different concepts on explicit examples.

The first non-integrable geometry that raised the interest of string theorists was the squashed 7-sphere with its weak G_2-structure, although the first steps in this direction were still marked by confusion about the different holonomy concepts. A good overview about G_2 in string theory is the survey article by M. Duff ([84]). It includes speculations about possible applications of weak Spin(9)-structures in dimension 16, which a priori are of too high dimension to be considered in physics. In dimension three, it is well known (see for example [220]) that the Strominger equation $\nabla\Psi = 0$ can be solved only on a compact Lie group with bi-invariant metric, and that the torsion of the invariant connection ∇ coincides with the Lie bracket. In dimension four, the Strominger model leads to a HKT structure (see Section 2.4 for more references), i.e., a hyper-Hermitian structure that is parallel with respect to ∇, and – in the compact case – the manifold is either a Calabi–Yau manifold or a Hopf surface [151]. Hence, the first interesting dimension for further mathematical investigations is five.

Obviously, besides the basic correspondence outlined here, there is still much more going on between special geometries and detailed properties of physical models constructed from them. Some weak geometries have been rederived by physicists looking for partial solutions by numerical analysis of ODE's and heavy special function machinery [117].

2 Metric connections with torsion

2.1 Types of connections and their lift into the spinor bundle

The notion of torsion of a connection was invented by Elie Cartan, and appeared for the first time in a short note at the Académie des Sciences de Paris in 1922 [63]. Although the article contains no formulas, Cartan observed that such a connection may or may not preserve geodesics, and initially turns his attention to those who do so. In this

sense, Cartan was the first to investigate this class of connections. He then goes on to explain in very general terms how the connection should be *adapted* to the geometry under consideration. We believe that this point of view should be taken into account in Riemannian geometry, too.

We give a short review of the eight classes of geometric torsion tensors. Consider a Riemannian manifold (M^n, g). The difference between its Levi-Civita connection ∇^g and any linear connection ∇ is a $(2,1)$-tensor field A,

$$\nabla_X Y = \nabla^g_X Y + A(X,Y), \quad X, Y \in TM^n.$$

The vanishing of the symmetric or the antisymmetric part of A has immediate geometric interpretations. The connection ∇ is torsion-free if and only if A is symmetric. The connection ∇ has the same geodesics as the Levi-Civita connection ∇^g if and only if A is skew-symmetric. Following Cartan, we study the algebraic types of the torsion tensor for a metric connection. Denote by the same symbol the $(3,0)$-tensor derived from a $(2,1)$-tensor by contraction with the metric. We identify TM^n with $(TM^n)^*$ using g from now on. Let $\mathcal{T}$ be the $n^2(n-1)/2$-dimensional space of all possible torsion tensors,

$$\mathcal{T} = \{T \in \otimes^3 TM^n \mid T(X,Y,Z) = -T(Y,X,Z)\} \cong \Lambda^2 TM^n \otimes TM^n.$$

A connection ∇ is metric if and only if A belongs to the space

$$\mathcal{A}^g := TM^n \otimes (\Lambda^2 TM^n) = \{A \in \otimes^3 TM^n \mid A(X,V,W) + A(X,W,V) = 0\}.$$

In particular, $\dim \mathcal{A}^g = \dim \mathcal{T}$, reflecting the fact that metric connections can be uniquely characterized by their torsion.

Proposition 2.1 ([65, p. 51], [232], [209]). *The spaces $\mathcal{T}$ and $\mathcal{A}^g$ are isomorphic as $\mathrm{O}(n)$ representations, an equivariant bijection being*

$$\begin{aligned} T(X,Y,Z) &= A(X,Y,Z) - A(Y,X,Z), \\ 2\,A(X,Y,Z) &= T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y). \end{aligned}$$

For $n \geq 3$, they split under the action of $\mathrm{O}(n)$ into the sum of three irreducible representations,

$$\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'.$$

The last module will also be denoted $\mathcal{A}'$ if viewed as a subspace of $\mathcal{A}^g$ and is equivalent to the Cartan product of representations $TM^n \otimes \Lambda^2 TM^n$,

$$\mathcal{T}' = \left\{T \in \mathcal{T} \mid \mathop{\mathfrak{S}}_{X,Y,Z} T(X,Y,Z) = 0,\ \textstyle\sum_{i=1}^n T(e_i, e_i, X) = 0 \text{ for all } X, Y, Z\right\}$$

for any orthonormal frame $e_1, \ldots, e_n$. For $n = 2$, $\mathcal{T} \cong \mathcal{A}^g \cong \mathbb{R}^2$ is $\mathrm{O}(2)$-irreducible.

The eight classes of linear connections are now defined by the possible components of their torsions T in these spaces. The nice lecture notes by Tricerri and Vanhecke [232] use a similar approach in order to classify homogeneous spaces by the algebraic

properties of the torsion of the canonical connection. They construct homogeneous examples of all classes, study their "richness" and give explicit formulas for the projections on every irreducible component of $\mathcal{T}$ in terms of $\mathrm{O}(n)$-invariants.

Definition 2.1 (Connection with vectorial torsion). The connection ∇ is said to have *vectorial torsion* if its torsion tensor lies in the first space of the decomposition in Proposition 2.1, i.e., if it is essentially defined by some vector field V on M. The tensors A and T can then be directly expressed through V as

$$A(X,Y) = g(X,Y)V - g(V,Y)X, \quad T(X,Y,Z) = g\big(g(V,X)Y - g(V,Y)X, Z\big).$$

These connections are particularly interesting on surfaces, in as much that *every* metric connection on a surface is of this type.

In [232], F. Tricerri and L. Vanhecke showed that if M is connected, complete, simply-connected and V is ∇-parallel, then (M, g) has to be isometric to the hyperbolic space. V. Miquel studied in [192] and [193] the growth of geodesic balls of such connections, but did not investigate the detailed shape of geodesics. The study of the latter was outlined in [15], whereas [12] and [152] are devoted to holonomy aspects and a possible role in superstring theory.

Notice that there is some similarity to Weyl geometry. In both cases, we consider a Riemannian manifold with a fixed vector field V on it ([62], [114]). A Weyl structure is a pair consisting of a conformal class of metrics and a *torsion-free non-metric connection* preserving the conformal structure. This connection is constructed by choosing a metric g in the conformal class and is then defined by the formula

$$\nabla^{\mathrm{w}}_X Y := \nabla^g_X Y + g(X,V)\cdot Y + g(Y,V)X - g(X,Y)V.$$

Weyl geometry deals with the geometric properties of these connections, but in spite of the resemblance, it turns out to be a rather different topic. Yet in special geometric situations it may happen that ideas from Weyl geometry can be useful.

Definition 2.2 (Connection with skew-symmetric torsion). The connection ∇ is said to have (*totally*) *skew-symmetric torsion* if its torsion tensor lies in the second component of the decomposition in Proposition 2.1, i.e., it is given by a 3-form. They are by now – for reasons to be detailed later – a well-established tool in superstring theory and weak holonomy theories (see for example [222], [185], [117], [75], [88], [84], [10] etc.). In Examples 2.2 to 2.5, we describe large classes of interesting manifolds that carry natural connections with skew-symmetric torsion. Geometrically, these connections can be characterized as those which are metric and geodesic-preserving. In contrast to the case of vectorial torsion, manifolds admitting invariant metric connections ∇ with ∇-parallel skew-symmetric torsion form a vast class that is worth a separate investigation ([20], [74], [212]).

Suppose now that we are given a metric connection ∇ with torsion on a Riemannian spin manifold (M^n, g) with spin bundle ΣM^n. We slightly modify our notation and

write ∇ as

$$\nabla_X Y := \nabla^g_X Y + A_X Y,$$

where A_X defines an endomorphism $TM^n \to TM^n$ for every X. The condition for ∇ to be metric

$$g(A_X Y, Z) + g(Y, A_X Z) = 0$$

means that A_X preserves the scalar product g, which can be expressed as $A_X \in \mathfrak{so}(n)$. After identifying $\mathfrak{so}(n)$ with $\Lambda^2(\mathbb{R}^n)$, A_X can be written relative to some orthonormal frame

$$A_X = \sum_{i<j} \alpha_{ij} e_i \wedge e_j.$$

Since the lift into $\mathfrak{spin}(n)$ of $e_i \wedge e_j$ is $E_i \cdot E_j/2$, A_X defines an element in $\mathfrak{spin}(n)$ and hence an endomorphism of the spinor bundle. In fact, we need not introduce a different notation for the lift of A_X. Rather, observe that if A_X is written as a 2-form,

(1) its action on a vector Y as an element of $\mathfrak{so}(n)$ is just $A_X Y = Y \lrcorner A_X$, so our connection takes on vectors the form

$$\nabla_X Y = \nabla^g_X Y + Y \lrcorner A_X,$$

(2) the action of A_X on a spinor ψ as an element of $\mathfrak{spin}(n)$ is just $A_X \psi = (1/2)\, A_X \cdot \psi$, where $\cdot$ denotes the Clifford product of a k-form by a spinor. The lift of the connection ∇ to the spinor bundle ΣM^n (again denoted by ∇) is thus given by

$$\nabla_X \psi = \nabla^g_X \psi + \frac{1}{2} A_X \cdot \psi.$$

We denote by $(-,-)$ the Hermitian product on the spinor bundle ΣM^n induced by g. When lifted to the spinor bundle, ∇ satisfies the following properties that are well known for the lift of the Levi-Civita connection. In fact, the proof easily follows from the corresponding properties for the Levi-Civita connection [96, p. 59] and the Hermitian product [96, p. 24].

Lemma 2.1. *The lift of any metric connection ∇ on TM^n into the spinor bundle ΣM^n satisfies*

$$\nabla_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot (\nabla_X \psi), \quad X(\psi_1, \psi_2) = (\nabla_X \psi_1, \psi_2) + (\psi_1, \nabla_X \psi_2).$$

Any spinorial connection with the second property is again called *metric*. The first property (chain rule for Clifford products) makes only sense for spinorial connections that are lifts from the tangent bundle, not for arbitrary spin connections.

Example 2.1 (Connection with vectorial torsion). For a metric connection with vectorial torsion given by $V \in TM$, $A_X = 2\, X \wedge V$, since

$$\begin{aligned} Y \lrcorner (2\, X \wedge V) &= 2(X \wedge V)(Y, -) = (X \otimes V)(Y, -) - (V \otimes X)(Y, -) \\ &= g(X, Y)V - g(Y, V)X. \end{aligned}$$

Example 2.2 (Connection with skew-symmetric torsion). For a metric connection with skew-symmetric torsion defined by some $T \in \Lambda^3(M)$, $A_X = X \lrcorner T$. Examples of manifolds with a geometrically defined torsion 3-form are given in the next section.

Example 2.3 (Connection defined by higher order differential forms). As example of a metric spinorial connection not induced from the tangent bundle, consider

$$\nabla_X \psi := \nabla^g_X \psi + (X \lrcorner \omega^k) \cdot \psi + (X \wedge \eta^l) \cdot \psi$$

for some forms $\omega^k \in \Lambda^k(M)$, $\eta^l \in \Lambda^l(M)$ $(k, l \geq 4)$. These are of particular interest in string theory as they are used for the description of higher dimensional membranes ([9], [205]).

Example 2.4 (General case). The class $\mathcal{A}'$ of Proposition 2.1 cannot be directly interpreted as vectors or forms of a given degree, but it is not complicated to construct elements in $\mathcal{A}'$ either. Connections of this type have not yet been investigated as a class of their own, but they are used as an interesting tool in several contexts – for example, in closed G_2-geometry ([57], [70]). The canonical connection of an almost Kähler manifold is also of this type.

What makes metric connections with torsion so interesting is the huge variety of geometric situations that they unify in a mathematically useful way. Let us illustrate this fact by some examples.

2.2 Naturally reductive spaces

Naturally reductive spaces are a key example of manifolds with a metric connection with skew-symmetric torsion.

Consider a Riemannian homogeneous space $M = G/H$. We suppose that M is *reductive*, i.e., the Lie algebra $\mathfrak{g}$ of G splits as vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$, where $\mathrm{Ad}\colon H \to \mathrm{SO}(\mathfrak{m})$ is the isotropy representation of M. We identify $\mathfrak{m}$ with T_0M and we pull back the Riemannian metric $\langle\ ,\ \rangle_0$ on T_0M to an inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$. By a theorem of Wang ([170, Chapter X, Theorem 2.1]), there is a one-to-one correspondence between the set of G-invariant metric affine connections and the set of linear mappings $\Lambda_{\mathfrak{m}}\colon \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ such that

$$\Lambda_{\mathfrak{m}}(hXh^{-1}) = \mathrm{Ad}(h)\Lambda_{\mathfrak{m}}(X)\mathrm{Ad}(h)^{-1} \quad \text{for } X \in \mathfrak{m} \text{ and } h \in H.$$

A homogeneous Riemannian metric g on M is said to be *naturally reductive* (with respect to G) if the map $[X, -]_{\mathfrak{m}}\colon \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric,

$$g([X, Y]_{\mathfrak{m}}, Z) + g(Y, [X, Z]_{\mathfrak{m}}) = 0 \quad \text{for all } X, Y, Z \in \mathfrak{m}.$$

The family of metric connections ∇^t defined by $\Lambda^t_{\mathfrak{m}}(X)Y := (1 - t)/2\,[X, Y]_{\mathfrak{m}}$ has then skew-symmetric torsion $T^t(X, Y) = -t[X, Y]_{\mathfrak{m}}$. The connection ∇^1 is

of particular interest and is called the *canonical connection*. Naturally reductive homogeneous spaces equipped with their canonical connection are a well studied (see for example [78]) generalization of symmetric spaces since they satisfy $\nabla^1 T^1 = \nabla^1 \mathcal{R}^1 = 0$, where $\mathcal{R}^1$ denotes the curvature tensor of ∇^1 (Ambrose–Singer, [25]; see [234, Theorem 2.3] for a converse).

Naturally reductive spaces have been classified in small dimensions by Kowalski, Tricerri and Vanhecke, partially in the larger context of commutative spaces (in the sense of Gel'fand): the 3-dimensional naturally reductive homogeneous spaces are $\mathrm{SU}(2)$, the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ and the Heisenberg group H_3, all with special families of left-invariant metrics ([232]). A simply connected four-dimensional naturally reductive space is either symmetric or decomposable as direct product ([176]). In dimension 5, it is either symmetric, decomposable or locally isometric to $\mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)$, $\mathrm{SO}(3) \times H_3/\mathrm{SO}(2)$ (or any of these with $\mathrm{SO}(3)$ replaced by $\mathrm{SL}(2,\mathbb{R})$), to the five-dimensional Heisenberg group H_5 or to the Berger sphere $\mathrm{SU}(3)/\mathrm{SU}(2)$ (or $\mathrm{SU}(2,1)/\mathrm{SU}(2)$), all endowed with special families of metrics ([178]).

Other standard examples of naturally reductive spaces are

- Geodesic spheres in two-point homogeneous spaces,with the exception of the complex and quaternionic Cayley planes [248], [233]
- Geodesic hyperspheres, horospheres and tubes around totally geodesic non-flat complex space forms, described and classified in detail by S. Nagai [194], [195], [196]
- Simply connected φ-symmetric spaces [45]. They are Sasaki manifolds with complete characteristic field for which reflections with respect to the integral curves of that field are global isometries.
- All known left-invariant Einstein metrics on compact Lie groups [78]. In fact, every simple Lie group apart from $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ carries at least one naturally reductive Einstein metric other than the bi-invariant metric. Similarly, large families of naturally reductive Einstein metrics on compact homogeneous spaces were constructed in [242]. In contrast, non-compact naturally reductive Einstein manifolds are necessarily symmetric [125].

2.3 Almost Hermitian manifolds

An *almost Hermitian manifold* (M^{2n}, g, J) is a manifold with a Riemannian metric g and a g-compatible almost complex structure $J : TM^{2n} \to TM^{2n}$. We denote by $\Omega(X,Y) := g(JX,Y)$ its Kähler form and by N the Nijenhuis tensor of J, defined by

$$\begin{aligned} N(X,Y) &:= [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] \\ &= (\nabla^g_X J)JY - (\nabla^g_Y J)JX + (\nabla^g_{JX} J)Y - (\nabla^g_{JY} J)X, \end{aligned}$$

where the second expression follows directly from the vanishing of the torsion of ∇^g and the identity

$$(\nabla^g_X J)(Y) = \nabla^g_X(JY) - J(\nabla^g_X Y). \tag{4}$$

The reader is probably acquainted with the *first canonical Hermitian connection*[1] (see the nice article [115] by Paul Gauduchon, which we strongly recommend for further reading on Hermitian connections)

$$\bar{\nabla}_X Y := \nabla^g_X Y + \frac{1}{2}(\nabla^g_X J)JY.$$

Indeed, the condition $\bar{\nabla} J = 0$ is equivalent to the identity (4), and the antisymmetry of the difference tensor $g((\nabla^g_X J)JY, Z)$ in Y and Z can for example be seen from the standard identity[2]

$$2\,g((\nabla^g_X J)Y, Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) + g(N(Y, Z), JX). \tag{5}$$

Sometimes, one finds the alternative formula $-1/2 J(\nabla^g_X J)Y$ for the difference tensor of $\bar{\nabla}$; but this is the same, since $J^2 = -1$ implies $\nabla^g_X J^2 = 0 = (\nabla^g_X J)J + J(\nabla_X J)$, i.e., $\nabla^g J \in \mathfrak{u}(n)^\perp \subset \mathfrak{so}(2n)$. Let us now express the difference tensor of the connection $\bar{\nabla}$ using the Nijenhuis tensor and the Kähler form. Since $\nabla^g_X \Omega(Y, Z) = g\big((\nabla^g_X J)Y, Z\big)$, the differential $d\Omega$ is just

$$d\Omega(X, Y, Z) = g\big((\nabla^g_X J)Y, Z\big) - g\big((\nabla^g_Y J)X, Z\big) + g\big((\nabla^g_Z J)X, Y\big). \tag{6}$$

Together with the expression for N in terms of covariant derivatives of J, this yields

$$\begin{aligned} g((\nabla^g_X J)JY, Z) = {} & g\big(N(X, Y), Z\big) + d\Omega(JX, JY, JZ) \\ & - g((\nabla^g_Y J)JZ, X) - g((\nabla^g_{JZ} J)Y, X). \end{aligned}$$

A priori, $(\nabla^g_X J)Y$ has no particular symmetry properties in X and Y, hence the last two terms cannot be simplified any further (in general, they are a mixture of the two other Cartan types). An exceptional situation occurs if M is nearly Kähler ($(\nabla^g_X J)X = 0$), for then $(\nabla^g_X J)Y = -(\nabla^g_Y J)X$ and the last two terms cancel each other. Furthermore, this antisymmetry property implies that the difference tensor is totally skew-symmetric, hence we can conclude:

Lemma 2.2. *On a nearly Kähler manifold (M^{2n}, g, J), the formula*

$$\bar{\nabla}_X Y := \nabla^g_X Y + \frac{1}{2}(\nabla^g_X J)JY = \nabla^g_X Y + \frac{1}{2}\,[N(X, Y) + d\Omega(JX, JY, J-)]$$

defines a Hermitian connection with totally skew-symmetric torsion.

This connection – the so-called *characteristic connection* – was first defined and studied by Alfred Gray (see [127, p. 304] and [129, p. 237]). It is a non-trivial result

[1] By definition, a connection ∇ is called *Hermitian* if it is metric and has ∇-parallel almost complex structure J.

[2] For a proof, see [170, Proposition 4.2]. Be aware of the different conventions in this book: Ω is defined with J in the second argument, the Nijenhuis tensor is twice our N and derivatives of k-forms differ by a multiple of $1/k$, see [169, Proposition 3.11].

of Kirichenko that it has $\overline{\nabla}$-parallel torsion ([167], see also [22] for a modern index-free proof). Furthermore, it is shown in [129] that any 6-dimensional nearly Kähler manifold is Einstein and of constant type, i.e., it satisfies

$$\|(\nabla^g_X J)(Y)\|^2 = \frac{\mathrm{scal}^g}{30}\left[\|X\|^2\cdot\|Y\|^2 - g(X,Y)^2 - g(X,JY)^2\right].$$

Together with Lemma 2.2, this identity yields by a direct calculation that *any 6-dimensional nearly Kähler manifold is also* $\overline{\nabla}$*-Einstein with* $\mathrm{Ric}^{\overline{\nabla}} = 2(\mathrm{scal}^g/15)\,g$ (see Theorem A.1 for the relation between Ricci tensors).

Now let us look for a Hermitian connection with totally skew-symmetric torsion on a larger class of Hermitian manifolds generalizing nearly Kähler manifolds.

Lemma 2.3 ([102]). *Let* (M^{2n}, g, J) *be an almost Hermitian manifold with skew-symmetric Nijenhuis tensor* $N(X,Y,Z) := g(N(X,Y),Z)$. *Then the formula*

$$g(\nabla_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2}\left[N(X,Y,Z) + d\Omega(JX, JY, JZ)\right]$$

defines a Hermitian connection with skew-symmetric torsion.

Besides nearly Kähler manifolds, Hermitian manifolds ($N = 0$) trivially fulfill the condition of the preceding lemma and ∇ coincides then with the Bismut connection; however, in the non-Hermitian situation, ∇ is *not* in the standard family of canonical Hermitian connections that is usually considered (see [115, 2.5.4]). Proposition 2 in this same reference gives the decomposition of the torsion of any Hermitian connection in its (p,q)-components and gives another justification for this precise form for the torsion. Later, we shall see that ∇ is the *only possible* Hermitian connection with skew-symmetric torsion and that the class of almost Hermitian manifolds with skew-symmetric Nijenhuis tensor is the largest possible where it is defined.

We will put major emphasis on almost Hermitian manifolds of dimension 6, although one will find some general results formulated independently of the dimension. Two reasons for this choice are that nearly Kähler manifolds are of interest only in dimension 6, and that 6 is also the relevant dimension in superstring theory.

2.4 Hyper-Kähler manifolds with torsion (HKT-manifolds)

We recall that a manifold M is called *hypercomplex* if it is endowed with three (integrable) complex structures I, J, K satisfying the quaternionic identities $IJ = -JI = K$. A metric g compatible with these three complex structures (a so-called *hyper-Hermitian* metric) is said to be *hyper-Kähler with torsion* or just a *HKT-metric* if the Kähler forms satisfy the identity

$$I\,d\Omega_I = J\,d\Omega_J = K\,d\Omega_K. \tag{7}$$

Despite the misleading name, these manifolds are not Kähler (and hence even less hyper-Kähler). HKT-metrics were introduced by Howe and Papadopoulos as target

spaces of some two-dimensional sigma models with $(4,0)$ supersymmetry with Wess–Zumino term [144]. Their mathematical description was given by Grantcharov and Poon in [126] and further investigated by several authors since then (see for example [239], [82], [204], [93], [94], [150]). From the previous example, we can conclude immediately that

$$g(\nabla_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2}\, d\Omega_I(IX, IY, IZ)$$

defines a metric connection with skew-symmetric torsion such that $\nabla I = \nabla J = \nabla K = 0$; one easily checks that ∇ is again the *only* connection fulfilling these conditions. Equation (7) implies that we could equally well have chosen J or K in the last term. In general, a hyper-Hermitian manifold will not carry an HKT-structure, except in dimension 4 where this is proved in [116]. Examples of homogeneous HKT-metrics can be constructed using a family of homogeneous hypercomplex structures associated with compact semisimple Lie groups constructed by Joyce [155]. Inhomogeneous HKT-structures exist for example on $S^1 \times S^{4n-3}$ [126]. The question of existence of suitable potential functions for HKT-manifolds was first raised and discussed in the context of super-conformal quantum mechanics by the physicists Michelson and Strominger [191] (a maximum principle argument shows that compact HKT-manifolds do not admit global potentials); Poon and Swann discussed potentials for some symmetric HKT-manifolds [203], while Banos and Swann were able to show local existence [32].

2.5 Almost contact metric structures

An odd-dimensional manifold M^{2n+1} is said to carry an *almost contact structure* if it admits a $(1,1)$-tensor field φ and a vector field ξ (sometimes called the *characteristic* or *Reeb vector field*) with dual 1-form η ($\eta(\xi) = 1$) such that $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$. Geometrically, this means that M has a preferred direction (defined by ξ) on which φ^2 vanishes, while φ behaves like an almost complex structure on any linear complement of ξ. An easy argument shows that $\varphi(\xi) = 0$ [44, Theorem 4.1]. If there exists in addition a φ-compatible Riemannian metric g on M^{2n+1}, i.e., satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then we say that $(M^{2n+1}, g, \xi, \eta, \varphi)$ carries an *almost contact metric structure* or that it is an *almost contact metric manifold*. The condition says that ξ is a vector field of unit length with respect to g and that g is φ-compatible in the sense of Hermitian geometry on the orthogonal complement $\xi^\perp$. Unfortunately, this relatively intuitive structural concept splits into a myriad of subtypes and leads to complicated equations in the defining data (ξ, η, φ), making the investigation of almost contact metric structures look rather unattractive at first sight (see [19], [66], [67], and [90] for a classification). Yet, they constitute a rich and particularly interesting class of non-integrable geometries, as they have no integrable analogue on Berger's list. An excellent general source

on contact manifolds with extensive references are the books by David Blair, [43] and [44] (however, the classification is not treated in these).

The fundamental form F of an almost contact metric structure is defined by $F(X,Y) = g(X,\varphi(Y))$, its Nijenhuis tensor is given by a similar, but slightly more complicated formula as in the almost Hermitian case and can also be written in terms of covariant derivatives of φ,

$$\begin{aligned} N(X,Y) &:= [\varphi(X),\varphi(Y)] - \varphi[X,\varphi(Y)] - \varphi[\varphi(X),Y] + \varphi^2[X,Y] + d\eta(X,Y)\xi \\ &= (\nabla^g_X\varphi)\varphi(Y) - (\nabla^g_Y\varphi)\varphi(X) + (\nabla^g_{\varphi(X)}\varphi)Y \\ &\quad - (\nabla^g_{\varphi(Y)}\varphi)X + \eta(X)\nabla^g_Y\xi - \eta(Y)\nabla^g_X\xi. \end{aligned}$$

Let us emphasize some particularly interesting cases. A manifold with an almost contact metric structure $(M^{2n+1}, g, \xi, \eta, \varphi)$ is called

(1) a *normal almost contact metric manifold* if $N = 0$,

(2) a *contact metric manifold* if $2F = d\eta$.

Furthermore, a contact metric structure is said to be a *K-contact metric structure* if ξ is in addition a Killing vector field, and a *Sasaki structure* if it is normal (it is then automatically K-contact, see [44, Corollary 6.3]). *Einstein–Sasaki manifolds* are just Sasaki manifolds whose φ-compatible Riemannian metric g is Einstein. Without doubt, the forthcoming monograph by Ch. Boyer and Kr. Galicki on Sasakian geometry [51] is set to become *the* standard reference for this area of contact geometry in the future; in the meantime, the reader will have to be contented with the shorter reviews [49] and [50].

Much less is known about metric connections on almost contact metric manifolds than on almost Hermitian manifolds. In fact, only the so-called *generalized Tanaka connection* (introduced by S. Tanno) has been investigated. It is a metric connection defined on the class of contact metric manifolds by the formula

$$\nabla^*_X Y := \nabla^g_X Y + \eta(X)\varphi(Y) - \eta(Y)\nabla^g_X\xi + (\nabla_X\eta)(Y)\xi$$

satisfying the additional conditions $\nabla^*\eta = 0$ (which is of course equivalent to $\nabla^*\xi = 0$), see [227] and [44, 10.4]. One easily checks that its torsion is not skew-symmetric, not even in the Sasaki case. In fact, from the point of view of non-integrable structures, it seems appropriate to require in addition $\nabla^*\varphi = 0$ (compare with the almost Hermitian situation).

Following a similar but more complicated line of arguments as in the almost Hermitian case, Th. Friedrich and S. Ivanov showed:

Theorem 2.1 ([102, Theorem 8.2]). *Let $(M^{2n+1}, g, \xi, \eta, \varphi)$ be an almost contact metric manifold. It admits a metric connection ∇ with totally skew-symmetric torsion T and $\nabla\eta = \nabla\varphi = 0$ if and only if the Nijenhuis tensor N is skew-symmetric and if ξ is a Killing vector field. Furthermore, $\nabla = \nabla^g + (1/2)T$ is uniquely determined by*

$$T = \eta\wedge d\eta + d^\varphi F + N - \eta\wedge(\xi \lrcorner N),$$

where $d^\varphi F$ stands for the φ-twisted derivative,

$$d^\varphi F(X,Y,Z) := -dF(\varphi(X),\varphi(Y),\varphi(Z)).$$

For a Sasaki structure, $N = 0$ and $2F = d\eta$ implies $d^\varphi F = 0$, hence T is given by the much simpler formula $T = \eta \wedge d\eta$. This connection had been noticed before, for example in [179]. In fact, one sees that $\nabla T = 0$ holds, hence Sasaki manifolds endowed with this connection are examples of non-integrable geometries with parallel torsion. A. Fino studied naturally reductive almost contact metric structures such that φ is parallel with respect to the canonical connection in [89]. In general, potentials are hardly studied in contact geometry (compare with the situation for HKT-manifolds), but a suitable analogue of the Kähler potential was constructed on Sasaki manifolds by M. Godlinski, W. Kopczynski and P. Nurowski [121].

2.6 3-Sasaki manifolds

Similarly to HKT-manifolds and quaternionic-Kähler manifolds, it makes sense to investigate configurations with three 'compatible' almost metric contact structures $(\varphi_i, \xi_i, \eta_i)$, $i = 1,2,3$ on (M^{2n+1}, g) for some fixed metric g. The compatibility condition may be formulated as

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \eta_j, \quad \xi_k = \varphi_i\xi_j = -\varphi_j\xi_i$$

for any even permutation (i,j,k) of $(1,2,3)$, and such a structure is called an *almost contact metric 3-structure*. By defining on the cone $M^{2n+1} \times \mathbb{R}$ three almost complex structures J_1, J_2, J_3, one sees that it carries an almost quaternionic structure and hence has dimension divisible by 4. Consequently, almost contact metric 3-structures exist only in dimensions $4n+3, n \in \mathbb{N}$, and it is no surprise that the structure group of its tangent bundle turns out to be contained in $\mathrm{Sp}(n) \times \{1\}$. What is surprising is the recent result by T. Kashiwada that if all three structures $(\varphi_i, \xi_i, \eta_i)$ are contact metric structures, they automatically have to be Sasakian [161]. A manifold with such a structure will be called a 3-*Sasaki*(*an*) *manifold*. An earlier result by T. Kashiwada claims that any 3-Sasaki manifold is Einstein [160]. The canonical example of a 3-Sasaki manifold is the sphere S^{4n+3} realized as a hypersurface in $\mathbb{H}^{n+1}$: each of the three almost complex structures forming the quaternionic structure of $\mathbb{H}^{n+1}$ applied to the exterior normal vector field of the sphere yields a vector field ξ_i $(i = 1,2,3)$ on S^{4n+3}, leading thus to three orthonormal vector fields on S^{4n+3}. Th. Friedrich and I. Kath showed that every compact simply connected 7-dimensional spin manifold with regular 3-Sasaki structure is isometric to S^7 or the Aloff–Wallach space $N(1,1) = \mathrm{SU}(3)/S_1$ (see [106] or [33]). By now, many non-homogeneous examples have been constructed and the homogeneous 3-Sasaki manifolds have been classified [52].

The analogy between 3-Sasaki manifolds and HKT-manifolds breaks down when one starts looking at connections, however. For an arbitrary 3-Sasaki structure with 1-forms η_i $(i = 1,2,3)$, each of the three underlying Sasaki structures yields one

possible choice of a metric connection ∇^i with $\nabla^i \eta_i = 0$ and torsion $T^i = \eta_i \wedge d\eta_i$ as detailed in Theorem 2.1. *However, these three connections do not coincide*; hence, the 3-Sasaki structure itself is not preserved by any metric connection with skew-symmetric torsion. A detailed discussion of these three connections and their spinorial properties in dimension 7 can be found in [10]. Nevertheless, 3-Sasaki manifolds have recently appeared and been investigated in the context of the AdS/CFT-correspondence by Martelli, Sparks and Yau [186].

2.7 Holonomy theory

Let (M^n, g) be a (connected) Riemannian manifold equipped with *any* connection ∇. For a curve $\gamma(s)$ from p to q, parallel transport along γ is the linear mapping $P_\gamma : T_pM \to T_qM$ such that the vector field $\mathcal{V}$ defined by

$$\mathcal{V}(q) := P_\gamma \mathcal{V}(p) \quad \text{along } \gamma$$

is parallel along γ, $\nabla\mathcal{V}(s)/ds = \nabla_{\dot\gamma}\mathcal{V} = 0$. P_γ is always an invertible endomorphism, hence, for a closed loop γ through $p \in M$, it can be viewed as an element of $\mathrm{GL}(n, \mathbb{R})$ (after choice of some basis). Consider the *loop space* $C(p)$ of all closed, piecewise smooth curves through p, and therein the subset $C_0(p)$ of curves that are homotopic to the identity. The set of parallel translations along loops in $C(p)$ or $C_0(p)$ forms a group acting on $\mathbb{R}^n \cong T_pM$, called the *holonomy group* $\mathrm{Hol}(p;\nabla)$ *of* ∇ or the *restricted holonomy group* $\mathrm{Hol}_0(p;\nabla)$ *of* ∇ at the point p. Let us now change the point of view from p to q, γ a path joining them; then $\mathrm{Hol}(q;\nabla) = P_\gamma \mathrm{Hol}(p;\nabla) P_\gamma^{-1}$ and similarly for $\mathrm{Hol}_0(p;\nabla)$. Hence, all holonomy groups are isomorphic, so we drop the base point from now on. Customary notation for them is $\mathrm{Hol}(M;\nabla)$ and $\mathrm{Hol}_0(M;\nabla)$. Their action on $R^n \cong T_pM$ shall be called the (*restricted*) *holonomy representation*.

In general, it is only known that ([169, Theorem IV.4.2])

(1) $\mathrm{Hol}(M;\nabla)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$,

(2) $\mathrm{Hol}_0(p)$ is the connected component of the identity of $\mathrm{Hol}(M;\nabla)$.

If one assumes in addition – as we will do through this text – that ∇ be metric, parallel transport becomes an isometry: for any two parallel vector fields $\mathcal{V}(s)$ and $\mathcal{W}(s)$, being metric implies

$$\frac{d}{ds} g\big(\mathcal{V}(s), \mathcal{W}(s)\big) = g\Big(\frac{\nabla\mathcal{V}(s)}{ds}, \mathcal{W}(s)\Big) + \Big(\mathcal{V}(s), \frac{\nabla\mathcal{W}(s)}{ds}\Big) = 0.$$

Hence, $\mathrm{Hol}(M;\nabla) \subset \mathrm{O}(n)$ and $\mathrm{Hol}_0(M;\nabla) \subset \mathrm{SO}(n)$. For convenience, we shall henceforth speak of the *Riemannian* (*restricted*) *holonomy group* if ∇ is the Levi-Civita connection, to distinguish it from holonomy groups in our more general setting.

Example 2.5. This is a good moment to discuss Cartan's first example of a space with torsion (see [63, p. 595]). Consider $\mathbb{R}^3$ with its usual Euclidean metric and the

connection

$$\nabla_X Y = \nabla^g_X Y - X \times Y,$$

corresponding, of course, to the choice $T = -2 \cdot e_1 \wedge e_2 \wedge e_3$. Cartan observed correctly that this connection has same geodesics than ∇^g, but induces a different parallel transport. Indeed, consider the z-axis $\gamma(t) = (0, 0, t)$, a geodesic, and the vector field V which, in every point $\gamma(t)$, consists of the vector $(\cos t, \sin t, 0)$. Then one checks immediately that $\nabla^g_{\dot\gamma} V = \dot\gamma \times V$, that is, the vector V is parallel transported according to a helicoidal movement. If we now transport the vector along the edges of a closed triangle, it will be rotated around three linearly independent axes, hence the holonomy algebra is $\mathfrak{hol}(\nabla) = \mathfrak{so}(3)$.

Example 2.6 (Holonomy of naturally reductive spaces). Consider a naturally reductive space $M^n = G/H$ as in Example 2.2 with its canonical connection ∇^1, whose torsion is $T^1(X, Y) := -[X, Y]_{\mathfrak{m}}$. Recall that $\mathrm{ad}\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ denotes its isotropy representation. The holonomy algebra $\mathfrak{hol}(\nabla^1)$ is the Lie subalgebra of $\mathrm{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m})$ generated by the images under ad of all projections of commutators $[X, Y]_{\mathfrak{h}}$ on $\mathfrak{h}$ for $X, Y \in \mathfrak{m}$,

$$\mathfrak{hol}(\nabla^1) = \mathrm{Lie}\big(\mathrm{ad}([X, Y]_{\mathfrak{h}})\big) \subset \mathrm{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m}).$$

For all other connections ∇^t in this family, the general expression for the holonomy is considerably more complicated [170, Theorem X.4.1].

Remark 2.1 (Holonomy and contact properties). As we observed earlier, all contact structures are non-integrable and therefore not covered by Berger's holonomy theorem. Via the cone construction, it is nevertheless possible to characterize them by a Riemannian holonomy property (see [52], [49]). Consider a Riemannian manifold (M^n, g) and its cone over the positive real numbers $N := \mathbb{R}_+ \times M^n$ with the warped product metric $g_N := dr^2 + r^2 g$. Then, (M^n, g) is

(1) *Sasakian* if and only if $\mathrm{Hol}(N; \nabla^g) \subset \mathrm{U}(\frac{n+1}{2})$, that is, its positive cone is Kähler,

(2) *Einstein–Sasakian* if and only if $\mathrm{Hol}(N; \nabla^g) \subset \mathrm{SU}(\frac{n+1}{2})$, that is, its positive cone is a (non-compact) Calabi–Yau manifold,

(3) 3-*Sasakian* if and only if $\mathrm{Hol}(N; \nabla^g) \subset \mathrm{Sp}(\frac{n+1}{4})$, that is, its positive cone is hyper-Kähler.

The Lie algebra of any holonomy group can be determined by computing all parallel transports of its curvature tensor (Ambrose–Singer, 1953 [24]). Yet, the practical use of this result is severely restricted by the fact that the properties of the curvature transformation of a metric connection with torsion are more complicated than the Riemannian ones. For example, $\mathcal{R}(U, V)$ is still skew-adjoint with respect to the metric g,

$$g(\mathcal{R}(U, V)W_1, W_2) = -g(\mathcal{R}(U, V)W_2, W_1),$$

but there is no relation between $g(\mathcal{R}(U_1, U_2)W_1, W_2)$ and $g(\mathcal{R}(W_1, W_2)U_1, U_2)$ in general (but see Remark 2.2 below); in consequence, the Bianchi identities are less

tractable, the Ricci tensor is not necessarily symmetric etc. As an example of these complications, we cite (see [151] for the case of skew-symmetric torsion):

Theorem 2.2 (First Bianchi identity).

(1) *A metric connection* ∇ *with vectorial torsion* $V \in TM^n$ *satisfies*

$$\mathop{\mathfrak{S}}_{X,Y,Z} \mathcal{R}(X,Y)Z = \mathop{\mathfrak{S}}_{X,Y,Z} dV(X,Y)Z.$$

(2) *A metric connection* ∇ *with skew-symmetric torsion* $T \in \Lambda^3(M^n)$ *satisfies*

$$\mathop{\mathfrak{S}}_{X,Y,Z} \mathcal{R}(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma_T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z),$$

where σ_T *is a* 4*-form that is quadratic in* T *defined by* $2\,\sigma_T = \sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T)$ *for any orthonormal frame* $e_1, \ldots, e_n$.

Remark 2.2. Consequently, if the torsion $T \in \Lambda^3(M^n)$ of a metric connection with skew-symmetric torsion happens to be ∇-parallel, $\mathop{\mathfrak{S}}_{X,Y,Z} \mathcal{R}(X,Y,Z,V)$ is a 4-form and thus antisymmetric. Since the cyclic sum over all four arguments of any 4-form vanishes, we obtain

$$\mathop{\mathfrak{S}}_{X,Y,Z,V} \Big[\mathop{\mathfrak{S}}_{X,Y,Z} \mathcal{R}(X,Y,Z,V) \Big] = 2\,\mathcal{R}(Z,X,Y,V) - 2\,\mathcal{R}(Y,V,Z,X) = 0,$$

as for the Levi-Civita connection. Thus the ∇-curvature tensor is invariant under swaps of the first and second pairs of arguments.

Extra care has to be taken when asking which properties of the Riemannian holonomy group are preserved. For example, there are many instances of irreducible manifolds with metric connections whose holonomy representation is *not* irreducible. This implies that no analogue of de Rham's splitting theorem can hold. Similarly, there is no theoretical argument ensuring the closure of the restricted holonomy group of a metric connection with torsion, although the examples that I know of this effect are rather pathological.

We are particularly interested in the vector bundle of (r,s)-tensors $T^{r,s}M$ over M^n, that of differential k-forms $\Lambda^k M$ and its spinor bundle ΣM (assuming that M is spin, of course). At some point $p \in M$, the fibers are just $(T_pM)^r \otimes (T_p^*M)^s$, $\Lambda^k T_p^*M$ or Δ_n, the n-dimensional spin representation (which has dimension $2^{[n/2]}$); an element of the fibre at some point will be called an *algebraic tensor, form, spinor* or just *algebraic vector* for short. Then, on any of these bundles,

(1) the holonomy representation induces a representation of $\mathrm{Hol}(M;\nabla)$ on each fibre (the "lifted holonomy representation");

(2) the metric connection ∇ induces a connection (again denoted by ∇) on these vector bundles (the "lifted connection") compatible with the induced metric (for tensors) or the induced Hermitian scalar product (for spinors), it is thus again metric;

(3) in particular, there is a notion of "lifted parallel transport" consisting of isometries, and its abstract holonomy representation on the fibres coincides with the lifted holonomy representation.

We now formulate the general principle underlying our study.

Theorem 2.3 (General Holonomy Principle). *Let M be a differentiable manifold and E a (real or complex) vector bundle over M endowed with (any!) connection ∇. The following three properties are equivalent:*

(1) *E has a global section α invariant under parallel transport, i.e., $\alpha(q) = P_\gamma(\alpha(p))$ for any path γ from p to q;*

(2) *E has a parallel global section α, i.e., $\nabla\alpha = 0$;*

(3) *at some point $p \in M$, there exists an algebraic vector $\alpha_0 \in E_p$ which is invariant under the holonomy representation on the fibre.*

The following two consequences are immediate, but of the utmost importance.

Corollary 2.1. (1) *The number of parallel global sections of E coincides with the number of trivial representations occurring in the holonomy representation on the fibres.*

(2) *The holonomy group* $\mathrm{Hol}(\nabla)$ *is a subgroup of the isotropy group $G_\alpha := \{g \in \mathrm{O}(n) \mid g^*\alpha = \alpha\}$ of any parallel global section α of E.*

This is a powerful tool for (dis-)proving existence of parallel objects.

Example 2.7. The fact that the determinant is an $\mathrm{SO}(n)$-invariant element in $\Lambda^n(\mathbb{R}^n)$ which is not $\mathrm{O}(n)$-invariant implies that a Riemannian manifold (M^n, g) is orientable if and only if the holonomy $\mathrm{Hol}(M;\nabla)$ of *any metric connection* ∇ is a subgroup of $\mathrm{SO}(n)$. Furthermore, the volume form is then ∇-parallel.

Remark 2.3. In fact, an arbitrary connection ∇ admits a ∇-parallel n-form (possibly with zeroes) if and only if $\sum g(\mathcal{R}(U,V)e_i, e_i) = 0$ for any orthonormal frame $e_1, \dots, e_n$. This property is weaker than the skew-adjointness of $\mathcal{R}(U,V)$ that holds for all metric connections; the holonomy is a subgroup of $\mathrm{SL}(n,\mathbb{R})$. In 1924, J. A. Schouten called such connections "inhaltstreue "Ubertragungen" (*volume-preserving connections*), see [213, p. 89]. This terminology seems not to have been used anymore afterwards[3].

The existence of parallel objects imposes restrictions on the curvature of the connection. For example, if a connection ∇ admits a parallel spinor ψ, we obtain by contracting the identity

$$0 = \nabla\nabla\psi = \sum_{i,j=1}^{n} \mathcal{R}^\nabla(e_i, e_j)e_i \cdot e_j \cdot \psi$$

[3] A *D'Atri space* is a Riemannian manifold whose local geodesic symmetries are volume-preserving. Although every naturally reductive space is a D'Atri space [77], the two notions are only loosely related.

the following integrability condition:

Proposition 2.2. *Let (M^n, g) be a Riemannian spin manifold, ∇ a metric connection with torsion $T \in \Lambda^3(M^n)$. A ∇-parallel spinor ψ satisfies*

$$\left[\frac{1}{2} X \lrcorner dT + \nabla_X T\right] \cdot \psi = \mathrm{Ric}^{\nabla}(X) \cdot \psi$$

In particular, the existence of a ∇^g-parallel spinor ($T = 0$) implies Ricci-flatness.

3 Geometric stabilizers

By the General Holonomy Principle, geometric representations with invariant objects are a natural source for parallel objects. This leads to the systematic investigation of *geometric stabilizers*, which we shall now discuss.

3.1 U(n) and SU(n) in dimension $2n$

A Hermitian metric $h(V, W) = g(V, W) - i g(JV, W)$ is invariant under $A \in \mathrm{End}(\mathbb{R}^{2n})$ if and only if A preserves the Riemannian metric g and the Kähler form $\Omega(V, W) := g(JV, W)$. Thus $\mathrm{U}(n)$ is embedded in $\mathrm{SO}(2n)$ as

$$\mathrm{U}(n) = \{A \in \mathrm{SO}(2n) \mid A^*\Omega = \Omega\}.$$

To fix ideas, choose a skew-symmetric endomorphism J of $\mathbb{R}^{2n}$ with square -1 in the normal form

$$J = \mathrm{diag}(j, j, j, \dots) \quad \text{with } j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then a complex $(n \times n)$-matrix $A = (a_{ij}) \in \mathrm{U}(n)$ is realized as a real $(2n \times 2n)$-matrix with (2×2)-blocks $\begin{bmatrix} \mathrm{Re}\, a_{ij} & -\mathrm{Im}\, a_{ij} \\ \mathrm{Im}\, a_{ij} & \mathrm{Re}\, a_{ij} \end{bmatrix}$. An adapted orthonormal frame is one such that J has the given normal form; $\mathrm{U}(n)$ consists then exactly of those endomorphisms transforming adapted orthonormal frames into adapted orthonormal frames. Allowing now complex coefficients, one obtains an $(n, 0)$-form Ψ by declaring

$$\Psi := (e_1 + i e_2) \wedge \cdots \wedge (e_{n-1} + i e_{2n}) =: \Psi^+ + i\Psi^-$$

in the adapted frame above. An element $A \in \mathrm{U}(n)$ acts on Ψ by multiplication with $\det A$.

Lemma 3.1. *Under the restricted action of* $\mathrm{U}(n)$, $\Lambda^{2k}(\mathbb{R}^{2n})$ *contains the trivial representation once; it is generated by* $\Omega, \Omega^2, \dots, \Omega^n$.

The action of $\mathrm{U}(n) \subset \mathrm{O}(2n)$ cannot be lifted to an action of $\mathrm{U}(n)$ inside $\mathrm{Spin}(2n)$ – reflecting the fact that not every Kähler manifold is spin. For the following arguments

though, it is enough to consider $\mathfrak{u}(n)$ inside $\mathfrak{spin}(2n)$. It then appears that $\mathfrak{u}(n)$ has no invariant spinors, basically because $\mathfrak{u}(n)$ has a one-dimensional center, generated precisely by Ω after identifying $\Lambda^2(\mathbb{R}^{2n}) \cong \mathfrak{so}(2n)$. Hence one-dimensional $\mathfrak{u}(n)$-representations are usually not trivial. More precisely, the complex $2n$-dimensional spin representation Δ_{2n} splits into two irreducible components $\Delta_{2n}^{\pm}$ described in terms of eigenspaces of $\Omega \in \mathfrak{u}(n)$. Set (see [96] and [164] for details on this decomposition of spinors)

$$S_r = \{\psi \in \Delta_{2n} \mid \Omega\psi = i(n-2r)\psi\}, \quad \dim S_r = \binom{n}{r},\ 0 \le r \le n.$$

S_r is isomorphic to the space of $(0,r)$-forms with values in S_0 (which explains the dimension),

$$S_r \cong \overline{\Lambda^{0,r}} \otimes S_0.$$

Since the spin representations decompose as

$$\Delta_{2n}^{+}\big|_{\mathfrak{u}(n)} \cong S_n \oplus S_{n-2} \oplus \cdots, \quad \Delta_{2n}^{-}\big|_{\mathfrak{u}(n)} \cong S_{n-1} \oplus S_{n-3} \oplus \cdots$$

we conclude immediately that they cannot contain a trivial $\mathfrak{u}(n)$-representation for n odd. For $n = 2k$ even, Ω has eigenvalue zero on S_k, but this space is an irreducible representation of dimension $\binom{2k}{k} \neq 1$, hence not trivial either. The representations S_0 and S_n are one-dimensional, but again not trivial under $\mathfrak{u}(n)$. If one restricts further to $\mathfrak{su}(n)$, they are indeed:

Lemma 3.2. *The spin representations $\Delta_{2n}^{\pm}$ contain no $\mathfrak{u}(n)$-invariant spinor. If one restricts further to $\mathfrak{su}(n)$, there are exactly two invariant spinors (both in Δ_{2n}^{+} for n even, one in each $\Delta_{2n}^{\pm}$ for n odd).*

All other spinors in $\Delta_{2n}^{\pm}$ have geometric stabilizer groups that do not act irreducibly on the tangent representation $\mathbb{R}^{2n}$. They can be described explicitly in a similar way; By de Rham's splitting theorem, they do not appear in the Riemannian setting.

To finish, we observe that the almost complex structure J (and hence Ω) can be recovered from the invariant spinor $\psi^+ \in \Delta_{2n}^{+}$ by $J(X)\psi^+ := iX \cdot \psi^+$ ($X \in TM$), a formula well known from the investigation of Killing spinors on 6-dimensional nearly Kähler manifolds (see [133] and [33, Section 5.2]).

Remark 3.1. The group $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ can be deduced from the previous discussion: $\mathrm{Sp}(n)$ with quaternionic entries $a+bj$ is embedded into $\mathrm{SU}(2n)$ by (2×2)-blocks $\left[\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right]$, and $\mathrm{SU}(2n)$ sits in $\mathrm{SO}(4n)$ as before. We shall not treat $\mathrm{Sp}(n)$- and quaternionic geometries in this exposition (but see [223], [224], [18], [126], [20], [204], [17], [189] for a first acquaintance).

3.2 U(n) and SU(n) in dimension $2n+1$

These G-structures arise from contact structures and are remarkable inasmuch they manifest a genuinely non-integrable behaviour – they do not occur in Berger's list because the action of $\mathrm{U}(n)$ on $\mathbb{R}^{2n+1}$ is not irreducible, hence any manifold with this action as Riemannian holonomy representation splits by de Rham's theorem. Given an almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \varphi)$, we may construct an adapted local orthonormal frame by choosing any $e_1 \in \xi^\perp$ and setting $e_2 = \varphi(e_1)$ (as well as fixing $e_{2n+1} = \xi$ once and for all); now choose any e_3 perpendicular to e_1, e_2, e_{2n+1} and set again $e_4 = \varphi(e_3)$ etc. With respect to such a basis, φ is given by

$$\varphi = \mathrm{diag}(j, j, \ldots, j, 0) \quad \text{with } j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and we conclude that the structural group of M^{2n+1} is reducible to $\mathrm{U}(n) \times \{1\}$. If we denote the fundamental form by F (see Example 2.5), then

$$\mathrm{U}(n) \times \{1\} = \{A \in \mathrm{SO}(2n+1) \mid A^* F = F\}.$$

The $\mathrm{U}(n) \times \{1\}$-action on $\mathbb{R}^{2n+1}$ inherits invariants from the $\mathrm{U}(n)$-action on $\mathbb{R}^{2n}$ in a canonical way; one then just needs to check that no new one appears. Hence, we can conclude:

Lemma 3.3. *Under the action of* $\mathrm{U}(n)$, $\Lambda^{2k}(\mathbb{R}^{2n+1})$ *contains the trivial representation once; it is generated by* $F, F^2, \ldots, F^n$.

The action of $\mathrm{U}(n) \subset \mathrm{O}(2n+1)$ cannot, in general, be lifted to an action of $\mathrm{U}(n)$ inside $\mathrm{Spin}(2n+1)$. As in the almost Hermitian case, let us thus study the $\mathfrak{u}(n)$ action on Δ_{2n+1}. The irreducible $\mathrm{Spin}(2n+1)$-module Δ_{2n+1} splits into $\Delta_{2n}^+ \oplus \Delta_{2n}^-$ under the restricted action of $\mathrm{Spin}(2n)$, and it decomposes accordingly into

$$\Delta_{2n+1} = S_0 \oplus \cdots \oplus S_n,$$

where

$$S_r = \{\psi \in \Delta_{2n+1} \mid F\psi = i(n-2r)\psi\} \quad \text{and} \quad \dim S_r = \binom{n}{r}, \quad 0 \le r \le n.$$

Hence, Δ_{2n+1} can be identified with $\Delta_{2n}^+ \oplus \Delta_{2n}^-$, yielding finally the following result:

Lemma 3.4. *The spin representation* Δ_{2n+1} *contains no* $\mathfrak{u}(n)$*-invariant spinor. If one restricts further to* $\mathfrak{su}(n)$*, there are precisely two invariant spinors.*

3.3 G_2 in dimension 7

While invariant 2-forms exist in all even dimensions and lead to the rich variety of almost Hermitian structures, the geometry of 3-forms played a rather exotic role

in classical Riemannian geometry until the nineties, as it occurs only in apparently random dimensions, most notably dimension seven. That G_2 is the relevant simple Lie group is a classical, although unfortunately not so well known result from invariant theory. A mere dimension count shows already this effect (see Table 3): the stabilizer of a generic 3-form ω^3

$$G^n_{\omega^3} := \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \omega^3 = A^*\omega^3\}$$

cannot be contained in the orthogonal group for $n \leq 6$, it must lie in some group between $\mathrm{SO}(n)$ and $\mathrm{SL}(n,\mathbb{R})$ (for $n = 3$, we even have $G^3_{\omega^3} = \mathrm{SL}(3,\mathbb{R})$). The

Table 3. Dimension count for possible geometries defined by 3-forms.

n	$\dim \mathrm{GL}(n,\mathbb{R}) - \dim \Lambda^3\mathbb{R}^n$	$\dim \mathrm{SO}(n)$
3	$9-1=8$	3
4	$16-4=12$	6
5	$25-10=15$	10
6	$36-20=16$	15
7	$49-35=14$	21
8	$64-56=8$	28

case $n = 7$ is the first dimension where $G^n_{\omega^3}$ can sit in $\mathrm{SO}(n)$. That this is indeed the case was shown as early as 1907 in the doctoral dissertation of Walter Reichel in Greifswald, supervised by F. Engel ([206]). More precisely, he computed a system of invariants for a 3-form in seven variables and showed that there are exactly two open $\mathrm{GL}(7,\mathbb{R})$-orbits of 3-forms. The stabilizers of any representatives ω^3 and $\tilde{\omega}^3$ of these orbits are 14-dimensional simple Lie groups of rank two, one compact and the other non-compact:

$$G^7_{\omega^3} \cong G_2 \subset \mathrm{SO}(7), \quad G^7_{\tilde{\omega}^3} \cong G_2^* \subset \mathrm{SO}(3,4).$$

Reichel also showed the corresponding embeddings of Lie algebras by explicitly writing down seven equations for the coefficients of $\mathfrak{so}(7)$ resp. $\mathfrak{so}(3,4)$. As in the case of almost Hermitian geometry, every author has his or her favourite normal 3-form with isotropy group G_2, for example,

$$\omega^3 := e_{127} + e_{347} - e_{567} + e_{135} - e_{245} + e_{146} + e_{236}.$$

An element of the second orbit with stabilizer the split form G_2^* of G_2 may be obtained by reversing any of the signs in ω^3.

Lemma 3.5. *Under G_2, one has the decomposition $\Lambda^3(\mathbb{R}^7) \cong \mathbb{R} \oplus \mathbb{R}^7 \oplus S_0(\mathbb{R}^7)$, where $\mathbb{R}^7$ denotes the 7-dimensional standard representation given by the embedding $G_2 \subset \mathrm{SO}(7)$ and $S_0(\mathbb{R}^7)$ denotes the traceless symmetric endomorphisms of $\mathbb{R}^7$ (of dimension 27).*

Now let us consider the spinorial picture, as G_2 can indeed be lifted to a subgroup of Spin(7). From a purely representation theoretic point of view, this case is trivial: $\dim \Delta_7 = 8$ and the only irreducible representations of G_2 of dimension ≤ 8 are the trivial and the 7-dimensional representations. Hence $8 = 1 + 7$ yields:

Lemma 3.6. *Under the restricted action of G_2, the 7-dimensional spin representation Δ_7 decomposes as $\Delta_7 \cong \mathbb{R} \oplus \mathbb{R}^7$.*

This lemma has an important consequence: the 'spinorial' characterization of G_2-manifolds.

Corollary 3.1. *Let (M^7, g) be a Riemannian manifold, ∇ a metric connection on its spin bundle. Then there exists a ∇-parallel spinor if and only if $\mathrm{Hol}(\nabla) \subset G_2$.*

One direction follows from the fact that G_2 is the stabilizer of an algebraic spinor, the converse from Lemma 3.6.

In fact, the invariant 3-form and the invariant algebraic spinor ψ are equivalent data. They are related (modulo an irrelevant constant) by

$$\omega^3(X, Y, Z) = \langle X \cdot Y \cdot Z \cdot \psi, \psi \rangle. \tag{8}$$

We now want to ask which subgroups $G \subset G_2$ admit other invariant algebraic spinors. Such a subgroup has to appear on Berger's list and its induced action on $\mathbb{R}^7$ (viewed as a subspace of Δ_7) has to contain one or more copies of the trivial representation. Thus, the only possibilities are SU(3) with $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ (standard SU(3)-action on $\mathbb{C}^3$) and SU(2) with $\mathbb{R}^7 \cong 3 \cdot \mathbb{R} \oplus \mathbb{C}^2$ (standard SU(2)-action on $\mathbb{C}^2$). Both indeed occur, with a total of 2 resp. 4 invariant spinors.

Remark 3.2. A modern account of Reichel's results can be found in the article [243] by R. Westwick; it is interesting (although it seems not to have had any further influence) that J. A. Schouten also rediscovered these results in 1931 [214]. A classification of 3-forms is still possible in dimensions 8 ([134], [134], [81]) and 9 ([240]), although the latter one is already of inexorable complexity. Based on these results, J. Bureš and J. Vanžura started recently the investigation of so-called *multisymplectic structures* ([237], [60], [59]). For a historically account on the discovery of G_2 and its impact on differential geometry we refer to [6].

3.4 Spin(7) in dimension 8

As we just learned from G_2 geometry, Spin(7) has an irreducible 8-dimensional representation isomorphic to Δ_7, hence it can be viewed as a subgroup of SO(8), and it does lift to Spin(8). By restricting to SO(7), Spin(7) certainly also has a 7-dimensional representation. What is so special in this dimension is that Spin(7) has *two* conjugacy classes in SO(8) that are interchanged by means of the triality automorphism; hence the decomposition of the spin representation depends on the (arbitrary) choice of one of these classes.

Lemma 3.7. *Under the restricted action of* Spin(7)*, the* 8*-dimensional spin representations decompose as* $\Delta_8^+ \cong \mathbb{R}^8 \cong \Delta_7$ *and* $\Delta_8^- \cong \mathbb{R} \oplus \mathbb{R}^7$ *for one choice of* $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$*; the other choice swaps* Δ_8^+ *and* Δ_8^-*.*

In particular, there is exactly one invariant spinor in Δ_8^+. Again, it corresponds one-to-one to an invariant form, of degree 4 in this case:

$$\beta^4(X, Y, Z, V) = \langle X \cdot Y \cdot Z \cdot V \cdot \psi, \psi \rangle.$$

Yet, Spin(7)-geometry in dimension eight is not just an enhanced version of G_2-geometry in dimension seven. Because $\dim \mathrm{GL}(8, \mathbb{R}) = 64 < 70 = \dim \Lambda^4(\mathbb{R}^8)$, there are no dense open orbits under the action of $\mathrm{GL}(8, \mathbb{R})$. Thus, there is no result in invariant theory similar to that of Reichel for G_2 in the background.

Let us fix the first choice for embedding Spin(7) in SO(8) made in Lemma 3.7. A second invariant spinor can either be in Δ_8^+ or in Δ_8^-. If it is in Δ_8^+, we are asking for a subgroup $G \subset \mathrm{Spin}(7)$ whose action on $\mathbb{R}^7$ contains the trivial representation once – obviously, G_2 is such a group. Under G_2, Δ_8^+ and Δ_8^- are isomorphic,

$$\Delta_8^+|_{G_2} \cong \Delta_8^-|_{G_2} \cong \mathbb{R} \oplus \mathbb{R}^7.$$

Thus, $\mathrm{SU}(3) \subset G_2 \subset \mathrm{Spin}(7)$ and $\mathrm{SU}(2) \subset G_2 \subset \mathrm{Spin}(7)$ are two further admissible groups with $2+2$ and $4+4$ invariant spinors. On the other hand, if we impose a second invariant spinor to live in Δ_8^-, we need a subgroup $G \subset \mathrm{Spin}(7)$ that has partially trivial action on $\mathbb{R}^7$, but not on $\Delta_8^+ \cong \mathbb{R}^8$. A straightforward candidate is $G = \mathrm{Spin}(6)$ with its standard embedding and $\mathbb{R}^7 = \mathbb{R}^6 \oplus \mathbb{R}$; the classical isomorphism $\mathrm{Spin}(6) = \mathrm{SU}(4)$ shows that G acts irreducibly on $\Delta_8^+ \cong \mathbb{C}^4$. The group SU(4) in turn has subgroups $\mathrm{Sp}(2) = \mathrm{Spin}(5)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4)$ that still act irreducibly on Δ_8^+, and act on Δ_8^- by

$$\Delta_8^-|_{\mathrm{Sp}(2)} \cong 3 \cdot \mathbb{R} \oplus \mathbb{R}^5, \quad \Delta_8^-|_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \cong 4 \cdot \mathbb{R} \oplus \mathbb{R}^4.$$

The results are summarized in Table 4. The resemblance between Tables 4 and 2 in Section 1.4 is no coincidence. A convenient way to describe Spin(9, 1) is to start with Spin(10) generated by elements $e_1, \ldots, e_{10}$ and acting irreducibly on Δ_{10}^+. The vector spaces Δ_{10}^+ and $\Delta_{9,1}^+$ can be identified, and Spin(9, 1) can be generated by $e_i^* := e_i$

for $i = 1, \dots, 9$ and $e_{10}^* := i\, e_{10}$. Elements $\omega \in \mathfrak{spin}(9,1)$ can thus be written

$$\omega = \sum_{1 \le i < j \le 10} \omega_{ij}\, \omega_{ij} e_i^* \wedge e_j^* = \sum_{i<j\le 9} \omega_{ij}\, e_i \wedge e_j + i \sum_{k \le 9} \omega_{k,10}\, e_k \wedge e_{10}$$

and we conclude that $\mathfrak{spin}(9,1)$ can be identified with $\mathfrak{spin}(9) \ltimes \mathbb{R}^9$. A spinor $\psi \in \Delta_{9,1}^+$ is stabilized by an element $\omega \in \mathfrak{spin}(9,1)$ if and only if

$$0 = \sum_{1 \le i < j \le 10} \omega_{ij}\, e_i^* \cdot e_j^* \cdot \psi = \sum_{i<j\le 9} \omega_{ij}\, e_i \cdot e_j \cdot \psi + i \sum_{k \le 9} \omega_{k,10}\, e_k \cdot e_{10} \cdot \psi.$$

In this last expression, both real and imaginary part have to vanish simultaneously, leading to 16 equations. A careful look reveals that they define $\mathfrak{spin}(7) \ltimes \mathbb{R}^8$, and the statements from Table 4 imply those from Table 2 since $\Delta_{9,1}^+ \cong \Delta_8^+ \oplus \Delta_8^-$.

Table 4. Possible stabilizers of invariant spinors in dimension 8.

group	invariant spinors in Δ_8^+	invariant spinors in Δ_8^-
Spin(7)	0	1
SU(4) $\cong$ Spin(6)	0	2
Sp(2) $\cong$ Spin(5)	0	3
SU(2) $\times$ SU(2) $\cong$ Spin(4)	0	4
G_2	1	1
SU(3)	2	2
SU(2)	4	4
$\{e\}$	8	8

Remark 3.3 (Weak PSU(3)-structures). Recently, Hitchin observed that 3-forms can be of interest in 8-dimensional geometry as well ([141]). The canonical 3-form on the Lie algebra $\mathfrak{su}(3)$ spans an open orbit under $\mathrm{GL}(8,\mathbb{R})$, and the corresponding 3-form on SU(3) is parallel with respect to the Levi-Civita connection of the bi-invariant metric. The Riemannian holonomy reduces to $\mathrm{SU}(3)/\mathbb{Z}_2 =: \mathrm{PSU}(3)$. More generally, manifolds modelled on this group lead to the investigation of closed and coclosed 3-forms that are not parallel, see also [245].

Sorting out the technicalities that we purposely avoided, one obtains Wang's classification of Riemannian parallel spinors. By de Rham's theorem, only irreducible

holonomy representations occur for the Levi-Civita connection. From Proposition 2.2, we already know that these manifolds are Ricci-flat.

Theorem 3.1 (Wang's Theorem, [241]). *Let (M^n, g) be a complete, simply connected, irreducible Riemannian manifold of dimension n. Let N denote the dimension of the space of parallel spinors with respect to the Levi-Civita connection. If (M^n, g) is non-flat and $N > 0$, then one of the following holds:*

(1) *$n = 2m$ ($m \geq 2$), the holonomy representation is the vector representation of $\mathrm{SU}(m)$ on $\mathbb{C}^m$, and $N = 2$ ("Calabi–Yau case").*

(2) *$n = 4m$ ($m \geq 2$), the holonomy representation is the vector representation of $\mathrm{Sp}(m)$ on $\mathbb{C}^{2m}$, and $N = m + 1$ ("hyper-Kähler case").*

(3) *$n = 7$, the holonomy representation is the unique 7-dimensional representation of G_2, and $N = 1$ ("parallel G_2- or Joyce case").*

(4) *$n = 8$, the holonomy representation is the spin representation of $\mathrm{Spin}(7)$, and $N = 1$ ("parallel $\mathrm{Spin}(7)$- or Joyce case").*

4 A unified approach to non-integrable geometries

4.1 Motivation

For G-structures defined by some tensor $\mathcal{T}$, it has been for a long time customary to classify the possible types of structures by the isotypic decomposition under G of the covariant derivative $\nabla^g \mathcal{T}$. The integrable case is described by $\nabla^g \mathcal{T} = 0$, all other classes of non-integrable G-structures correspond to combinations of non-vanishing contributions in the isotypic decomposition and are described by some differential equation in $\mathcal{T}$. This was carried out in detail for example for almost Hermitian manifolds (Gray/Hervella [130]), for G_2-structures in dimension 7 (Fernández/Gray [87]), for Spin(7)-structures in dimension 8 (Fernández [85]) and for almost contact metric structures (Chinea/Gonzales [66]).

In this section, we shall present a simpler and unified approach to non-integrable geometries. The theory of principal fibre bundles suggests that the difference Γ between the Levi-Civita connection and the canonical G-connection induced on the G-structure is a good measure for how much the given G-structure fails to be integrable. By now, Γ is widely known as the *intrinsic torsion* of the G-structure (see Section 1.5 for references). Although this is a "folklore" approach, it is still not as popular as it could be. Our presentation will follow the main lines of [99]. We will see that it easily reproduces the classical results cited above with much less computational work whilst having the advantage of being applicable to geometries not defined by tensors. Furthermore, it allows a uniform and clean description of those classes of geometries admitting G-connections with totally skew-symmetric torsion, and led to the discovery of new interesting geometries.

4.2 G-structures on Riemannian manifolds

Let $G \subset \mathrm{SO}(n)$ be a closed subgroup of the orthogonal group and decompose the Lie algebra $\mathfrak{so}(n)$ into the Lie algebra $\mathfrak{g}$ of G and its orthogonal complement $\mathfrak{m}$, i.e., $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$. Denote by $\mathrm{pr}_{\mathfrak{g}}$ and $\mathrm{pr}_{\mathfrak{m}}$ the projections onto $\mathfrak{g}$ and $\mathfrak{m}$, respectively. Consider an oriented Riemannian manifold (M^n, g) and denote its frame bundle by $\mathcal{F}(M^n)$; it is a principal $\mathrm{SO}(n)$-bundle over M^n. By definition, a G-structure on M^n is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi-Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi-Civita connection to $\mathcal{R}$ and decompose it with respect to the splitting $\mathfrak{g} \oplus \mathfrak{m}$:

$$Z\big|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a tensorial 1-form of type Ad, i.e., a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_G \mathfrak{m}$. By now, it has become standard to call Γ the *intrinsic torsion* of the G-structure (see Section 1.5 for references). The G-structure $\mathcal{R}$ on (M^n, g) is called *integrable* if Γ vanishes, for this means that it is preserved by the Levi-Civita connection and that $\mathrm{Hol}(\nabla^g)$ is a subgroup of G. All G-structures with $\Gamma \neq 0$ are called *non-integrable*; the basic classes of non-integrable G-structures are defined – via the decomposition of Γ – as the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. For an orthonormal frame $e_1, \ldots, e_n$ adapted to the reduction $\mathcal{R}$, the connection forms $\omega_{ij} := g(\nabla^g e_i, e_j)$ of the Levi-Civita connection define a 1-form $\Omega := (\omega_{ij})$ with values in the Lie algebra $\mathfrak{so}(n)$ of all skew-symmetric matrices. The form Γ can then be computed as the $\mathfrak{m}$-projection of Ω,

$$\Gamma = \mathrm{pr}_{\mathfrak{m}}(\Omega) = \mathrm{pr}_{\mathfrak{m}}(\omega_{ij}).$$

Interesting is the case in which G happens to be the isotropy group of some tensor $\mathcal{T}$. Suppose that there is a faithful representation $\varrho\colon \mathrm{SO}(n) \to \mathrm{SO}(V)$ and a tensor $\mathcal{T} \in V$ such that

$$G = \big\{ g \in \mathrm{SO}(n) \mid \varrho(g)\mathcal{T} = \mathcal{T} \big\}.$$

The Riemannian covariant derivative of $\mathcal{T}$ is then given by $\nabla^g \mathcal{T} = \varrho_*(\Gamma)(\mathcal{T})$, where $\varrho_*\colon \mathfrak{so}(n) \to \mathfrak{so}(V)$ is the differential of the representation. As a tensor, $\nabla^g \mathcal{T}$ is an element of $\mathbb{R}^n \otimes V$. The algebraic G-types of $\nabla^g \mathcal{T}$ define the algebraic G-types of Γ and vice versa. Indeed, we have

Proposition 4.1 ([99, Proposition 2.1]). *The G-map $\mathbb{R}^n \otimes \mathfrak{m} \to \mathbb{R}^n \otimes \mathrm{End}(V) \to \mathbb{R}^n \otimes V$ given by $\Gamma \to \rho_*(\Gamma)(\mathcal{T})$ is injective.*

An easy argument in representation theory shows that for $G \neq \mathrm{SO}(n)$, the G-representation $\mathbb{R}^n$ does always appear as summand in the G-decomposition of $\mathbb{R}^n \otimes \mathfrak{m}$. Geometrically, this module accounts precisely for conformal transformations of G-structures. Let $(M^n, g, \mathcal{R})$ be a Riemannian manifold with a fixed geometric structure and denote by $\hat{g} := e^{2f} \cdot g$ a conformal transformation of the metric. There is a natural

identification of the frame bundles

$$\mathcal{F}(M^n, g) \cong \mathcal{F}(\widehat{M}^n, \hat{g})$$

and a corresponding G-structure $\widehat{\mathcal{R}}$. At the infinitesimal level, the conformal change is defined by the 1-form df, corresponding to an $\mathbb{R}^n$-part in Γ.

We shall now answer the question under which conditions a given G-structure admits a metric connection ∇ with skew-symmetric torsion preserving the structure. For this, consider for any orthonormal basis e_i of $\mathfrak{m}$ the G-equivariant map

$$\Theta\colon \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}, \quad \Theta(T) := \sum_i (e_i \lrcorner T) \otimes e_i.$$

Theorem 4.1 ([102, Proposition 4.1]). *A G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of a Riemannian manifold admits a connection ∇ with skew-symmetric torsion if and only if the 1-form Γ belongs to the image of Θ,*

$$2\,\Gamma = -\Theta(T) \quad \textit{for some } T \in \Lambda^3(\mathbb{R}^n).$$

In this case the 3-form T is the torsion form of the connection.

Definition 4.1. A metric G-connection ∇ with torsion T as in Theorem 4.1 will be called a *characteristic connection* and denoted by ∇^c, $T =: T^c$ is called the *characteristic torsion*. By construction, the holonomy $\mathrm{Hol}(\nabla^c)$ is a subgroup of G.

Thus, not every G-structure admits a characteristic connection. If that is the case, T^c is unique for all geometries we have investigated so far, and it can easily be expressed in terms of the geometric data (almost complex structure etc.). Henceforth, we shall just speak of *the* characteristic connection. Due to its properties, it is an excellent substitute for the Levi-Civita connection, which in these situations is not adapted to the underlying geometric structure.

Remark 4.1. The canonical connection ∇^c of a naturally reductive homogeneous space is an example of a characteristic connection that satisfies in addition $\nabla^c T^c = \nabla^c \mathcal{R}^c = 0$; in this sense, geometric structures admitting a characteristic connection such that $\nabla^c T^c = 0$ constitute a natural generalization of naturally reductive homogeneous spaces. As a consequence of the General Holonomy Principle (Corollary 2.1), $\nabla^c T^c = 0$ implies that the holonomy group $\mathrm{Hol}(\nabla^c)$ lies in the stabilizer G_{T^c} of T^c.

With this technique, we shall now describe special classes of non-integrable geometries, some new and others previously encountered. We order them by increasing dimension.

4.3 Almost contact metric structures

At this stage, almost contact metric structures challenge any expository article because of the large number of classes. Qualitatively, the situation is as follows. The first

classifications of these structures proceeded in analogy to the Gray–Hervella set-up for almost Hermitian manifolds (see Section 4.5) by examining the space of tensors with the same symmetry properties as the covariant derivative of the fundamental form F (see Section 2.5) and decomposing it under the action of the structure group $G = \mathrm{U}(n) \times \{1\}$ using invariant theory. Because the G-action on $\mathbb{R}^{2n+1}$ is already not irreducible, this space decomposes into four G-irreducibles for $n = 1$, into ten summands for $n = 2$ and into twelve for $n \geq 3$, leading eventually to 2^4, 2^{10} and 2^{12} possible classes of almost contact metric structures ([19], [66], [67]). Obviously, most of these classes do not carry names and are not studied, and the result being what it is, the investigation of such structures is burdened by technical details and assumptions. From the inner logic of non-integrable geometries, it makes not so much sense to base their investigation on the covariant derivative $\nabla^g F$ of F or some other fundamental tensor, as the Levi-Civita connection does not preserve the geometric structure. This accounts for the technical complications that one faces when following this approach.

For this section, we decided to restrict our attention to dimension five, this being the most relevant for the investigation of non-integrable geometries (in dimension seven, it is reasonable to study contact structures simultaneously with G_2-structures). Besides, this case will illustrate the power of the intrinsic torsion concept outlined above. We shall use throughout that $\mathbb{R}^5 = \mathbb{R}^4 \oplus \mathbb{R}$ with standard $\mathrm{U}(2)$-action on the first term and trivial action on the second term. Let us look at the decompositions of the orthogonal Lie algebras in dimension 4 and 5. First, we have

$$\mathfrak{so}(4) = \Lambda^2(\mathbb{R}^4) = \mathfrak{u}(2) \oplus \mathfrak{n}^2.$$

Here $\mathfrak{n}^2$ is $\mathrm{U}(2)$-irreducible, while $\mathfrak{u}(2)$ splits further into $\mathfrak{su}(2)$ and the span of Ω. Combining this remark with the characterization of these subspaces via the complex structure J defining $\mathfrak{u}(2)$, we obtain

$$\mathfrak{u}(2) = \{\omega \in \Lambda^2(\mathbb{R}^4) \mid J\omega = \omega\} = \mathfrak{su}(2) \oplus \mathbb{R} \cdot \Omega,$$
$$\mathfrak{n}^2 = \{\omega \in \Lambda^2(\mathbb{R}^4) \mid J\omega = -\omega\}.$$

In particular, $\Lambda^2(\mathbb{R}^4)$ is the sum of three $\mathrm{U}(2)$-representations of dimensions 1, 2 and 3. For $\mathfrak{so}(5)$, we deduce immediately that

$$\mathfrak{so}(5) = \Lambda^2(\mathbb{R}^4 \oplus \mathbb{R}) = \Lambda^2(\mathbb{R}^4) \oplus \mathbb{R}^4 = \mathfrak{u}(2) \oplus (\mathbb{R}^4 \oplus \mathfrak{n}^2) =: \mathfrak{u}(2) \oplus \mathfrak{m}^6.$$

Thus, the intrinsic torsion Γ of a 5-dimensional almost metric contact structure is an element of the representation space

$$\mathbb{R}^5 \otimes \mathfrak{m}^6 = (\mathbb{R}^4 \oplus \mathbb{R}) \otimes (\mathbb{R}^4 \oplus \mathfrak{n}^2) = \mathfrak{n}^2 \oplus \mathbb{R}^4 \oplus (\mathbb{R}^4 \otimes \mathfrak{n}^2) \oplus (\mathbb{R}^4 \otimes \mathbb{R}^4).$$

The last term splits further into trace-free symmetric, trace and antisymmetric part, written for short as

$$\mathbb{R}^4 \otimes \mathbb{R}^4 = S_0^2(\mathbb{R}^4) \oplus \mathbb{R} \oplus \Lambda^2(\mathbb{R}^4).$$

The 9-dimensional representation $S_0^2(\mathbb{R}^4)$ is again a sum of two irreducible ones of dimensions 3 and 6, but we do not need this here. To decompose the representation

$\mathbb{R}^4 \otimes \mathfrak{n}^2$, we observe that the U(2)-equivariant map $\Theta\colon \Lambda^3(\mathbb{R}^4) \to \mathbb{R}^4 \otimes \mathfrak{n}^2$ (see Section 4.2) has 4-dimensional irreducible image isomorphic to $\Lambda^3\mathbb{R}^4$ (which is again of dimension 4); its complement is an inequivalent irreducible U(2)-representation of dimension 4 which we call V_4. Consequently,

$$\mathbb{R}^5 \otimes \mathfrak{m}^6 = \mathbb{R} \oplus \mathfrak{n}^2 \oplus \mathbb{R}^4 \oplus S^2_0(\mathbb{R}^4) \oplus \Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4) \oplus V_4.$$

Taking into account the further splitting of $S^2_0(\mathbb{R}^4) \oplus \Lambda^2(\mathbb{R}^4)$, this is the sum of 10 irreducible U(2)-representations as claimed. On the other side,

$$\Lambda^3(\mathbb{R}^5) = \Lambda^3(\mathbb{R}^4 \oplus \mathbb{R}) = \Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4).$$

We found a unique copy of this 10-dimensional space in the 30-dimensional space $\mathbb{R}^5 \otimes \mathfrak{m}^6$. Thus, we conclude from Theorem 4.1:

Proposition 4.2. *A 5-dimensional almost metric contact structure $(M^5, g\xi, \eta, \varphi)$ admits a unique characteristic connection if and only if its intrinsic torsion is of class $\Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)$.*

In dimension 5, skew-symmetry of the Nijenhuis tensor N implies that it has to be zero, hence in the light of the more general Theorem 2.5, the almost metric contact manifolds of class $\Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)$ should coincide with the almost metric contact structures with $N = 0$ and ξ a Killing vector field. That this is indeed the case follows from the classifications cited above. This class includes for example all *quasi-Sasakian manifolds* ($N = 0$ and $dF = 0$), see [168].

Example 4.1. Consider $\mathbb{R}^5$ with 1-forms

$$2e_1 = dx_1, \quad 2e_2 = dy_1, \quad 2e_3 = dx_2, \quad 2e_4 = dy_2,$$
$$4e_5 = 4\eta = dz - y_1 dx_1 - y_2 dx_2,$$

metric $g = \sum_i e_i \otimes e_i$, and almost complex structure φ defined in $\langle\xi\rangle^\perp$ by

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = e_4, \quad \varphi(e_4) = -e_3, \quad \varphi(e_5) = 0.$$

Then $(\mathbb{R}^5, g, \eta, \varphi)$ is a Sasakian manifold, and the torsion of its characteristic connection is of type $\Lambda^2(\mathbb{R}^4)$ ([89, Example 3.D]) and explicitly given by

$$T^c = \eta \wedge d\eta = 2(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5.$$

This example is in fact a left-invariant metric on a 5-dimensional Heisenberg group with $\mathrm{scal}^g = -4$ and $\mathrm{scal}^{\nabla^c} = \mathrm{scal}^g - 3\|T\|^2/2 = -16$ (see Theorem A.1). In a left-invariant frame, spinors are simply functions $\psi\colon \mathbb{R}^5 \to \Delta_5$ with values in the 5-dimensional spin representation. In [104], it is shown that there exist two ∇^c-parallel spinors ψ_i with the additional property $F \cdot \psi_i = 0$ $(i = 1, 2)$. This implies $T \cdot \psi_i = 0$, an equation of interest in superstring theory (see Section 5.4). It turns out that these spinors are constant, hence the same result holds for all compact quotients $\mathbb{R}^5/\Gamma$ (Γ a discrete subgroup).

We recommend the articles [89] and [90] for a detailed investigation of the representation theory of almost metric contact structures (very much in the style of the book [210]) – in particular, the decomposition of the space of possible torsion tensors $\mathcal{T}$ of metric connections (see Proposition 2.1) under $\mathrm{U}(n)$ is being related to the possible classes for the intrinsic torsion.

4.4 SO(3)-structures in dimension 5

These structures were discovered by Th. Friedrich in a systematic investigation of possible G-structures for interesting non-integrable geometries (see [99]); until that moment, it was generally believed that contact structures were the only remarkable G-structures in dimension 5.

The group SO(3) has a unique, real, irreducible representation in dimension 5. We consider the corresponding non-standard embedding $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ as well as the decomposition

$$\mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{m}^7.$$

It is well known that the SO(3)-representation $\mathfrak{m}^7$ is the unique, real, irreducible representation of SO(3) in dimension 7. We decompose the tensor product into irreducible components

$$\mathbb{R}^5 \otimes \mathfrak{m}^7 = \mathbb{R}^3 \oplus \mathbb{R}^5 \oplus \mathfrak{m}^7 \oplus \mathrm{E}^9 \oplus \mathrm{E}^{11}.$$

There are five basic types of SO(3)-structures on 5-dimensional Riemannian manifolds. The symmetric spaces $\mathrm{SU}(3)/\mathrm{SO}(3)$ and $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ are examples of 5-dimensional Riemannian manifolds with an integrable SO(3)-structure ($\Gamma = 0$). On the other hand, 3-forms on $\mathbb{R}^5$ decompose into

$$\Lambda^3(\mathbb{R}^5) = \mathbb{R}^3 \oplus \mathfrak{m}^7.$$

In particular, a conformal change of an SO(3)-structure does *not* preserve the property that the structure admits a connection with totally skew-symmetric torsion.

M. Bobienski and P. Nurowski investigated SO(3)-structures in their articles [47] and [46]. In particular, they found a ternary symmetric form describing the reduction to SO(3) and constructed many examples of non-integrable SO(3)-structures with non-vanishing intrinsic torsion. Recently, P. Nurowski suggested a link to Cartan's work on isoparametric surfaces in spheres, and predicted the existence of similar geometries in dimensions 8, 14 and 26; we refer the reader to [199] for details. The case of SO(3)-structures illustrates that new classes of non-integrable geometries are still to be discovered beyond the well-established ones, and that their study reveals deeper connections between areas which used to be far from each other.

4.5 Almost Hermitian manifolds in dimension 6

We begin with the Gray–Hervella classification of almost Hermitian manifolds and the consequences for the characteristic connection. Although most of these results hold in all even dimensions, we shall henceforth restrict our attention to the most interesting case, namely dimension 6.

Let us consider a 6-dimensional almost Hermitian manifold (M^6, g, J), corresponding to a U(3)-structure inside SO(6). We decompose the Lie algebra into $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}$ and remark that the U(3)-representation in $\mathbb{R}^6$ is the real representation underlying $\Lambda^{1,0}$. Similarly, $\mathfrak{m}$ is the real representation underlying $\Lambda^{2,0}$. We decompose the complexification under the action of U(3):

$$\left(\mathbb{R}^6 \otimes \mathfrak{m}\right)^{\mathbb{C}} = \left(\Lambda^{1,0} \otimes \Lambda^{2,0} \oplus \Lambda^{1,0} \otimes \Lambda^{0,2}\right)^{\mathbb{C}}_{\mathbb{R}}.$$

The symbol $(\dots)^{\mathbb{C}}_{\mathbb{R}}$ means that we understand the complex representation as a real representation and complexify it. Next we split the complex U(3)-representations

$$\Lambda^{1,0} \otimes \Lambda^{2,0} = \mathbb{C}^3 \otimes \Lambda^2(\mathbb{C}^3) = \Lambda^{3,0} \oplus \mathrm{E}^8,$$

$$\Lambda^{1,0} \otimes \Lambda^{0,2} = \mathbb{C}^3 \otimes \Lambda^2(\bar{\mathbb{C}}^3) = \mathbb{C}^3 \otimes \Lambda^2(\mathbb{C}^3)^* = (\mathbb{C}^3)^* \oplus \mathrm{E}^6.$$

E^6 and E^8 are irreducible U(3)-representations of complex dimensions 6 and 8, respectively. Finally we obtain

$$\mathbb{R}^6 \otimes \mathfrak{m} = \Lambda^{3,0} \oplus \mathrm{E}^8 \oplus \mathrm{E}^6 \oplus (\mathbb{C}^3)^* =: \mathcal{W}_1^{(2)} \oplus \mathcal{W}_2^{(16)} \oplus \mathcal{W}_3^{(12)} \oplus \mathcal{W}_4^{(6)}.$$

Consequently, $\mathbb{R}^6 \otimes \mathfrak{m}$ splits into four irreducible representations of real dimensions 2, 16, 12 and 6, that is, there are four basic classes and a total of sixteen classes of U(3)-structures on 6-dimensional Riemannian manifolds, a result known as Gray/Hervella-classification ([130]). Recently, F. Martín Cabrera established the defining differential equations for these classes solely in terms of the intrinsic torsion (see [187]), as we shall state them for G_2-manifolds in the next section. In case we restrict the structure group to SU(3), the orthogonal complement $\mathfrak{su}(3)^\perp$ is now 7- instead of 6-dimensional, and we obtain

$$\mathbb{R}^6 \otimes \mathfrak{su}(3)^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

where $\mathcal{W}_5$ is isomorphic to $\mathcal{W}_4 \cong (\mathbb{C}^3)^*$. Furthermore, $\mathcal{W}_1$ and $\mathcal{W}_2$ are not irreducible anymore, but they split into $\mathcal{W}_1 = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- = \mathbb{R} \oplus \mathbb{R}$ and $\mathcal{W}_2 = \mathcal{W}_2^+ \oplus \mathcal{W}_2^- = \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ (see [69], [187]). Table 5 summarizes some remarkable classes of U(3)-structures in dimension 6. Most of these have by now well-established names, while there is still some confusion for others; these can be recognized by the parentheses indicating the different names to be found in the literature. In the last column, we collected characterizations of these classes (where several are listed, these are to be understood as equivalent characterizations, not as simultaneous requirements). Observe that we included in the last line a remarkable class of SU(3)-structures, the so called *half-flat* SU(3)-structures (Ψ^+ is the real part of the $(3,0)$-form defined by J, see Section 3.1). The name is chosen in order to suggest that half of all $\mathcal{W}$-components vanish

Table 5. Some types of U(3)- and SU(3)-structures in dimension six.

name	class	characterization
nearly Kähler manifold	$\mathcal{W}_1$	a) $(\nabla^g_X J)(X) = 0$ b) $\exists$ real Killing spinor
almost Kähler manifold	$\mathcal{W}_2$	$d\Omega = 0$
balanced (almost Hermitian) or (Hermitian) semi-Kähler manifold	$\mathcal{W}_3$	$N = 0$ and $\delta\Omega = 0$
locally conformally Kähler manifold	$\mathcal{W}_4$	$N = 0$ and $d\Omega = \Omega \wedge \theta$ (θ: Lee form)
quasi-Kähler manifold	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$\nabla^g_X \Omega(Y,Z) + \nabla^g_{JX} \Omega(JY,Z) = 0$
Hermitian manifold	$\mathcal{W}_3 \oplus \mathcal{W}_4$	$N = 0$
(almost-)semi-Kähler or (almost) cosymplectic manifold	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	a) $\delta\Omega = 0$, b) $\Omega \wedge d\Omega = 0$
KT- or $\mathcal{G}_1$-manifold	$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	a) N is skew-symmetric b) $\exists$ char. connection ∇^c
half-flat SU(3)-manifold	$\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$	$\Omega \wedge d\Omega = 0$ and $d\Psi^+ = 0$

for these structures. Relying on results of [141], S. Chiossi and S. Salamon described in [69] explicit metrics with Riemannian holonomy G_2 on the product of any half-flat SU(3)-manifold with a suitable interval. A construction of half-flat SU(3)-manifolds as T^2-principal fibre bundles over Kählerian 4-manifolds goes back to Goldstein and Prokushkin [124] and was generalized by Li, Fu and Yau [181], [110]. We refer to Section 5.2 for examples of half-flat SU(3)-structures on nilmanifolds.

The following result may easily be deduced from Theorem 4.1 and the decomposition of 3-forms into isotypic U(3)-representations,

$$\Lambda^3(\mathbb{R}^6) = \Lambda^{3,0} \oplus \mathrm{E}^6 \oplus (\mathbb{C}^3)^* = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

Theorem 4.2 ([102]). *An almost Hermitian 6-manifold (M^6, g, J) admits a characteristic connection ∇^c if and only if it is of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, i.e., if its Nijenhuis*

tensor N is skew-symmetric. Furthermore, ∇^c is unique and given by the expression

$$g(\nabla^c_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2}\,[N(X,Y,Z) + d\Omega(JX, JY, JZ)]$$

Remark 4.2 (Parallel torsion). In Example 2.3, it had been observed that the characteristic torsion of nearly Kähler manifolds is always parallel (Kirichenko's Theorem). Another interesting class of almost Hermitian $\mathcal{G}_1$-manifolds with this property are the so-called *generalized Hopf structures*, that is, locally conformally Kähler manifolds (class $\mathcal{W}_4$, sometimes abbreviated lcK-manifolds) with parallel *Lee form* $\theta := \delta\Omega \circ J \neq 0$ (in fact, $\nabla^g\theta = 0$ and $\nabla^c T^c = 0$ are equivalent conditions for $\mathcal{W}_4$-manifolds). Besides the classical Hopf manifolds, they include for example total spaces of flat principal S^1-bundles over compact 5-dimensional Sasaki manifolds (see [235], [236] for details); generalized Hopf structures are never Einstein. We recommend the book by S. Dragomir and L. Ornea [83] as a general reference and the articles [35], [111] for complex lcK-surfaces.

In his thesis, N. Schoemann investigates almost Hermitian structures with parallel skew-symmetric torsion in dimension 6. A full classification of the possible algebraic types of the torsion form is worked out, and based on this a systematic description of the possible geometries is given. In addition numerous new examples are constructed (and, partially, classified) on naturally reductive spaces (including compact spaces with closed torsion form) and on nilmanifolds (see [22] and [212]).

Remark 4.3 (Almost Kähler manifolds). The geometry of almost Kähler manifolds is strongly related to famous problems in differential geometry. W. Thurston was the first to construct an explicit compact symplectic manifold with $b_1 = 3$, hence that does not admit a Kähler structure [228]. E. Abbena generalized this example and gave a natural associated metric which makes it into an almost Kähler non-Kähler manifold [1]; many more examples of this type have been constructed since then.

In 1969, S. I. Goldberg conjectured that a compact almost Kähler–Einstein manifold is Kähler [123]. In this generality, the conjecture is still open. K. Sekigawa proved it under the assumption of non-negative scalar curvature [216], and it is known that the conjecture is false for non-compact manifolds: P. Nurowski and M. Przanowski gave the first example of a 4-dimensional Ricci-flat almost-Kähler non-Kähler manifold [200], J. Armstrong showed some non-existence results [28], while V. Apostolov, T. Drăghici and A. Moroianu constructed non-compact counterexamples to the conjecture in dimensions ≥ 6 [27]. Different partial results with various additional curvature assumptions are now available. The integrability conditions for almost Kähler manifolds were studied in full generality in [26] and [166].

Remark 4.4 (Nearly Kähler manifolds). We close this section with some additional remarks on nearly Kähler manifolds. Kirichenko's Theorem ($\nabla^c T^c = 0$) implies that $\mathrm{Hol}(\nabla^c) \subset \mathrm{SU}(3)$, that the first Chern class of the tangent bundle $c_1(TM^6, J)$ vanishes, M^6 is spin and that the metric is Einstein. The only known examples are homogeneous metrics on S^6, $\mathbb{CP}^3$, $S^3 \times S^3$ and on the flag manifold $F(1,2) =$

$\mathrm{U}(3)/\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$, although (many?) more are expected to exist. It was shown that these exhaust all nearly Kähler manifolds that are locally homogeneous (see [61]) or satisfying $\mathrm{Hol}(\nabla^c) \neq \mathrm{SU}(3)$ (see [36]). A by now classical result asserts that a 6-dimensional spin manifold admits a real Killing spinor if and only if it is nearly Kähler (see [101], [133] and [33]). Finally, more recent structure theorems justify why nearly Kähler manifolds are only interesting in dimension 6: any complete simply connected nearly Kähler manifold is locally a Riemannian product of Kähler manifolds, twistor spaces over Kähler manifolds and 6-dimensional nearly Kähler manifolds (see [197], [198]).

4.6 G_2-structures in dimension 7

We consider 7-dimensional Riemannian manifolds equipped with a G_2-structure. Since G_2 is the isotropy group of a 3-form ω of general type, a G_2-structure is a triple (M^7, g, ω) consisting of a 7-dimensional Riemannian manifold and a 3-form ω of general type at any point. We decompose the G_2-representation (see [102])

$$\mathbb{R}^7 \otimes \mathfrak{m} = \mathbb{R} \oplus \Lambda^2_{14} \oplus \Lambda^3_{27} \oplus \mathbb{R}^7 =: \mathcal{X}_1^{(1)} \oplus \mathcal{X}_2^{(14)} \oplus \mathcal{X}_3^{(27)} \oplus \mathcal{W}_4^{(7)},$$

and, consequently, there are again four basic classes and a total of sixteen classes G_2-structures (namely, parallel G_2-manifolds and 15 non-integrable G_2-structures). This result is known as the Fernández/Gray-classification of G_2-structures (see [87]); some important classes are again summarized in tabular form, see Table 6. The different classes of G_2-structures can be characterized by differential equations. They can be written in a unified way as

$$d\omega = \lambda \cdot *\omega + \frac{3}{4}\theta \wedge \omega + *\tau_3, \quad \delta\omega = - * d * \omega = - * (\theta \wedge *\omega) + *(\tau_2 \wedge \omega),$$

where λ is a scalar function corresponding to the $\mathcal{X}_1$-part of the intrinsic torsion Γ, τ_2, τ_3 are 2- resp. 3-forms corresponding to its $\mathcal{X}_2$ resp. $\mathcal{X}_3$-part and θ is a 1-form describing its $\mathcal{X}_4$-part, which one sometimes calls the *Lee form* of the G_2-structure. This accounts for some of the characterizations listed in Table 6. For example, a G_2-structure is of type $\mathcal{X}_1$ (*nearly parallel* G_2-structure) if and only if there exists a number λ (it has to be constant in this case) such that $d\omega = \lambda * \omega$ holds. Again, this condition is equivalent to the existence of a real Killing spinor and the metric has to be Einstein [107]; more recently, the Riemannian curvature properties of arbitrary G_2-manifolds have been discussed in detail by R. Cleyton and S. Ivanov [72]. G_2-structures of type $\mathcal{X}_1 \oplus \mathcal{X}_3$ (*cocalibrated* G_2-structures) are characterized by the condition that the 3-form is coclosed, $\delta\omega^3 = 0$. Under the restricted action of G_2, one obtains the following isotypic decomposition of 3-forms on $\mathbb{R}^7$:

$$\Lambda^3(\mathbb{R}^7) = \mathbb{R} \oplus \Lambda^3_{27} \oplus \mathbb{R}^7 = \mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4.$$

Table 6. Some types of G_2-structures in dimension seven.

name	class	characterization
nearly parallel G_2-manifold	$\mathcal{X}_1$	a) $d\omega = \lambda * \omega$ for some $\lambda \in \mathbb{R}$ b) $\exists$ real Killing spinor
almost parallel or closed (or calibrated symplectic) G_2-manifold	$\mathcal{X}_2$	$d\omega = 0$
balanced G_2-manifold	$\mathcal{X}_3$	$\delta\omega = 0$ and $d\omega \wedge \omega = 0$
locally conformally parallel G_2-manifold	$\mathcal{X}_4$	$d\omega = \frac{3}{4}\theta \wedge \omega$ and $d * \omega = \theta \wedge *\omega$ for some 1-form θ
cocalibrated (or semi-parallel or cosymplectic) G_2-manifold	$\mathcal{X}_1 \oplus \mathcal{X}_3$	$\delta\omega = 0$
locally conformally (almost) parallel G_2-manifold	$\mathcal{X}_2 \oplus \mathcal{X}_4$	$d\omega = \frac{3}{4}\theta \wedge \omega$
G_2T-manifold	$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	a) $d * \omega = \theta \wedge *\omega$ for some 1-form θ b) $\exists$ char. connection ∇^c

This explains the first part of the following theorem and the acronym 'G_2T-manifolds' for this class: it stands for 'G_2 with (skew) torsion'. The explicit formula for the characteristic torsion may be derived directly from the properties of ∇^c.

Theorem 4.3 ([102], Theorem 4.8). *A 7-dimensional manifold (M^7, g, ω) with a fixed G_2-structure $\omega \in \Lambda^3(M^7)$ admits a characteristic connection ∇^c if and only if it is of class $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$, i.e., if there exists a 1-form θ such that $d * \omega = \theta \wedge *\omega$. Furthermore, ∇^c is unique, admits (at least) one parallel spinor and is given by the expression*

$$\nabla^c_X Y := \nabla^g_X Y + \frac{1}{2}\Big[- *d\omega - \frac{1}{6} g(d\omega, *\omega)\omega + *(\theta \wedge \omega)\Big].$$

This last remarkable property is a direct consequence of our investigation of geometric stabilizers, as explained in Corollary 3.1. For a nearly parallel G_2-manifold,

the ∇^c-parallel spinor coincides with the Riemannian Killing spinor and the manifold turns out to be ∇^c-Einstein [102]. But there is no general argument identifying Killing spinors with parallel spinors. The characteristic connection of a Sasaki-Einstein manifold does not necessarily admit parallel spinors ([102], [104]), though there always exist Killing spinors [105]. A 3-Sasaki manifold admits three real Killing spinors, but it does not admit a characteristic connection in any reasonable sense (each Sasaki structure has a characteristic connection, but it does not preserve the other two Sasaki structures), see Section 2.6 and [10]. This reflects the fact that Sasaki-Einstein and 3-Sasaki manifolds do not fit too well into the general framework of G-structures.

Remark 4.5 (Parallel torsion). For a nearly parallel G_2-manifold, the explicit formula from Theorem 4.3 implies that the characteristic torsion T^c is proportional to ω, hence it is trivially ∇^c-parallel. For the larger class of cocalibrated G_2-manifolds (class $\mathcal{X}_1 \oplus \mathcal{X}_3$), the case of parallel characteristic torsion has been investigated systematically by Th. Friedrich (see [100]). Again, many formerly unknown examples have been constructed, for example, from deformations of η-Einstein Sasaki manifolds, from S^1-principal fibre bundles over 6-dimensional Kähler manifolds or from naturally reductive spaces.

4.7 Spin(7)-structures in dimension 8

Let us consider Spin(7)-structures on 8-dimensional Riemannian manifolds. The subgroup $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ is the real Spin(7)-representation $\Delta_7 = \mathbb{R}^8$, the complement $\mathfrak{m} = \mathbb{R}^7$ is the standard 7-dimensional representation and the Spin(7)-structures on an 8-dimensional Riemannian manifold M^8 correspond to the irreducible components of the tensor product

$$\mathbb{R}^8 \otimes \mathfrak{m} = \mathbb{R}^8 \otimes \mathbb{R}^7 = \Delta_7 \otimes \mathbb{R}^7 = \Delta_7 \oplus \mathrm{K} = \mathbb{R}^8 \oplus \mathrm{K},$$

where K denotes the kernel of the Clifford multiplication $\Delta_7 \otimes \mathbb{R}^7 \to \Delta_7$. It is well known that K is an irreducible Spin(7)-representation, i.e., there are two basic classes of Spin(7)-structures (the Fernández classification of Spin(7)-structures, see [85]). For 3-forms, we find the isotypic decomposition

$$\Lambda^3(\mathbb{R}^8) = \Delta_7 \oplus \mathrm{K},$$

showing that $\Lambda^3(\mathbb{R}^8)$ and $\mathbb{R}^8 \otimes \mathfrak{m}$ are isomorphic. Theorem 4.1 yields immediately that *any* Spin(7)*-structure on an* 8*-dimensional Riemannian manifold admits a unique connection with totally skew-symmetric torsion*. The explicit formula for its characteristic torsion may be found in [149].

5 Weitzenböck formulas for Dirac operators with torsion

5.1 Motivation

The question whether or not the characteristic connection of a G-structure admits parallel tensor fields differs radically from the corresponding problem for the Levi-Civita connection. In particular, one is interested in the existence of parallel spinor fields, interpreted in superstring theory as supersymmetries of the model. The main analytical tool for the investigation of parallel spinors is the Dirac operator and several remarkable identities for it. We discuss two identities for the square of the Dirac operator. While the first one is straightforward and merely of computational difficulty, the second relies on comparing the Dirac operator corresponding to the connection with torsion T with the spinorial Laplace operator corresponding to the connection with torsion $3T$. Such an argument has been used in the literature at several places. The first was probably S. Slebarski ([218], [219]) who noticed that on a naturally reductive space, the connection with torsion one-third that of the canonical connection behaves well under fibrations; S. Goette applied this property to the computation of the η-invariant on homogeneous spaces [122]. J.-P. Bismut used such a rescaling for proving an index theorem for Hermitian manifolds [42]. It is implicit in Kostant's work on a 'cubic Dirac operator', which can be understood as an identity in the Clifford algebra for the symbol of the Dirac operator of the rescaled canonical connection on a naturally reductive space ([173], [4]).

5.2 The square of the Dirac operator and parallel spinors

Consider a Riemannian spin manifold (M^n, g, T) with a 3-form $T \in \Lambda^3(M^n)$ as well as the one-parameter family of linear metric connections with skew-symmetric torsion ($s \in \mathbb{R}$),

$$\nabla^s_X Y := \nabla^g_X Y + 2s\, T(X, Y, -).$$

In particular, the superscript $s = 0$ corresponds to the Levi-Civita connection and $s = 1/4$ to the connection with torsion T considered before. As before, we shall also sometimes use the superscript g to denote the Riemannian quantities corresponding to $s = 0$. These connections can all be lifted to connections on the spinor bundle ΣM^n, where they take the expression

$$\nabla^s_X \psi := \nabla^g_X \psi + s(X \lrcorner T) \cdot \psi.$$

Two important elliptic operators may be defined on ΣM^n, namely, the Dirac operator and the spinor Laplacian associated with the connection ∇^s:

$$D^s := \sum_{k=1}^{n} e_k \cdot \nabla^s_{e_k} = D^0 + 3sT, \quad \Delta^s(\psi) = (\nabla^s)^* \nabla^s \psi = -\sum_{k=1}^{n} \nabla^s_{e_k} \nabla^s_{e_k} \psi + \nabla^s_{\nabla^g_{e_i} e_i} \psi.$$

By a result of Th. Friedrich and S. Sulanke [109], the Dirac operator D^∇ associated with any metric connection ∇ is formally self-adjoint if and only if the ∇-divergence $\mathrm{div}^\nabla(X) := \sum_i g(\nabla_{e_i} X, e_i)$ of any vector field X coincides with its Riemannian ∇^g-divergence. Writing $\nabla = \nabla^g + A$, this is manifestly equivalent to $\sum_i g(A(e_i, X), e_i) = 0$ and trivially satisfied for metric connections with totally skew-symmetric torsion[4].

Shortly after P. Dirac introduced the Dirac operator, E. Schrödinger noticed the existence of a remarkable formula for its square [215]. Of course, since the concept of spin manifold had not yet been established, all arguments of that time were of local nature, but contained already all important ingredients that would be established in a more mathematical way later. By the sixties and the seminal work of Atiyah and Singer on index theory for elliptic differential operators, Schrödinger's article was almost forgotten and the formula rediscovered by A. Lichnerowicz [182]. In our notation, the *Schrödinger–Lichnerowicz formula* states that

$$(D^0)^2 = \Delta^0 + \frac{1}{4}\mathrm{scal}^0.$$

Our goal is to derive useful relations for the square of D^s. In order to state the first formula, let us introduce the first order differential operator

$$\mathcal{D}^s\psi := \sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \cdot \nabla^s_{e_k}\psi = \mathcal{D}^0\psi + s\sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \cdot (e_k \lrcorner \mathrm{T}) \cdot \psi. \qquad (9)$$

Theorem 5.1 ([102, Theorems 3.1, 3.3]). *Let (M^n, g, ∇^s) be an n-dimensional Riemannian spin manifold with a metric connection ∇^s of skew-symmetric torsion $4s \cdot T$. Then, the square of the Dirac operator D^s associated with ∇^s acts on an arbitrary spinor field ψ as*

$$(D^s)^2\psi = \Delta^s(\psi) + 3s\, dT \cdot \psi - 8s^2\, \sigma_T \cdot \psi + 2s\, \delta T \cdot \psi - 4s\, \mathcal{D}^s\psi + \frac{1}{4}\,\mathrm{scal}^s \cdot \psi. \quad (10)$$

Furthermore, the anticommutator of D^s and T is

$$D^s \circ T + T \circ D^s = dT + \delta T - 8s\,\sigma_T - 2\,\mathcal{D}^s. \qquad (11)$$

scal^s denotes the scalar curvature of the connection ∇^s. Remark that $\mathrm{scal}^0 = \mathrm{scal}^g$ is the usual scalar curvature of the underlying Riemannian manifold (M^n, g) and that the relation $\mathrm{scal}^s = \mathrm{scal}^0 - 24s^2\|T\|^2$ holds. Moreover, the divergence δT can be taken with respect to any connection ∇^s from the family, hence we do not make a notational difference between them (see Proposition A.2).

This formula for $(D^s)^2$ has the disadvantage of still containing a first order differential operator with uncontrollable spectrum as well as several 4-forms that are difficult to treat algebraically, hence it is not suitable for deriving vanishing theorems.

[4] One checks that it also holds for metric connections with vectorial torsion, but not for connections of Cartan type $\mathcal{A}'$.

It has however a nice application in the study of ∇^s-parallel spinors for *different* values of s. As a motivation, let us consider the following example:

Example 5.1. Let G be a simply connected Lie group, g a bi-invariant metric and consider the torsion form $T(X,Y,Z) := g([X,Y],Z)$. The connections $\nabla^{\pm 1/4}$ are flat [170], hence they both admit non-trivial parallel spinor fields.

Such a property for the connections with torsions $\pm T$ is required in some superstring models. Theorem 5.1 now implies that there cannot be many values s admitting ∇^s-parallel spinors.

Theorem 5.2 ([10, Theorem 7.1]). *Let (M^n, g, T) be a compact spin manifold, ∇^s the family of metric connections defined by T as above. For any ∇^s-parallel spinor ψ, the following formula holds:*

$$64\, s^2 \int_{M^n} \langle \sigma_T \cdot \psi,\, \psi \rangle + \int_{M^n} \mathrm{scal}^s \cdot \|\psi\|^2 = 0.$$

If the mean value of $\langle \sigma_T \cdot \psi,\, \psi \rangle$ does not vanish, the parameter s is given by

$$s = \frac{1}{8} \int_{M^n} \langle dT \cdot \psi,\, \psi \rangle \Big/ \int_{M^n} \langle \sigma_T \cdot \psi,\, \psi \rangle.$$

If the mean value of $\langle \sigma_T \cdot \psi,\, \psi \rangle$ vanishes, the parameter s depends only on the Riemannian scalar curvature and on the length of the torsion form,

$$0 = \int_{M^n} \mathrm{Scal}^s = \int_{M^n} \mathrm{scal}^g - 24 s^2 \int_{M^n} \|T\|^2.$$

Finally, if the 4-forms dT and σ_T are proportional (for example, if $\nabla^{1/4} T = 0$), there are at most three parameters with ∇^s-parallel spinors.

Remark 5.1. The property that dT and σ_T are proportional is more general than requiring parallel torsion. For example, it holds for the whole family of connections ∇^t on naturally reductive spaces discussed in Sections 2.2 and 5.3, but its torsion is ∇^t-parallel only for $t = 1$.

Example 5.2. On the 7-dimensional Aloff–Wallach space $N(1,1) = \mathrm{SU}(3)/S^1$, one can construct a non-flat connection such that ∇^{s_0} *and* ∇^{-s_0} admit parallel spinors for suitable s_0, hence showing that both cases from Theorem 5.2 can actually occur in non-trivial situations. On the other hand, it can be shown that on a 5-dimensional Sasaki manifold, only the characteristic connection ∇^c can have parallel spinors [10].

Inspired by the homogeneous case (see Section 5.3), we were looking for an alternative comparison of $(D^s)^2$ with the Laplace operator of some *other* connection $\nabla^{s'}$ from the same family. Since $(D^s)^2$ is a symmetric second order differential operator with metric principal symbol, a very general result by P. B. Gilkey claims that there

exists a connection ∇ and an endomorphism E such that $(D^s)^2 = \nabla^*\nabla + E$ [120]. Based on the results of Theorem 5.1, one shows:

Theorem 5.3 (Generalized Schrödinger–Lichnerowicz formula, [10, Theorem 6.2]). *The spinor Laplacian Δ^s and the square of the Dirac operator $D^{s/3}$ are related by*

$$(D^{s/3})^2 = \Delta^s + s\, d^*T + \frac{1}{4}\operatorname{scal}^g - 2s^2\,\|T\|^2.$$

We observe that $D^{s/3}$ appears basically by quadratic completion. Under additional assumptions on T, this result can be used as a basis for eigenvalue estimates for $D^{s/3}$ – see [13].

Theorem 5.4 ([10, Theorem 6.3]). *Let (M^n, g, T) be a compact, Riemannian spin manifold of non positive scalar curvature,* $\operatorname{scal}^g \leq 0$. *If there exists a solution $\psi \neq 0$ of the equations*

$$\nabla_X\psi = \nabla^g_X\psi + \frac{1}{2}(X \lrcorner T)\cdot\psi = 0, \quad \langle d^*T\cdot\psi, \psi\rangle \leq 0,$$

the 3-form and the scalar curvature vanish, $T = 0 = \operatorname{scal}^g$, and ψ is parallel with respect to the Levi-Civita connection.

Theorem 5.4 applies, in particular, to Calabi–Yau or Joyce manifolds, where we know that ∇^g-parallel spinors exist by Wang's Theorem (Theorem 3.1). Let us perturb the connection ∇^g by a suitable 3-form (for example, a closed one). Then the new connection ∇ does not admit ∇-parallel spinor fields: the Levi-Civita connection and its parallel spinors are thus, in some sense, rigid. Nilmanifolds are a second family of examples where the theorem applies. A further family of examples arises from certain naturally reductive spaces with torsion form T proportional to the torsion form of the canonical connection, see [4]. From the high energy physics point of view, a parallel spinor is interpreted as a supersymmetry transformation. Hence the physical problem behind the above question (which in fact motivated our investigations) is really whether a free "vacuum solution" can also carry a non-vacuum supersymmetry, and how the two are related.

Naturally, Theorem 5.4 raises the question to which extent compactness is really necessary. We shall now show that it is by using the equivalence between the inclusion $\operatorname{Hol}(\nabla) \subset G_2$ and the existence of a ∇-parallel spinor for a metric connection with skew-symmetric torsion known from Corollary 3.1. For this, it is sufficient to find a 7-dimensional Riemannian manifold (M^7, g) whose Levi-Civita connection has a parallel spinor (hence is Ricci-flat, in particular), but also admits a ∇-parallel spinor for some other metric connection with skew-symmetric torsion.

Gibbons *et al.* produced non-complete metrics with Riemannian holonomy G_2 in [119]. Those metrics have among others the interesting feature of admitting a 2-step nilpotent isometry group N acting on orbits of codimension one. By [68] such metrics are locally conformal to homogeneous metrics on rank-one solvable

extension of N, and the induced SU(3)-structure on N is half-flat. In the same article all half-flat SU(3) structures on 6-dimensional nilpotent Lie groups whose rank-one solvable extension is endowed with a conformally parallel G_2 structure were classified. Besides the torus, there are exactly six instances, which we considered in relation to the problem posed. It turns out that four metrics of the six only carry integrable G_2 structures, thus reproducing the pattern of the compact situation, whilst one admits complex solutions, a physical interpretation for which is still lacking. The remaining solvmanifold (Sol, g) – which has *exact* Riemannian holonomy G_2 – provides a positive answer to both questions posed above, hence becoming the most interesting. The Lie algebra associated to this solvmanifold has structure equations

$$[e_i, e_7] = \tfrac{3}{5}me_i, \quad i = 1, 2, 5, \quad [e_j, e_7] = \tfrac{6}{5}me_j, \quad j = 3, 4, 6,$$
$$[e_1, e_5] = -\tfrac{2}{5}me_3, \quad [e_2, e_5] = -\tfrac{2}{5}me_4, \quad [e_1, e_2] = -\tfrac{2}{5}me_6.$$

The homogeneous metric it bears can be also seen as a G_2 metric on the product $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T}$ is the total space of a T^3-bundle over another 3-torus. For the sake of an easier formulation of the result, we denote by ∇^T the metric connection with torsion T.

Theorem 5.5 ([7, Theorem 4.1]). *The equation $\nabla^T \Psi = 0$ admits 7 solutions for some 3-form, namely:*

a) *A two-parameter family of pairs $(T_{r,s}, \Psi_{r,s}) \in \Lambda^3(\mathrm{Sol}) \times \Sigma(\mathrm{Sol})$ such that $\nabla^{T_{r,s}} \Psi_{r,s} = 0$; for $r = s$ the torsion $T_{r,r} = 0$ and $\Psi_{r,r}$ is a multiple of the ∇^g-parallel spinor.*

b) *Six 'isolated' solutions occurring in pairs, $(T_i^\varepsilon, \Psi_i^\varepsilon) \in \Lambda^3(\mathrm{Sol}) \times \Sigma(\mathrm{Sol})$ for $i = 1, 2, 3$ and $\varepsilon = \pm$.*

All these G_2 structures admit exactly one parallel spinor, and for

$|r| \neq |s|$: $\omega_{r,s}$ is of general type $\mathbb{R} \oplus S^2_0\mathbb{R}^7 \oplus \mathbb{R}^7$,

$r = s$: $\omega_{r,r}$ is ∇^g-parallel,

$r = -s$: the G_2 class has no $\mathbb{R}$-part.

Here $\omega_{r,s}$ denotes the defining 3-form of the G_2-structure, see equation (8).

Remark 5.2. A routine computation establishes that $\langle dT \cdot \Psi, \Psi \rangle < 0$ for all solutions found in Theorem 5.5, except for the integrable case $r = s$ of solution a) where it vanishes trivially since $T = 0$.

Remark 5.3. The interaction between explicit Riemannian metrics with holonomy G_2 on non-compact manifolds and the non-integrable G_2-geometries as investigated with the help of connections with torsion was up-to-now limited to "cone-type arguments", i.e., a non-integrable structure on some manifold was used to construct an integrable structure on a higher dimensional manifold (like its cone, an so on). It is thus a natural

question whether the same Riemannian manifold (M, g) can carry structures of both type *simultaneously*. This appears to be a remarkable property, of which the above example is the only known instance. To emphasize this, consider that the projective space $\mathbb{C}P^3$ with the well-known Kähler–Einstein structure and the nearly Kähler one inherited from triality does not fit the picture, as they refer to different metrics.

5.3 Naturally reductive spaces and Kostant's cubic Dirac operator

On arbitrary manifolds, only Weitzenböck formulas that express D^2 through the Laplacian are available. On homogeneous spaces, it makes sense to look for expressions for D^2 of *Parthasarathy type*, that is, in terms of Casimir operators. Naturally reductive spaces $M^n = G/H$ with their family of metric connections ($X, Y \in \mathfrak{m}$)

$$\nabla^t_X Y := \nabla^g_X Y - \frac{t}{2}[X, Y]_{\mathfrak{m}}$$

were in fact investigated prior to the more general case described in the previous section. As symmetric spaces are good toy models for integrable geometries, homogeneous non-symmetric spaces are a very useful field for 'experiments' in non-integrable geometry. Furthermore, many examples of such geometries are in fact homogeneous. We will show that the main achievement in [173] was to realize that, for the parameter value $t = 1/3$, the square of D^t may be expressed in a very simple way in terms of Casimir operators and scalars only ([173, Theorem 2.13], [221, 10.18]). It is a remarkable generalization of the classical Parthasarathy formula for D^2 on symmetric spaces (formula (1) in this chapter, see [201]). We shall speak of the *generalized Kostant–Parthasarathy formula* in the sequel. S. Slebarski used the connection $\nabla^{1/3}$ to prove a "vanishing theorem" for the kernel of the twisted Dirac operator, which can be easily recovered from Kostant's formula (see [180, Theorem 4]). His articles [218] and [219] contain several formulas of Weitzenböck type for D^2, but none of them is of Parthasarathy type.

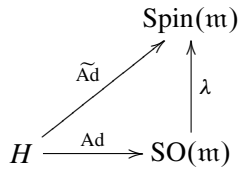
In fact, Kostant needed his operator not for the investigation of non-integrable geometries, but as a tool for explaining a distinguished property of the representations of Spin(9) discovered by the physicists Ramond and Pengpan [132]. In representation theory this opened the possibility to realize infinite-dimensional representations in kernels of twisted Dirac operators on homogeneous spaces ([145], [190]), as it had been carried out on symmetric spaces in the seventies.

In order to exploit the full power of harmonic analysis, it is necessary to extend the naturally reductive metric $\langle\ ,\ \rangle$ on $\mathfrak{m}$ to the whole Lie algebra $\mathfrak{g}$ of G. By a classical theorem of B. Kostant, there exists a unique $\mathrm{Ad}(G)$ invariant, symmetric, non degenerate, bilinear form Q on $\mathfrak{g}$ such that

$$Q(\mathfrak{h} \cap \mathfrak{g}, \mathfrak{m}) = 0 \quad \text{and} \quad Q|_{\mathfrak{m}} = \langle\ ,\ \rangle$$

if G acts effectively on M^n and $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$, which we will tacitly assume from now on [172]. In general, Q does not have to be positive definite; if it is, the metric

is called *normal homogeneous*. Assume furthermore that there exists a homogeneous spin structure on M, i.e., a lift $\widetilde{\mathrm{Ad}}\colon H \to \mathrm{Spin}(\mathfrak{m})$ of the isotropy representation such that the diagram



commutes, where λ denotes the spin covering. Moreover, denote by $\widetilde{\mathrm{ad}}$ the corresponding lift into $\mathfrak{spin}(\mathfrak{m})$ of the differential $\mathrm{ad}\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ of Ad. Let $\kappa\colon \mathrm{Spin}(\mathfrak{m}) \to \mathrm{GL}(\Delta_{\mathfrak{m}})$ be the spin representation, and identify sections of the spinor bundle $\Sigma M^n = G \times_{\widetilde{\mathrm{Ad}}} \Delta_{\mathfrak{m}}$ with functions $\psi\colon G \to \Delta_{\mathfrak{m}}$ satisfying

$$\psi(gh) = \kappa(\widetilde{\mathrm{Ad}}(h^{-1}))\psi(g).$$

The Dirac operator takes for $\psi \in \Sigma M^n$ the form

$$D^t\psi = \sum_{i=1}^{n} e_i(\psi) + \frac{1-t}{2}\, H \cdot \psi,$$

where H is the third degree element in the Clifford algebra $\mathrm{Cl}(\mathfrak{m})$ of $\mathfrak{m}$ induced from the torsion,

$$H := \frac{3}{2} \sum_{i<j<k} \big\langle [e_i, e_j]_{\mathfrak{m}}, e_k \big\rangle\, e_i \cdot e_j \cdot e_k.$$

This fact suggested the name "cubic Dirac operator" to B. Kostant. Two expressions appear over and over again for naturally reductive spaces: these are the $\mathfrak{m}$- and $\mathfrak{h}$-parts of the Jacobi identity,

$$\begin{aligned} \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) &:= [X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}, \\ \mathrm{Jac}_{\mathfrak{h}}(X,Y,Z) &:= [X,[Y,Z]_{\mathfrak{h}}] + [Y,[Z,X]_{\mathfrak{h}}] + [Z,[X,Y]_{\mathfrak{h}}]. \end{aligned}$$

Notice that the summands of $\mathrm{Jac}_{\mathfrak{h}}(X,Y,Z)$ automatically lie in $\mathfrak{m}$ by the assumption that M is reductive. The Jacobi identity for $\mathfrak{g}$ implies $\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) + \mathrm{Jac}_{\mathfrak{h}}(X,Y,Z) = 0$. In fact, since the torsion is given by $T^t(X,Y) = -t[X,Y]_{\mathfrak{m}}$, one immediately sees that $\langle \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z), V\rangle$ is just $-\sigma_{T^t}(X,Y,Z,V)$ as defined before. From the explicit formula for T^t and the property $\nabla^1 T^1 = 0$, it is a routine computation to show that (see [4, Lemmas 2.3, 2.5])

$$\begin{aligned} \nabla^t_Z T^t(X,Y) &= \frac{1}{2} t(t-1)\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z), \\ d\,T^t(X,Y,Z,V) &= -2t\, \langle \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z), V\rangle\,. \end{aligned}$$

In particular, $d\,T^t$ and σ_{T^t} are always proportional (see Remark 5.1). The first formula implies $X \lrcorner \nabla^t_X T^t = 0$, hence $\delta^t T^t = 0$ and it equals the Riemannian divergence $\delta^g T^t$ by Proposition A.2 of the Appendix. Since the $\mathrm{Ad}(G)$-invariant extension Q

of $\langle\ ,\ \rangle$ is not necessarily positive definite when restricted to $\mathfrak{h}$, it is more appropriate to work with dual rather than with orthonormal bases. So pick bases x_i, y_i of $\mathfrak{h}$ ($i = 1, \dots, \dim \mathfrak{h}$) which are dual with respect to $Q_{\mathfrak{h}}$, i.e., $Q_{\mathfrak{h}}(x_i, y_j) = \delta_{ij}$. The (lift into the spin bundle of the) Casimir operator of the full Lie algebra $\mathfrak{g}$ is now the sum of a second order differential operator (its $\mathfrak{m}$-part) and a constant element of the Clifford algebra (its $\mathfrak{h}$-part)

$$\Omega_{\mathfrak{g}}(\psi) = -\sum_{i=1}^{n} e_i^2(\psi) - \sum_{j=1}^{\dim \mathfrak{h}} \widetilde{\mathrm{ad}}(x_j) \circ \widetilde{\mathrm{ad}}(y_j) \cdot \psi \quad \text{for } \psi \in \Sigma M^n.$$

In order to prove the generalized Kostant–Parthasarathy formula for the square of D^t, similar technical prerequisites as in Section 5.2 are needed, but now expressed with respect to representation theoretical quantities instead of analytical ones. We refer to [4] for details and will rather formulate the final result without detours. Observe that the dimension restriction below ($n \geq 5$) is not essential, for small dimensions a similar formula holds, but it looks slightly different.

Theorem 5.6 (Generalized Kostant–Parthasarathy formula, [4, Theorem 3.2]). *For $n \geq 5$, the square of D^t is given by*

$$\begin{aligned}(D^t)^2\psi = \Omega_{\mathfrak{g}}(\psi) &+ \frac{1}{2}(3t-1)\sum_{i,j,k} \big\langle [e_i, e_j]_{\mathfrak{m}}, e_k \big\rangle\, e_i \cdot e_j \cdot e_k(\psi) \\ &- \frac{1}{2}\sum_{i<j<k<l} \Big\langle e_i, \mathrm{Jac}_{\mathfrak{h}}(e_j, e_k, e_l) + \frac{9(1-t)^2}{4}\mathrm{Jac}_{\mathfrak{m}}(e_j, e_k, e_l)\Big\rangle \\ &\qquad \cdot e_i \cdot e_j \cdot e_k \cdot e_l \cdot \psi \\ &+ \frac{1}{8}\sum_{i,j} Q_{\mathfrak{h}}([e_i, e_j], [e_i, e_j])\psi + \frac{3(1-t)^2}{24}\sum_{i,j} Q_{\mathfrak{m}}([e_i, e_j], [e_i, e_j])\psi.\end{aligned}$$

Qualitatively, this result is similar to equation (10) of Theorem 5.1, although one cannot be deduced directly from the other. Again, the square of the Dirac operator is written as the sum of a second order differential operator (the Casimir operator), a first order differential operator, a four-fold product in the Clifford algebra and a scalar part (recall that $\delta^t T^t = 0$, hence this term has no counterpart here). An immediate consequence is the special case $t = 1/3$:

Corollary 5.1 (The Kostant–Parthasarathy formula for $t = 1/3$). *For $n \geq 5$ and $t = 1/3$, the general formula for $(D^t)^2$ reduces to*

$$\begin{aligned}(D^{1/3})^2\psi = \Omega_{\mathfrak{g}}(\psi) &+ \frac{1}{8}\Big[\sum_{i,j} Q_{\mathfrak{h}}([e_i, e_j], [e_i, e_j]) \\ &+ \frac{1}{3}\sum_{i,j} Q_{\mathfrak{m}}([e_i, e_j], [e_i, e_j])\Big]\psi\end{aligned}$$

$$= \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8}\Big[\mathrm{scal}^{1/3} + \frac{1}{9}\sum_{i,j} Q_{\mathfrak{m}}([e_i, e_j], [e_i, e_j])\Big]\psi.$$

Remark 5.4. In particular, one immediately recovers the classical Parthasarathy formula for a symmetric space, since then all scalar curvatures coincide and $[e_i, e_j] \in \mathfrak{h}$. In fact, compared with Theorem 5.3, Corollary 5.1 has the advantage of containing no 4-form action on the spinor and the draw-back that the Casimir operator of a naturally reductive space is not necessarily a non-negative operator (see below).

As in the classical Parthasarathy formula, the scalar term as well as the eigenvalues of $\Omega_{\mathfrak{g}}(\psi)$ may be expressed in representation theoretical terms if G (and hence M) is compact.

Lemma 5.1 ([4, Lemma 3.6]). *Let G be compact, $n \geq 5$, and denote by $\varrho_{\mathfrak{g}}$ and $\varrho_{\mathfrak{h}}$ the half sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then the Kostant–Parthasarathy formula for $(D^{1/3})^2$ may be restated as*

$$(D^{1/3})^2\psi = \Omega_{\mathfrak{g}}(\psi) + \big[Q(\varrho_{\mathfrak{g}}, \varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}}, \varrho_{\mathfrak{h}})\big]\,\psi = \Omega_{\mathfrak{g}}(\psi) + \big\langle \varrho_{\mathfrak{g}} - \varrho_{\mathfrak{h}}, \varrho_{\mathfrak{g}} - \varrho_{\mathfrak{h}}\big\rangle\,\psi.$$

In particular, the scalar term is positive independently of the properties of Q.

We can formulate our first conclusion from Corollary 5.1:

Corollary 5.2 ([4, Corollary 3.1]). *Let G be compact. If the operator $\Omega_{\mathfrak{g}}$ is non-negative, the first eigenvalue $\lambda_1^{1/3}$ of the Dirac operator $D^{1/3}$ satisfies the inequality*

$$\big(\lambda_1^{1/3}\big)^2 \geq Q(\varrho_{\mathfrak{g}}, \varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}}, \varrho_{\mathfrak{h}}).$$

Equality occurs if and only if there exists an algebraic spinor in $\Delta_{\mathfrak{m}}$ which is fixed under the lift $\kappa(\widetilde{\mathrm{Ad}}H)$ of the isotropy representation.

Remark 5.5. This eigenvalue estimate is remarkable for several reasons. Firstly, for homogeneous non symmetric spaces, it is sharper than the classical Parthasarathy formula. For a symmetric space, one classically obtains $\lambda_1^2 \geq \mathrm{scal}/8$. But since the Schrödinger–Lichnerowicz formula yields immediately $\lambda_1^2 \geq \mathrm{scal}/4$, the lower bound in the classical Parthasarathy formula is never attained. In contrast, there exist many examples of homogeneous non symmetric spaces with constant spinors. Secondly, it uses a lower bound which is *always* strictly positive; for many naturally reductive metrics with negative scalar curvature a pure curvature bound would be of small interest. Our previously discussed generalizations of the Schrödinger–Lichnerowicz yield no immediate eigenvalue estimate. S. Goette derived in [122, Lemma 1.17] an eigenvalue estimate for normal homogeneous naturally reductive metrics, but it is also not sharp.

Remark 5.6. Since D^t is a G-invariant differential operator on M by construction, Theorem 5.6 implies that the linear combination of the first order differential operator

and the multiplication by the element of degree four in the Clifford algebra appearing in the formula for $(D^t)^2$ is again G invariant for all t. Hence, the first order differential operator

$$\widetilde{\mathcal{D}}\psi := \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \, Z_i \cdot Z_j \cdot Z_k(\psi)$$

has to be a G-invariant differential operator, a fact that cannot be seen directly by any simple arguments. It has no analogue on symmetric spaces and certainly deserves further separate investigations. Of course, it should be understood as a 'homogeneous cousin' of the more general operator $\mathcal{D}$ defined in equation (9).

Typically, the canonical connection of a naturally reductive homogeneous space M can be given an alternative geometric characterization – for example, as the unique metric connection with skew-symmetric torsion preserving a given G-structure. Once this is done, $D^{1/3}$, scal^g and $\|\mathrm{T}\|^2$ are geometrically invariant objects, whereas $\Omega_{\mathfrak{g}}$ still heavily relies on the concrete realization of the homogeneous space M as a quotient. At the same time, the same interesting G-structures exist on many non-homogeneous manifolds. Hence it was our goal to find a tool similar to $\Omega_{\mathfrak{g}}$ which has more intrinsic geometric meaning and which can be used in both situations just described [11].

We consider a Riemannian spin manifold (M^n, g, ∇) with a metric connection ∇ and skew-symmetric torsion T. Denote by Δ_T the spinor Laplacian of the connection ∇.

Definition 5.1. The *Casimir operator* of (M^n, g, ∇) is the differential operator acting on spinor fields by

$$\begin{aligned}\Omega &:= (D^{1/3})^2 + \frac{1}{8}(dT - 2\sigma_T) + \frac{1}{4}\delta(T) - \frac{1}{8}\mathrm{scal}^g - \frac{1}{16}\|\mathrm{T}\|^2 \\ &= \Delta_T + \frac{1}{8}(3\,dT - 2\sigma_T + 2\delta(T) + \mathrm{scal}).\end{aligned}$$

Remark 5.7. A naturally reductive space $M^n = G/H$ endowed with its canonical connection satisfies $dT = 2\sigma_T$ and $\delta T = 0$, hence $\Omega = \Omega_{\mathfrak{g}}$ by Theorem 5.1. For connections with $dT \neq 2\sigma_T$ and $\delta T \neq 0$, the numerical factors are chosen in such a way to yield an overall expression proportional to the scalar part of the right hand side of equation (10).

Example 5.3. For the Levi-Civita connection ($T = 0$) of an arbitrary Riemannian manifold, we obtain

$$\Omega = (D^g)^2 - \frac{1}{8}\mathrm{scal}^g = \Delta^g + \frac{1}{8}\mathrm{scal}^g.$$

The second equality is just the classical Schrödinger–Lichnerowicz formula for the Riemannian Dirac operator, whereas the first one is – in case of a symmetric space – the classical Parthasarathy formula.

Example 5.4. Consider a 3-dimensional manifold of constant scalar curvature, a constant $a \in \mathbb{R}$ and the 3-form $T = 2\,a\,dM^3$. Then

$$\Omega = (D^g)^2 - a\,D^g - \frac{1}{8}\,\mathrm{scal}^g.$$

The kernel of the Casimir operator corresponds to eigenvalues $\lambda \in \mathrm{Spec}(D^g)$ of the Riemannian Dirac operator such that

$$8\,(\lambda^2 - a\,\lambda) - \mathrm{scal}^g = 0.$$

In particular, the kernel of Ω is in general larger then the space of ∇-parallel spinors. Indeed, such spinors exist only on space forms. More generally, fix a real-valued smooth function f and consider the 3-form $T := f \cdot dM^3$. If there exists a ∇-parallel spinor

$$\nabla^g_X \psi + (X \lrcorner T) \cdot \psi = \nabla^g_X \psi + f \cdot X \cdot \psi = 0,$$

then, by a theorem of A. Lichnerowicz (see [183]), f is constant and (M^3, g) is a space form.

Let us collect some elementary properties of the Casimir operator.

Proposition 5.1 ([11, Proposition 3.1]). *The kernel of the Casimir operator contains all ∇-parallel spinors.*

Proof. By Theorem 5.1, one of the integrability conditions for a ∇-parallel spinor field ψ is

$$\big(3\,dT - 2\,\sigma_T + 2\,\delta(T) + \mathrm{scal}\big) \cdot \psi = 0. \qquad \square$$

If the torsion form T is ∇-parallel, the formulas for the Casimir operator simplify. Indeed, in this case we have (see the Appendix)

$$d\mathrm{T} = 2\,\sigma_{\mathrm{T}}, \quad \delta(\mathrm{T}) = 0,$$

and the Ricci tensor Ric of ∇ is symmetric. Using the formulas of Section 5.2 (in particular, Theorems 5.1 and 5.3), we obtain a simpler expression for the Casimir operator.

Proposition 5.2 ([11, Proposition 3.2]). *For a metric connection with parallel torsion ($\nabla T = 0$), the Casimir operator can equivalently be written as:*

$$\begin{aligned}\Omega &= (D^{1/3})^2 - \frac{1}{16}\big(2\,\mathrm{scal}^g + \|T\|^2\big)\\ &= \Delta_T + \frac{1}{16}\big(2\,\mathrm{scal}^g + \|T\|^2\big) - \frac{1}{4}\,T^2\\ &= \Delta_T + \frac{1}{8}\big(2\,dT + \mathrm{scal}\big).\end{aligned}$$

Integrating these formulas, we obtain a vanishing theorem for the kernel of the Casimir operator.

Proposition 5.3 ([11, Proposition 3.3]). *Assume that M is compact and that ∇ has parallel torsion T. If one of the conditions*

$$2\,\mathrm{scal}^g \leq -\|\mathrm{T}\|^2 \quad \text{or} \quad 2\,\mathrm{scal}^g \geq 4\,\mathrm{T}^2 - \|\mathrm{T}\|^2,$$

holds, the Casimir operator is non-negative in $\mathrm{L}^2(S)$.

Example 5.5. For a naturally reductive space $M = G/H$, the first condition can never hold, since by Lemma 5.1, $2\,\mathrm{scal}^g + \|\mathrm{T}\|^2$ is strictly positive. In concrete examples, the second condition typically singles out the normal homogeneous metrics among the naturally reductive ones.

Proposition 5.4 ([11, Proposition 3.4]). *If the torsion form is ∇-parallel, the Casimir operator Ω and the square of the Dirac operator $(D^{1/3})^2$ commute with the endomorphism* T,

$$\Omega \circ \mathrm{T} = \mathrm{T} \circ \Omega, \quad (D^{1/3})^2 \circ \mathrm{T} = \mathrm{T} \circ (D^{1/3})^2.$$

The endomorphism T acts on the spinor bundle as a symmetric endomorphism with *constant* eigenvalues.

5.4 Some remarks on the common sector of type II superstring theory

The mathematical model discussed in the common sector of type II superstring theory (also sometimes referred to as type I supergravity) consists of a Riemannian manifold (M^n, g), a metric connection ∇ with totally skew-symmetric torsion T and a non-trivial spinor field Ψ. Putting the full Ricci tensor aside for starters and assuming that the dilaton is constant, there are three equations relating these objects:

$$\nabla\Psi = 0, \quad \delta(T) = 0, \quad T \cdot \Psi = \mu \cdot \Psi. \qquad (*)$$

The spinor field describes the supersymmetry of the model. It has been our conviction throughout this chapter that this is the most important of the equations, as non-existence of ∇-parallel spinors implies the breakdown of supersymmetry. Yet, interesting things can be said if looking at all equations simultaneously. Since ∇ is a metric connection with totally skew-symmetric torsion, the divergences $\delta^\nabla(T) = \delta^g(T)$ of the torsion form coincide (see Proposition A.2). We denote this unique 2-form simply by $\delta(T)$. The third equation is an algebraic link between the torsion form T and the spinor field Ψ. Indeed, the 3-form T acts as an endomorphism in the spinor bundle and the last equation requires that Ψ is an eigenspinor for this endomorphism. Generically, $\mu = 0$ in the physical model; but the mathematical analysis becomes more transparent if we first include this parameter. A priori, μ may be an arbitrary function. Since T acts on spinors as a symmetric endomorphism, μ has to be real. Moreover, we will see that only real, constant parameters μ are possible. Recall that the conservation law $\delta(T) = 0$ implies that the Ricci tensor Ric^∇ of the connection ∇ is symmetric, see the Appendix. Denote by scal^∇ the ∇-scalar curvature and by scal^g the scalar

curvature of the Riemannian metric. Based on the results of Section 5.2, the existence of the ∇-parallel spinor field yields the so called integrability conditions, i.e., relations between μ, T and the curvature tensor of the connection ∇.

Theorem 5.7 ([14, Theorem 1.1]). *Let $(M^n, g, \nabla, T, \Psi, \mu)$ be a solution of $(*)$ and assume that the spinor field Ψ is non-trivial. Then the function μ is constant and we have*

$$\|T\|^2 = \mu^2 - \frac{\mathrm{scal}^\nabla}{2} \geq 0, \quad \mathrm{scal}^g = \frac{3}{2}\mu^2 + \frac{\mathrm{scal}^\nabla}{4}.$$

Moreover, the spinor field Ψ is an eigenspinor of the endomorphism defined by the 4-form dT,

$$dT \cdot \Psi = -\frac{\mathrm{scal}^\nabla}{2} \cdot \Psi.$$

Since μ has to be constant, equation $T \cdot \Psi = \mu \cdot \Psi$ yields:

Corollary 5.3. *For all vectors X, one has*

$$(\nabla_X T) \cdot \Psi = 0.$$

The set of equations $(*)$ is completed in the common sector of type II superstring theory by the condition $\mathrm{Ric}^\nabla = 0$ and the requirement $\mu = 0$. In [4], it had been shown that the existence of a non-trivial solution of this system implies $T = 0$ on *compact* manifolds. Theorem 5.7 enables us to prove the same result without compactness assumption and under the much weaker curvature assumption $\mathrm{scal}^\nabla = 0$:

Corollary 5.4. *Assume that there exists a spinor field $\Psi \neq 0$ satisfying the equations $(*)$. If $\mu = 0$ and $\mathrm{scal}^\nabla = 0$, the torsion form T has to vanish.*

This result underlines the strength of the algebraic identities in Theorem 5.7. Physically, this result may either show that the dilaton is a necessary ingredient for $T \neq 0$ (while it is not for $T = 0$) or that the set of equations is too restrictive (it is derived from a variational principle).

Remark 5.8. In the common sector of type II string theories, the "Bianchi identity" $dT = 0$ is often required in addition. It does not affect the mathematical structure of the equations $(*)$, hence we do not include it into our discussion.

On a naturally reductive space, even more is true. The generalized Kostant–Parthasarathy formula implies for the family of connections ∇^t:

Theorem 5.8 ([4, Theorem 4.3]). *If the operator $\Omega_{\mathfrak{g}}$ is non-negative and if ∇^t is not the Levi-Civita connection, there do not exist any non trivial solutions to the equations*

$$\nabla^t \psi = 0, \quad T^t \cdot \psi = 0.$$

The last equation in type II string theory deals with the Ricci tensor Ric^∇ of the connection. Usually one requires for constant dilaton that the Ricci tensor has to vanish (see [118]). The result above, however, indicates that this condition may be too strong. Understanding this tensor as an energy-momentum tensor, it seems to be more convenient to impose a weaker condition, namely

$$\mathrm{div}(\mathrm{Ric}^\nabla) = 0.$$

A subtle point is however the fact that there are a priori two different divergence operators. The first operator div^g is defined by the Levi-Civita connection of the Riemannian metric, while the second operator div^∇ is defined by the connection ∇. By Lemma A.2, they coincide if Ric is symmetric, that is, if $\delta T = 0$. This is for example satisfied if $\nabla T = 0$. We can then prove:

Corollary 5.5. *Let $(M^n, g, \nabla, T, \Psi, \mu)$ be a manifold with metric connection defined by T and assume that there exists a spinor $0 \neq \psi \in \Sigma M^n$ such that*

$$\nabla\psi = 0, \quad \nabla T = 0, \quad T \cdot \psi = \mu \cdot \psi.$$

Then all scalar curvatures are constant and the divergence of the Ricci tensor vanishes, $\mathrm{div}(\mathrm{Ric}^\nabla) = 0$.

This is one possible way to weaken the original set of equations in such a way that the curvature condition follows from the other ones, as it is the case for $T = 0$ – there, the existence of a ∇^g-parallel spinor implies $\mathrm{Ric}^g = 0$. Of course, only physics can provide a definite answer whether these or other possible replacements are 'the right ones'.

Incorporating a non-constant dilaton $\Phi \in C^\infty(M^n)$ is more subtle. The full set of equations reads in this case

$$\mathrm{Ric}^\nabla + \frac{1}{2}\delta T + 2\nabla^g d\Phi = 0, \quad \delta T = 2\,\mathrm{grad}(\Phi) \lrcorner T, \quad \nabla\psi = 0, \quad (2\,d\Phi - T)\cdot\psi = 0.$$

In some geometries, it is possible to interpret it as a partial conformal change of the metric. In dimension 5, this allows the proof that Φ basically has to be constant:

Theorem 5.9 ([104]). *Let $(M^5, g, \xi, \eta, \varphi)$ be a normal almost contact metric structure with Killing vector field ξ, ∇^c its characteristic connection and Φ a smooth function on M^5. If there exists a spinor field $\psi \in \Sigma M^5$ such that*

$$\nabla^c\psi = 0, \quad (2\,d\Phi - T)\cdot\psi = 0,$$

then the function Φ is constant.

In higher dimension, the picture is less clear, basically because a clean geometric interpretation of Φ is missing.

Appendix

Compilation of remarkable identities for connections with skew-symmetric torsion

We collect in this appendix some more or less technical formulas that one needs in the investigation of metric connections with skew-symmetric torsion. In order to keep this exposition readable, we decided to gather them in a separate section.

We tried to provide at least one reference with full proofs for every stated result; however, no claim is made whether these are the articles where these identities appeared for the first time. In fact, many of them have been derived and rederived by authors when needed, some had been published earlier but the authors had not considered it worth to publish a proof etc.

In this section, the connection ∇ is normalized as

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2}T(X,Y,*), \quad \nabla_X \psi = \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T)\cdot\psi.$$

It then easily follows that the Dirac operators are related by $D^\nabla = D^g + (3/4)T$.

Definition A.1. Recall that for any 3-form T, an algebraic 4-form σ_T quadratic in T may be defined by $2\,\sigma_T = \sum_{i=1}^n (e_i \lrcorner T)\wedge(e_i \lrcorner T)$, where $e_1,\dots,e_n$ denotes an orthonormal frame. Alternatively, σ_T may be written without reference to an orthonormal frame as

$$\begin{aligned}\sigma_T(X,Y,Z,V) = {} & g(T(X,Y),T(Z,V)) + g(T(Y,Z),T(X,V)) \\ & + g(T(Z,X),T(Y,V)).\end{aligned}$$

We first encountered σ_T in the first Bianchi identity for metric connections with torsion T (Theorem 2.2).

Proposition A.1 ([4, Proposition 3.1]). *Let T be a 3-form, and denote by the same symbol its associated $(2,1)$-tensor. Then its square inside the Clifford algebra has no contribution of degree 6 and 2, and its scalar and fourth degree part are given by*

$$T_0^2 = \frac{1}{6}\sum_{i,j=1}^n \|T(e_i,e_j)\|^2 =: \|T\|^2, \quad T_4^2 = -2\cdot\sigma_T.$$

Lemma A.1 ([4, Lemma 2.4]). *If ω is an r-form and ∇ any connection with torsion, then*

$$\begin{aligned}(d\omega)(X_0,\dots,X_r) = {} & \sum_{i=0}^r (-1)^i (\nabla_{X_i}\omega)(X_0,\dots,\widehat{X}_i,\dots,X_r) \\ & - \sum_{0\le i<j\le r} (-1)^{i+j}\omega(T(X_i,X_j),X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r).\end{aligned}$$

Corollary A.1 ([151]). *For a metric connection ∇ with torsion T, the exterior derivative of T is given by*

$$dT(X,Y,Z,V) = \overset{X,Y,Z}{\mathfrak{S}} [(\nabla_X T)(Y,Z,V)] - (\nabla_V T)(X,Y,Z) + 2\,\sigma_T(X,Y,Z,V).$$

In particular, $dT = 2\sigma_T$ if $\nabla T = 0$.

Proposition A.2 ([10, Proposition 5.1]). *Let ∇ be a connection with skew-symmetric torsion and define the ∇-divergence of a differential form ω as $\delta^\nabla\omega := -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i}\omega$. Then, for any exterior form ω, the following formula holds:*

$$\delta^\nabla\omega = \delta^g\omega - \frac{1}{2}\cdot\sum_{i,j=1}^n (e_i \lrcorner e_j \lrcorner T)\wedge(e_i \lrcorner e_j \lrcorner \omega).$$

In particular, for the torsion form itself, we obtain $\delta^\nabla T = \delta^g T =: \delta T$.

Corollary A.2. *If $\nabla T = 0$, its divergence vanishes, $\delta^g T = \delta^\nabla T = 0$.*

We define the divergence for a $(0,2)$-tensor S as $\mathrm{div}^\nabla(S)(X) := \sum_i (\nabla_{e_i} S)(X, e_i)$ and denote by div^g the divergence with respect to the Levi-Civita connection ∇^g. Then

$$\mathrm{div}^g(S)(X) - \mathrm{div}^\nabla(S)(X) = -\frac{1}{2}\sum_{i,j=1}^n S(e_i,\, e_j)\,\mathrm{T}(e_i,\, X,\, e_j) = 0$$

because S is symmetric while T is antisymmetric, and we conclude immediately:

Lemma A.2 ([14, Lemma 1.1]). *If ∇ is a metric connection with totally skew-symmetric torsion and S a symmetric 2-tensor, then* $\mathrm{div}^g(S) = \mathrm{div}^\nabla(S)$.

Theorem A.1 ([151]). *The Riemannian curvature quantities and the ∇-curvature quantities are related by*

$$\begin{aligned}
\mathcal{R}^g(X,Y,Z,V) &= \mathcal{R}^\nabla(X,Y,Z,V) - \frac{1}{2}(\nabla_X T)(Y,Z,V) + \frac{1}{2}(\nabla_Y T)(X,Z,V) \\
&\quad - \frac{1}{4}g(T(X,Y),T(Z,V)) - \frac{1}{4}\sigma_T(X,Y,Z,V) \\
\mathrm{Ric}^g(X,Y) &= \mathrm{Ric}^\nabla(X,Y) + \frac{1}{2}\delta T(X,Y) - \frac{1}{4}\sum_{i=1}^{\dim M} g(T(e_i,X),T(e_i,Y)) \\
\mathrm{scal}^\nabla &= \mathrm{scal}^g - \frac{3}{2}\,\|T\|^2
\end{aligned}$$

In particular, Ric^∇ *is symmetric if and only if $\delta T = 0$:* $\mathrm{Ric}^\nabla(X,Y) - \mathrm{Ric}^\nabla(Y,X) = -\delta T(X,Y)$.

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Chapter 10

Connections with totally skew-symmetric torsion and nearly-Kähler geometry

Paul-Andi Nagy

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1 Introduction

For a given Riemannian manifold (M^n, g), the holonomy group $\mathrm{Hol}(M, g)$ of the metric g contains fundamental geometric information. If the manifold is locally irreducible, as it can always be assumed locally by the well-known de Rham splitting theorem, requiring that $\mathrm{Hol}(M, g)$ is smaller that $\mathrm{SO}(n)$ has strong geometric implications at the level of the geometric structure supported by M (see [52] for an account of the classification of irreducible Riemannian holonomies). Nowadays, this has been extended to cover the classification of possible holonomy groups of torsion-free affine connections (see [9], [10], [44] for an overview).

From a somewhat different perspective, in the past years there has been increased interest in the theory of metric connections with totally skew-symmetric torsion on Riemannian manifolds. The holonomy group of such a connection is once again an insightful object which can be used to describe the geometry in many situations. One general reason supporting this fact is that the de Rham splitting theorem fails to hold in this generalised setting and the counterexamples to this extent are more intricate than just Riemannian products. Additional motivation for the study of connections in the above mentioned class comes from string theory, where this requirement is part of various models (see [1], [54] for instance).

Actually, some of these directions go back to the work of A.Gray, in the seventies, in connection to the notion of weak holonomy [31], [3], aimed to generalise that of Riemannian holonomy. The study of nearly-Kähler manifolds initiated by Gray in [30], [32], [33] has been instrumental in the theory of geometric structures of non-integrable type. As it is well known, for almost Hermitian structures a classification has been first given by Gray and Hervella in [34], which proved to be quite suitable for generalisation.

Recently, general G-structures of non-integrable type have been studied in [55], [26], [17], [25], by taking into account the algebraic properties of the various torsion modules. Using results from holonomy theory they have indicated how to read off various intersections of G-modules interesting geometric properties when in presence, say, of a connection with totally skew-symmetric torsion. Aspects of such an approach are contained in the Gray–Hervella classification and its analogues for other classical groups, we shall comment on later. Note that a central place in [17] is occupied by connections with parallel skew-symmetric torsion, studied since then by many authors.

By combining these two points of view, let us return to the Gray–Hervella classification and recall that one of its subclasses, that of nearly-Kähler structures, has a priori parallel torsion for a connection with totally skew-symmetric torsion [40]. This suggests perhaps that geometric properties of the intrinsic torsion might by obtained in more general context.

In this chapter we will at first investigate some of the properties of almost Hermitian structures in the Gray–Hervella class $W_1 + W_3 + W_4$, also referred to – as we shall in what follows – as the class $\mathcal{G}_1$. It is best described [26] as the class of almost Hermitian structures admitting a metric and Hermitian connection with totally skew-symmetric torsion. Yet, this property is equivalent [26] with requiring the Nijenhuis tensor of the almost complex structure to be a 3-form when evaluated using the Riemannian metric of the structure. There are several subclasses of interest, for example the class $W_3 + W_4$ consisting of Hermitian structures, the class W_4 of locally conformally Kähler metrics and the class W_1 of nearly-Kähler structures, which will occupy us in the second half of the exposition. One general question which arises is how far the class $W_1 + W_3 + W_4$ is, for instance, from the subclass $W_3 + W_4$ of Hermitian structures.

The chapter is organised as follows. In Section 2 we present a number of algebraic facts related to representations of the unitary group and also some background material on algebraic curvature tensors. We also briefly review some basic facts from almost-

Hermitian geometry, including a short presentation of the Hermitian connections of relevance for what follows. At the end of Section 2 we introduce the notion of Hermitian Killing form with respect to a Hermitian connection with torsion. Although this is a straightforward variation on the Riemannian notion it will serve an explanatory rôle in the next section.

Section 3 is devoted to the study of the properties of the torsion tensor of an almost Hermitian structure in the class $\mathcal{G}_1$. More precisely we prove:

Theorem 1.1. *Let (M^{2m}, g, J) be an almost-Hermitian structure of type $\mathcal{G}_1$ and let D be the unique Hermitian connection whose torsion tensor T^D belongs to $\Lambda^3(M)$. Then*

$$D_X \psi^+ = -\frac{1}{2}(X \lrcorner dT^D)_{\lambda^3}$$

for all X in TM, where ψ^+ is proportional to the Nijenhuis form of J and the subscript indicates orthogonal projection on $\lambda^3(M)$, the bundle of real valued forms of type $(0,3)+(3,0)$.

For unexplained notation and the various numerical conventions we refer the reader to Section 2. In the language of that section, this can be rephrased to say that ψ^+ is a Hermitian Killing form of type $(3,0)+(0,3)$. Theorem 1.1 is extending to the wider class of $\mathcal{G}_1$-structures the well-known [40] parallelism of the Nijenhuis tensor of a nearly-Kähler structure. We also give necessary and sufficient conditions for the parallelism of the Nijenhuis tensor, in a $\mathcal{G}_1$-context and specialise Theorem 1.1 to the subclass of $W_1 + W_4$ manifolds which has been recently subject of attention [13], [16], [47] in dimension 6. Section 3 ends with the explicit computation of the most relevant components of the curvature tensor of the connection D above, in a ready to use form.

Section 4 contains a survey of available classification results in nearly-Kähler geometry. Although this is based on [48], [49] we have adopted a slightly different approach, which emphasises the rôle of the non-Riemannian holonomy system which actually governs the way various holonomy reductions are performed geometrically. We have outlined, step by step, the procedure reducing a holonomy system of nearly-Kähler type to an embedded irreducible Riemannian holonomy system, which is the key argument in the classification result which is stated below.

Theorem 1.2 ([49]). *Let (M^{2m}, g, J) be a complete SNK-manifold. Then M is, up to finite cover, a Riemannian product whose factors belong to the following classes:*

(i) *homogeneous SNK-manifolds;*

(ii) *twistor spaces over positive quaternionic Kähler manifolds;*

(iii) 6*-dimensional SNK-manifolds.*

We have attempted to make the text as self-contained as possible, for the convenience of the reader.

2 Almost Hermitian geometry

An almost Hermitian structure (g, J) on a manifold M consists in a Riemannian metric g on M and a compatible almost complex structure J. That is, J is an endomorphism of the tangent bundle to M such that $J^2 = -1_{TM}$ and

$$g(JX, JY) = g(X, Y)$$

for all X, Y in TM. It follows that the dimension of M is even, to be denoted by $2m$ in the subsequent. Any almost Hermitian structure comes equipped with its so-called Kähler form $\omega = g(J\cdot, \cdot)$ in Λ^2. Since ω is everywhere non-degenerate the manifold M is naturally oriented by the top degree form ω^m.

We shall give now a short account on some of the $\mathrm{U}(m)$-modules of relevance for our study. Let $\Lambda^\star$ denote the space of differential p-forms on M, to be assumed real-valued, unless stated otherwise.

We consider the operator $\mathcal{J}\colon \Lambda^p \to \Lambda^p$ acting on a p-form α by

$$(\mathcal{J}\alpha)(X_1, \ldots, X_p) = \sum_{k=1}^{p} \alpha(X_1, \ldots, JX_k, \ldots, X_p)$$

for all $X_1, \ldots, X_p$ in TM. For future use let us note that alternatively

$$\mathcal{J}\alpha = \sum_{i=1}^{2m} e_i^\flat \wedge (Je_i \lrcorner \alpha)$$

for all α in $\Lambda^\star$, where $\{e_i,\ 1 \leq i \leq 2m\}$ is some local orthonormal frame on M. The almost complex structure J can also be extended to $\Lambda^\star$ by setting

$$(J\alpha)(X_1, \ldots, X_p) = \alpha(JX_1, \ldots, JX_p)$$

for all X_i, $1 \leq i \leq p$ in TM.

Let us recall that the musical isomorphism identifying vectors and 1-forms is given by $X \in TM \mapsto X^\flat = g(X, \cdot) \in \Lambda^1$. Note that in our present conventions one has $JX^\flat = -(JX)^\flat$ whenever X belongs to TM.

Now $\mathcal{J}$ acts as a derivation on $\Lambda^\star$ and gives the complex bi-grading of the exterior algebra in the following sense. Let $\lambda^{p,q}$ be given as the $-(p-q)^2$-eigenspace of $\mathcal{J}^2$. One has an orthogonal, direct sum decomposition

$$\Lambda^s = \bigoplus_{p+q=s} \lambda^{p,q},$$

called the bidegree splitting of Λ^s, $0 \leq s \leq 2m$. Note that $\lambda^{p,q} = \lambda^{q,p}$. Of special importance in our discussion are the spaces $\lambda^p = \lambda^{p,0}$; forms α in λ^p are such that the assignment

$$(X_1, \ldots, X_p) \to \alpha(JX_1, X_2, \ldots, X_p)$$

is still an alternating form which equals $p^{-1}\mathcal{J}\alpha$. We now consider the operator $L\colon \Lambda^\star \to \Lambda^\star$ given as multiplication with the Kähler form ω, together with its adjoint, to be denoted by $L^\star$. Forms in

$$\Lambda_0^\star = \operatorname{Ker} L^\star$$

are called primitive and one has

$$[L^\star, L] = (m-p)1_{\Lambda^p}$$

on the space of p-forms. Note that the inclusion $\lambda^p \subseteq \Lambda_0^p$ holds whenever $p \geq 0$.

Let $\lambda^p \otimes_1 \lambda^q$ be the space of tensors $Q\colon \lambda^p \to \lambda^q$ which satisfy

$$Q\mathbb{J} = -\mathbb{J}Q,$$

where $\mathbb{J}\colon \lambda^p \to \lambda^p$ is given by $\mathbb{J} = p^{-1}\mathcal{J}$. We also define $\lambda^p \otimes_2 \lambda^q$ as the space of tensors $Q\colon \lambda^p \to \lambda^q$ such that $Q\mathbb{J} = \mathbb{J}Q$. The following lemma is easy to verify.

Lemma 2.1. *Let $a\colon \lambda^p \otimes \lambda^q \to \Lambda^{p+q}$ be the total alternation map. Then:*

(i) *the image of the restriction of a to $\lambda^p \otimes_1 \lambda^q \to \Lambda^{p+q}$ is contained in $\lambda^{p,q}$;*

(ii) *the image of the restriction of a to $\lambda^p \otimes_2 \lambda^q \to \Lambda^{p+q}$ is contained in λ^{p+q};*

(iii) *the total alternation map $a\colon \lambda^p \otimes_1 \lambda^q \to \Lambda^{p+q}$ is injective for $p \neq q$;*

(iv) *the kernel of $a\colon \lambda^p \otimes \lambda^q \to \Lambda^{p+q}$ is contained in $\lambda^p \otimes_2 \lambda^q$.*

In this chapter most of our computations will involve forms of degree up to 4. For further use we recall that one has

$$\mathcal{J} = J$$

on $\lambda^{1,2}$ and also that

$$J_{\lambda^{2,2}\oplus\lambda^4} = 1_{\lambda^{2,2}\oplus\lambda^4}, \quad J_{\lambda^{1,3}} = -1_{\lambda^{1,3}}.$$

We will now briefly introduce a number of algebraic operators which play an important rôle in the study of connections with totally skew-symmetric torsion. If α belongs to Λ^2 and φ is in $\Lambda^\star$ we define the commutator [1]

$$[\alpha, \varphi] = \sum_{i=1}^{2m} (e_i \lrcorner \alpha) \wedge (e_i \lrcorner \varphi),$$

where $\{e_i, 1 \leq i \leq 2m\}$ is a local orthonormal frame in TM. Note that

$$[\omega, \varphi] = -\mathcal{J}\varphi$$

for all φ in $\Lambda^\star$. It is also useful to mention that

$$[\lambda^{1,1}, \lambda^3] \subseteq \lambda^3, \quad [\lambda^{1,1}, \lambda^{1,2}] \subseteq \lambda^{1,2}, \quad [\lambda^2, \lambda^3] \subseteq \lambda^{1,2}. \tag{2.1}$$

[1] This is proportional to the commutator in the Clifford algebra bundle $\mathrm{Cl}(M)$ of M.

Lastly, for any couple of forms (φ_1, φ_2) in $\Lambda^p \times \Lambda^q$ we define their product $\varphi_1 \bullet \varphi_2$ in Λ^{p+q-2} by

$$\varphi_1 \bullet \varphi_2 = \sum_{k=1}^{2m} (e_k \lrcorner \varphi_1) \wedge (e_k \lrcorner \varphi_2),$$

for some local orthonormal frame $\{e_k, 1 \leq k \leq 2m\}$ on M. We end this section by listing a few properties of the product $\bullet$ for low degree forms, to be used in what follows.

Lemma 2.2. *The following hold:*

(i) $\varphi_1 \bullet \varphi_2$ *belongs to* $\lambda^{2,2} \oplus \lambda^{1,3}$ *for all* φ_1, φ_2 *in* $\lambda^{1,2}$*;*

(ii) $\varphi \bullet J\varphi$ *is in* $\lambda^{1,3}$ *for all* φ *in* $\lambda^{1,2}$*;*

(iii) $\varphi \bullet \psi$ *belongs to* $\lambda^{1,3} \oplus \lambda^4$ *whenever* φ *is in* $\lambda^{1,2}$ *and all* ψ *in* λ^3*;*

(iv) $\psi_1 \bullet \psi_2$ *belongs to* $\lambda^{2,2}$ *for all* ψ_1, ψ_2 *in* λ^3*;*

(v) *if* ψ *is in* λ^3 *we have that* $\psi \bullet J\psi = 0$.

The proofs consist in a simple verification which is left to the reader.

2.1 Curvature tensors in the almost Hermitian setting

We shall present here a number of basic facts concerning algebraic curvature tensors we shall need later on. Since this is intended to be mainly at an algebraic level, our context will be that of a given Hermitian vector space (V^{2m}, g, J). Let us recall that the Bianchi map $b_1 \colon \Lambda^2 \otimes \Lambda^2 \to \Lambda^1 \otimes \Lambda^3$ is defined by

$$(b_1 R)_x = \sum_{i=1}^{2m} e_i^\flat \wedge R(e_i, x)$$

for all R in $\Lambda^2 \otimes \Lambda^2$ and for all x in V. The space of algebraic curvature tensors on V is given by

$$\mathcal{K}(\mathfrak{so}(2m)) = (\Lambda^2 \otimes \Lambda^2) \cap \mathrm{Ker}(b_1)$$

and it is worth observing that $\mathcal{K}(\mathfrak{so}(2m)) = S^2(\Lambda^2) \cap \mathrm{Ker}(a)$. Restricting the group to the unitary one makes appear the space of Kähler curvature tensors given by

$$\mathcal{K}(\mathfrak{u}(m)) = (\lambda^{1,1} \otimes \lambda^{1,1}) \cap \mathrm{Ker}(b_1) = S^2(\lambda^{1,1}) \cap \mathrm{Ker}(a).$$

We shall briefly present some well-known facts related to the space of Kähler curvature tensors, with proofs given mostly for the sake of completeness. First of all, we have an inclusion of $\lambda^{2,2}$ into $S^2(\lambda^{1,1})$ given by

$$\Omega \mapsto \widehat{\Omega},$$

where $\widehat{\Omega}(x,y) = -\frac{1}{4}(\Omega(x,y) + \Omega(Jx,Jy))$ for all x, y in V. This is mainly due to the fact that forms of type $(2,2)$ are J-invariant. A short computation yields

$$b_1(\widehat{\Omega}) = \Omega \tag{2.2}$$

for all Ω in $\lambda^{2,2}$. There is also an embedding of $\lambda^{1,3}$ into $\lambda^2 \otimes \lambda^{1,1}$ given by $\Omega \mapsto \breve{\Omega}$, where

$$\breve{\Omega}(x,y) = \Omega(x,y) - \Omega(Jx,Jy)$$

for all x, y in V. It is easily verified that

$$(b_1\breve{\Omega})_x = Jx \lrcorner \not{g}\Omega - 2x \lrcorner \Omega \tag{2.3}$$

for all x, y in V and for all Ω in $\lambda^{1,3}$.

Similarly to the well-known splitting $S^2(\Lambda^2) = \mathcal{K}(\mathfrak{so}(2m)) \oplus \Lambda^4$ we have:

Proposition 2.1. *There is an orthogonal, direct sum splitting*

$$S^2(\lambda^{1,1}) = \mathcal{K}(\mathfrak{u}(m)) \oplus \lambda^{2,2}.$$

Explicitly, any Q in $S^2(\lambda^{1,1})$ can be uniquely written as $Q = R + \widehat{\Omega}$ for some Kähler curvature tensor R and some Ω in $\lambda^{2,2}$.

Proof. Let Q belong to $S^2(\lambda^{1,1})$. It satisfies $Q(Jx,Jy) = Q(x,y)$ for all x, y in V and since Q belongs to $S^2(\Lambda^2)$ it splits as

$$Q = R + \Omega,$$

where R is in $\mathcal{K}(\mathfrak{so}(2m))$ and Ω belongs to Λ^4. In particular

$$R(Jx,Jy) + \Omega(Jx,Jy) = R(x,y) + \Omega(x,y) \tag{2.4}$$

for all x, y in V.

Since Q belongs to $\lambda^{1,1} \otimes \lambda^{1,1}$ we have that $(b_1 Q)x$ belongs to $\lambda^{1,2}$ for all x in V. But R is a curvature tensor, hence $(b_1 Q)x = -3x \lrcorner \Omega$, leading to $x \lrcorner \Omega$ in $\lambda^{1,2}$ for all x in V. It is then easy to see that Ω must be an element of $\lambda^{2,2}$. It follows by direct verification that R_Ω given by

$$R_\Omega(x,y) = \Omega(Jx,Jy) - \frac{1}{3}\Omega(x,y)$$

belongs to $\mathcal{K}(\mathfrak{so}(2m))$. Setting now $R_0 = R - \frac{3}{4}R_\Omega$ it is easy to see from (2.4) that R_0 belongs to $\mathcal{K}(\mathfrak{u}(m))$ and our claim follows after appropriate rescaling from

$$Q(x,y) = R_0(x,y) + \frac{3}{4}\big[\Omega(Jx,Jy) + \Omega(x,y)\big]$$

for all x, y in V. □

Our next and last goal in this section is to have an explicit splitting of certain elements of $\Lambda^2 \otimes \lambda^{1,1}$ along $\Lambda^2 = \lambda^{1,1} \oplus \lambda^2$. More explicitly, we will look at this in

the special case of a tensor $\mathcal{R}$ in $\Lambda^2 \otimes \lambda^{1,1}$ such that

$$\begin{aligned}\mathcal{R}(x,y,z,u) - \mathcal{R}(z,u,x,y) = \gamma_x(y,z,u) - \gamma_y(x,z,u) \\ - \gamma_z(u,x,y) + \gamma_u(z,x,y)\end{aligned} \tag{2.5}$$

holds for all x, y, z, u in V. Here γ belongs to $\Lambda^1 \otimes \Lambda^3$ and in what follows we shall use the notation

$$T = a(\gamma)$$

for the total alternation of γ. The tensor $\mathcal{R}$ is the algebraic model for the curvature tensor of a Hermitian connection with totally skew-symmetric torsion, which will be our object of study later on in the chapter.

Remark 2.1. A complete decomposition of $\mathcal{K}(\mathfrak{so}(2m))$ into irreducible components under the action of $\mathrm{U}(m)$ has been given by Tricerri and Vanhecke in [56]. Further information concerning the splitting of $\Lambda^2 \otimes \Lambda^2$, again under the action of $\mathrm{U}(m)$ is given in detail in [20]. While the material we shall present next can be equivalently derived from these references, it is given both for self-containedness and also as an illustration that one can directly proceed, in the case of a connection with torsion, to directly split its curvature tensor without reference to the Riemann one. As a general observation we also note that the procedure involves only control of the orthogonal projections onto the relevant $\mathrm{U}(m)$-submodules.

Recall that for any α in Λ^2 its orthogonal projection on $\lambda^{1,1}$ is given by

$$\alpha_{\lambda^{1,1}} = \frac{1}{2}(\alpha + J\alpha).$$

To obtain the decomposition of $\mathcal{R}$ we need the following preliminary lemma.

Lemma 2.3. *Let η in $\lambda^1 \otimes \lambda^{1,2}$ be given. We consider the tensors H_1, H_2 in $\Lambda^2 \otimes \lambda^{1,1}$ given by*

$$H_1(x,y) = (x \lrcorner \eta_y - y \lrcorner \eta_x)_{\lambda^{1,1}} \quad \textit{and} \quad H_2(x,y) = H_2(Jx, Jy)$$

for all x, y in V. Then:

(i) $2(b_1 H_1)_x = 3\eta_x + J\eta_{Jx} + x \lrcorner a(\eta) - Jx \lrcorner a(J\eta)$;

(ii) $2(b_1 H_2)_x = \eta_x + 3J\eta_{Jx} + Jx \lrcorner a^c(\eta) + x \lrcorner a^c(J\eta)$

for all x in V. Here $J\eta$ in $\lambda^1 \otimes \lambda^{1,2}$ is defined by $(J\eta)_x = J\eta_x$ for all x in V, and the complex alternation map $a^c : \lambda^1 \otimes \lambda^{1,2} \to \Lambda^4$ is given by

$$a^c(\zeta) = \sum_{k=1}^{2m} e_k^\flat \wedge \eta_{Je_k}.$$

Proof. (i) Let $\{e_k, 1 \leq k \leq 2m\}$ be an orthonormal basis in V. Directly from its definition, the tensor H_1 is given by

$$2H_1(x,y) = x \lrcorner \eta_y - Jx \lrcorner J\eta_y - y \lrcorner \eta_x + Jy \lrcorner J\eta_x,$$

hence

$$2(b_1 H_1)_x = 4\eta_x - \sum_{k=1}^{2m} e_k^\flat \wedge (x \lrcorner \eta_{e_k}) + \sum_{k=1}^{2m} e_k^\flat \wedge (Jx \lrcorner J\eta_{e_k})$$

since $\mathcal{J} = J$ on $\lambda^{1,2}$. Now

$$x \lrcorner a(\eta) = \eta_x - \sum_{k=1}^{2m} e_k^\flat \wedge (x \lrcorner \eta_{e_k})$$

and

$$Jx \lrcorner a(J\eta) = J\eta_{Jx} - \sum_{k=1}^{2m} e_k^\flat \wedge (Jx \lrcorner J\eta_{e_k})$$

for all x in V and the claim follows immediately.

(ii) It is enough to use (i) when replacing η by $J\eta$. □

We split now

$$\gamma = \gamma^{1,2} + \gamma^3$$

along $\Lambda^1 \otimes \Lambda^3 = (\Lambda^1 \otimes \lambda^{1,2}) \oplus (\Lambda^1 \otimes \lambda^3)$ and also

$$T = T^{2,2} + T^{1,3} + T^4$$

along the bidegree decomposition of Λ^4 in order to be able to introduce some of the components of the tensor $\mathcal{R}$. These are the tensor $\mathcal{R}^a$ in $\lambda^{1,1} \otimes \lambda^{1,1}$ given by

$$\mathcal{R}^a(x,y) = (\gamma_x y - \gamma_y x)_{\lambda^{1,1}} + (\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}} - \frac{1}{2}(T^{2,2}(x,y) + T^{2,2}(Jx,Jy))$$

for all x, y in V, and the tensor $\mathcal{R}^m$ in $\lambda^2 \otimes \lambda^{1,1}$ defined by

$$\mathcal{R}^m(x,y) = (\gamma_x y - \gamma_y x)_{\lambda^{1,1}} - (\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}} - \frac{1}{2}(T^{1,3}(x,y) - T^{1,3}(Jx,Jy))$$

for all x, y in V. Here we use $\gamma_x y$ as a shorthand for $y \lrcorner \gamma_x$, whenever x, y are in V.

The promised decomposition result for $\mathcal{R}$ is achieved mainly by projection of (3.5) onto

$$\Lambda^2 \otimes \lambda^{1,1} = (\lambda^{1,1} \otimes \lambda^{1,1}) \oplus (\lambda^{1,1} \otimes \lambda^2)$$

while taking into account that $\mathcal{R}$ belongs to $\Lambda^2 \otimes \lambda^{1,1}$.

Theorem 2.1. *Let $\mathcal{R}$ belong to $\Lambda^2 \otimes \lambda^{1,1}$ such that (2.5) is satisfied. We have a decomposition*

$$\mathcal{R} = \mathcal{R}^K + \widehat{\Omega} + \frac{1}{2}\mathcal{R}^a + \mathcal{R}^m,$$

where $\mathcal{R}^K$ belongs to $\mathcal{K}(\mathfrak{u}(m))$ and Ω is in $\lambda^{2,2}$. Moreover:

(i) *R^a belongs to $\Lambda^2(\lambda^{1,1})$ and satisfies the Bianchi identity*

$$(b_1 R^a)_x = -2(\gamma_x^{1,2} + J\gamma_{Jx}^{1,2}) - \frac{1}{2}x \lrcorner A_1 - \frac{1}{2}Jx \lrcorner A_2$$

for all x in V. The forms A_1 and A_2 are explicitly given by

$$A_1 = a(\gamma^{1,2}) + a^c(J\gamma^{1,2}) - 4T^{2,2},$$
$$A_2 = a^c(\gamma^{1,2}) - a(J\gamma^{1,2}).$$

(ii) *The Bianchi identity for R^m in $\lambda^2 \otimes \lambda^{1,1}$ is*

$$(b_1 R^m)_x = -(\gamma_x^{1,2} - J\gamma_{Jx}^{1,2}) - \frac{1}{2}x \lrcorner A_3 + \frac{1}{2}Jx \lrcorner A_4$$

for all x in V, where

$$A_3 = a(\gamma^{1,2}) - a^c(J\gamma^{1,2}) - 2T^{1,3},$$
$$A_4 = a(J\gamma^{1,2}) + a^c(\gamma^{1,2}) - \mathcal{J}T^{1,3}.$$

Proof. We first show that R^a belongs to $\Lambda^2(\lambda^{1,1})$. Directly from the definition of the map a we have that the tensor

$$Q(x, y) := \gamma_x y - \gamma_y x - \frac{1}{2}T(x, y)$$

belongs to $\Lambda^2(\Lambda^2)$, so after projection on $\lambda^{1,1} \otimes \lambda^{1,1}$ it follows that

$$(\gamma_x y - \gamma_y x)_{\lambda^{1,1}} + (\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}} - \frac{1}{2}(T(x, y) + T(Jx, Jy))_{\lambda^{1,1}}$$

is an element of $\Lambda^2(\lambda^{1,1})$. We conclude by recording that

$$(T(x, y) + T(Jx, Jy))_{\lambda^{1,1}} = T^{2,2}(x, y) + T^{2,2}(Jx, Jy)$$

since forms in $\lambda^{1,3}$ are J-anti-invariant whilst those in $\lambda^{2,2} \oplus \lambda^4$ are J-invariant and moreover any φ in λ^4 satisfies $\varphi(J\cdot, J\cdot, \cdot, \cdot) = -\varphi(\cdot, \cdot, \cdot, \cdot)$.

The next step is to notice that (2.5) actually says that $\mathcal{R} - Q$ belongs to $S^2(\Lambda^2)$ and to project again on $\lambda^{1,1} \otimes \lambda^{1,1}$. By the argument above it follows that

$$\mathcal{R}(x, y) + \mathcal{R}(Jx, Jy) - R^a(x, y)$$

is in $S^2(\lambda^{1,1})$, thus by Proposition 2.1 we can write

$$\mathcal{R}(x, y) + \mathcal{R}(Jx, Jy) - R^a(x, y) = 2R^K(x, y) + 2\widehat{\Omega}(x, y) \tag{2.6}$$

for all x, y in V, where Ω is in $\lambda^{2,2}$ and R^K belongs to $\mathcal{K}(\mathfrak{u}(m))$.

Now, rewriting (2.5) as

$$\mathcal{R}(x, y, z, u) - \mathcal{R}(z, u, x, y) = 2(\gamma_x(y, z, u) - \gamma_y(x, z, u)) - T(x, y, z, u)$$

we obtain after projection on $\lambda^2 \otimes \lambda^{1,1}$ that

$$\mathcal{R}(x,y) - \mathcal{R}(Jx,Jy) = 2(\gamma_x y - \gamma_y x)_{\lambda^{1,1}} - 2(\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}} - (T(x,y) - T(Jx,Jy))_{\lambda^{1,1}}$$

for all x, y in V. Since $\lambda^4 \oplus \lambda^{2,2}$ consists in J-invariant forms, it is easy to see that

$$(T(x,y) - T(Jx,Jy))_{\lambda^{1,1}} = T^{1,3}(x,y) - T^{1,3}(Jx,Jy),$$

in other words

$$\frac{1}{2}(\mathcal{R}(x,y) - \mathcal{R}(Jx,Jy)) = \mathcal{R}^m(x,y) \tag{2.7}$$

for all x, y in V. The splitting of $\mathcal{R}$ follows now from (2.6) and (2.7) and from the fact that the component of γ on $\lambda^1 \otimes \lambda^3$ is not seen by the projection on $\lambda^{1,1}$.

To finish the proof, it remains only to prove the Bianchi identities for $\mathcal{R}^a$ and $\mathcal{R}^m$. Using Lemma 2.3 and (2.2) it is easy to get that

$$(b_1 R^a)_x = -2(\gamma_x^{1,2} + J\gamma_{Jx}^{1,2}) + 2x \lrcorner T^{2,2} - \frac{1}{2} x \lrcorner (a(\gamma^{1,2}) + a^c(J\gamma^{1,2})) - \frac{1}{2} Jx \lrcorner (a^c(\gamma^{1,2}) - a(J\gamma^{1,2}))$$

for all x in V. The Bianchi identity for $\mathcal{R}^m$ is proved along the same lines and therefore left to the reader. □

Remark 2.2. Underlying Theorem 2.1 are the following isomorphisms of $\mathfrak{u}(m)$-modules. The first is

$$b_1 \colon \Lambda^2(\lambda^{1,1}) \to (\lambda^1 \otimes_1 \lambda^{1,2}) \cap \mathrm{Ker}(a),$$

where

$$\lambda^1 \otimes_1 \lambda^{1,2} = \{\zeta \in \lambda^1 \otimes \lambda^{1,2} : \zeta_{Jx} = -J\zeta_x \text{ for all } x \in V\}.$$

The second is given by

$$b_1 \colon \lambda^2 \otimes \lambda^{1,1} \to \lambda^1 \otimes_2 \lambda^{1,2},$$

where $\lambda^1 \otimes_2 \lambda^{1,2} = \{\zeta \in \lambda^1 \otimes \lambda^{1,2} : \zeta_{Jx} = J\zeta_x \text{ for all } x \in V\}$.

This makes that in practice it is not necessary to work with the somewhat involved expressions for R^a, R^m but rather with their Bianchi contractions, which are tractable.

2.2 The Nijenhuis tensor and Hermitian connections

Let (M^{2m}, g, J) be an almost Hermitian manifold. Recall that the Nijenhuis tensor of the almost complex structure J is defined by

$$N_J(X,Y) = [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY]$$

for all vector fields X, Y on M. It satisfies

$$N_J(X,Y) + N_J(Y,X) = 0,$$
$$N_J(JX,JY) = -N_J(X,Y),$$
$$N_J(JX,Y) = -JN_J(X,Y)$$

for all X, Y in TM. By evaluating N_J using the Riemannian metric g we can also form the tensor N^J defined by

$$N^J_X(Y,Z) = g(N_J(Y,Z),X)$$

for all X, Y, Z in TM, which therefore belongs to $\lambda^1 \otimes_2 \lambda^2$. When N_J vanishes identically, J is said to be integrable and gives M the structure of a complex manifold.

Denoting by ∇ the Levi-Civita connection attached to the Riemannian metric g we form the tensor ∇J in $\lambda^1 \otimes \lambda^2$. Then one can alternatively compute the Nijenhuis tensor as

$$N_J(X,Y) = -\big[(\nabla_{JX}J)Y - (\nabla_{JY}J)X\big] + J\big[(\nabla_X J)Y - (\nabla_Y J)X\big]. \tag{2.8}$$

It is now a good moment to recall that an almost Hermitian structure (g,J) is called Kähler if and only if $\nabla J = 0$, or equivalently $\nabla\omega = 0$. This implies the integrability of J by making use of (2.8) and also that ω is a symplectic form, in the sense that $d\omega = 0$ where d denotes the exterior derivative. It turns out that for an arbitrary almost complex structure (g,J), the tensors $d\omega$ in Λ^3 and N^J in $\lambda^1 \otimes_2 \lambda^2$ form a full set of obstructions to having (g,J) Kähler as the following general fact shows.

Proposition 2.2. *For any almost Hermitian structure (g,J) on M we have*

$$2\nabla_X\omega = -N^J_{JX} + X \lrcorner d\omega + JX \lrcorner Jd\omega \tag{2.9}$$

for all X in TM.

Proof. Although this is a standard fact we give the proof for self-containedness. From the definition of the exterior derivative we have

$$d\omega(X,Y,Z) = (\nabla_X\omega)(Y,Z) - (\nabla_Y\omega)(X,Z) + (\nabla_Z\omega)(X,Y)$$

for all X, Y, Z in TM. Since $\nabla_X\omega$ is in λ^2 for all X in TM we obtain

$$\begin{aligned} &d\omega(X,Y,Z) - d\omega(JX,JY,Z) \\ &\quad = (\nabla_X\omega)(Y,Z) - (\nabla_Y\omega)(X,Z) \\ &\qquad - (\nabla_{JX}\omega)(JY,Z) + (\nabla_{JY}\omega)(JX,Z) + 2(\nabla_Z\omega)(X,Y) \end{aligned}$$

for all X, Y, Z in TM. This can be rewritten by using (2.8) as

$$d\omega(X,Y,Z) - d\omega(JX,JY,Z) = \langle N_J(X,Y), JZ\rangle + 2(\nabla_Z\omega)(X,Y)$$

whenever X, Y, Z belong to TM, and the claim follows. □

Therefore $d\omega = 0$ and $N_J = 0$ result in $\nabla J = 0$, in other words in (g,J) being Kähler.

We shall call a linear connection on the tangent bundle to M Hermitian if it respects both the metric and the almost complex structure. In the framework of almost Hermitian geometry two connections play a distinguished rôle. The first is called the first canonical Hermitian connection and it is defined by

$$\overline{\nabla}_X = \nabla_X + \eta_X,$$

where η in $\lambda^1 \otimes \lambda^2$ is given by $\eta_X = \frac{1}{2}(\nabla_X J)J$. The tensor η is called the intrinsic torsion tensor of the $\mathrm{U}(m)$-structure on M induced by (g, J). The connection $\overline{\nabla}$ is minimal [28], in the sense of minimising the norm within the space of almost Hermitian connections.

To introduce the second Hermitian connection we need some preliminaries. First of all let us decompose the 3-form $d\omega = d^{1,2}\omega + d^3\omega$ along the splitting $\Lambda^3 = \lambda^{1,2} \oplus \lambda^3$. Then:

Lemma 2.4. *Let (M^{2m}, g, J) be almost Hermitian. Then $a(N^J) = 4Jd^3\omega$.*

Proof. Using (2.9) we obtain

$$d\omega = a(\nabla\omega) = -\sum_{i=1}^{2m} e_i^\flat \wedge N^J_{Je_i} + 3d\omega + \mathcal{J}(Jd\omega),$$

where $\{e_i,\ 1 \leq i \leq 2n\}$ is some local orthonormal frame. Since N^J belongs to $\lambda^1 \otimes_2 \lambda^2$ it is easy to see that $\mathcal{J}(a(N^J)) = 3\sum_{i=1}^{2m} e_i^\flat \wedge N^J_{Je_i}$ and the claim follows by taking into account that $\mathcal{J} = J$ on $\lambda^{1,2}$. □

Since $\lambda^3 \subseteq \lambda^1 \otimes_2 \lambda^3$ it follows that the Nijenhuis tensor splits as

$$N^J_X = \hat{N}^J_X + \frac{4}{3} X \lrcorner Jd^3\omega$$

for all X in TM, where the tensor $\hat{N}^J$ belongs to the irreducible $\mathrm{U}(m)$-module

$$W_2 = (\lambda^1 \otimes_2 \lambda^2) \cap \mathrm{Ker}(a).$$

Proposition 2.3. *Let (M^{2m}, g, J) be almost Hermitian. The linear connection D defined by $D_X = \nabla_X + \zeta_X$, where*

$$2\zeta_X = X \lrcorner Jd^{1,2}\omega - \frac{1}{3} X \lrcorner Jd^3\omega + \frac{1}{2}\hat{N}^J_X,$$

is Hermitian.

Proof. That D is metric is clear since ζ_X is a two form for all X in TM. To see that $D\omega = 0$ it is enough to show that $\nabla_X\omega + [\zeta_X, \omega] = 0$ or equivalently

$$\nabla_X\omega + \mathcal{J}\zeta_X = 0$$

for all X in TM. But

$$2\mathcal{J}\zeta_X = \mathcal{J}(X \lrcorner Jd^{1,2}\omega) - \frac{1}{3}\mathcal{J}(X \lrcorner Jd^3\omega) + \frac{1}{2}\mathcal{J}\widehat{N}^J_X$$
$$= -X \lrcorner d^{1,2}\omega - JX \lrcorner Jd^{1,2}\omega - \frac{2}{3}X \lrcorner d^3\omega + \widehat{N}^J_{JX}$$

after taking into account that $\widehat{N}^J$ belongs to $\lambda^1 \otimes_2 \lambda^2$. Therefore, after taking into account (2.9),

$$2\nabla_X\omega + 2\mathcal{J}\zeta_X = -\left[\widehat{N}^J_{JX} + \frac{4}{3}JX \lrcorner Jd^3\omega\right] + X \lrcorner d\omega + JX \lrcorner Jd\omega$$
$$- X \lrcorner d^{1,2}\omega - JX \lrcorner Jd^{1,2}\omega - \frac{2}{3}X \lrcorner d^3\omega + \widehat{N}^J_{JX} = 0$$

after a straightforward computation, and the result follows. □

It follows from Proposition 2.2 and Lemma 2.4 that the intrinsic torsion tensor of the almost Hermitian structure (g, J) is completely determined by

$$(d\omega, \widehat{N}^J) \text{ in } \Lambda^3 \oplus W_2.$$

Since $\Lambda^3 \oplus W_2$ has four irreducible components under the action of $\mathrm{U}(m)$ the Gray–Hervella classification [34] singles out 16-classes of almost Hermitian manifolds. Similar classification results are available for the groups G_2 [22], [16] and Spin(7) [21] as well as for quaternionic structures [41] and SU(3)-structures on 6-dimensional manifolds [14]. The case of Spin(9)-structures on 16-dimensional manifolds has been equally treated in [24]. For PSU(3)-structures on 8-dimensional manifolds and SO(3)-structures in dimension 5 the decomposition of the intrinsic torsion tensor has been studied in [37], [58] and [8].

2.3 Admissible totally skew-symmetric torsion

In this section we shall start to specialise our discussion to a particular class of almost Hermitian manifolds, to be characterized in terms of the torsion tensor of the Hermitian connection D. We recall that the torsion tensor of a linear connection on M, e.g. D, is the tensor T^D in $\Lambda^2 \otimes \Lambda^1$ given by

$$T^D(X, Y) = D_X Y - D_Y X - [X, Y]$$

for all vector fields X, Y on M. In the case of the connection $\overline{\nabla}$, the torsion will be denoted simply by T.

The following result of Friedrich and Ivanov clarifies in which circumstances an almost Hermitian structure admits a Hermitian connection with totally skew-symmetric torsion.

Theorem 2.2 ([26]). *Let (M^{2m}, g, J) be almost Hermitian. There exists a Hermitian connection with torsion in Λ^3 if and only if N^J is a 3-form. If the latter holds, the connection is unique and equals D.*

Almost Hermitian manifolds with totally skew-symmetric Nijenhuis tensor form the so-called Gray–Hervella class $W_1 + W_3 + W_4$ and are usually called $\mathcal{G}_1$ almost Hermitian structures. As it follows from the discussion above an almost Hermitian structure (g, J) belongs to the class $\mathcal{G}_1$ if and only if $\widehat{N}^J = 0$. Because of its uniqueness, the connection D in Theorem 2.2 will be referred to as the characteristic connection of the $\mathcal{G}_1$-manifold (M^{2m}, g, J).

In the remaining part of this section we will work on a given $\mathcal{G}_1$-manifold (M^{2m}, g, J) and we will derive a number of facts to be used later on. We have

$$N^J = \frac{4}{3} J d^3\omega,$$

$$2\zeta = T^D = J d^{1,2}\omega - \frac{1}{3} J d^3\omega.$$

It is also worth noting that (2.9) is then updated to

$$2\nabla_X \omega = X \lrcorner d^{1,2}\omega + JX \lrcorner J d^{1,2}\omega + \frac{2}{3} X \lrcorner d^3\omega$$

for all X in TM. Under a shorter and perhaps more tractable form this reads

$$\nabla_X \omega = X \lrcorner t + JX \lrcorner Jt + X \lrcorner \psi^+ \tag{2.10}$$

for all X in TM, where the 3-forms t in $\lambda^{1,2}$ and ψ^+ in λ^3 are given by

$$t = \frac{1}{2} d^{1,2}\omega, \quad \psi^+ = \frac{1}{3} d^3\omega.$$

In the same spirit we can also re-express $d\omega$ and the torsion form as

$$d\omega = 2t + 3\psi^+ \tag{2.11}$$

and

$$T^D = 2Jt + \psi^-, \tag{2.12}$$

where the 3-form ψ^- in λ^3 is given by $\psi^- = \psi^+(J\cdot, \cdot, \cdot)$.

Remark 2.3. (i) When $m = 2$ any almost Hermitian structure of class $\mathcal{G}_1$ is automatically Hermitian, that is $N_J = 0$. This is due to the vanishing of λ^3 in dimension 4. In what follows we shall therefore assume that $m \geq 3$.

(ii) From (2.10) it is easy to see that almost Hermitian structures in the class $\mathcal{G}_1$ are alternatively described as those satisfying $(\nabla_{JX} J)JX = (\nabla_X J)X$ for all X in TM.

When J is integrable it is easy to see that $T^D = 2Jd\omega$ belongs to Λ^3 (actually $\lambda^{1,2}$ after taking into account Lemma 2.4). In this case the connection D is referred to as the Bismut connection [7].

Almost quaternionic or almost hyper-Hermitian structures admitting a structure preserving connection with totally skew-symmetric torsion have been introduced and given various characterisations in [38], [29], [39], [43]. The resulting geometries are known under the names QKT (quaternion-Kähler with torsion) and HKT (hyper-Kähler with torsion).

2.4 Hermitian Killing forms

In this section, briefly recalling the definition of Killing forms in Riemannian geometry we shall present a variation of that notion, more suitable to the almost Hermitian setting. This is done to prepare the ground to present, after proving the results in the next section, the special relationship binding together almost Hermitian structure of type $\mathcal{G}_1$ and this kind of generalised Killing form.

Let (M^{2m}, g, J) be almost Hermitian in the class $\mathcal{G}_1$. Using the metric connection D we can form the associated exterior differential d_D. It is given as in the case of the usual exterior derivative d by

$$d_D = \sum_{i=1}^{2m} e_i^\flat \wedge D_{e_i}.$$

Given that D is a Hermitian connection it is easy to see that

$$[\mathcal{J}, d_D] = (-1)^p J d_D J \tag{2.13}$$

holds on Λ^p. Its dual reads

$$[\mathcal{J}, d_D^\star] = (-1)^p J d_D^\star J \tag{2.14}$$

on Λ^p, where $d_D^\star \colon \Lambda^\star \to \Lambda^\star$ is the formal adjoint of d_D. Note that $d_D^\star$ is computed at a point m of M by

$$d_D^\star = -\sum_{i=1}^{2m} e_i \lrcorner D_{e_i},$$

where $\{e_i, 1 \leq i \leq 2m\}$ is a local frame around m, geodesic at m w.r.t. the connection D. A straightforward implication of (2.13) is that

$$d_D \varphi \in \lambda^{p+1,q} \oplus \lambda^{p,q+1}$$

for all φ in $\lambda^{p,q}$. Denoting by $\varphi_{\lambda^{p,q}}$ the orthogonal projection of φ in $\Lambda^\star$ on $\lambda^{p,q}$ we split

$$d_D = \partial_D + \bar{\partial}_D,$$

where $\partial_D \varphi = (d_D \varphi)_{\lambda^{p+1,q}}$ and $\bar{\partial}_D \varphi = (d_D \varphi)_{\lambda^{p,q+1}}$ for all φ in $\lambda^{p,q}$.

Lastly, we mention that the Kähler identities

$$[d_D, L^\star] = (-1)^p J d_D^\star J \quad \text{and} \quad [d_D^\star, L] = (-1)^{p+1} J d_D J \tag{2.15}$$

hold whenever φ belongs to Λ^p. We can now make the following

Definition 2.1. A form φ in $\lambda^{p,q}$ is a Hermitian Killing form if and only if

$$D_X\varphi = (X \lrcorner A)_{\lambda^{p,q}}$$

for some form A in $\lambda^{p,q+1} \oplus \lambda^{p+1,q}$ and for all X in TM.

This is much in analogy with the concept of twistor form from Riemannian geometry, see [53] for details. We shall just recall that that a differential form φ in $\Lambda^p(N)$, where (N^n, h) is some Riemannian manifold, is called a twistor form or conformal Killing form if

$$\nabla_X\varphi = \frac{1}{p+1} X \lrcorner d\varphi + \frac{1}{n-p+1} X^\flat \wedge d^\star\varphi$$

holds, for all X in TM. Moreover φ is called a Killing form if it also coclosed, that is $d^\star\varphi = 0$.

Remark 2.4. Hermitian Killing 1-forms are dual to Killing vector fields for the metric g. This is essentially due to the fact that D has totally skew-symmetric torsion.

In the rest of this section we shall make a number of elementary observations on Hermitian Killing forms in $\lambda^\star$, the most relevant case for our aims.

Proposition 2.4. *Let φ in λ^p, $p \geq 2$ be a Hermitian Killing form. The following hold:*

(i) $D_X\varphi = (X \lrcorner A)_{\lambda^p}$ *for all X in TM, where $A = \partial\varphi + \frac{1}{p+1}\bar{\partial}\varphi$;*

(ii) $\mathcal{J}\varphi$ *is a Hermitian Killing form.*

Proof. (i) From the definition we have

$$D_X\varphi = (X \lrcorner A)_{\lambda^p}$$

for all X in TM, where A is in $\lambda^{p,1} \oplus \lambda^{p+1}$. We split $A = B + C$ where B and C belong to $\lambda^{p,1}$ and λ^{p+1} respectively. Since for all X in TM we have

$$(X \lrcorner B)_{\lambda^p} = \frac{1}{2(p-1)}(JX \lrcorner \mathcal{J}B + (p-1)X \lrcorner B),$$
$$(X \lrcorner C)_{\lambda^p} = X \lrcorner C,$$

we find, after taking the appropriate contractions that

$$d_D\varphi = B + (p+1)C$$

and the claim follows.

(ii) After applying $\mathcal{J}$ to the Hermitian Killing equation satisfied by φ, it is enough to notice that

$$\mathcal{J}(X \lrcorner B)_{\lambda^p} = \frac{p}{p-1}(X \lrcorner \mathcal{J}B)_{\lambda^p}$$

holds, together with

$$\mathcal{J}(X \lrcorner C) = \frac{p}{p+1} X \lrcorner \mathcal{J}C$$

for all X in TM. □

Note that, unlike Riemannian Killing forms, Hermitian Killing ones need not be necessarily coclosed. This can be easily verified by means of (2.14) and (2.15). The following identity is useful in order to better understand the notion of Hermitian Killing form.

Lemma 2.5. *Let φ in λ^p, $p \geq 1$ be given. Then:*

$$D_{JX}\varphi - D_X \mathbb{J}\varphi = -2(X \lrcorner \partial_D \mathbb{J}\varphi)_{\lambda^p}$$

for all X in TM.

Proof. Let us consider the tensor γ in $\lambda^1 \otimes_1 \lambda^p$ given by

$$\gamma_X = D_{JX}\varphi - D_X \mathbb{J}\varphi$$

for all X in TM. For notational convenience we define $B = \partial_D \mathbb{J}\varphi$ in $\lambda^{1,p}$ and note, as in the proof of Proposition 2.4 that $X \in TM \mapsto (X \lrcorner B)_{\lambda^p}$ belongs to $\lambda^1 \otimes_1 \lambda^p$ as well. Now a straightforward calculation shows that

$$a(X \in TM \mapsto (X \lrcorner B)_{\lambda^p}) = B.$$

At the same time one has

$$a(\gamma) = -2\partial_D \mathbb{J}\varphi$$

hence $\gamma_X = -2(X \lrcorner \partial_D \mathbb{J}\varphi)_{\lambda^p}$ for all X in TM by making use of Lemma 2.1 (iii) when $p \neq 1$. When $p = 1$ the claim follows by a simple direct verification which is left to the reader. □

Let us now give an equivalent characterisation of Hermitian Killing forms.

Proposition 2.5. *The following hold:*

(i) *a form φ in λ^p, $p \geq 2$ is a Hermitian Killing form if and only if the component of $D\varphi$ on $\lambda^1 \otimes_2 \lambda^p$ is determined by $\bar{\partial}_D \mathbb{J}\varphi$, that is*

$$D_{JX}\varphi + D_X \mathbb{J}\varphi = \frac{2}{p+1} X \lrcorner \bar{\partial}_D \mathbb{J}\varphi$$

for all X in TM;

(ii) *a form φ in λ^m is a Hermitian Killing form if and only if*

$$|\varphi| D\varphi = \frac{1}{2} d|\varphi|^2 \otimes \varphi + \frac{1}{2} J d|\varphi|^2 \otimes \mathbb{J}\varphi.$$

Proof. (i) We have

$$D_X \mathbb{J}\varphi = \frac{1}{2}(D_{JX}\varphi + D_X \mathbb{J}\varphi) - \frac{1}{2}(D_{JX}\varphi - D_X \mathbb{J}\varphi)$$
$$= (X \lrcorner \partial_D \varphi)_{\lambda^3} + \frac{1}{2}(D_{JX}\varphi + D_X \mathbb{J}\varphi)$$

for all X in TM, and the claim follows immediately from (i) in Proposition 2.4.

(ii) If φ is in λ^m, we have that $\bar{\partial}_D(\mathbb{J}\varphi) = 0$ since $\lambda^{m+1} = 0$. From (i) we know that φ is a Hermitian Killing form if and only if

$$D_{JX}\varphi + D_X \mathbb{J}\varphi = 0 \tag{2.16}$$

holds for any X in TM. Let U be the open subset of M given by $U = \{m \in M : \varphi_m \neq 0\}$ and let Z be the complement of U in M. We claim that (2.16) holds iff it holds on U. Indeed (2.16) holds trivially on $\mathrm{int}(Z)$ and $U \cup \mathrm{int}(Z)$ is dense in M.

Suppose now that φ in λ^m is a Hermitian Killing form. Then $\varphi, \mathbb{J}\varphi$ is a basis of $\lambda^m_{|U}$, hence

$$D\varphi = a \otimes \varphi + b \otimes \mathbb{J}\varphi$$

for a couple of 1-forms a, b on U. The first is determined by $|\varphi|a = \frac{1}{2} d|\varphi|^2$. On the other hand the fact that φ is a Hermitian form implies that $b = Ja$, hence our claim is proved on U and by the density argument above on M. The converse statement is also clear from the previous observations. □

An immediate consequence of (ii) in the proposition above is that a Hermitian Killing form in λ^m is *parallel* with respect to the connection D as soon as it has constant length. It is now a good moment to provide some examples of Hermitian Killing forms.

Proposition 2.6. *Let (M^{2m}, g, J) be a Kähler manifold and let ζ_k, $k = 1, 2$ be holomorphic Killing vector fields, that is*

$$L_{\zeta_k} g = 0 \quad \textit{and} \quad L_{\zeta_k} J = 0 \quad \textit{for } k = 1, 2.$$

The form $\varphi = (\zeta_1^\flat \wedge \zeta_2^\flat)_{\lambda^2}$ is a Hermitian Killing form.

Proof. The Killing equation yields

$$\nabla_X \zeta_k^\flat = \frac{1}{2} X \lrcorner d\zeta_k^\flat$$

for all X in TM and $k = 1, 2$. After a few manipulations we get

$$\nabla_X(\zeta_1^\flat \wedge \zeta_2^\flat) = \frac{1}{2}(X \lrcorner d\zeta_1^\flat) \wedge \zeta_2^\flat + \frac{1}{2}\zeta_1^\flat \wedge (X \lrcorner d\zeta_2^\flat)$$
$$= \frac{1}{2} X \lrcorner d(\zeta_1^\flat \wedge \zeta_2^\flat) - \frac{1}{2}\langle \zeta_2, X\rangle d\zeta_1^\flat + \frac{1}{2}\langle \zeta_1, X\rangle d\zeta_2^\flat$$

for all X in TM. Since our given vector fields are also holomorphic, the forms $d\zeta_k^\flat$, $k = 1, 2$ belong to $\lambda^{1,1}$ and the claim is proved by projecting onto λ^2 while using the Definition 2.1. We note that $\bar{\partial}\varphi = 0$ because one has, as is well known, $d(J\zeta_k^\flat) = 0$ for $k = 1, 2$. □

We continue by giving an exterior algebra characterisation of Hermitian Killing forms, in the context of a $\mathcal{G}_1$-manifold (M^{2m}, g, J).

Proposition 2.7. *Let $p \geq 3$ be odd. Then:*

(i) *if φ in λ^p is a Hermitian Killing form, then*

$$d_D^\star(\varphi \wedge \mathbb{J}\varphi) = (d_D^\star\varphi) \wedge \mathbb{J}\varphi - \varphi \wedge d_D^\star \mathbb{J}\varphi - \frac{2}{p+1}\varphi \bullet \bar{\partial}_D \mathbb{J}\varphi;$$

(ii) *if φ in λ^p satisfies*

$$d_D^\star(\varphi \wedge \mathbb{J}\varphi) = (d_D^\star\varphi) \wedge \mathbb{J}\varphi - \varphi \wedge d_D^\star \mathbb{J}\varphi - \frac{2}{p+1}\varphi \bullet \bar{\partial}_D \mathbb{J}\varphi$$

and it is everywhere nondegenerate, then φ is a Hermitian Killing form.

Proof. We have

$$\begin{aligned} e_i \lrcorner D_{e_i}(\varphi \wedge \mathbb{J}\varphi) &= e_i \lrcorner (D_{e_i}\varphi \wedge \mathbb{J}\varphi + \varphi \wedge D_{e_i}\mathbb{J}\varphi) \\ &= (e_i \lrcorner D_{e_i}\varphi) \wedge \mathbb{J}\varphi - D_{e_i}\varphi \wedge (e_i \lrcorner \mathbb{J}\varphi) \\ &\quad + (e_i \lrcorner \varphi) \wedge D_{e_i}\mathbb{J}\varphi - \varphi \wedge (e_i \lrcorner D_{e_i}\mathbb{J}\varphi). \end{aligned}$$

After summation, we get

$$\begin{aligned} -d_D^\star(\varphi \wedge \mathbb{J}\varphi) &= -(d_D^\star\varphi) \wedge \mathbb{J}\varphi + \varphi \wedge d_D^\star \mathbb{J}\varphi \\ &\quad - \sum_{k=1}^{2m} D_{e_i}\varphi \wedge (e_i \lrcorner \mathbb{J}\varphi) + \sum_{k=1}^{2m} (e_i \lrcorner \varphi) \wedge D_{e_i}\mathbb{J}\varphi. \end{aligned}$$

But

$$\begin{aligned} \sum_{k=1}^{2m} D_{e_i}\varphi \wedge (e_i \lrcorner \mathbb{J}\varphi) &= \sum_{k=1}^{2m} D_{e_i}\varphi \wedge (Je_i \lrcorner \varphi) \\ &= -\sum_{k=1}^{2m} D_{Je_i}\varphi \wedge (e_i \lrcorner \varphi) = -\sum_{k=1}^{2m} (e_i \lrcorner \varphi) \wedge D_{Je_i}\varphi \end{aligned}$$

whence

$$\begin{aligned} -d_D^\star(\varphi \wedge \mathbb{J}\varphi) &= -(d_D^\star\varphi) \wedge \mathbb{J}\varphi + \varphi \wedge d_D^\star \mathbb{J}\varphi \\ &\quad + \sum_{k=1}^{2m} e_i \lrcorner \varphi \wedge (D_{Je_i}\varphi + D_{e_i}\mathbb{J}\varphi). \end{aligned} \tag{2.17}$$

(i) If φ is a Hermitian Killing form one uses (2.17) and (i) in Proposition 2.5 to obtain the conclusion.

(ii) In this case, using (2.17) again we have that

$$\sum_{i=1}^{2m}(e_i \lrcorner \varphi) \wedge \gamma_{e_i} = 0,$$

where γ in $\lambda^1 \otimes_2 \lambda^p$ is given by

$$\gamma_X = D_{JX}\varphi + D_X \mathbb{J}\varphi - \frac{2}{p+1} X \lrcorner \bar{\partial}_D \mathbb{J}\varphi$$

for all X in TM. We define now $\hat{\gamma}$ in $\lambda^{p-1} \otimes \lambda^p$ by

$$\hat{\gamma}(X_1, \ldots, X_{p-1}) = \gamma_{X_1 \lrcorner \cdots \lrcorner X_{p-1} \lrcorner \varphi}$$

whenever $X_k, 1 \leq k \leq p-1$ belong to TM. It is easy to verify that $\hat{\gamma}$ is in $\lambda^{p-1} \otimes_1 \lambda^p$, and since $a(\hat{\gamma}) = 0$ Lemma 2.1 (iii) yields $\hat{\gamma} = 0$. Because φ is everywhere non-degenerate we obtain that γ vanishes and we conclude by Proposition 2.5 (i). □

To end this section let us present another way of characterizing Hermitian Killing forms, this time under the form of a product rule with respect to the exterior differential d_D.

Proposition 2.8. *Let $p \geq 3$ be odd. We have:*

(i) *any Hermitian Killing form φ in λ^p satisfies*

$$-d_D(\varphi \bullet \varphi) = d_D\varphi \bullet \varphi + d_D\mathbb{J}\varphi \bullet \mathbb{J}\varphi - \frac{2}{p+1} J(\varphi \bullet \bar{\partial}_D \mathbb{J}\varphi);$$

(ii) *if φ in λ^p satisfies*

$$-d_D(\varphi \bullet \varphi) = d_D\varphi \bullet \varphi + d_D\mathbb{J}\varphi \bullet \mathbb{J}\varphi - \frac{2}{p+1} J(\varphi \bullet \bar{\partial}_D \mathbb{J}\varphi)$$

and it is nowhere degenerate, then φ is a Hermitian Killing form.

Proof. We will mainly use Proposition 2.7 and the Kähler identities. Indeed, let us observe that

$$L^\star(\varphi \wedge \mathbb{J}\varphi) = -\varphi \bullet \varphi.$$

Using (2.15) it follows that

$$d_D L^\star(\varphi \wedge \mathbb{J}\varphi) = L^\star d_D(\varphi \wedge \mathbb{J}\varphi) + (J d_D^\star J)(\varphi \wedge \mathbb{J}\varphi)$$

hence

$$-d_D(\varphi \bullet \varphi) = L^\star d_D(\varphi \wedge \mathbb{J}\varphi) + J d_D^\star(\varphi \wedge \mathbb{J}\varphi).$$

It is easy to see that

$$L^\star(d_D\varphi \wedge \mathbb{J}\varphi) = (L^\star d_D\varphi) \wedge \mathbb{J}\varphi + d_D\varphi \bullet \varphi,$$
$$-L^\star(\varphi \wedge d_D\mathbb{J}\varphi) = -(L^\star d_D\mathbb{J}\varphi) \wedge \varphi + d_D\mathbb{J}\varphi \bullet \mathbb{J}\varphi,$$

hence we get further

$$-d_D(\varphi\bullet\varphi) = (L^\star d_D\varphi)\wedge\mathbb{J}\varphi + d_D\varphi\bullet\varphi \\ -(L^\star d_D\mathbb{J}\varphi)\wedge\varphi + d_D\mathbb{J}\varphi\bullet\mathbb{J}\varphi + Jd_D^\star(\varphi\wedge\mathbb{J}\varphi).$$

Suppose now that φ is a Hermitian Killing form. Then, by using (i) in Proposition 2.7 we get

$$d_D^\star(\varphi\wedge\mathbb{J}\varphi) = d_D^\star\varphi\wedge\mathbb{J}\varphi - \varphi\wedge d_D^\star(\mathbb{J}\varphi) - \frac{2}{p+1}\varphi\bullet\bar{\partial}_D(\mathbb{J}\varphi)$$

while the second part of the Kähler identities (2.15) provides us with

$$-L^\star d_D(\mathbb{J}\varphi) = d_D^\star\varphi, \\ L^\star d_D\varphi = d_D^\star\mathbb{J}\varphi.$$

The claim in (i) follows now by a simple computation. The converse statement in (ii) is proved by using methods similar to those employed for (ii) in Proposition 2.7, and it is therefore left to the reader. □

3 The torsion within the $\mathcal{G}_1$ class

3.1 $\mathcal{G}_1$-structures

In the following (M^{2m}, g, J) will be almost-Hermitian, in the class $\mathcal{G}_1$. The connection D acts on any form φ in $\Lambda^\star$ according to

$$D_X\varphi = \nabla_X\varphi + \frac{1}{2}[X\lrcorner T^D, \varphi]$$

for all X in TM. Based on this fact it is straightforward to check that the differential d_D is related to d by

$$d_D\varphi = d\varphi - \sum_{k=1}^{2m}(e_k\lrcorner T^D)\wedge(e_k\lrcorner\varphi)$$

for all φ in $\Lambda^\star$. This yields, in the particular case of 3-forms the following comparison formula

$$d_D\varphi = d\varphi - T^D\bullet\varphi \tag{3.1}$$

for all φ in Λ^3.

Let now R be the curvature tensor of the metric g with the convention that $R(X,Y) = -\nabla^2_{X,Y} + \nabla^2_{Y,X}$ for all X, Y in TM. In the same time we consider the curvature tensor R^D of the connection D where the same convention applies. We shall now recall that the first Bianchi identity for the connection D takes the following form.

Lemma 3.1. *For all X in TM,*

$$b_1(R^D)_X = D_X T^D + \frac{1}{2} X \lrcorner dT^D.$$

Proof. It is easy to see, directly from the definition, that the curvature tensors of the Levi-Civita connection and of the connection D are related by

$$R^D(X,Y) = R(X,Y) - \frac{1}{2}\big[Y \lrcorner D_X T^D - X \lrcorner D_Y T^D\big] + \frac{1}{4}\varepsilon^{T^D}(X,Y) \qquad (3.2)$$

for all X, Y in TM. Here we have set

$$\varepsilon^{T^D}(X,Y) = [T^D_X, T^D_Y] - 2T^D_{T^D_X Y}.$$

Since R satisfies the algebraic Bianchi identity and

$$b_1(\varepsilon^{T^D})_X = 2X \lrcorner (T^D \bullet T^D),$$

the assertion follows eventually by setting $X = e_k$, taking the wedge product with e_k in (3.2) and summing over $1 \le k \le 2m$. Note that in the process one also uses the comparison formula (3.1). □

Let us gather now some information on the differentials of the components of the torsion form T^D.

Proposition 3.1. *Let (M^{2m}, g, J) belong to the class $\mathcal{G}_1$. The following hold:*

(i) $\partial_D t = 0$;

(ii) $3\partial_D \psi^+ + 2\bar{\partial}_D t = -4(t \bullet Jt) - 4(Jt \bullet \psi^+)_{\lambda^{1,3}}$;

(iii) $\bar{\partial}_D \psi^+ = -\frac{8}{3}(Jt \bullet \psi^+)_{\lambda^4}$.

Proof. Since $d\omega = 2t + 3\psi^+$ is closed it follows that

$$2dt + 3d\psi^+ = 0.$$

Rewritten by means of the connection D this identity yields, when also using the comparison formula (3.1),

$$2(d_D t + T^D \bullet t) + 3(d_D \psi^+ + T^D \bullet \psi^+) = 0.$$

We now take into account that $T^D = 2Jt + \psi^-$ to arrive, after expansion of the product and use of Lemma 2.2 (v) at

$$2d_D t + 4Jt \bullet t + 2\psi^- \bullet t + 3d_D \psi^+ + 6Jt \bullet \psi^+ = 0.$$

An elementary computation yields $J(\psi^- \bullet t) = Jt \bullet \psi^+$, in particular

$$(\psi^- \bullet t)_{\lambda^4} = (Jt \bullet \psi^+)_{\lambda^4},$$
$$(\psi^- \bullet t)_{\lambda^{1,3}} = -(Jt \bullet \psi^+)_{\lambda^{1,3}}.$$

It suffices thus to use Lemma 2.2 to obtain the proof of the claims after identifying the various components in the equation above along the bidegree decomposition of Λ^4. □

We are now ready to prove the main result of this section.

Theorem 3.1. *Let (M^{2m}, g, J) be almost Hermitian, of type $\mathcal{G}_1$. Then ψ^- is a Hermitian Killing form, that is*

$$D_X\psi^- = \left[X \lrcorner (\partial_D\psi^- + \frac{1}{4}\bar{\partial}_D\psi^-)\right]_{\lambda^3}$$

for all X in TM.

Proof. Since D is a Hermitian connection we have $[R^D(X,Y),\omega] = 0$, in other words

$$\sum_{i=1}^{2m} R^D(X,Y)e_i \wedge (Je_i)^\flat = 0$$

for all X, Y in TM. Setting $X = e_k$ and taking the exterior product with e_k we find

$$\sum_{i,k=1}^{2m} e_k^\flat \wedge R^D(e_k,Y)e_i \wedge (Je_i)^\flat = 0.$$

Obviously, this is equivalent with

$$\sum_{i,k=1}^{2m} e_i \lrcorner (e_k^\flat \wedge R^D(e_k,Y)) \wedge (Je_i)^\flat = \sum_{i=1}^{2m} R^D(e_i,Y) \wedge (Je_i)^\flat$$

or further, by using Lemma 3.1,

$$-\mathcal{J}(D_XT^D + \frac{1}{2}X \lrcorner d_DT^D) = \sum_{i=1}^{2m} R^D(e_i,Y) \wedge (Je_i)^\flat$$

for all Y in TM. Since D is Hermitian, we have that $R^D(X,Y)$ is in $\lambda^{1,1}$, hence the right-hand side above is in $\lambda^{1,2}$ given that $\Lambda^1 \wedge \lambda^{1,1} \subseteq \lambda^{1,2}$. Thus

$$\left(D_XT^D + \frac{1}{2}X \lrcorner dT^D\right)_{\lambda^3} = 0,$$

and it follows that

$$D_X\psi^- = -\frac{1}{2}(X \lrcorner dT^D)_{\lambda^3} \tag{3.3}$$

for all X in TM. Therefore, ψ^- is a Hermitian Killing form and the claim follows by using Proposition 2.4. □

We have the following immediate consequence of the above result.

Theorem 3.2. *Let (M^{2m}, g, J) have totally skew-symmetric Nijenhuis tensor. Then*

$$DN_J = 0 \quad \text{if and only if} \quad dT^D \text{ belongs to } \lambda^{2,2}.$$

In particular N_J is parallel if T^D is a closed 3-form.

Proof. This is a direct consequence of (3.3) and of the properties of the projection on λ^3. □

Remark 3.1. It also follows from (3.3) when combined with Theorem 3.1 that dT^D belongs to $\lambda^{2,2}$ if and only if $d_D\psi^+ = 0$.

In dimension 6, the fact that ψ^+ is a Hermitian Killing form is described by (ii) in Proposition 2.5. As observed in [13], it amounts then to the local parallelism of ψ^+ w.r.t. the characteristic connection, after performing a conformal transformation in order to normalise the length of ψ^+ to a constant. In dimension 6 structures of type $\mathcal{G}_1$ with $DT^D = 0$ have been classified in [4]. Classification results are also available [2] under the same assumption in the Hermitian case, provided that the holonomy of D is contained in $S^1 \times \mathrm{U}(m-1)$.

3.2 The class $W_1 + W_4$

In this section we shall record some of the additional features of the geometry of almost Hermitian manifolds in the Gray–Hervella class $W_1 + W_4$. It can be described as the subclass of $\mathcal{G}_1$ having the property that t has no component on Λ^3_0, or alternatively

$$t = \theta \wedge \omega. \tag{3.4}$$

Here θ is a 1-form on M, called the Lee form of the almost-Hermitian structure (g, J). It can be recovered directly from the Kähler form of (g, J) by

$$(m-1)J\theta = d^\star\omega,$$

in particular we have that $d^\star(J\theta) = 0$.

Proposition 3.2. *Let (M^{2m}, g, J), $m \geq 3$ be almost-Hermitian in the class $W_1 + W_4$. The following hold:*

(i) $\partial_D\theta = 0$;

(ii) $-\partial_D\psi^+ = 2(J\theta \wedge \psi^- + \theta \wedge \psi^+) + \frac{2}{3}(\bar{\partial}_D\theta - 2\zeta \lrcorner \psi^-) \wedge \omega$;

(iii) $\frac{1}{4}\bar{\partial}_D\psi^+ = \theta \wedge \psi^+ - J\theta \wedge \psi^-$.

Proof. All claims follows from Proposition 3.1, applied to our present situation, that is $t = \theta \wedge \omega$. Therefore, (i) is immediate from (i) in the previously cited proposition.

Through direct computation we find

$$(J\theta \wedge \omega) \bullet \psi^+ = -(\zeta \lrcorner \psi^-) \wedge \omega + 3J\theta \wedge \psi^-$$
$$(J\theta \wedge \omega) \bullet (\theta \wedge \omega) = 0.$$

Our last two assertions follow now from Proposition (3.1), after projection of the formulae above onto $\Lambda^4 = \lambda^{2,2} \oplus \lambda^{1,3} \oplus \lambda^4$. □

In dimension 6 it has been shown in [13], [16] that an almost Hermitian manifold in the class $W_1 + W_4$ has closed Lee form, i.e., $d\theta = 0$. Theorem 3.1 combined with Proposition 3.2 seems a good starting point to investigate up to what extent structures of $W_1 + W_4$ are closed in arbitrary dimensions but we shall not pursue this direction here.

3.3 The $\mathfrak{u}(m)$-decomposition of curvature

In this section we shall investigate, for further use, the splitting of the curvature tensor of the Hermitian connection D. Our main goal is to identify explicitly the "non-Kähler" part of the tensor R^D and to give it an explicit expression in terms of the torsion form T^D of the characteristic connection. Note that for almost quaternionic-Hermitian or G_2-structures (not necessarily with skew-symmetric torsion) similar results have been obtained in [42], [15]. We will mainly use that

$$\begin{aligned} R^D(X,Y,Z,U) - R^D(Z,U,X,Y) \\ = -\frac{1}{2}\big[(D_X T^D)_Y(Z,U) - (D_Y T^D)_X(Z,U)\big] \\ + \frac{1}{2}\big[(D_Z T^D)_U(X,Y) - (D_U T^D)_Z(X,Y)\big] \end{aligned} \tag{3.5}$$

holds for all X, Y, Z, U in TM, as it easily follows from the difference formula (3.2), after verifying that ε^{T^D} belongs to $S^2(\Lambda^2)$.

Theorem 3.3. *Let (M^{2m}, g, J) be almost Hermitian of type $\mathcal{G}_1$. We have*

$$R^D = R^K + \widehat{\Omega} + \frac{1}{2}R^a + R^m,$$

where R^K belongs to $\mathcal{K}(\mathfrak{u}(m))$. Moreover:

(i) *R^a belongs to $\Lambda^2(\lambda^{1,1})$ and satisfies the Bianchi identity*

$$\frac{1}{2}(b_1 R^a)_X = D_X(Jt) - D_{JX}t - \frac{1}{2}X \lrcorner \partial_D(Jt)$$

for all X in TM.

(ii) *The Bianchi identity for R^m in $\lambda^2 \otimes \lambda^{1,1}$ is*

$$(b_1 R^m)_X = D_X(Jt) + D_{JX}t + \frac{1}{4}(JX \lrcorner \mathcal{J}\partial_D\psi^- - 2X \lrcorner \partial_D\psi^-)$$

for all X in TM.

(iii) *Ω in $\lambda^{2,2}$ is given by*

$$\Omega = \frac{3}{2}\partial_D(Jt) + 2(Jt \bullet Jt)_{\lambda^{2,2}} + \frac{1}{2}\psi^- \bullet \psi^-.$$

Proof. This is a direct application of Theorem 2.1 which can be used, as it follows from (3.5), for the tensor $\mathcal{R} = R^D$ and

$$\gamma = -\frac{1}{2} DT^D = -D(Jt) - \frac{1}{2} D\psi^-.$$

Then we have $\gamma^{1,2} = -D(Jt)$ and $\gamma^3 = -\frac{1}{2} D\psi^-$, and in order to prove our claims we only need to determine the various antisymmetrisations of $\gamma^{1,2}$. Also note that in this context

$$T = -\frac{1}{2} d_D T^D = -d_D(Jt) - \frac{1}{2} d_D \psi^-,$$

in particular $T^{2,2} = -\partial_D(Jt)$.

(i) Now, it is easy to see that

$$\begin{aligned} a(\gamma^{1,2}) &= -d_D(Jt), \\ a(J\gamma^{1,2}) &= d_D t, \\ a^c(\gamma^{1,2}) &= -Jd_D t, \\ a^c(J\gamma^{1,2}) &= -Jd_D Jt. \end{aligned}$$

Hence,

$$\begin{aligned} A_1 &= -d_D(Jt) - Jd_D(Jt) - 4T^{2,2} = -d_D(Jt) - Jd_D(Jt) + 4\partial_D(Jt) \\ &= -2\partial_D(Jt) + 4\partial_D(Jt) = 2\partial_D(Jt) \end{aligned}$$

and

$$A_2 = -Jd_D t - d_D t = -2\partial_D t = 0$$

by (i) of Proposition 3.1.

(ii) Since $T^{1,3} = -\bar{\partial}_D(Jt) - \frac{1}{2}\partial_D\psi^-$ we have

$$A_3 = -d_D(Jt) + Jd_D Jt + 2\bar{\partial}_D(Jt) + \partial_D\psi^- = \partial_D\psi^-$$

and

$$A_4 = d_D t - Jd_D t + \mathcal{J}\bar{\partial}_D(Jt) + \frac{1}{2}\mathcal{J}\partial_D\psi^- = \frac{1}{2}\mathcal{J}\partial_D\psi^-$$

after making use of (2.13). The claim in (ii) now follows.

(iii) We apply the Bianchi operator to

$$R^D = R^K + \widehat{\Omega} + R^a + R^m$$

and find after making use of (i), (ii) and of (2.2) that

$$b_1(R^D)_X = X \lrcorner \Omega + 2D_X(Jt) - \frac{1}{2} X \lrcorner \partial_D(Jt) + \frac{1}{4}(JX \lrcorner \mathcal{J}\partial_D\psi^- - 2X \lrcorner \partial_D\psi^-)$$

for all X in TM. Plugging into this the Bianchi identity for D in Lemma 3.1, combined with the fact that ψ^- is a Hermitian Killing form as asserted in Theorem 3.1 yields

further

$$\frac{1}{2}(X \lrcorner dT^D)_{\lambda^{1,2}} = X \lrcorner \left[\Omega - \frac{1}{2}\partial_D(Jt)\right] + \frac{1}{4}(JX \lrcorner \not{J}\partial_D\psi^- - 2X \lrcorner \partial_D\psi^-)$$
$$= X \lrcorner \left[\Omega - \frac{1}{2}\partial_D(Jt)\right] - (X \lrcorner \partial_D\psi^+)_{\lambda^{1,2}}$$

for all X in TM. After identifying components on $\lambda^1 \otimes_1 \lambda^{1,2}$ and $\lambda^1 \otimes_2 \lambda^{1,2}$ respectively (see also Remark 2.1 for the definition of these spaces) we find that

$$\frac{1}{2}(dT^D)_{\lambda^{2,2}} = \Omega - \frac{1}{2}\partial_D(Jt).$$

Now $dT^D = d_D T^D + T^D \bullet T^D$ by the comparison formula (3.1) thus by projection on $\lambda^{2,2}$ we find

$$(dT^D)_{\lambda^{2,2}} = 2\partial_D(Jt) + (T^D \bullet T^D)_{\lambda^{2,2}}$$
$$= 2\partial_D(Jt) + 4(Jt \bullet Jt)_{\lambda^{2,2}} + \psi^- \bullet \psi^-$$

after expansion of the product and use of Lemma 2.2 (iii). The claim in (iii) follows now immediately. □

Remark 3.2. The curvature decomposition in Theorem 3.3 can be still refined, given that the $\mathrm{U}(m)$-modules

$$\mathcal{K}(\mathfrak{u}(m)), \quad \lambda^{2,2}, \quad \Lambda^2(\lambda^{1,1}) \quad \text{and} \quad \lambda^2 \otimes \lambda^{1,1}$$

are not irreducible. Although this is an algebraically simple procedure, the computations at the level of the derivative DT^D of the torsion form become somewhat involved and will not be presented here. We just illustrate the situation in the simpler case of $W_1 + W_4$ below.

Theorem 3.3 has various applications, a class of which consists in giving torsion interpretation of curvature conditions imposed on the tensors R^D or R. As an example in this direction we have the following:

Proposition 3.3. *Let (M^{2m}, g, J) belong to the class $\mathcal{G}_1$. The curvature tensor R^D is Hermitian, that is*

$$R^D(JX, JY) = R^D(X, Y)$$

for all X, Y in TM, if and only if

$$D_{JX}t + D_X(Jt) = \frac{2}{3}(X \lrcorner \bar{\partial}_D(Jt))_{\lambda^{1,2}}$$

for all X in TM and

$$2\bar{\partial}_D(Jt) = 3\partial_D\psi^-.$$

Proof. By Theorem 3.3 the tensor R^D is Hermitian if and only if $R^m = 0$ which in turn happens if and only if $b_1 R^m = 0$. Therefore, by (ii) in Theorem 3.3 the curvature

of D is Hermitian if and only if

$$D_X(Jt) + D_{JX}t + \frac{1}{4}(JX \lrcorner \not{\partial}_D\psi^- - 2X \lrcorner \partial_D\psi^-) = 0$$

for all X in TM. Taking the alternation above we find

$$2\bar{\partial}_D(Jt) - 3\partial_D\psi^- = 0.$$

We conclude by recalling that

$$(X \lrcorner A)_{\lambda^{1,2}} = \frac{1}{4}(2X \lrcorner A - JX \lrcorner \not{J}A) \tag{3.6}$$

for all X in TM, where A belongs to $\lambda^{1,3}$. □

Remark 3.3. In the case when the curvature tensor of the connection D is Hermitian the fact that $2\bar{\partial}_D(Jt) = 3\partial_D\psi^-$ combined with (ii) in Proposition 3.1 further yields

$$\bar{\partial}_D(Jt) = -(t \bullet Jt) - (Jt \bullet \psi^+)_{\lambda^{1,3}}.$$

For the subclass $W_1 \oplus W_4 \subseteq \mathcal{G}_1$ more information on the curvature tensor of the connection D is available and it will actually turn out that the components R^a and R^m have simple algebraic expressions. We define

$$S^{2,-}(M) = \{S \in S^2(M) : SJ + JS = 0\}.$$

This is embedded in $\lambda^2 \otimes \lambda^{1,1}$ via $S \mapsto \mathring{S}$ where

$$\mathring{S}(X,Y) = \frac{1}{2}((SJX)^\flat \wedge Y^\flat + X^\flat \wedge (SJY)^\flat + (SX)^\flat \wedge (JY)^\flat + (JX)^\flat \wedge (SY)^\flat)$$

for all X, Y in TM. One verifies that

$$(b_1\mathring{S})_X = (SX)^\flat \wedge \omega \tag{3.7}$$

for all X in TM. We also have an embedding $\lambda^{1,3} \hookrightarrow \lambda^2 \otimes \lambda^{1,1}$ given by $\Omega \mapsto \widetilde{\Omega}$ where we define

$$\widetilde{\Omega}(X,Y) = \frac{1}{4}(\Omega(JX,JY) - \Omega(X,Y))$$

for all X, Y in TM. Elementary considerations ensure that this is well defined and subject to

$$(b_1\widetilde{\Omega})_X = (X \lrcorner \Omega)_{\lambda^{1,2}} \tag{3.8}$$

whenever X belongs to TM. As a last piece of notation let the symmetrized action of D on 1-forms be defined by

$$(\mathring{D}\alpha)(X,Y) = \frac{1}{2}((D_X\alpha)Y + (D_Y\alpha)X)$$

whenever α is in λ^1 and X, Y belong to TM.

Proposition 3.4. *Let (M^{2m}, g, J) belong to the class $W_1 + W_4$. If θ denotes the Lee form of (g, J) the following hold:*

(i) $\Omega = \left[\frac{3}{2}\partial_D(J\theta) + 2|\theta|^2\omega - 4\theta \wedge J\theta\right] \wedge \omega + \frac{1}{2}\psi^- \bullet \psi^-$;

(ii) $R^a = \partial_D(J\theta) \otimes \omega - \omega \otimes \partial_D(J\theta)$;

(iii) $R^m = -\bar{\partial}_D(J\theta) \otimes \omega + \mathring{S}_\theta - \widetilde{\partial_D\psi^-}$, *where* S_θ *in* $S^{2,-}(M)$ *is defined by* $S_\theta = (1 - J)\mathring{D}(J\theta)$.

Proof. Recall that for the class $W_1 + W_4$ we have that $t = \theta \wedge \omega$. Taking this into account we will apply now Theorem 3.3.

(i) follows from (iii) in Theorem 3.3 when observing that

$$(Jt \bullet Jt)_{\lambda^{2,2}} = |\theta|^2\omega \wedge \omega - 2\omega \wedge \theta \wedge J\theta.$$

(ii) First of all we note that

$$D_X(J\theta) - D_{JX}\theta = X \lrcorner \partial_D(J\theta)$$

for all X in TM. The quickest way to see this is to observe that the tensor q in $\lambda^1 \otimes \lambda^1$ defined by $q(X) = D_X(J\theta) - D_{JX}\theta - X \lrcorner \partial_D(J\theta)$ for all X in TM is actually in $\lambda^1 \otimes_1 \lambda^1$ and satisfies $a(q) = a^c(q) = 0$, by using also that $\partial_D\theta = 0$ (cf. Proposition 3.2 (i)). Our claim now follows by observing that (iii) in Lemma 2.1 continues to hold when $p = q = 1$.

Using now (i) in Theorem 3.3 we get

$$(b_1 R^a)_X = (X \lrcorner \partial_D(J\theta)) \wedge \omega - \partial_D(J\theta) \wedge (X \lrcorner \omega)$$

for all X in TM.

Now for any φ in $\lambda^{1,1}$ we consider the element $\omega \otimes \varphi - \varphi \otimes \omega$ in $\Lambda^2(\lambda^{1,1})$, which is explicitly given by

$$(\omega \otimes \varphi - \varphi \otimes \omega)(X, Y) = \omega(X, Y)\varphi - \varphi(X, Y)\omega$$

for all X, Y in TM. A straightforward computation following the definitions yields

$$b_1(\partial_D(J\theta) \otimes \omega - \omega \otimes \partial_D(J\theta))_X = (X \lrcorner \partial_D(J\theta)) \wedge \omega - \partial_D(J\theta) \wedge (X \lrcorner \omega)$$

for all X in TM, and we conclude by using the injectivity of the Bianchi map $b_1 \colon \Lambda^2(\Lambda^2) \to \Lambda^1 \otimes \Lambda^3$.

(iii) By (iii) in Theorem 3.3 we have

$$(b_1 R^m)_X = \left[D_X(J\theta) + D_{JX}\theta\right] \wedge \omega - (X \lrcorner \partial_D\psi^-)_{\lambda^{1,2}}$$

after also making use of (3.6). Since

$$D_X(J\theta) + D_{JX}\theta = X \lrcorner \bar{\partial}_D(J\theta) + (S_\theta X)^\flat$$

it follows by means of (3.7), (3.8), that $R^m + \bar{\partial}_D(J\theta) \otimes \omega - \mathring{S}_\theta + \widetilde{\partial_D\psi^-}$ belongs to $\mathrm{Ker}(b_1) \cap (\lambda^2 \otimes \lambda^{1,1})$. It therefore vanishes and the proof is finished. □

Finally, we remark that R^m can be given a more detailed expression by taking into account (ii) of Proposition 3.2.

4 Nearly-Kähler geometry

We shall begin here our survey of nearly-Kähler geometry. This is the class of almost Hermitian structures introduced by the following.

Definition 4.1. Let (M^{2m}, g, J) be an almost Hermitian manifold. It is called nearly-Kähler (NK for short) if and only if

$$(\nabla_X J)X = 0$$

whenever X belongs to TM.

This can be easily rephrased to say that an almost Hermitian structure (g, J) is nearly-Kähler if and only if its Kähler form is subject to

$$\nabla_X \omega = \frac{1}{3} X \lrcorner d\omega \tag{4.1}$$

for all X in TM. In other words, the Kähler form of any NK-structure is a Killing form and conversely U. Semmelmann [53] shows that any almost Hermitian structure with this property must be nearly-Kähler. Therefore, nearly-Kähler structures belong to the class W_1 in the sense that the 3-form t in Proposition 2.2 vanishes identically.

From now on we shall work on a given nearly-Kähler manifold (M^{2m}, g, J). For many of the properties of an NK-structure are best expressed by means of its first canonical Hermitian connection, it is important to note then the coincidence of the canonical connection and D, that is

$$D = \bar{\nabla}.$$

For this reason, the torsion tensor T^D will be denoted simply by T in what follows. It is given by

$$T^D = \psi^-$$

and hence belongs to λ^3. Therefore, in dimension 4 NK-structures are Kähler and we shall assume from now on that $m \geq 3$. The Nijenhuis tensor of the almost complex J is computed from (2.8) by

$$N^J = -4\psi^-.$$

As an immediate consequence one infers that the almost complex structure of an NK-manifold is integrable if and only if the structure is actually a Kähler one. This observation motivates the following

Definition 4.2. Let (M^{2m}, g, J) be an NK-structure. It is called strict iff $\nabla_X J = 0$ implies that $X = 0$ for all X in TM.

Therefore the non-degeneracy of any of the forms ψ^+ and ψ^- is equivalent with the strictness of the NK-structure (g, J). An important property of NK-structures is contained in the following result.

Theorem 4.1. *Let (M^{2m}, g, J) be a nearly-Kähler manifold. Then*

$$\bar{\nabla}\psi^{\pm} = 0$$

in other words the Nijenhuis tensor of J is parallel w.r.t. the canonical Hermitian connection.

This has been proved first by Kirichenko [40] and a short proof can be found in [5]. It also follows from our Theorem 3.1, which is unifying this type of property in the class $\mathcal{G}_1$.

Let us introduce now the symmetric tensor r in S^2M given by

$$\langle rX, Y\rangle = \langle X \lrcorner \psi^+, Y \lrcorner \psi^+\rangle$$

for all X, Y in TM. It is easily seen to be J-invariant and if (M^{2m}, g, J) is strict then r is non-degenerate. Moreover, from Theorem 4.1 it also follows that

$$\bar{\nabla} r = 0.$$

Corollary 4.1 ([48]). *Any NK-manifold is locally the product of a Kähler manifold and a strict nearly-Kähler one.*

Proof. Consider the $\bar{\nabla}$-parallel distribution $\mathcal{V} := \{V \in TM : \psi^+_V = 0\}$. Since the torsion of the connection $\bar{\nabla}$ vanishes in direction of $\mathcal{V}$ the latter must be parallel for the Levi-Civita and the result follows by using the de Rham splitting theorem. □

It follows that locally and also globally if our original manifold is simply connected we can restrict to the study of strict nearly-Kähler structures (SNK for short). Note however that in dimension 6, any NK-structure which is not Kähler must be strict.

The parallelism of the torsion tensor w.r.t. $\bar{\nabla}$ is eventually reflected in the properties its curvature tensor. Indeed

Proposition 4.1 ([30], [33]). *The following hold:*

(i) $\bar{R}(X, Y, Z, U) = \bar{R}(Z, U, X, Y)$ *for all X, Y, Z, U in TM;*

(ii) $\bar{R}(JX, JY) = \bar{R}(X, Y)$ *for all X, Y in TM;*

(iii) $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = [\psi^-_X, \psi^-_Y]Z - \psi^-_{\psi^-_X Y} Z$ *for all X, Y, Z in TM.*

Proof. (i) is immediate from (3.5) and the parallelism of the torsion, whilst (ii) follows from (i) and the fact that $\bar{R}$ belongs to $\Lambda^2 \otimes \lambda^{1,1}$. The last claim follows for instance from Lemma 3.1 □

Remark 4.1. Proposition 4.1 continues to hold when the nearly-Kähler metric is allowed to have signature. This has been exploited in [18] to classify NK-structures compatible with a flat pseudo-Riemannian metric.

To establish first order properties of SNK-structures we will have a look at the Ricci tensor of such metrics. The Hermitian Ricci tensor of (g, J) is defined by

$$\langle \overline{\mathrm{Ric}} X, Y\rangle = \sum_{i=1}^{2m} \bar{R}(X, e_i, Y, e_i)$$

for all X, Y in TM, where $\{e_i, 1 \leq i \leq 2m\}$ is some local orthonormal frame. By making use of Proposition 4.1 we find that $\overline{\mathrm{Ric}}$ is actually symmetric and J-invariant. The Hermitian Ricci tensor is related to the usual Riemannian one by

$$\overline{\mathrm{Ric}} = \mathrm{Ric} - \frac{3}{4} r \tag{4.2}$$

as implied by the general curvature comparison formula (3.2).

Theorem 4.2 ([48]). *Let (M^{2m}, g, J) be a strict nearly-Kähler manifold. The following hold:*

(i) *the Ricci tensor of the metric g is parallel w.r.t. $\bar{\nabla}$, that is $\bar{\nabla}\mathrm{Ric} = 0$;*

(ii) Ric *is positive definite.*

Proof. The proof of both relies on the explicit computation of the Ricci tensor of the metric g. From the parallelism of ψ^+, after derivation and use of the Ricci identity for the connection with torsion $\bar{\nabla}$ we find

$$[\bar{R}(X, Y), \psi^+] = 0 \tag{4.3}$$

for all X, Y in TM. In a local orthonormal frame $\{e_i, 1 \leq i \leq 2m\}$ this reads

$$\sum_{i=1}^{2m} \bar{R}(X, Y) e_i \wedge \psi^+_{e_i} = 0$$

for all X, Y in TM. We now set $Y = e_k$, take the interior product with e_k above to find, after summation over $1 \leq k \leq 2m$ and some straightforward manipulations that

$$\psi^+_{\overline{\mathrm{Ric}} X} = \sum_{1 \leq k, i \leq 2m} \bar{R}(X, e_k) e_i \wedge \psi^+_{e_i, e_k}.$$

Since ψ^+ is a form, the sum in the right-hand side equals

$$\frac{1}{2} \sum_{1 \leq k, i \leq 2m} (\bar{R}(X, e_k) e_i - \bar{R}(X, e_i) e_k) \wedge \psi^+_{e_i, e_k}$$

$$= \frac{1}{2} \sum_{1 \leq k, i \leq 2m} \left([\psi^+_X, \psi^+_{e_k}] e_i - \psi^+_{\psi^+_X e_k} e_i \right) \wedge \psi^+_{e_i, e_k} + \frac{1}{2} \sum_{1 \leq k, i \leq 2m} \bar{R}(e_i, e_k) X \wedge \psi^+_{e_i, e_k}$$

after using the Bianchi identity for $\bar{R}$. Now the last sum vanishes since ψ^+ is J-anti-invariant in the first arguments whereas $\bar{R}$ is J-invariant and our frame can be chosen to be Hermitian. After neglecting terms which are J-invariant in e_i, e_k in the algebraic sum above we end up with

$$\psi^+_{\overline{\mathrm{Ric}}X} = \frac{1}{2} \sum_{1\leq k,i\leq 2m} \psi^+_X \psi^+_{e_k} e_i \wedge \psi^+_{e_i,e_k}$$

for all X in TM. Using now the definition of the tensor r a straightforward manipulation yields

$$\psi^+_{\overline{\mathrm{Ric}}X} = -\frac{1}{2} \sum_{k=1}^{2m} \psi^+_X e_k \wedge (re_k)^\flat \tag{4.4}$$

for all X in TM. By derivation and using the parallelism of ψ^+ it follows that

$$\psi^+_{(\bar{\nabla}_X \overline{\mathrm{Ric}})Y} = 0$$

for all X, Y in TM. (i) follows now from the fact that ψ^+ is nondegenerate and the comparison fact in (4.2). For the claim in (ii) we refer the reader to [48]. □

Using Myer's theorem it follows from the above that complete SNK-manifolds must be compact with finite fundamental group. Therefore, in the compact case one can restrict attention, up to a finite cover, to simply connected nearly-Kähler manifolds.

Another important object is the first Chern form of the almost Hermitian structure (g, J) defined by

$$8\pi\gamma_1(X, Y) = \sum_{i=1}^{2m} \bar{R}(X, Y, e_i, Je_i)$$

for all X, Y in TM. Since $\bar{R}$ is J-invariant we have that γ_1 belongs to $\lambda^{1,1}$ so one can write $4\pi\gamma_1 = \langle CJ\cdot, \cdot\rangle$ for some C in $S^2(TM)$ such that $CJ = JC$. A straightforward computation using the first Bianchi identity yields the relation

$$C = \overline{\mathrm{Ric}} - r,$$

hence C must be parallel w.r.t. the canonical connection, that is

$$\bar{\nabla} C = 0$$

by means of Theorem 4.2. Since γ_1 is a closed form, as it follows from the second Bianchi identity for $\bar{\nabla}$, this results in the algebraic obstruction

$$-C(\psi^+_X Y) = \psi^+_X CY + \psi^+_{CX} Y \tag{4.5}$$

for all X, Y in TM. Note that this can be given a direct algebraic proof by observing that $[\gamma_1, \psi^+] = 0$, as it follows from (4.3) when taking into account that $\bar{R}$ belongs to $S^2(\Lambda^2)$.

4.1 The irreducible case

To obtain classification results for strict nearly-Kähler manifolds we shall start from examining the holonomy representation of the canonical Hermitian connection. At a point x of M where (M^{2m}, g, J) is some SNK-manifold this is the representation

$$\mathrm{Hol}_x(\bar{\nabla})\colon T_xM \to T_xM$$

obtained by parallel transport w.r.t. $\bar{\nabla}$ along loops about x. The holonomy representation is Hermitian, for $\bar{\nabla}$ is a Hermitian connection and this gives rise to two different notions of irreducibility as indicated by the following well-known result from representation theory.

Proposition 4.2. *Let (V^{2m}, g, J) be a Hermitian vector space and let (G, V) be a Hermitian representation of some group G. If we write $V^{\mathbb{C}}$ for the complex vector space obtain from V by setting $iv = Jv$ for all v in V, the following cases can occur:*

(i) *(G, V) is irreducible.*

(ii) *$(G, V^{\mathbb{C}})$ is irreducible but not (G, V). In this case V splits orthogonally as $V = L \oplus JL$ for some G-invariant subspace L of V.*

(iii) *$(G, V^{\mathbb{C}})$ is reducible.*

In this section we shall deal with the instances when the holonomy representation of $\bar{\nabla}$ is irreducible in the sense of (i) or (ii) in the Proposition 4.2. The first is actually covered by the following powerful result of R. Cleyton and A. Swann.

Theorem 4.3 ([17]). *Let (N^n, g) be Riemannian such that there exists a metric connection D such that the following hold:*

(i) *the torsion tensor T of D belongs to Λ^3;*

(ii) *$DT = 0$ and T does not vanish identically.*

If the holonomy representation of D is irreducible then D is an Ambrose–Singer connection in the sense that $DR^D = 0$ where R^D denotes the curvature tensor of the connection D, provided that $n \neq 6, 7$.

The two exceptions in the result above correspond actually to nearly-parallel G_2-structures in dimensions 7 (see [27] for an account) and SNK-structures in dimension 6.

In the situation in the theorem above (N^n, g) is a locally homogeneous space (see [57]) for more details). Theorem 4.3 is proved by making use of general structure results on Berger algebras and formal curvature tensors spaces, for irreducible representations of compact Lie algebras (see also [44] for the non-compact case). We can now state the following.

Theorem 4.4. *Let (M^{2m}, g, J) be a strict nearly-Kähler manifold. Then either:*

(i) *$\bar{\nabla}$ is an Ambrose–Singer connection; or*

(ii) $m = 3$*; or*

(iii) *the holonomy representation of* $\overline{\nabla}$ *is reducible over* $\mathbb{C}$.

Proof. By making use of Theorem 4.3 we see that the case (i) in Proposition 4.2 corresponds to (i) in our statement. To finish the proof let us suppose that at some point x of M we have an orthogonal splitting

$$T_xM = L_x \oplus JL_x$$

for some $\mathrm{Hol}_x(\overline{\nabla})$-invariant subspace of T_xM. Using parallel transport L_x extends to a $\overline{\nabla}$-parallel distribution L of TM such that

$$TM = L \oplus JL.$$

It follows that $\overline{R}(L, JL) = 0$ by also using Proposition 4.1 (ii). Since $\overline{R}$ belongs to $S^2(\lambda^{1,1})$, this means that $\overline{R}$ is an algebraic expression in ψ^+, which can be explicitly determined from the first Bianchi identity (see [49] for details). It follows that that $\overline{\nabla}\,\overline{R} = 0$ and then $\overline{\nabla}$ is an Ambrose–Singer connection. □

5 When the holonomy is reducible

In this section we present classification results for SNK-structures in the case when the holonomy representation of $\overline{\nabla}$ is complex reducible. We begin by setting up some terminology which is aimed to gain some understanding concerning the relation between the algebraic properties of the torsion form of an SNK-structure and the geometry of the underlying Riemannian manifold.

5.1 Nearly-Kähler holonomy systems

We start by the following definition which extracts the more peculiar facts from NK-geometry which relate to the holonomy of the canonical Hermitian connection.

Definition 5.1. A strict nearly-Kähler holonomy system $(V^{2m}, g, J, \psi^+, R)$ is constituted of the following data:

(i) a Hermitian vector space (V^{2m}, g, J);

(ii) a nondegenerate 3-form ψ^+ in λ^3;

(iii) a tensor R in $\lambda^{1,1} \otimes \lambda^{1,1}$ of the form

$$R = R^K + \widehat{\Omega},$$

where R^K in $\mathcal{K}(\mathfrak{u}(m))$ is such that

$$[R(x, y), \psi^+] = 0 \tag{5.1}$$

holds for any x, y in V. Moreover the form Ω in $\lambda^{2,2}$ must be given by $\Omega = \frac{1}{2}\psi^+ \bullet \psi^+$.

Geometrically, the assumption of having the form ψ^+ nondegenerate amounts to working on an SNK-manifold (see also Definition 4.2 and Corollary 4.1). In what follows we shall work on a given strict nearly-Kähler holonomy system to be denoted by $(V^{2m}, g, J, \psi^+, R)$. As with Riemannian holonomy systems, the liaison with the holonomy algebra of the canonical connection of a geometric SNK-structure is through the Lie subalgebra

$$\mathfrak{h} := \mathrm{Lie}\{R(x, y) : x, y \in V\}$$

of $\lambda^{1,1}$.

Definition 5.2. An SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ is complex reducible if the representation $(\mathfrak{h}, V^{\mathbb{C}})$ is reducible.

Another object of relevance here is the isotropy algebra $\mathfrak{g}$ of the form ψ^+ defined by

$$\mathfrak{g} = \{\alpha \in \Lambda^2 : [\alpha, \psi^+] = 0\}.$$

It is easy to see, starting from (2.1) and then using an invariance argument together with the non-degeneracy of ψ^+ that

$$\mathfrak{g} \subseteq \lambda^{1,1}.$$

Moreover, the condition (5.1) in Definition 5.1 reads $\mathfrak{h} \subseteq \mathfrak{g}$.

We are interested here in the structure of complex invariant subspaces of the metric representation $(\mathfrak{h}, V)$. The following definition singles out three main classes of subspaces of relevance for our situation.

Definition 5.3. A proper, J-invariant subspace $\mathcal{V}$ of V is said to be (w.r.t. ψ^+):

(i) isotropic if $\psi^+(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$;

(ii) null if $\psi^+(\mathcal{V}, \mathcal{V}) = 0$;

(iii) special if it is null and $\psi^+(H, H) = \mathcal{V}$, where H is the orthogonal complement of $\mathcal{V}$ in V.

Remark 5.1. Any 2-dimensional, J-invariant subspace of V is null w.r.t. ψ^+. Moreover, in dimension 6, any two dimensional J-invariant subspace is special w.r.t. ψ^+. However, we are interested here in isotropic or special subspaces which are invariant w.r.t. a particular Lie group or Lie algebra.

A useful criterion to prove that a subspace is special is the following.

Lemma 5.1. *Let (V^{2m}, g, J) be a Hermitian vector space and let ψ^+ be non-degenerate in λ^3. If $\mathcal{V} \subseteq V$ is null w.r.t. ψ^+ and such that $\psi^+(H, H) \subseteq \mathcal{V}$, then:*

(i) $\mathcal{V}$ *is special w.r.t.* ψ^+*;*

(ii) $\psi^+(\mathcal{V}, H) = H$.

Proof. (i) First of all let us notice that

$$\psi^+(\mathcal{V}, H) \subseteq H$$

since $\psi^+(\mathcal{V}, H)$ is orthogonal to $\mathcal{V}$, as it follows from the fact that $\mathcal{V}$ is null. Let $\mathcal{V}_0 := \psi^+(H, H) \subseteq \mathcal{V}$ and let $\mathcal{V}_1$ be the orthogonal complement of $\mathcal{V}_0$ in $\mathcal{V}$. From the definition of $\mathcal{V}_1$ it follows that $\psi^+(H, \mathcal{V}_1)$ is orthogonal to H thus $\psi^+(H, \mathcal{V}_1) = 0$. But $\psi^+(\mathcal{V}, \mathcal{V}_1) = 0$ as well since $\mathcal{V}$ is null, in other words $\psi^+(\mathcal{V}_1, \cdot) = 0$ and we conclude that $\mathcal{V}_1 = 0$ using that ψ^+ is nondegenerate. This proves (i).

(ii) is proved by an argument similar to that in (i), which we leave to the reader. □

We end this section with the following:

Definition 5.4. An SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ is said to split as

$$V = V_1 \oplus V_2$$

if V admits an $\mathfrak{h}$-invariant, orthogonal and J-invariant splitting $V = V_1 \oplus V_2$ such that

$$\psi^+ \text{ belongs to } \lambda^3(V_1) \oplus \lambda^3(V_2).$$

It is clear that if an SNK-holonomy system splits as $V = V_1 \oplus V_2$ then each of $(V_k, g_{|V_k}, J_{|V_k}, \psi^+_{|V_k})$, $k = 1, 2$ is again an SNK-holonomy system. Note that factors of dimension ≤ 4 are not permitted by this definition since in dimension 2 or 4 there are no non-zero holomorphic 3-forms. Also note that this type of decomposition of an SNK-holonomy system corresponds exactly to local products of SNK-manifolds.

5.2 The structure of the form ψ^+

We are now ready to have a look at the structure of the form ψ^+ when a complex holonomy reduction is given. The starting point of our approach to the classification problem of reducible SNK-holonomy systems is:

Proposition 5.1. *Let* $\mathcal{V}$ *be a proper,* J*-invariant subspace of* $(\mathfrak{h}, V)$*. If* H *denotes the orthogonal complement of* $\mathcal{V}$ *in* V*, the following hold:*

(i) $(\psi^+_x \circ \psi^+_v)w = 0$ *for all* x *in* H *and* v*,* w *in* $\mathcal{V}$*;*

(ii) $(\psi^+_x \circ \psi^+_y)z$ *belongs to* H *whenever* x*,* y*,* z *are in* H*;*

(iii) $(\psi^+_v \circ \psi^+_w)x$ *belongs to* H *for all* x *in* H *and* v*,* w *in* $\mathcal{V}$*;*

(iv) $(\psi^+_x \circ \psi^+_y)v$ *is in* $\mathcal{V}$*, for all* x*,* y *in* H *and all* v *in* $\mathcal{V}$.

Proof. Directly from the first Bianchi identity for R, combined with the fact that $R(\mathcal{V}, H) = 0$ we get

$$R(x, y, v, w) = \langle [\psi_v^+, \psi_w^+]x, y\rangle - \langle \psi_v^+ w, \psi_x^+, y\rangle \tag{5.2}$$

for all x in H, y in V and v, w in $\mathcal{V}$. All claims are now easy consequences of the fact that R belongs to $S^2(\lambda^{1,1})$ and ψ^+ is in λ^3, see [49] for details. □

We can show that any complex reducible SNK-holonomy system contains, up to products, a null invariant sub-space.

Proposition 5.2. *Let (V^{2m}, g, ψ^+, R) be a complex reducible SNK-holonomy system. Then V splits as*

$$V = V_1 \oplus V_2,$$

where V_2 contains a null invariant space.

Proof. The proof is completed in two steps we shall outline below.

Step 1: Existence of an isotropic invariant subspace.

The reducibility of $(\mathfrak{h}, V)$ implies the existence of an invariant splitting

$$V = E \oplus F$$

which is moreover orthogonal and stable under J. Let F_0 be the subspace of F spanned by $\{(\psi_v^+ w)_F : v, w \text{ in } E\}$, where the subscript indicates orthogonal projection. Using Proposition 5.1 (i) we get

$$\psi_x^+ \psi_y^+ ((\psi_v^+ w)_F) + \psi_x^+ \psi_y^+ ((\psi_v^+ w)_E) = 0$$

for all x, y in F and whenever v, w are in E. But the first summand is in E by Proposition 5.1 (iv) while the second is in F by (ii) of the same proposition. therefore both summands vanish individually, and a positivity argument yields then

$$\psi^+(F, F_0) = 0.$$

In particular F_0 is null and from (5.1) and the $\mathfrak{h}$-invariance of E and F we also get that F_0 is $\mathfrak{h}$-invariant. Now $F_0 \neq V$ since $\psi^+ \neq 0$ and if $F_0 = 0$ we have that E is isotropic, that is $\psi^+(E, E) \subseteq E$.

Step 2: Existence of an invariant nullspace.

Using Step 1, we can find a $\mathfrak{h}$-invariant subspace , say $\mathcal{V}$ in V, which is isotropic in the sense that $\psi^+(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$. Let us consider now the $\mathfrak{h}$-invariant tensor $r_1 : \mathcal{V} \to \mathcal{V}$ given by

$$\langle r_1 v, w\rangle = \mathrm{tr}_{\mathcal{V}}(\psi_v^+ \circ \psi_w^+)$$

for all v, w in $\mathcal{V}$. Obviously, this is symmetric and J-invariant. It follows that there is an orthogonal, $\mathfrak{h}$-invariant splitting

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1,$$

where $\mathcal{V}_0 = \mathrm{Ker}(r_1)$. Since $\mathcal{V}$ is isotropic, r_1 is given by $r_1 = -\sum_{k=1}^{d}(\psi^+_{v_k})^2 v$ for all v in $\mathcal{V}$, where $d := \dim_{\mathbb{R}} \mathcal{V}$ and $\{v_k, 1 \leq k \leq d\}$ is an orthonormal basis in $\mathcal{V}$. Then Proposition 5.1 (i) implies that $\psi^+_x(r_1 v) = 0$ for all x in H and for all v in $\mathcal{V}$, in other words

$$\psi^+(H, \mathcal{V}_1) = 0.$$

But the definition of $\mathcal{V}_0$ yields that $\psi^+(\mathcal{V}, \mathcal{V}_0) = 0$, in particular $\mathcal{V}_0$ is null. We now form the $\mathfrak{h}$ and J-invariant subspace $H_1 = \mathcal{V}_0 \oplus H_0$ which is easily seen to satisfy that

$$\psi^+(H_1, H_1) \subseteq H_1,\ \psi^+(\mathcal{V}_1, \mathcal{V}_1) \subseteq \mathcal{V}_1,\ \psi^+(\mathcal{V}_1, H_1) = 0.$$

In other words $V = \mathcal{V}_1 \oplus H_1$ is a splitting of our holonomy system in the sense of Definition 5.4 and the result is proved. □

This can be furthermore (see [49] for details) refined to:

Proposition 5.3. *Let $(V^{2m}, g, J, \psi^+, R)$ be an SNK-holonomy system. If it contains a proper invariant nullspace, it splits as*

$$V = V_1 \oplus V_2,$$

where V_2 contains a special invariant subspace.

Summarising the results obtained up to now we obtain, after an easy induction argument on the irreducible components of $(\mathfrak{h}, V)$:

Theorem 5.1. *Let $(V^{2m}, g, J, \psi^+, R)$ be a complex reducible SNK-holonomy system. Then V is product of SNK-holonomy systems belonging to one of the following classes:*

(i) *irreducible SNK-holonomy systems;*

(ii) *SNK-holonomy systems which contain a special invariant subspace.*

To advance with the classification of our holonomy systems we need therefore only to discuss the second class present in the theorem above. We first observe that given a special invariant subspace in some SNK-holonomy system one can explicitly determine the curvature along the special subspace.

Proposition 5.4. *Let $(V^{2m}, g, J, \psi^+, R)$ be an SNK-holonomy system containing a special invariant subspace $\mathcal{V}$. Then:*

$$R(\psi^+_x y, v_1, v_2, v_3) = \langle [\psi^+_{v_1}, [\psi^+_{v_2}, \psi^+_{v_3}]]x, Jy \rangle$$

whenever x, y are in $H = \mathcal{V}^\perp$ and for all v_k, $1 \leq k \leq 3$ in $\mathcal{V}$.

Actually, Proposition 5.4 singles out a second Lie algebra of relevance for us, obtained as follows. Consider the subspace $\mathfrak{p} \subseteq \lambda^2(H)$ given by

$$\mathfrak{p} := \{\psi^+_v : v \text{ in } \mathcal{V}\}$$

together with the subspace $\mathfrak{q}$ of $\lambda^{1,1}(H)$ given by $\mathfrak{q} := [\mathfrak{p}, \mathfrak{p}]$.

Proposition 5.5. *The following hold:*

(i) $\mathfrak{p} \cap \mathfrak{q} = 0$;

(ii) $\mathfrak{r} = \mathfrak{p} \oplus \mathfrak{q}$ *is a Lie subalgebra of* $\Lambda^2(H)$.

Proof. From Proposition 5.4 we get

$$[\psi^+_{v_1}, [\psi^+_{v_2}, \psi^+_{v_3}] = \psi^+_{R(v_1,v_2)v_3}$$

for all v_k, $1 \leq k \leq 3$ in $\mathcal{V}$. That is $\mathfrak{p}$ is a Lie triple system in the sense that $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \subseteq \mathfrak{p}$ hence $\mathfrak{p} + \mathfrak{q}$ is a Lie algebra (see [36] for details). To prove (i), let us pick z in $\mathfrak{p} \cap \mathfrak{q}$. Then $z = \psi^+_v$ for some v in $\mathcal{V}$ and moreover $[z, \mathfrak{p}] \subseteq \mathfrak{p}$. In particular $[\psi^+_v, \psi^+_{Jv}] = \psi^+_w$ for some w in $\mathcal{V}$, or equivalently $2(\psi^+_v)^2 = \psi^+_{Jw}$. But the left-hand side of this equality is symmetric whilst the right-hand is skew-symmetric which yields easily that $v = 0$. □

The Lie algebra $\mathfrak{r}$ is best though of as the holonomy algebra of a symmetric space of compact type. In the realm of NK-geometry this will turn out to be precisely the case.

5.3 Reduction to a Riemannian holonomy system

We shall consider in what follows an SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ containing a special invariant sub-space $\mathcal{V}$, with orthogonal complement to be denoted by H.

At this moment need a finer notion of irreducibility for an SNK-holonomy system containing an $\mathfrak{h}$-invariant special subspace. Let us now define $R^H : \Lambda^2(H) \to \Lambda^2(H)$ by

$$R^H(x, y) = R(x, y) + \psi^+_{\psi^+_x y}$$

for all x, y in H. Note this is well defined because $\psi^+(H, H) \subseteq \mathcal{V}$ and $\psi^+(\mathcal{V}, H) = H$ and also because H is $\mathfrak{h}$-invariant. Moreover, the Bianchi identity for R ensures that R^H is an algebraic curvature tensor on H, that is R^H belongs to $\mathcal{K}(\mathfrak{so}(H))$. The second Lie subalgebra of $\Lambda^2(H)$ of interest for us is

$$\mathfrak{h}^H = \mathrm{Lie}\{R^H(x, y) : x, y \text{ in } H\}.$$

Definition 5.5. $(H, g, \mathfrak{h}^H)$ is called the Riemannian holonomy system associated to $(V^{2m}, g, J, \psi^+, R)$. It is irreducible if the metric representation $(\mathfrak{h}^H, H)$ is irreducible.

The main result in this section is to prove that we can reduce, up to products in the sense of Definition 5.4 – hence of Riemannian type, the study of special SNK-holonomy to the case when the associated Riemannian holonomy system is irreducible.

Theorem 5.2. *Let $(V^{2m}, g, J, \psi^+, R)$ be an SNK-holonomy system containing an $\mathfrak{h}$-invariant subspace $\mathcal{V}$. Then V splits as*

$$V = V_1 \oplus \cdots \oplus V_q,$$

a product of special SNK-holonomy systems, such that each factor has irreducible associated Riemannian holonomy system.

Proof. Let $\mathcal{V}$ be a special $\mathfrak{h}$-invariant subspace and let us orthogonally decompose

$$H = H_1 \oplus H_2$$

in invariant subspaces for the action of $\mathfrak{h}^H$. As a straightforward consequence of having R in $S^2(\lambda^{1,1})$ we note that

$$R^H(\psi_v^+) = \frac{1}{2}\psi_{rv}^+ \tag{5.3}$$

on H, for all v in $\mathcal{V}$. Since $R^H(H, H, H_1, H_2) = 0$ by assumption and since r is invertible on $\mathcal{V}$ it follows that $\psi_{\mathcal{V}}^+ H_1$ and H_2 are orthogonal, hence

$$\psi_{\mathcal{V}}^+ H_k \subseteq H_k, \quad k = 1, 2 \tag{5.4}$$

given that $\psi_{\mathcal{V}}^+ H \subseteq H$. It also follows that $\psi^+(H_1, H_2) = 0$. We will need now several preliminary steps.

Step 1: H_k are J-invariant for $k = 1, 2$.

From (5.4) it follows that $r^{\mathcal{V}}(H_k) \subseteq H_k$, $k = 1, 2$ where $r^{\mathcal{V}}\colon H \to H$ is defined by

$$r^{\mathcal{V}} x = -\sum_{k=1}^{d} (\psi_{v_k}^+)^2 x$$

for all x in H, where $\{v_k, 1 \leq k \leq d = dim_{\mathbb{R}}\mathcal{V}\}$ is some orthonormal basis in $\mathcal{V}$. But $r^{\mathcal{V}}$ has no kernel, as an easy consequence of the fact that ψ^+ is non-degenerate and $\mathcal{V}$ is special. It follows that $r^{\mathcal{V}}_{|H_k}\colon H_k \to H_k$ is injective, hence surjective, that is

$$r^{\mathcal{V}}(H_k) = H_k$$

for $k = 1, 2$. Again from (5.4) we find that $r^{\mathcal{V}}(JH_k) \subseteq H_k$ hence after applying $(r^{\mathcal{V}})^{-1}$ (which as we have seen preserves H_k) we get that $JH_k \subseteq H_k$ for $k = 1, 2$.

Step 2: A first decomposition of V.

Let $\mathcal{V}' := \mathcal{V} \oplus H_1$ and consider the orthogonal and J-invariant splitting

$$V = \mathcal{V}' \oplus H_2.$$

Since $R^H(H_1, H_2, H, H) = 0$ and also because $\psi^+(H_1, H_2) = 0$ we find that $R(H_1, H_2, H, H) = 0$. In other words we have $R(H, H)H_2 \subseteq H_2$ and also $R(H, \mathcal{V})H_2 = 0$ since $\mathcal{V}$ is $\mathfrak{h}$-invariant and R is symmetric. Now the Bianchi identity

for R, combined again with the $\mathfrak{h}$-invariance of $\mathcal{V}$ and its nullity gives

$$R(v,w)x = [\psi_v^+, \psi_w^+]x$$

for all v, w in $\mathcal{V}$ and for all x in H. Using again (5.4) we find that $R(\mathcal{V},\mathcal{V})H_2 \subseteq H_2$ so H_2 is actually $\mathfrak{h}$-invariant.

Step 3: The decomposition of $\mathcal{V}$.

Thus we may apply Proposition 5.1 (i) to the splitting $V = \mathcal{V}' \oplus H_2$ whence

$$\psi_x^+ \psi_v^+ w = 0$$

for all x in H_2 and whenever v, w belong to $\mathcal{V}'$. Let us define $\mathcal{V}_k = \psi^+(H_k, H_k) \subseteq \mathcal{V}$ for $k = 1, 2$ and notice these are J-invariant. Then

$$\psi^+(H_2, \mathcal{V}^1) = 0$$

in particular $\mathcal{V}^1$ and $\mathcal{V}^2$ are orthogonal, after taking the scalar product with elements in H_2. Therefore

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$$

after using that $\psi^+(H,H) = \mathcal{V}$ and $\psi^+(H_1, H_2) = 0$.

Now since H_1 and H_2 play equal rôles, by repeating the arguments above we also get that $\psi^+(\mathcal{V}^2, H_1) = 0$.

Step 4: Proof of the theorem.

We consider the orthogonal and J-invariant splitting

$$V = V_1 \oplus V_2,$$

where $V_k := \mathcal{V}^k \oplus H_k$, $k = 1, 2$. From Step 3 it follows that ψ^+ belongs to

$$\lambda^3(V_1) \oplus \lambda^3(V_2)$$

and moreover that H_k are invariant under $\mathfrak{h}$ for $k = 1, 2$. Let us prove that, say $\mathcal{V}^1$, is $\mathfrak{h}$-invariant too. From (5.1) we get

$$R(x,y)(\psi_{x_1}^+ x_2) = \psi_{R(x,y)x_1}^+ x_2 + \psi_{x_1}^+ R(x,y)x_2$$

for all x, y in V and x_1, x_2 in H_1 and for all x, y in H. Since H is $\mathfrak{h}$-invariant and we have seen that $\psi^+(H, H_1) = \psi^+(H_1, H_1) = \mathcal{V}^1$, the invariance of of $\mathcal{V}^1$ follows and that of $\mathcal{V}^2$ is proved analogously. We conclude that V^k are invariant under $\mathfrak{h}$, for $k = 1, 2$.

The claim follows now by induction on the number of irreducible components of $(\mathfrak{h}^H, H)$, provided we show that $\mathfrak{h}^H$ does not fix any vector in H. But this is implied by (5.3), for such a vector, say x_0 in H would satisfy then $\psi^+(\mathcal{V}, x_0) = 0$. Taking scalar products with elements in H this yields $\psi^+(x_0, H) = 0$ as $\psi^+(H,H) \subseteq \mathcal{V}$. This is to say that $\psi_{x_0}^+ = 0$ hence $x_0 = 0$ since ψ^+ is non-degenerate. This proves the absence of fixed points for the representation $(\mathfrak{h}^H, H)$ and the proof is finished. □

More properties of the non-Riemannian holonomy representation $(\mathfrak{h}, V)$ can now be formulated.

Proposition 5.6. *Let (V^{2m}, g, J, ψ^+) be a special SNK-holonomy system containing an invariant subspace $\mathcal{V}$ and such that the Riemannian holonomy system $(\mathfrak{h}^H, H)$ is irreducible. Then the representation $(\mathfrak{h}, \mathcal{V})$ is irreducible over $\mathbb{C}$.*

Proof. Let us suppose that $\mathcal{V}$ is not irreducible and split $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$ as orthogonal sum of J-stable and $\mathfrak{h}$-invariant subspaces. Then $R(\psi^+_x y, v, v_1, v_2) = 0$ for all x, y in H, v in $\mathcal{V}$ and for all v_k in $\mathcal{V}^k$, $k = 1, 2$. Using Proposition 5.4 this leads to

$$[\psi^+_v, [\psi^+_{v_1}, \psi^+_{v_2}]] = 0$$

whenever v belongs to $\mathcal{V}$ and v_1, v_2 are in $\mathcal{V}^1$, $\mathcal{V}^2$ respectively. An easy invariance argument which can be found in [49], page 492 yields

$$\psi^+_{v_1}\psi^+_{v_2} = 0 \tag{5.5}$$

for all v_k in $\mathcal{V}^k$, $k = 1, 2$. We form $H^k := \psi_{\mathcal{V}^k} H$, $k = 1, 2$ which are therefore orthogonal, J-invariant and such that $H = H^1 \oplus H^2$. The spaces H^k, $k = 1, 2$ are actually invariant under $\mathfrak{h}$ a fact which follows from (5.1) and the $\mathfrak{h}$-invariance of $\mathcal{V}^k$, $k = 1, 2$. Now by using (5.5) we obtain that

$$\psi^+_{\psi^+_x y} H^k \subseteq H^k$$

for all x, y in H, since $\psi^+_{\mathcal{V}^k} H \subseteq H$ and $\psi^+(H, H) \subseteq \mathcal{V}$. This means, see also the definition of the tensor R^H, that H^k, $k = 1, 2$ are invariant under $\mathfrak{h}^H$ hence they cannot be both proper since $(\mathfrak{h}^H, H)$ is irreducible. If $H^1 = 0$ for instance, a routine argument leads to $\mathcal{V}^1 = 0$, a contradiction and the proof is finished. □

5.4 Metric properties

Let (V, g, J, ψ^+, R) be an SNK-holonomy system with special invariant subspace $\mathcal{V}$ and such that the associated representation $(\mathfrak{h}^H, H)$ is irreducible. Let us define the tensors Ric, $\overline{\text{Ric}}$ and C exactly as in the beginning of this section, and note they are still subject to (4.4) and (4.5) as it follows from the corresponding proofs. Their invariance under $\mathfrak{h}$ is an easy consequence of (4.4). We shall prove here that one reduces the discussion to the case when C has only a few eigenvalues but skip most of the technical details. The eigenspaces of the tensor C will be used to construct invariant subspaces of the representation $(\mathfrak{h}, H)$ which is not necessarily irreducible, in contrast to $(\mathfrak{h}, V)$ which is irreducible by Proposition 5.6.

Proposition 5.7. *The tensor C has at most* 3*-eigenvalues.*

Proof. Directly from its definition C preserves the splitting $V = \mathcal{V} \oplus H$. Since C is $\mathfrak{h}$-invariant and $(\mathfrak{h}, \mathcal{V})$ is irreducible it follows that $C_{|\mathcal{V}} = \lambda 1_{\mathcal{V}}$ for some real number λ. Using (4.5) we find that the symmetric and J-invariant tensor $S\colon H \to H$, $S := C_{|H} + \frac{\lambda}{2}$ satisfies

$$S(\psi^+_v x) = -\psi^+_v S x,$$

hence

$$S^2(\psi_v^+ x) = \psi_v^+ S^2 x \tag{5.6}$$

for all x in H and for all v in $\mathcal{V}$. Since C is $\mathfrak{h}$-invariant, so is S^2, and in fact the latter turns out to be $\mathfrak{h}^H$-invariant, after using (5.6). It follows that S^2 is a multiple of the identity and the claim follows. □

Let us consider the tensor $r^{\mathcal{V}}\colon H \to H$ given by

$$r^{\mathcal{V}} = -\sum_{k=1}^{2d} (\psi_{v_k}^+)^2,$$

where $2d = \dim_{\mathbb{R}} \mathcal{V}$ and $\{v_k, 1 \le k \le 2d\}$ is some orthonormal basis in $\mathcal{V}$. It is easy to see that $r^{\mathcal{V}}$ is positive, J-commuting, symmetric and $\mathfrak{h}$-invariant and moreover that $r_{|H} = 2r^{\mathcal{V}}$.

When C has only one eigenvalue, then $C = 0$ and one can show [49] by using (4.4) that both Ric and r must actually by diagonal on V.

It remains to investigate the cases when C has two, respectively three eigenvalues which are described below.

Proposition 5.8. *The following hold:*

(i) *If C has exactly two eigenvalues then there exists $k > 0$ such that $r^{\mathcal{V}} = k1_H$ and moreover the eigenvalues together with the corresponding eigenspaces for the tensors r, C,* Ric *are given in the following table:*

Eigenvalue	r	Ric	C	*Eigenspace*
λ_1	$\frac{n-d}{d}k$	$\frac{n+7d}{4d}k$	$\frac{4(n-3d)}{d}k$	$\mathcal{V}$
λ_2	$2k$	$\frac{n+2d}{2d}k$	$-\frac{2(n-3d)}{d}k$	H

(ii) *If C has exactly 3-eigenvalues we have a $\mathfrak{h}$-invariant, orthogonal and J-invariant splitting*

$$V = \mathcal{V} \oplus H_1 \oplus H_2$$

such that $\psi^+(H_1, H_2) = \psi^+(H_2, H_2) = 0$ and $\psi^+(H_1, H_2) = \mathcal{V}$.

(iii) *In all cases, including when $C = 0$, we have*

$$\mathrm{Ric}^H = \mu g_{|H}$$

for some $\mu > 0$, where Ric^H *denotes the Ricci contraction of R^H.*

We refer the reader to [49] for details of the proof and also mention that in case (ii) above it is also possible to display the relations between the eigenvalues of all relevant symmetric endomorphisms.

5.5 The final classification

By Theorem 5.2 it is enough to consider an SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ with a special invariant subspace $\mathcal{V}$ and such that the Riemannian holonomy system $(\mathfrak{h}^H, H)$ is irreducible, in the notations of the previous section. We shall combine here some well-known results for irreducible Riemannian holonomy systems and the information which is available in our present context. To progress in this direction we will first make clear the relationship between the Lie algebra $\mathfrak{r}$ and the Riemannian holonomy algebra $\mathfrak{h}^H$.

Proposition 5.9. *The following hold:*

(i) $\mathfrak{r}$ *is an ideal in* $\mathfrak{h}^H$*;*

(ii) $R^H = \frac{\nu}{2} 1_{\mathfrak{r}}$ *on* $\mathfrak{r}$ *for some* $\nu > 0$ *such that* $r_{|\mathcal{V}} = \nu 1_{\mathcal{V}}$.

Proof. (i) That $\mathfrak{p} \subseteq \mathfrak{h}^H$ follows from (5.3) hence $\mathfrak{q} = [\mathfrak{p}, \mathfrak{p}]$ is contained in $\mathfrak{h}^H$ as well. To see that $\mathfrak{r}$ is an ideal we use (5.1) to observe that

$$[R(x, y), \psi_v^+]$$

belongs to $\mathfrak{p}$ for all x, y in H and for all v in $\mathcal{V}$. The definition of R^H and that of $\mathfrak{r}$ now yields

$$[R^H(x, y), \psi_v^+]$$

in $\mathfrak{r}$ and it easy to conclude by using the Jacobi identity and Proposition 5.5.

(ii) First of all let us observe that r preserves $\mathcal{V}$, since the latter is special. The $\mathfrak{h}$-invariance of r, together with the irreducibility of $(\mathfrak{h}, \mathcal{V})$ implies the existence of a constant $\nu > 0$ such that $r_{|\mathcal{V}} = \nu 1_{\mathcal{V}}$. For the action of R^H on $\mathfrak{p}$ our claim follows now from (5.3). To prove it on $\mathfrak{q}$ we observe we proceed as follows. Given that R belongs to $S^2(\Lambda^2)$ we can alternatively rewrite (5.1) under the form

$$R(\psi_{x_1}^+ x_2, x_3) + R(x_2, \psi_{x_1}^+ x_3) = R(x_1, \psi_{x_2}^+ x_3)$$

whenever x_k are in $V, k = 1, 2, 3$. Using this for $x_1 = v, x_2 = \psi_w^+ e_i, x_3 = e_i$ where v, w are in $\mathcal{V}$ and $\{e_i\}$ is some orthonormal basis in H yields after summation and rearrangement of terms

$$\sum_i R(e_i, [\psi_v^+, \psi_w^+] e_i) = R(v, rw)$$

that is $R([\psi_v^+, \psi_w^+]) = \frac{\nu}{2}[\psi_v^+, \psi_w^+]$ on H, after also using (5.2). After a straightforward invariance argument this leads to $R^H([\psi_v^+, \psi_w^+]) = \frac{\nu}{2}[\psi_v^+, \psi_w^+]$ and the claim is proved. □

Now we recall that for a given metric representation $(\mathfrak{l}, W)$ of some Lie algebra $\mathfrak{l}$ we have the Berger list of irreducible Riemannian holonomies, and proceed to the

final classification result, hence proving Theorem 1.1 in the introduction.

$\mathfrak{l}$	W
$\mathfrak{so}(n)$	$\mathbb{R}^n$
$\mathfrak{u}(m)$	$\mathbb{R}^{2m}$
$\mathfrak{su}(m)$	$\mathbb{R}^{2m}$
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R}^{4m}$
$\mathfrak{sp}(m)$	$\mathbb{R}^{4m}$
$\mathfrak{spin}(7)$	$\mathbb{R}^8$
$\mathfrak{g}_2$	$\mathbb{R}^7$

Theorem 5.3. *Let (M^{2m}, g, J) be a complete SNK-manifold. Then M is, up to finite cover, a Riemannian product whose factors belong to the following classes:*

(i) *homogeneous SNK-manifolds;*

(ii) *twistor spaces over positive quaternionic Kähler manifolds;*

(iii) 6*-dimensional SNK-manifolds.*

Proof. First of all, because M must be compact with finite fundamental group we may assume up to a finite cover that it is simply connected. Pick now a point x in M and consider the holonomy representation $(\mathrm{Hol}_x(\overline{\nabla}), T_xM)$. If this is irreducible over $\mathbb{R}$ or complex irreducible but real reducible we conclude by Theorem 4.4.

If the holonomy representation of $\overline{\nabla}$ is reducible over $\mathbb{C}$ at x we consider the SNK-holonomy system $(T_xM, g_x, J_x, \psi_x^+, \bar{R}_x)$. By the theorem of Ambrose–Singer we have that $\mathfrak{h} \subseteq \mathfrak{hol}_x(\overline{\nabla})$. Also note that when *starting* the whole reduction procedure leading to Theorem 5.2 from some $\mathfrak{hol}_x(\overline{\nabla})$ complex invariant sub-space we end up with a splitting

$$T_xM = V_1 \oplus \cdots \oplus V_p$$

with the properties in Theorem 5.2 and which is furthermore $\mathfrak{hol}_x(\overline{\nabla})$-invariant together with the special sub-spaces contained in each factor. Using parallel transport w.r.t. to $\overline{\nabla}$ this extends to a $\overline{\nabla}$-parallel decomposition, which is actually ∇-parallel since the torsion of $\overline{\nabla}$ is split along the decomposition. The splitting theorem of de Rham applies and our study is reduced to that of SNK-manifolds such that at each point, we have an invariant special subspace having the property that the associated holonomy system is irreducible in the sense of Definition 5.5. Then we show (see [49]) that M is the total space of a Riemannian submersion

$$\pi : (M, g) \to (N^{2n}, h)$$

whose fibres are totally geodesic and w.r.t. the induced metric and almost complex structure are simply connected, irreducible and compact Hermitian symmetric spaces. The tensor R^H defined in Section 5.3 projects onto the Riemann curvature tensor of h making of $\mathfrak{h}^H$ a subalgebra of $\mathfrak{hol}(N,h)$ at each point. Using this, one shows that the Riemannian manifold (N^{2n},h) is irreducible and Proposition 5.8 (iii) gives that h is Einstein of positive scalar curvature.

If the tensor C has three eigenvalues a short manipulation of the second Bianchi identity for $\overline{\nabla}$ combined with the algebraic structure of the splitting of TM given in (ii) of Proposition 5.8 gives that $\overline{\nabla}$ is an Ambrose–Singer connection.

Let us discuss now the case when C has two eigenvalues or it vanishes.

If (N^{2n},h) is a symmetric space the relations of O'Neill (see [6]) for the Riemannian submersion $\pi: M \to N$ essentially say that the curvature tensor $\bar{R}$ is an explicit expression of ψ^+, R^H and the curvature of the fibre (which is a symmetric space) along the $\overline{\nabla}$-parallel splitting $TM = \mathcal{V} \oplus H$. This information results easily in having $\overline{\nabla}$ an Ambrose–Singer connection again.

Suppose now that (N^{2n},h) is not a symmetric space. Then the holonomy representation of (N,h), at the Lie algebra level, corresponds to the representation $(\mathfrak{h}^H, H)$ and as it is well known (see [52]), must be one of the entries in the Berger list above. We shall mainly use now Proposition 5.9. The Lie algebras $\mathfrak{su}(m)$, $\mathfrak{sp}(m)$, $\mathfrak{spin}(7)$, $\mathfrak{g}_2$ are excluded because their curvature tensors are Ricci-flat. Supposing that $\mathfrak{h}^H = \mathfrak{u}(n)$, given that the ideal $\mathfrak{r}$ is at least two-dimensional we can only have $\mathfrak{r} = \mathfrak{su}(n), \mathfrak{u}(n)$. But using Proposition 5.9 combined with the fact that a curvature tensor of Kähler-Einstein type is completely determined by its restriction to $\mathfrak{su}(m)$ we find that (N^{2n},h) is symmetric a contradiction. Similar arguments together with results from [5] enable us to conclude in the case when $\mathfrak{h}^H = \mathfrak{so}(2n)$ or when $\mathfrak{r} = \mathfrak{sp}(m)$. The only case remaining is when $\mathfrak{h}^H = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ and $\mathfrak{r} = \mathfrak{sp}(1)$ which implies that $\mathcal{V}$ is of rank 2. Hence the holonomy group of $\overline{\nabla}$ is contained in $\mathrm{U}(1) \times \mathrm{U}(2m)$ and the fact that (M^{2m}, g, J) is a twistor space over a positive quaternionic-Kähler manifold has been proved in [48]. □

6 Concluding remarks

Due to considerations of space and time we have omitted to present some important facts related to nearly-Kähler geometry. The first aspect is related to the construction of examples, which is implicit in our present treatment. Given a quaternion-Kähler manifold (M^{4m}, g, Q) of positive scalar curvature the Salamon twistor construction [51] yields a Kähler manifold (Z,h,J) such that there is a Riemannian submersion with totally geodesic, complex fibres

$$S^2 \hookrightarrow (Z,h) \to (M,g).$$

Using this structure and a canonical variation of the metric h it has been shown [19] that Z admits an SNK-metric. Enlarging the context to that of Riemannian submersions with complex, totally geodesic fibres from Kähler manifolds one draws a similar conclusion [48] (see also [50] for some related facts). This observation can also be used to describe the homogeneous SNK-manifolds which appear in Theorem 5.3 as twistor spaces in the sense of [11] over symmetric spaces of compact type.

We have also omitted to discuss NK-structures in dimension 6, which have a rich geometry, although not yet fully understood. If (M^6, g, J) is a strict NK manifold the structure group automatically reduces to SU(3), hence $c_1(M, J) = 0$ [33]. Moreover the metric g must be Einstein [33], of positive scalar curvature. There is also a 1-1 correspondence between 6-dimensional manifolds admitting real Killing spinors and SNK-structures in dimension 6 [35].

In the compact case the only known examples are homogeneous, and various characterisations of these instances are available [12], [45]. However, compact examples with two conical singularities have been very recently constructed in [23] building on the fact that one can suitably rotate the SU(2)-structure of a Sasakian–Einstein 5-manifold and then extend it to a SU(3)-structure of NK-type through a generalised cone construction.

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Chapter 11

Homogeneous nearly Kähler manifolds

Jean-Baptiste Butruille

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Introduction

Probably the first known example of a nearly Kähler manifold is the round sphere in dimension 6, equipped with its well-known non integrable almost complex structure, introduced in [21]. The resulting *almost Hermitian structure* is invariant for the action of G_2 on S^6 coming from the octonions (we look at S^6 as the unit sphere in the imaginary set $\Im \subset \mathbb{O}$). Thus $S^6 \simeq G_2/\mathrm{SU}(3)$ is an example of a 6-*dimensional homogeneous nearly Kähler manifold* as we consider in the present exposition. Notice also that the representation of SU(3) on the tangent spaces – the isotropy representation – is irreducible. Then Joseph A. Wolf in his book *Spaces of constant curvature* [46] discovered a class of isotropy irreducible homogeneous spaces G/H that generalizes S^6, where G is a compact Lie group and H, a maximal connected subgroup, centralizing an element of order 3. Later on, Wolf and Gray [46] investigated the homogeneous spaces defined by Lie group automorphisms i.e. such that H is the fixed point set of some automorphism $s\colon G \to G$. They asked the following question: which of these spaces admit an invariant almost Hermitian structure with good properties? In particular, nearly Kähler manifolds are associated, through their work, to a type of homogeneous spaces – the 3-symmetric spaces – corresponding to s of order 3. Since their study was quite general, they felt legitimate to make a conjecture, which I reformulate using the terminology of the later article [24] by Gray alone:

Conjecture 1 (Gray and Wolf). Every nearly Kähler homogeneous manifold is a 3-symmetric space equipped with its canonical almost complex structure.

Another way to construct examples is by twistor theory. The twistor space Z of a self-dual 4-manifold has a natural complex structure by [3], but also a non integrable almost complex structure (see [16]). The latter may be completed into a nearly Kähler structure when the base is furthermore Einstein, with positive scalar curvature. The same construction holds for the twistor spaces of the positive quaternion-Kähler manifolds (see [2], [36]) or of certain symmetric spaces (as explained for instance in [40]). In this case Z is a 3-symmetric space again.

Next, in the '70s [22], [25] and more recently (we mention, as a very incomplete list of references on the topic, [38], [6], [36] and [35]), nearly Kähler manifolds have been studied for themselves. Some very interesting properties were discovered, especially in dimension 6, that give them a central role in the study of special geometries with torsion. These properties can all be interpreted in the setting of *weak holonomy*. Nearly Kähler 6-dimensional manifolds are otherwise called *weak holonomy* SU(3). Many definitions of this notion have been proposed. One is by spinors [4]; others by differential forms [41], [10]; the original one by Gray [23] was found incorrect (see [1]). Finally, Cleyton and Swann [42], [14], [15] have worked on a definition based on the torsion of the canonical connection of a G-structure. They obtain a theorem which can be applied to our problem to show that nearly Kähler manifolds whose canonical connection has *irreducible* holonomy are either 3-symmetric or 6-dimensional. This constitutes an advance towards and gives a new reason to believe Gray and Wolf's conjecture. However, the theorem of Cleyton and Swann does not say anything about dimension 6. Moreover, when the holonomy is reducible, we do not have a de Rham-like theorem, like in the torsion-free situation. It was Nagy's main contribution to this issue to show that we can always lead back, in this case, to the twistor situation. Using this, he was able to reduce Conjecture 1 to dimension 6 [35].

This is where we resume his work. We classify 6-dimensional nearly Kähler homogeneous spaces and show that they are all 3-symmetric.

Theorem 1. *Nearly Kähler, 6-dimensional, Riemannian homogeneous manifolds are isometric to a finite quotient of G/H equipped with the naturally reductive metric induced by the Killing form of G, where the groups G, H are given in the following list:*

- $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ *and* $H = \mathrm{SU}(2)$ (*diagonal*).
- $G = G_2$ *and* $H = \mathrm{SU}(3)$. *In this case G/H is the round 6-sphere.*
- $G = \mathrm{Sp}(2)$ *and* $H = \mathrm{SU}(2)\mathrm{U}(1)$. *Then, $G/H \simeq \mathbb{C}P^3$, the 3-dimensional complex projective space.*
- $G = \mathrm{SU}(3)$, $H = \mathrm{U}(1) \times \mathrm{U}(1)$ *and $G/H \simeq \mathbb{F}^3$ is the space of flags of $\mathbb{C}^3$.*

Each of the spaces $S^3 \times S^3$, $\mathbb{C}P^3$ and $\mathbb{F}^3$ carries a unique invariant nearly Kähler structure, up to homothety. The invariant nearly Kähler almost complex structures on the sphere S^6 are all conjugated to each other by an isometry. They form a space isomorphic to $\mathbb{R}P^7$.

As a corollary we obtain

Theorem 2. *Conjecture 1 is true in dimension 6 and thus, by the work of Nagy, in all dimensions.*

The proof is a case by case study. We start (Lemma 5.1) by making a list of the pairs (G, H) such that G/H is likely, for topological reasons, to admit an invariant nearly Kähler structure. The homogeneous spaces that appear in this list are the four compact 6-dimensional examples of 3-symmetric spaces found in [46]. Then we show, for each space, that there exists no other homogeneous nearly Kähler structure than the canonical almost complex structure on it.

As a consequence, this chapter is mainly focused on examples. However, the four spaces in question are quite representative of a number of features in nearly Kähler geometry. Thus, it might be read as a sort of survey on the topic. In Section 1, we introduce the notions and speak of 3-symmetric spaces in general. Section 2 contains the most difficult point in the proof of Theorem 1. We look for left-invariant nearly Kähler structures on $S^3 \times S^3$. For this we had to use the algebra of differential forms on the manifold, nearly Kähler manifolds in dimension 6 being characterized, by the work of Reyes Carrión [38] or Hitchin [31], by a differential system on the canonical SU(3)-structure. Section 3 is devoted to two homogeneous spaces $\mathbb{C}P^3$ and $\mathbb{F}^3$ that are the twistor spaces of two 4-dimensional Riemannian manifolds: respectively S^4 and $\mathbb{C}P^2$. We take the opportunity to specify the relation between nearly Kähler geometry and twistor theory. Weak holonomy stands in the background of Section 4 on the 6-sphere. Indeed, we may derive this notion from that of *special holonomy* trough the construction of the Riemannian cone as in [4]. Six-dimensional nearly Kähler – or weak holonomy SU(3) – structures on S^6 are in one-to-one correspondence with constant 3-forms, inducing a reduction of the holonomy to G_2 on $\mathbb{R}^7$. Finally, in Section 5, we provide the missing elements for the proof of Theorems 1 and 2.

We should mention here the remaining conjecture on nearly Kähler manifolds:

Conjecture 2. Every *compact* nearly Kähler manifold is a 3-symmetric space.

That this conjecture is still open means in particular that the homogeneous spaces presented along this chapter are the only known compact (or equivalently, complete) examples in dimension 6. Again by the work of Nagy [35], it may be separated in two restricted conjectures. The first one relates to a similar conjecture on quaternion-Kähler manifolds and symmetric spaces, for which there are many reasons to believe that it is true (to begin with, it was solved by Poon and Salamon [37] in dimension 8 and recently by Haydeé and Rafael Herrera [28] in dimension 12). The second may be formulated: the only compact, simply connected, *irreducible* (with respect to the holonomy of the intrinsic connection), 6-*dimensional*, nearly Kähler manifold is the sphere S^6 – and concerns the core of the nearly Kähler geometry: the fundamental explanation of the rareness of such manifolds or the difficulty to produce non-homogeneous examples.

1 Preliminaries: nearly Kähler manifolds and 3-symmetric spaces

Nearly Kähler manifolds are a type of *almost Hermitian manifolds* i.e. $2n$-dimensional real manifolds with a $\mathrm{U}(n)$-structure (a $\mathrm{U}(n)$-reduction of the frame bundle) or equivalently, with a pair of tensors (g, J) or (g, ω), where g is a Riemannian metric, J an almost complex structure compatible with g in the sense that

$$g(JX, JY) = g(X, Y) \quad \text{for all } X, Y \in TM$$

(J is orthogonal with respect to g pointwise) and ω is a differential 2-form, called the *Kähler form*, related to g, J by

$$\omega(X, Y) = g(JX, Y) \quad \text{for all } X, Y \in TM.$$

Associated with g there is the well-known Levi-Civita connection, ∇, metric preserving and torsion-free. But nearly Kähler manifolds, as every almost Hermitian manifolds, have another natural connection $\overline{\nabla}$, called the *intrinsic connection* or the *canonical Hermitian connection*, which shall be of considerable importance in the sequel. Let $\mathfrak{so}(M)$ be the bundle of skew-symmetric endomorphisms of the tangent spaces (the adjoint bundle of the metric structure). The set of metric connections of (M, g) is an affine space $\mathcal{SO}$ modelled on the space of sections of $T^*M \otimes \mathfrak{so}(M)$. Then the set $\mathcal{U}$ of Hermitian connections (i.e. connections which preserve both the metric and the almost complex structure or the Kähler form) is an affine subspace of $\mathcal{SO}$ with vector space $\Gamma(T^*M \otimes \mathfrak{u}(M))$, where $\mathfrak{u}(M)$ is the subbundle of $\mathfrak{so}(M)$ formed by the endomorphisms which commute with J (or in other words, the adjoint bundle of the $\mathrm{U}(n)$-structure). Finally, we denote by $\mathfrak{u}(M)^\perp$ the orthogonal complement of $\mathfrak{u}(M)$ in $\mathfrak{so}(M)$, identified with the bundle of skew-symmetric endomorphisms of TM, anti-commuting with J.

Definition 1.1. The canonical Hermitian connection $\overline{\nabla}$ is the orthogonal projection of $\nabla \in \mathcal{SO}$ on $\mathcal{U}$. Equivalently, it is the unique Hermitian connection such that $\nabla - \overline{\nabla}$ is a 1-form with values in $\mathfrak{u}(M)^\perp$.

The difference $\eta = \nabla - \overline{\nabla}$ is known explicitly:

$$\eta_X = \frac{1}{2} J \circ (\nabla_X J) \quad \text{for all } X \in TM.$$

It measures the failure of the $\mathrm{U}(n)$-structure to admit a torsion-free connection, in other words its *torsion* or its 1-jet (see [8]). Thus, it can be used to classify almost Hermitian manifolds as in [26]. For example, Kähler manifolds are defined by ∇ itself being a Hermitian connection: $\nabla = \overline{\nabla}$. Equivalently, since η determines both $d\omega$ and the Nijenhuis tensor N, ω is closed and J is integrable.

Definition 1.2. Let M be an almost Hermitian manifold. The following conditions are equivalent and define a nearly Kähler manifold:

(i) the torsion of $\overline{\nabla}$ is totally skew-symmetric,

(ii) $(\nabla_X J)X = 0$ for all $X \in TM$,

(iii) $\nabla_X \omega = \frac{1}{3} \iota_X d\omega$ for all $X \in TM$,

(iv) $d\omega$ is of type

$$(3,0) + (0,3)$$

and N is totally skew-symmetric.

The following result, due to Kirichenko [32], is the base of the partial classification by Nagy in [35] of nearly Kähler manifolds.

Proposition 1.3 (Kirichenko). *For a nearly Kähler manifold, the torsion of the intrinsic connection is totally skew-symmetric* (*by definition*) *and parallel:*

$$\overline{\nabla} \eta = 0.$$

Moreover, this is equivalent to $\overline{\nabla} \nabla \omega = 0$ or $\overline{\nabla} d\omega = 0$ because $\overline{\nabla}$ is Hermitian.

Now, suppose that the curvature of $\overline{\nabla}$ is also parallel: $\overline{\nabla}\, \overline{R} = 0$. Then M is *locally homogeneous* or an *Ambrose–Singer manifold*. Besides, the associated *infinitesimal model* is always *regular* (for a definition of these notions, see [15], citing [43]) and thus, if it is simply connected, M is an homogeneous space. Examples obtained in this way belong to a particular class of homogeneous manifolds: the 3-*symmetric spaces*, defined by Gray [24], which shall interest us in the rest of this section. As expected, the 3-symmetric spaces are a generalization of *symmetric* spaces:

Definition 1.4. A 3-symmetric space is a homogeneous space $M = G/H$, where G has an automorphism s of order 3 (instead of an involution, for a symmetric space) such that

$$G_0^s \subset H \subset G^s, \tag{1}$$

where $G^s = \{g \in G \mid s(g) = g\}$ is the fixed points set of s and G_0^s is the identity component of G^s.

Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H, respectively. For a symmetric space, the eigenspace $\mathfrak{m}$ for the eigenvalue -1 of $ds\colon \mathfrak{g} \to \mathfrak{g}$ (the derivative of s) is an $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$, so that symmetric spaces are always *reductive*. Conversely, for a reductive homogeneous space, we define an endomorphism f of $\mathfrak{g}$ by $f|_{\mathfrak{h}} = \mathrm{Id}|_{\mathfrak{h}}$ and $f|_{\mathfrak{m}} = -\mathrm{Id}|_{\mathfrak{m}}$, which integrates into a Lie group automorphism of G if and only if

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{2}$$

For a 3-symmetric space, things are slightly more complicated because ds has now three eigenvalues, 1, $j = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $j^2 = \bar{j} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, two of which are complex. The corresponding eigenspaces decomposition is

$$g^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}_j \oplus \mathfrak{m}_{j^2}.$$

Setting $\mathfrak{m} = (\mathfrak{m}_j \oplus \mathfrak{m}_{j^2}) \cap \mathfrak{g}$ we get

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}, \tag{3}$$

so 3-symmetric spaces are also reductive. As a consequence, invariant tensors, for the left action of G on M, are represented by constant, $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$. For example, an invariant almost complex structure is identified, first with the subbundle $T^+M \subset T^{\mathbb{C}}M$ of $(1,0)$-vectors, then with a decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^- \quad \text{where } \mathrm{Ad}(H)\mathfrak{m}^+ \subset \mathfrak{m}^+ \text{ and } \mathfrak{m}^- = \overline{\mathfrak{m}^+}. \tag{4}$$

Definition 1.5. The canonical almost complex structure of a 3-symmetric space is the invariant almost complex structure associated to the decomposition $\mathfrak{m}_j \oplus \mathfrak{m}_{j^2}$.

In other words, the restriction $ds\colon \mathfrak{m} \to \mathfrak{m}$ represents an invariant tensor S of M satisfying

(i) $S^3 = \mathrm{Id}$,

(ii) for all $x \in M$, 1 is not an eigenvalue of S_x.

One can then write S as a (non trivial) third root of unity:

$$S = -\frac{1}{2}\mathrm{Id} + \frac{\sqrt{3}}{2}J, \tag{5}$$

where J is the canonical almost complex structure of M.

Similarly, an $\mathrm{Ad}(H)$-invariant scalar product g on $\mathfrak{m}$ defines an invariant metric on M, also denoted by g, and the pair (M, g) is called a *Riemannian* 3-symmetric space if and only if g, J are compatible.

Conversely a decomposition like (4) comes from an automorphism of order 3 if and only if $\mathfrak{h}$, $\mathfrak{m}^+$, $\mathfrak{m}^-$ satisfy

$$[\mathfrak{m}^+, \mathfrak{m}^+] \subset \mathfrak{m}^-, \quad [\mathfrak{m}^-, \mathfrak{m}^-] \subset \mathfrak{m}^+ \quad \text{and} \quad [\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{h}^{\mathbb{C}} \tag{6}$$

instead of (2). Now, conditions involving the Lie bracket might be interpreted, on a reductive homogeneous space, as conditions on the torsion and the curvature of the normal connection $\widehat{\nabla}$. The latter is defined as the H-connection on G whose horizontal distribution is $G \times \mathfrak{m} \subset TG \simeq G \times \mathfrak{g}$.

Lemma 1.6. *The torsion $\widehat{T}$ and the curvature $\widehat{R}$ of the normal connection $\widehat{\nabla}$, viewed as constant tensors, are respectively the $\mathfrak{m}$-valued 2-form and the $\mathfrak{h}$-valued 2-form on $\mathfrak{m}$ given by*

$$\widehat{T}(X,Y) = -[X,Y]^{\mathfrak{m}} \quad \textit{and} \quad \widehat{R}_{X,Y} = [X,Y]^{\mathfrak{h}} \quad \textit{for all } X, Y \in \mathfrak{m}.$$

Proposition 1.7. *A reductive almost Hermitian homogeneous space $M = G/H$ is a 3-symmetric space if and only if it is quasi-Kähler and the intrinsic connection $\overline{\nabla}$ coincides with $\widehat{\nabla}$.*

An almost Hermitian manifold is said *quasi-Kähler* or $(2,1)$-*symplectic* iff $d\omega$ has type $(3,0)+(0,3)$ or equivalently η (or ∇J) is a section of $\mathbf{\Lambda}^1 \otimes \mathfrak{u}(M)^\perp \cap [\![\boldsymbol{\lambda}^{2,0}]\!] \otimes TM$ where $[\![\boldsymbol{\lambda}^{2,0}]\!] \subset \mathbf{\Lambda}^2$ is the bundle of real 2-forms of type $(2,0)+(0,2)$.

Proof. By Lemma 1.6, (6) is equivalent to

$$\widehat{T}(\mathfrak{m}^+,\mathfrak{m}^+) \subset \mathfrak{m}^-, \quad \widehat{T}(\mathfrak{m}^-,\mathfrak{m}^-) \subset \mathfrak{m}^+, \quad \widehat{T}(\mathfrak{m}^+,\mathfrak{m}^-) = \{0\}$$
$$\widehat{R}(\mathfrak{m}^+,\mathfrak{m}^+) = \widehat{R}(\mathfrak{m}^-,\mathfrak{m}^-) = \{0\}.$$

The first line implies that $\hat{\eta} = \nabla - \widehat{\nabla} \in \mathbf{\Lambda}^1 \otimes \mathfrak{u}(M)^\perp$ (for a metric connection, the torsion and the difference with the Levi-Civita connection are in one-to-one correspondence), i.e. $\widehat{\nabla}$ is the canonical Hermitian connection. It also implies that $\eta = \hat{\eta} \in [\![\boldsymbol{\lambda}^{2,0}]\!] \otimes TM$ (note that the bundles $\mathfrak{u}(M)^\perp \simeq [\![\boldsymbol{\lambda}^{2,0}]\!]$ are isomorphic through the operation of raising, or lowering, indices). The second line is automatically satisfied for a quasi-Kähler manifold, see [17]. □

There is also a local version of Proposition 1.7, as we announced after Proposition 1.3.

Theorem 1.8. *An almost Hermitian manifold M is locally 3-symmetric if and only if it is quasi-Kähler and the torsion and the curvature of the intrinsic connection $\overline{\nabla}$ satisfy*

$$\overline{\nabla} T = 0 \quad \textit{and} \quad \overline{\nabla}\, \overline{R} = 0.$$

The definition of a *locally 3-symmetric space* given in [24] relates to the existence of a family of *local cubic isometries* $(s_x)_{x\in M}$ such that, for all $x \in M$, x is an isolated fixed point of s_x (for a 3-symmetric space, the automorphism s provides such a family, moreover the isometries are globally defined).

Then the requirement that M, the Riemannian 3-symmetric space, is a nearly Kähler manifold (which is more restrictive than quasi-Kähler) translates to a structural condition on the homogeneous space.

Definition 1.9. A reductive Riemannian homogeneous space is called naturally reductive iff the scalar product g on $\mathfrak{m}$ representing the metric satisfies

$$g([X,Y],Z) = -g([X,Z],Y) \quad \text{for all } X,Y,Z \in \mathfrak{m}.$$

Equivalently, the torsion $\widehat{T}$ of the normal connection is totally skew-symmetric. Now, for a 3-symmetric space, the intrinsic connection coincides with the normal connection by Proposition 1.7. Thus, we get the following result from [24]:

Proposition 1.10 (Gray). *A Riemannian 3-symmetric space equipped with its canonical almost complex structure is nearly Kähler if and only if it is naturally reductive.*

Remark. Let $M = G/H$ be a compact inner 3-symmetric space such that G is compact, simple. (NB: A 3-symmetric space is called *inner* if the automorphism s of

order 3 coincides with an inner automorphism.) The Killing form B of G is negative definite so it induces a scalar product $q = -B$ on $\mathfrak{g}$. Then the summand $\mathfrak{m}$, associated to the eigendecomposition of $ds = \mathrm{Ad}(h)$, is orthogonal to $\mathfrak{h}$ and the restriction of q to $\mathfrak{m}$ defines a naturally reductive metric that makes M a nearly Kähler manifold.

2 The case of $S^3 \times S^3$: the natural reduction to SU(3)

In [33], Ledger and Obata gave a procedure to construct a nearly Kähler 3-symmetric space for each compact Lie group G. The Riemannian product $G \times G \times G$ has an obvious automorphism of order 3, given by the cyclic permutation, whose fixed point set is the diagonal subgroup $\Delta G = \{(x, x, x) \mid x \in G\} \simeq G$. The resulting homogeneous space is naturally isomorphic to $G \times G$: to fix things, we shall identify (x, y) with $[x, y, 1]$. In other words, we get a 3-symmetric structure on $G \times G$, invariant for the action:

$$((h_1, h_2, h_3), (x, y)) \mapsto (h_1 x h_3^{-1}, h_2 y h_3^{-1}).$$

Now, let q be an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$, representing a bi-invariant metric on G. We choose, for the $\mathrm{Ad}(\Delta G)$-invariant complement of $\delta\mathfrak{g}$ (the Lie algebra of ΔG) in $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$, $\mathfrak{m} = \{0\}\oplus\mathfrak{g}\oplus\mathfrak{g}$, the sum of the *last* two factors, so that the restriction of $q \oplus q \oplus q$ to $\mathfrak{m}$ defines a naturally reductive metric g which is *not* the bi-invariant metric of $G \times G$. Indeed, the vector $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g} \simeq T_e(G \times G)$ is identified with $(0, Y - X, X) \in \mathfrak{m}$ so we have the explicit formula $g_e((X, Y), (X', Y')) = q(Y - X, Y' - X') + q(X, X')$.

Following this procedure and setting $G = \mathrm{SU}(2) \simeq S^3$, we obtain a 6-dimensional example: $S^3 \times S^3$. In this section we will look for nearly Kähler structures on $S^3 \times S^3$ invariant for the smaller group

$$\begin{aligned} \mathrm{SU}(2) \times \mathrm{SU}(2) &\hookrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \\ (h_1, h_2) &\mapsto (h_1, h_2, 1), \end{aligned}$$

or for the left action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on itself. Such a structure is then simply given by constant tensors on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The latter is a 6-dimensional vector space so, for example, the candidates for the metric belong to a 21-dimensional space $S^2(\mathfrak{su}(2)\oplus\mathfrak{su}(2))$. Thus, the calculations involving the Levi-Civita connection as in Definition 1.2 are too hard and we shall look for another strategy.

Nearly Kähler manifolds in dimension 6 are special. In particular they are always Einstein [25], spin, and possess a Killing spinor [27]. But the most important feature, for us, is a natural reduction to SU(3). Indeed, an SU(3)-structure in dimension 6, unlike an almost Hermitian structure, may be defined, without the metric, only by means of differential forms. This should be compared to the fact that a G_2-structure is determined by a special differential 3-form on a 7-manifold.

In first instance, an SU(3)-manifold is an almost Hermitian manifold with a complex volume form (a complex 3-form of type $(3,0)$ and constant norm) Ψ. This complex 3-form can be decomposed into real and imaginary parts: $\Psi = \psi + i\phi$ where

$$\iota_X \psi = \iota_{JX}\phi \quad \text{for all } X \in TM. \tag{7}$$

As a consequence, one of the *real* 3-forms ψ or ϕ, together with J, determines the reduction of the manifold to SU(3). Now, for a nearly Kähler manifold, such a form is naturally given by the differential of the Kähler form. Indeed, because of our preliminaries, $d\omega$ has type

$$(3,0) + (0,3)$$

(see Definition 1.2) and constant norm, since it is parallel for the metric connection $\bar{\nabla}$.

Definition 2.1. The natural SU(3)-structure of a 6-dimensional nearly Kähler manifold is defined by

$$\psi := \frac{1}{3} d\omega,$$

where ω is the Kähler form.

On the other hand, Hitchin has observed in [31] that a differential 2-form ω and a differential 3-form ψ, satisfying certain algebraic conditions, are enough to define a reduction of the manifold to SU(3). In particular they determine the metric g and the almost complex structure J. Indeed, SU(3) may be seen as the intersection of two groups, $\mathrm{Sp}(3,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{C})$, which are themselves the stabilizers of two exteriors forms on $\mathbb{R}^6$. The symplectic group $\mathrm{Sp}(3,\mathbb{R})$ stabilizes a non-degenerate 2-form of course. The second group, $\mathrm{GL}(6,\mathbb{R})$, has two open orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\Lambda^3\mathbb{R}^6$. The stabilizer of the forms in the first orbit is $\mathrm{SL}(3,\mathbb{C})$. To define the action of the latter, we must see $\mathbb{R}^6$ as the complex vector space $\mathbb{C}^3$. Consequently, if a differential 3-form ψ belongs to $\mathcal{O}_1$ at each point, it determines an almost complex structure J on M. Then ω, J determine g under certain conditions.

We need to write this explicitly (after [30], [31]). For $\psi \in \mathbf{\Lambda}^3$, we define $K \in \mathrm{End}(TM) \otimes \mathbf{\Lambda}^6$ by

$$K(X) = A(\iota_X \psi \wedge \psi),$$

where $A\colon \mathbf{\Lambda}^5 \to TM \otimes \mathbf{\Lambda}^6$ is the isomorphism induced by the exterior product. Then $\tau(\psi) = \frac{1}{6}\mathrm{tr}\, K^2$ is a section of $(\mathbf{\Lambda}^6)^2$ and it can be shown that

$$K^2 = \mathrm{Id} \otimes \tau(\psi).$$

The 3-form ψ belongs to $\mathcal{O}_1$ at each point if and only if

$$\tau(\psi) < 0. \tag{8}$$

Then, for $\kappa = \sqrt{-\tau(\psi)}$,

$$J = \frac{1}{\kappa} K$$

is an almost complex structure on M. Moreover, the 2-form ω is of type $(1,1)$ with respect to J if and only if

$$\omega \wedge \psi = 0. \tag{9}$$

Finally, ω has to be non degenerate

$$\omega \wedge \omega \wedge \omega \neq 0, \tag{10}$$

and g has to be positive:

$$(X, Y) \mapsto g(X, Y) = \omega(X, JY) > 0. \tag{11}$$

This is at the algebraic level. At the geometric level, Salamon and Chiossi [13] have shown that the 1-jet of the SU(3)-structure (or the intrinsic torsion) is completely determined by the differentials of ω, ψ, ϕ. In particular, nearly Kähler manifolds are viewed, in this section, as SU(3)-manifolds satisfying a first order differential system involving these three forms:

$$\begin{cases} d\omega = 3\psi, \\ d\phi = -2\mu\, \omega \wedge \omega, \end{cases} \tag{12}$$

where $\mu \in \mathbb{R}^+$. This differential system was first written by Reyes Carrión in [38].

As a consequence, looking for a nearly Kähler structure on a manifold is the same as looking for a pair of forms (ω, ψ) satisfying (8)–(11) together with (12) or, considering the particular form of (12), for a 2-form ω only, satisfying a highly non linear second order differential equation. We shall resolve this system on the space of invariant 2-forms of $S^3 \times S^3$.

We work with a class of co-frames $(e_1, e_2, e_3, f_1, f_2, f_3)$, called *cyclic co-frames*, satisfying

(i) $(e_1, e_2, e_3, f_1, f_2, f_3)$ is invariant for the action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on itself,

(ii) the 1-forms e_i (resp. f_i), $i = 1, 2, 3$ vanish on the tangent space of the first (resp. the second) factor,

(iii) $de_i = e_{i+1} \wedge e_{i+2}$ where the subscripts are viewed as elements of $\mathbb{Z}_3$. Similarly, $df_i = f_{i+1} \wedge f_{i+2}$.

The group of isometries of the sphere $S^3 \simeq \mathrm{SU}(2)$, equipped with its round bi-invariant metric is SO(4), with isotropy subgroup SO(3). Moreover the isotropy representation lifts to the adjoint representation of $\mathrm{SU}(2) \simeq \mathrm{Spin}(3)$. We denote $(u, l) \mapsto u.l$ the action of SO(3) on the *dual* $\mathfrak{su}(2)^*$ of the Lie algebra. Then $\mathrm{SO}(3) \times \mathrm{SO}(3)$ acts transitively on the set of cyclic co-frames by

$$(u, v).(e_1, e_2, e_3, f_1, f_2, f_3) \mapsto (u.e_1, u.e_2, u.e_3, v.f_1, v.f_2, v.f_3). \tag{13}$$

In other words, two such co-frames are exchanged by a diffeomorphism of $S^3 \times S^3$ and more precisely by an isometry of the canonical metric.

Now, any invariant 2-form may be written

$$\omega = \sum_{i=1}^{3} a_i e_{i+1} \wedge e_{i+2} + \sum_{i=1}^{3} b_i f_{i+1} \wedge f_{i+2} + \sum_{i,j=1}^{3} c_{i,j} e_i \wedge f_j. \tag{14}$$

Let A be the column vector of the a_i, B the column vector of the b_i and C the square matrix $(c_{i,j})_{i,j=1,2,3}$. The latter is subject to the following transformation rule in a change of cyclic co-frame (13):

$$C \mapsto MC\,{}^tN, \tag{15}$$

where the 3×3 matrices M (resp. N) represent u (resp. v) in the old base.

We have the first essential simplification of (14):

Lemma 2.2. *Let ω be a non degenerate invariant 2-form on $S^3 \times S^3$. Then ω, $\psi = \frac{1}{3} d\omega$ satisfy (9) if and only if there exists a cyclic co-frame $(e_1, e_2, e_3, f_1, f_2, f_3)$ such that*

$$\omega = \lambda_1 e_1 \wedge f_1 + \lambda_2 e_2 \wedge f_2 + \lambda_3 e_3 \wedge f_3, \tag{16}$$

where $\lambda_i \in \mathbb{R}^*$ for all $i = 1, 2, 3$.

Proof. Starting from (14), we calculate $\omega \wedge \omega \wedge \omega$. The 2-form ω is non degenerate if and only if

$${}^tACB + \det C \neq 0. \tag{17}$$

Then we calculate $\psi = \frac{1}{3} d\omega$ using the relations (iii) in the definition of a cyclic co-frame. We find that $\omega \wedge \psi = 0$ is equivalent to ${}^tAC = CB = 0$. Reintroducing these equations in (17), we get $\det C \neq 0$, i.e. C is nonsingular, and so $A = B = 0$.

Secondly, we can always suppose that C is diagonal. Indeed, we write C as the product of a symmetric matrix S and an orthogonal matrix $O \in \mathrm{SO}(3)$. We diagonalize S: there exists an orthogonal matrix P such that $S = {}^tPDP$ where D is diagonal. Thus $C = {}^tPD(PO)$ and by (15), ω can always be written in the form (16) where $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. □

This is a key lemma that will constitute the base of our next calculations.

Lemma 2.3. *Let ω be the invariant 2-form given by (16) in a cyclic co-frame. Then ω, $\psi = \frac{1}{3} d\omega$ define an $\mathrm{SU}(3)$-structure on $S^3 \times S^3$ if and only if*

(i) $(\lambda_1 - \lambda_2 - \lambda_3)(-\lambda_1 + \lambda_2 - \lambda_3)(-\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3) < 0$,

(ii) $\lambda_1 \lambda_2 \lambda_3 > 0$.

Proof. The first condition is simply (8). Indeed,

$$3\psi = \lambda_1(e_{23} \wedge f_1 - e_1 \wedge f_{23}) + \lambda_2(e_{31} \wedge f_2 - e_2 \wedge f_{31}) + \lambda_3(e_{12} \wedge f_3 - e_3 \wedge f_{12})$$

and we calculate

$$81\tau(\psi) = (\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2\lambda_1^2\lambda_2^2 - 2\lambda_2^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2) \otimes \mathrm{vol}^2 \tag{18}$$

where $\mathrm{vol} = e_{123} \wedge f_{123}$. Now, the polynomial of degree 4 in the λ_i's in (18) factors into (i) of Lemma 2.3.

The second condition comes from the positivity of the metric. Note that the product $\lambda_1\lambda_2\lambda_3 = \det C$ is independent on the choice of the co-frame $(e_1, e_2, e_3, f_1, f_2, f_3)$, the determinant of the matrices M, N, in (15), being equal to 1. First, we compute the almost complex structure in the dual frame $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$:

$$JX_i = \alpha_i X_i + \beta_i Y_i, \quad JY_i = -\beta_i X_i - \alpha_i Y_i, \tag{19}$$

where

$$\alpha_i = \frac{1}{k}(\lambda_i^2 - \lambda_{i+1}^2 - \lambda_{i+2}^2) \quad \text{and} \quad \beta_i = -\frac{2}{k}\lambda_{i+1}\lambda_{i+2},$$
$$9\kappa = k\,\mathrm{vol}.$$

Now, $X \mapsto \omega(X, JX)$ is the sum of three quadratic forms of degree 2:

$$q_i : (x_i, y_i) \mapsto \frac{2\lambda_i}{k}\big(\lambda_{i+1}\lambda_{i+2}x_i^2 - (\lambda_i^2 - \lambda_{i+1}^2 - \lambda_{i+2}^2)x_i y_i + \lambda_{i+1}\lambda_{i+2}y_i^2\big).$$

The discriminant of these forms is $k^2 > 0$ so they are definite and their sign is given by $\lambda_1\lambda_2\lambda_3$. □

Using (19) it is easy to compute ϕ, by (7), and translate the second line of (12) into a system of equations *in the* λ_i*'s*. We refer to [11] for a detailed proof.

Lemma 2.4. *Let ω be the invariant 2-form given by (16) in a cyclic co-frame. Then ω, $\psi = \frac{1}{3}d\omega$ induce a nearly Kähler structure on $S^3 \times S^3$ if and only if there exists $\mu \in \mathbb{R}$ such that, for all $i = 1, 2, 3$,*

$$c = \lambda_i^2(\lambda_i^2 - \lambda_{i+1}^2 - \lambda_{i+2}^2), \tag{20}$$

where

$$c = -2\mu k \det C.$$

Finally we can conclude:

Proposition 2.5. *There exists a unique (up to homothety, up to a sign) left-invariant nearly Kähler structure on $S^3 \times S^3$, corresponding to Ledger and Obata's construction of a 3-symmetric space.*

Proof. Thanks to the preparatory work, we only need to solve the system (20) of equations of degree 4 in the λ_i's. Let $\Lambda = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. For all $i = 1, 2, 3$, λ_i^2 is a solution of the unique equation of degree 2:

$$2x^2 - \Lambda x - c = 0. \tag{21}$$

Suppose that λ_1^2, λ_2^2 are two distinct solutions of (21) and $\lambda_3^2 = \lambda_2^2$, for example. The sum of the roots $\lambda_1^2 + \lambda_2^2$ equals $\frac{\Lambda}{2}$. But then we also have $\Lambda = \lambda_1^2 + 2\lambda_2^2$, by definition of Λ. We immediately get $\lambda_1 = 0$, i.e. ω is degenerate, a contradiction. Thus, the λ_i's

must be equal, up to a sign. The positivity of the metric, (11) or Lemma 2.3 (ii), implies that either the three signs are positive, or only one of them. These two solutions are in fact the same: one is obtained from the other by a rotation of angle π in the first factor. Finally, one can always write ω, for a left-invariant nearly Kähler structure,

$$\omega = \lambda(e_1 \wedge f_1 + e_2 \wedge f_2 + e_3 \wedge f_3)$$

where $\lambda = \lambda_1 = \lambda_2 = \lambda_3 \in \mathbb{R}^{+*}$.

We also have $k = \lambda^2\sqrt{3}$,

$$JX_i = \frac{1}{\lambda^2\sqrt{3}}(-X_i + 2Y_i), \quad JY_i = \frac{1}{\lambda^2\sqrt{3}}(-2X_i + Y_i). \tag{22}$$

Now, (22) coincides with the canonical almost complex structure of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\Delta\mathrm{SU}(2)$. Indeed, the automorphism of order 3, $s : (h_1, h_2, h_3) \mapsto (h_2, h_3, h_1)$, induces the endomorphism $S : (X, Y) \mapsto (Y - X, -X)$ on $\mathfrak{m} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Then, by (5), J is identified with

$$J : (X, Y) \mapsto \frac{1}{\sqrt{3}}(2Y - X, -2X + Y).$$

This is nothing else than (22) with $\lambda = 1$ for an appropriate choice of the base.

NB: $c = 2\mu\lambda^5\sqrt{3}$ so by (20), $\mu = \frac{1}{2\lambda\sqrt{3}}$ is inversely proportional to the norm of ω. □

3 Twistor spaces: the complex projective space $\mathbb{C}P^3$ and the flag manifold $\mathbb{F}^3$

The twistor space Z of a 4-dimensional, Riemannian, oriented manifold (N, h) is equipped with two natural almost complex structures. The first, J_+, studied by Atiyah, Hitchin and Singer [3], is integrable as soon as N is *self-dual*, i.e one half of the Weyl tensor of h vanishes, while the second J_-, which was first considered by Salamon and Eels in [16], is never integrable. Now, on Z, varying the scalar curvature of the fibre, there are also a 1-parameter family of metrics $(g_t)_{t\in]0,+\infty[}$ such that the twistor fibration over (N, h) is a Riemannian submersion and for all t, $(g_t, J_\pm)$ is an almost Hermitian structure on Z. A natural problem is then to look at the type of this almost Hermitian structure. We are particularly interested in the case where N is compact, self-dual and Einstein.

Theorem 3.1 (Hitchin, Eels and Salamon). *Let (N, h) be a compact, Riemannian, oriented, self-dual, Einstein 4-manifold, Z its twistor space and $(g_t)_{t\in[0,+\infty[}$, the twistor metrics. There exists a choice of the parameter t such that the scalar curvature of the fibre of $\pi : Z \to N$ is proportional to t and*

(i) *(Z, g_2, J_+) is Kähler,*

(ii) (Z, g_1, J_-) *is nearly Kähler.*

This provides us with two compact, homogeneous, nearly Kähler structures in dimension 6 on the complex projective space $\mathbb{C}P^3$ and the flag manifold $\mathbb{F}^3$, the twistor spaces of S^4 and $\mathbb{C}P^2$, respectively. Moreover, we shall see that they correspond to a 3-symmetric structure. The goal of this section is to prove that these are the only invariant nearly Kähler structures on the above mentioned spaces.

This is quite easy for $\mathbb{C}P^3$. The complex projective space is seen, in this context, as $\mathrm{Sp}(2)/\mathrm{U}(1)\mathrm{Sp}(1)$. More generally, $\mathbb{C}P^{2q+1}$ is isomorphic to $\mathrm{Sp}(q+1)/\mathrm{U}(1)\mathrm{Sp}(q)$. Indeed, $\mathrm{Sp}(q+1)$ acts transitively on $\mathbb{C}^{2q+2} \simeq \mathbb{H}^{q+1}$, preserving the complex lines, and the isotropy subgroup at $x \in \mathbb{C}P^{2q+1}$ fixes also jx and acts on the orthogonal of $\{x, jx\}$, identified with $\mathbb{H}^q$, as $\mathrm{Sp}(q)$. Representing $\mathfrak{sp}(q)$ as usual in the set of $q \times q$ matrices, the embedding of $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{sp}(1)$, the Lie algebra of $H = \mathrm{U}(1)\mathrm{Sp}(1)$, in $\mathfrak{g} = \mathfrak{sp}(2)$ is given by the composition of the natural maps $\mathfrak{u}(1) \hookrightarrow \mathfrak{sp}(1)$ and the identity of $\mathfrak{sp}(1)$, followed by the diagonal map $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \hookrightarrow \mathfrak{sp}(2)$. Thus, a natural choice of complement of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{v}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \mid a \in \mathbb{H} \right\} \quad \text{and} \quad \mathfrak{v} = \left\{ \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \mid b = jx + ky,\ x, y \in \mathbb{R} \right\}.$$

These two subsets are $\mathrm{Ad}(H)$-invariant so their sum too and $\mathfrak{m}$ may be identified with the isotropy representation of $\mathrm{Sp}(2)/\mathrm{U}(1)\mathrm{Sp}(1)$. The restriction of $\mathrm{Ad}(H)$ to $\mathfrak{p}$ is irreducible because the induced representation of $\mathrm{Sp}(1)$ is isomorphic to the standard one on $\mathbb{H}$. Similarly, the restriction of $\mathrm{Ad}(H)$ to $\mathfrak{v}$ induces the standard representation of $\mathrm{U}(1)$ on $\mathbb{C}$. As a consequence, $\mathfrak{m}$ has exactly two irreducible summands and the set of invariant metrics on $\mathbb{C}P^3$ has dimension 2. Moreover, we can gain one degree of freedom by working "up to homothety". Finally, we get a 1-parameter family of metrics which may be identified with $(g_t)_{t\in[0,+\infty[}$. On the other hand, $\mathbb{C}P^3$ has $2^2 = 4$ invariant almost complex structures according to [46], Theorem 4.3: $\pm J_+$ and $\pm J_-$. Thus, we are in the hypothesis of Muskarov's work [34]. The conclusion is, as in Theorem 3.1,

Proposition 3.2. *The homogeneous space* $\mathbb{C}P^3 \simeq \mathrm{Sp}(2)/\mathrm{U}(1)\mathrm{Sp}(1)$ *has a unique – up to homothety, up to a sign – invariant nearly Kähler structure* (g_1, J_-)*, associated to the twistor fibration of* S^4*.*

Things are more complicated for $\mathbb{F}^3$ because the isotropy representation has three irreducible summands. We could adapt Muskarov's proof [34] for this case, as was done in [11], calculating ∇J for every g with Levi-Civita connection ∇ in the 3-dimensional space of invariant metrics, and every invariant almost complex structure J, or we can look more carefully at the structure of $\mathbb{F}^3$, as we will do now.

The flag manifold is the space of pairs (l, p) where $p \subset \mathbb{C}^3$ is a complex plane and $l \subset p$ is a complex line. It is isomorphic to G/H where $G = \mathrm{U}(3)$ and $H = S^1 \times S^1 \times S^1$. Indeed, we see $\mathrm{U}(3)$ as the space of unitary bases of $\mathbb{C}^3$. Then the

map $(u_1, u_2, u_3) \mapsto (l, p)$, where $l = \mathbb{C}u_1$ and $p = \mathbb{C}u_1 \oplus \mathbb{C}u_2$, is an H-principal bundle over $\mathbb{F}^3$ with total space G.

Now, the maps $\pi_a : (l, p) \mapsto \mathbb{C}u_a, a = 1, 2, 3$ are well-defined (because $\mathbb{C}u_1 = l$; $\mathbb{C}u_2 = l^\perp$ is the orthogonal line of l in p and finally $\mathbb{C}u_3 = p^\perp$ is the orthogonal of p) and give three different $\mathbb{C}P^1$-fibrations from $\mathbb{F}^3$ to $\mathbb{C}P^2$ (or, if we identify the base spaces, three different realizations of $\mathbb{F}^3$ as an almost complex submanifold of $\mathcal{Z}$, the twistor space of $\mathbb{C}P^2$). This has the following geometrical interpretation: on the fibre of π_3, l varies inside p; on the fibre of π_1, it is the plane p that varies around the line l; finally, on the fibre of π_2, both l and p vary while $l^\perp$ is fixed. These fibrations are all *twistor* fibrations over an Einstein self-dual 4-manifold. Let I_+, J_+, K_+ and I_-, J_-, K_- be the associated almost complex structures, as in Proposition 3.1. What is remarkable is that the Kählerian structures are all distinct but the non-integrable nearly Kähler structures coincide: $I_- = J_- = K_-$. This observation was already made by Salamon in [40], Section 6. We shall prove this at the infinitesimal level. As a consequence, the four almost complex structures described above, and their opposites, exhaust all the $8 = 2^3$ invariant almost complex structures on $\mathbb{F}^3 \simeq \mathrm{U}(3)/(S^1)^3$.

Let $\mathfrak{g} = \mathfrak{u}(3)$ be the set of the trace-free, anti-Hermitian, 3×3 matrices. Then $\mathfrak{h} = 3\mathfrak{u}(1)$ is identified with the subgroup of the diagonal matrices and the set $\mathfrak{m}$ of the matrices with zeros on the diagonal is an obvious $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. Denote

$$\langle a, b, c\rangle = \begin{pmatrix} 0 & -\bar{a} & b \\ a & 0 & -\bar{c} \\ -\bar{b} & c & 0 \end{pmatrix}$$

for all $a, b, c \in \mathbb{C}$. We have

$$\mathrm{Ad}_h\langle a, b, c\rangle = \langle e^{i(s-t)}a, e^{i(t-r)}b, e^{i(r-s)}c\rangle \quad \text{for all } h = \begin{pmatrix} e^{ir} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & e^{it} \end{pmatrix} \in H \tag{23}$$

It is easily seen from (23) that the isotropy representation decomposes into

$$\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q} \oplus \mathfrak{r} \tag{24}$$

where

$$\begin{aligned} \mathfrak{p} &= \{\langle a, 0, 0\rangle \mid a \in \mathbb{C}\}, \\ \mathfrak{q} &= \{\langle 0, b, 0\rangle \mid b \in \mathbb{C}\}, \\ \mathfrak{r} &= \{\langle 0, 0, c\rangle \mid c \in \mathbb{C}\}. \end{aligned}$$

Each of these 2-dimensional subspaces, $\mathfrak{a}$, has a natural scalar product $g_\mathfrak{a}$ and a natural complex structure $J_\mathfrak{a}$. For example, on $\mathfrak{p}$,

$$g_\mathfrak{p}(\langle a, 0, 0\rangle, \langle a', 0, 0\rangle) = \mathrm{Re}(a\bar{a}') \quad \text{and} \quad J_\mathfrak{p}\langle a, 0, 0\rangle = \langle ia, 0, 0\rangle.$$

Moreover, we denote by $\mathfrak{p}^+ \oplus \mathfrak{p}^-$ the decomposition of $\mathfrak{p}_\mathbb{C}$ associated to $J_\mathfrak{p}$ (and similarly for $\mathfrak{q}$, $\mathfrak{r}$). The relations between the three subspaces $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$ are given by

the Lie brackets:

$$[\langle a,0,0\rangle,\langle 0,b,0\rangle]=\langle 0,0,ab\rangle,\quad \text{etc.} \tag{25}$$

and

$$[\langle a,0,0\rangle,\langle a',0,0\rangle]=\operatorname{diag}(iy,-iy,0)\in\mathfrak{h},\quad \text{where } y=2\operatorname{Im}(a\bar{a}'),\quad \text{etc.} \tag{26}$$

From (25) and (26), it is easy to calculate

$$[\mathfrak{p}^+,\mathfrak{q}^+]\subset\mathfrak{r}^-,\quad [\mathfrak{p}^+,\mathfrak{r}^-]\subset\mathfrak{q}^+\quad\text{and}\quad [\mathfrak{p}^+,\mathfrak{p}^+]=\{0\},\quad\text{etc.}$$

Thus, $\mathfrak{p}^+\oplus\mathfrak{q}^+\oplus\mathfrak{r}^-$ is a subalgebra of $\mathfrak{m}_{\mathbb{C}}$, corresponding to an invariant *complex* structure on $\mathbb{F}^3$: K_+. In the same way, I_+, J_+ are represented by $\mathfrak{p}^-\oplus\mathfrak{q}^+\oplus\mathfrak{r}^+$, $\mathfrak{p}^+\oplus\mathfrak{q}^-\oplus\mathfrak{r}^+$, respectively. On the contrary, $\mathfrak{m}^+=\mathfrak{p}^+\oplus\mathfrak{q}^+\oplus\mathfrak{r}^+$ is *not* a subalgebra. However, it satisfies (6) with $\mathfrak{m}^-=\overline{\mathfrak{m}^+}=\mathfrak{p}^-\oplus\mathfrak{q}^-\oplus\mathfrak{r}^-$. Thus, $\mathbb{F}^3$ is a 3-symmetric space with $\mathfrak{m}^+$ as canonical almost complex structure. By Proposition 1.7, each pair $(g,\pm J_-)$, where g is a generic invariant metric,

$$g=rg_{\mathfrak{p}}+sg_{\mathfrak{q}}+tg_{\mathfrak{r}},$$

is a $(2,1)$-symplectic homogeneous structure on $\mathbb{F}^3$. Moreover, since I_+, J_+, K_+ are integrable, every invariant (strictly) nearly Kähler structure on $\mathbb{F}^3$ has that form, where g is naturally reductive (see Proposition 1.10). It is not hard to see that this corresponds to $r=s=t$.

Proposition 3.3. *The nearly Kähler structures associated to the three natural twistor fibrations over $\mathbb{C}P^2$ on $\mathbb{F}^3\simeq \mathrm{U}(3)/(S^1)^3$ coincide. Moreover, every invariant nearly Kähler structure on $\mathbb{F}^3$ is proportional to this one (or to its opposite).*

Remark. The decomposition (24) is still the irreducible decomposition for $\mathrm{SU}(3)\subset\mathrm{U}(3)$, so the results remain valid for this smaller group of isometries. This observation will be useful in Section 5.

4 Weak holonomy and special holonomy: the case of the sphere S^6

While $S^3\times S^3$ is considered the hardest case because the isotropy is reduced to $\{0\}$, the case of S^6 is apparently the easiest because the isotropy is maximal, $H=\mathrm{SU}(3)$. The isotropy representation is the standard representation of $\mathrm{SU}(3)$ in dimension 6, in particular it is irreducible so there exists only one metric up to homothety and also one almost complex structure up to a sign preserved by H. However, a difficulty occurs since on S^6 unlike on the other spaces considered in sections 2 or 3, the metric doesn't determine the almost complex structure:

Proposition 4.1. *Let (M,g) be a complete Riemannian manifold of dimension 6, not isomorphic to the round sphere. If there exists an almost complex structure J on M*

such that (M, g, J) *is nearly Kähler* (*non Kählerian*) *then it is unique. Moreover, in this case,* J *is invariant by the isometry group of* g.

This can be proved using the spinors (by [27], a 6-dimensional Riemannian manifold admits a nearly Kähler structure if and only if it carries a real Killing spinor): see [11], Proposition 2.4, and the reference therein, [5], Proposition 1, p. 126, or by a "cone argument" (see below) as in [44], Proposition 4.7.

On the contrary, on the sphere S^6 equipped with its round metric g_0, there exist infinitely many compatible nearly Kähler structures:

Proposition 4.2. *The set* $\mathcal{J}$ *of almost complex structures* J *such that* (S^6, g_0, J) *is nearly Kähler, is isomorphic to* $\mathrm{SO}(7)/G_2 \simeq \mathbb{R}P^7$.

Corollary 4.3. *Nearly Kähler structures compatible with the canonical metric on* S^6, *or almost complex structures* $J \in \mathcal{J}$, *are all conjugated by the isometry group* $\mathrm{SO}(7)$ *of* g_0.

To show this, we shall use a theorem of Bär [4]: the Riemannian cone of a nearly Kähler manifold has holonomy contained in G_2. However, in order to remain faithful to the point of view of differential forms adopted in this chapter we prefer the presentation by Hitchin [31] of this fact. According to Section 2, a nearly Kähler structure is determined by a pair of differential forms (ω, ψ) satisfying (12) as well as algebraic conditions (8)–(11). Moreover there exists, around each point, an orthonormal co-frame $(e_1, \ldots, e_6)$ such that

$$\omega = e_{12} + e_{34} + e_{56}$$

and

$$\psi = e_{135} - e_{146} - e_{236} - e_{245},$$

where $e_{12} = e_1 \wedge e_2$, $e_{135} = e_1 \wedge e_3 \wedge e_5$, etc. Now, the cone of (M, g) is the Riemannian manifold $(\bar{M}, \bar{g})$, where $\bar{M} = M \times \mathbb{R}^+$ and $\bar{g} = r^2 g + dr^2$ in the coordinates (x, r). We define a section ρ of $\Lambda^3 \bar{M}$ by

$$\rho = r^2 dr \wedge \omega + r^3 \psi \tag{27}$$

Let $u_0 = dr$ and for all $i = 1, \ldots, 6$, $u_i = \frac{1}{r} e_i$.

$$\rho = u_{012} + u_{034} + u_{056} + u_{135} - u_{146} - u_{236} - u_{245}$$

Thus, ρ is a generic 3-form, inducing a G_2-structure on the 7-manifold $\bar{M}$ such that $(u_0, \ldots, u_6)$ is an orthonormal co-frame for the underlying Riemannian structure i.e. $\bar{g}$ is the metric determined by ρ, given the inclusion of G_2 in $\mathrm{SO}(7)$. Moreover, if we denote by $*$ the Hodge dual of $\bar{g}$,

$$*\rho = -r^3 dr \wedge \phi + \frac{\mu}{2} r^4 \omega \wedge \omega.$$

Then (12) implies

$$\begin{cases} d\rho = 0 \\ d * \rho = 0 \end{cases}$$

By Gray–Fernandez [18], the last couple of equations is equivalent to $\nabla^{\bar{g}}\rho = 0$, where $\nabla^{\bar{g}}$ is the Levi-Civita connection of $\bar{g}$. In other words, the holonomy of $(\bar{M}, \bar{g})$ is contained in G_2.

Conversely, a parallel, generic 3-form on $\bar{M}$ can always be written as in (27) where (ω, ψ) define a nearly Kähler SU(3)-structure on M.

We are now ready to prove Proposition 4.2.

Proof. The Riemannian cone of the 6-sphere is the Euclidean space $\mathbb{R}^7$. According to the discussion above, a nearly Kähler structure on S^6 compatible with g_0, defines a parallel or equivalently, a constant 3-form on $\mathbb{R}^7$. This form must have the appropriate algebraic type, i.e. be an element of the open orbit $\mathcal{O} \simeq \mathrm{GL}(7,\mathbb{R})/G_2 \subset \Lambda^3\mathbb{R}^7$. But it must also induce the good metric (the cone metric) on $\mathbb{R}^7$. Finally the 3-forms that parametrize $\mathcal{J}$ belong to a subset of $\mathcal{O}$, isomorphic to $\mathrm{SO}(7)/G_2$. □

The homogeneous nearly Kähler structure on S^6 is defined using the octonions. The octonian product $(x, y) \mapsto x.y$ may be described in the following way. First, the 8-dimensional real vector space $\mathbb{O}$ decomposes into $\mathbb{R} \oplus \Im$. The subspace $\Im \simeq \mathbb{R}^7$ is called the space of imaginary octonions and equipped, for our purposes, with an inner product $(x, y) \mapsto \langle x, y\rangle$. Secondly, rules (i) to (iv) below are satisfied with respect to this decomposition:

(i) $1.1 = 1$,

(ii) for all $x \in \Im$, $1.x = x$,

(iii) for all imaginary quaternion x of norm 1, $x.x = -1$,

(iv) finally, for all orthogonal $x, y \in \Im$, $x.y = P(x, y)$ where $P : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ is the 2-*fold vector cross product*. The latter satisfies itself $(x, y, z) \mapsto \langle P(x, y), z\rangle$ is a 3-form. In particular $P(x, y)$ is orthogonal to x and y.

Now, let S^6 be the unit sphere in $\Im \simeq \mathbb{R}^7$. The tangent space at $x \in S^6$ is identified with the subspace of $\mathbb{R}^7$ orthogonal to x. Then J is defined by

$$\begin{aligned} J_x : T_xM &\longrightarrow T_xM, \\ y &\longmapsto x.y. \end{aligned}$$

This is a well defined almost complex structure because $x.y$ is orthogonal to x by rule (iv) and $J^2 = -\mathrm{Id}$ by rules (ii), (iii). Moreover J is compatible with the metric induced by $\langle\cdot,\cdot\rangle$ on S^6 (the round metric g_0).

Conversely, starting from $J \in \mathcal{J}$, we may rebuild an octonian product on $\mathbb{R}^7$ by bilinearity. Then J is invariant for the associated group of automorphisms, isomorphic to G_2. Consequently, all nearly Kähler structures on the round sphere are of the same kind (homogeneous). They correspond to different choices of embeddings of

G_2 into the group of isometries of (S^6, g_0). These embeddings are parametrized by $\mathrm{SO}(7)/G_2 \simeq \mathbb{R}P(7)$ thus the above discussion gives an alternative proof of Proposition 4.2.

The same question was raised by Friedrich in [19]. His proof is very similar to our first one though it uses the Hodge Laplacian instead of the differential system (12). Moreover, another proof is mentioned, that uses the Killing spinors.

Finally, a third or fifth way of looking at this is the following: each SU(3)-structure (ω_x, ψ_x) on a tangent space T_xS^6, $x \in S^6$, may be extended in a unique way to a nearly Kähler structure on the whole manifold. Let ρ be the constant 3-form on $\mathbb{R}^7$ whose value at $(x, 1)$ is

$$\rho_{(x,1)} = dr \wedge \omega_x + \psi_x.$$

Then ρ is parallel for the Levi-Civita connection of the flat metric and in return, $\omega = \iota_{\partial_r}\rho$, $\psi = \frac{1}{3}d\omega$ determine a nearly Kähler structure on S^6 whose values at x coincide with ω_x, ψ_x, consistent with our notations. Now, SU(3)-structures on a 6-dimensional vector space are parametrized by $\mathrm{SO}(6)/\mathrm{SU}(3)$ so this constitutes a (differential) geometric proof of the isomorphism

$$\frac{\mathrm{SO}(6)}{\mathrm{SU}(3)} \simeq \frac{\mathrm{SO}(7)}{G_2} \simeq \mathbb{R}P(7) \simeq \mathcal{J}.$$

5 Classification of 3-symmetric spaces and proof of the theorems

In this section, we draw all the useful conclusions of the facts gathered in the previous sections about nearly Kähler manifolds and synthesize all the results to achieve the proof of Theorems 1 and 2.

As such, the conjecture of Gray and Wolf is not easy to settle because the notion of a nearly Kähler manifold, or even of a 3-symmetric space, are too rich. Indeed, the classification of 3-symmetric spaces [46] discriminates between three types $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ that correspond to quite different geometries:

$\mathfrak{A}$. The first type consists of twistor spaces of symmetric spaces. Indeed, the situation described in the beginning of Section 3 has a wider application than the 6-dimensional twistor spaces of Einstein self-dual 4-manifolds. First, the study of quaternion-Kähler manifolds, i.e. Riemannian manifolds whose holonomy is contained in $\mathrm{Sp}(q)\mathrm{Sp}(1)$, provides an analog of that situation in dimension $4q$, $q \geq 2$. Such manifolds still admit a twistor space $Z \to M$ with fibre $\mathbb{C}P^1$ and two almost complex structures J_+ and J_- related by

$$J_+|_V = -J_-|_V, \quad J_+|_H = J_-|_H,$$

where V is the vertical distribution, tangent to the fibres, and H is the horizontal distribution of the Levi-Civita connection of the base, such that the first one, J_+, is

always integrable (see for example [39]) while the second, J_-, is non-integrable. Moreover, for a *positive* quaternion-Kähler manifold, there exist two natural metrics g_1 and g_2 such that (g_2, J_+) is a Kählerian structure and (g_1, J_-) is a nearly Kähler structure on Z (compare with Proposition 3.1). This includes the twistor spaces of the Wolf spaces: the compact, symmetric, quaternion-Kähler manifolds. Now, this construction can be extended to a larger class of inner symmetric spaces G/K (see [40]). The total space G/H is a *generalized flag manifold* (i.e. H is the centralizer of a torus in G) and a 3-symmetric space. In particular H contains a maximal torus of G (or has maximal rank) but it is not a maximal subgroup since the inclusion $H \subset K$ is strict.

$\mathfrak{B}$. The 3-symmetric spaces of the second type are those studied by Wolf [45], Theorem 8.10.9, p. 280. They are characterized by H being the connected centralizer of an element of order 3. Thus, H is not the centralizer of a torus anymore. However it still has maximal rank, i.e. the 3-symmetric space is *inner*, and is furthermore maximal (for an explicit description, using the extended Dynkin diagram of $\mathfrak{h}$, see [45], [46] or [12]).

$\mathfrak{C}$. Finally, the 3-symmetric spaces of the third type have rank $H <$ rank G. Equivalently M is an outer 3-symmetric space. This includes two exceptional spaces – $\mathrm{Spin}(8)/G_2$ and $\mathrm{Spin}(8)/\mathrm{SU}(3)$ – and the infinite family $G \times G \times G/G$ of Section 2.

This division, which was obtained in [46] by algebraic means (or group theory), has a profound geometrical interpretation. Indeed, the three classes can be characterized by the type of the isotropy representation:

$\mathfrak{A}$. In the first case, the vertical distribution V and the horizontal distribution H of $G/H \to G/K$ are invariant by the left action of G. Moreover, by definition of the natural almost complex structures, they are stable by $J_\pm$. Thus, the isotropy representation is *complex reducible* (we identify J_- with the multiplication by i on the tangent spaces).

$\mathfrak{B}$. The exceptional spaces that constitute the second class of 3-symmetric spaces are known to be isotropy *irreducible*. Moreover, by [45], Corollary 8.13.5, these are the only non-symmetric isotropy irreducible homogeneous spaces G/H such that H has maximal rank.

$\mathfrak{C}$. Finally, for the spaces of type $\mathfrak{C}$, the isotropy representation is reducible but not complex reducible. Let J be the canonical almost complex structure of G/H, viewed as a constant tensor on $\mathfrak{m}$. There exists an invariant subset $\mathfrak{n}$ such that $\mathfrak{m}$ decomposes into $\mathfrak{n} \oplus J\mathfrak{n}$. This situation is called *real reducible* by Nagy [35].

Remark. The dimension 6 is already representative of Gray and Wolf's classification. Indeed, we have seen that $\mathbb{C}P^3$ and $\mathbb{F}^3$ are the twistor spaces of S^4 and $\mathbb{C}P^2$, respectively. Secondly, the sphere $S^6 \simeq G_2/\mathrm{SU}(3)$ is isotropy irreducible (see Section 4). And thirdly, $S^3 \times S^3$ belongs to the infinite family of class $\mathfrak{C}$.

Now, for a general nearly Kähler manifold, we cannot look at the isotropy representation anymore. However we must remember Proposition 1.7 that the normal connection coincides with the intrinsic connection for a 3-symmetric space and so the isotropy representation is equivalent to the *holonomy representation* of $\overline{\nabla}$. The question then becomes, what can we say about the geometry of a nearly Kähler manifold whose holonomy is respectively: *complex reducible*, *irreducible* or *real reducible*?

a. Belgun and Moroianu, carrying out a program of Reyes Carrión [38], p. 57 (especially Proposition 4.24), have shown, in [6], that the holonomy of a 6-dimensional nearly Kähler manifold M is complex reducible (or equivalently the holonomy group of $\overline{\nabla}$ is contained in $\mathrm{U}(2) \subset \mathrm{SU}(3)$) if and only if M is the twistor space of a positive self-dual Einstein 4-manifold. Thus, by the result of Hitchin [30], the only compact, simply connected, complex reducible, nearly Kähler manifolds in dimension 6 are $\mathbb{C}P^3$ and $\mathbb{F}^3$. P.A. Nagy generalized this result in higher dimensions. Let M be a complete, irreducible (in the Riemannian sense) nearly Kähler manifold of dimension $2n$, $n \geq 4$, such that the holonomy representation of $\overline{\nabla}$ is complex reducible. Then M is the twistor space of a quaternion-Kähler manifold or of a locally symmetric space.

b. The irreducible case relates to weak holonomy. Cleyton and Swann [14], [15] have shown an analog of Berger's theorem on special holonomies [7] for the special geometries *with torsion*. By this we mean G-manifolds, or real manifolds of dimension m, equipped with a G-structure, $G \subset \mathrm{SO}(m)$, such that the Levi-Civita connection of the underlying metric structure is *not* a G-connection, or equivalently the torsion η of the intrinsic connection is not identically zero. More precisely they made the hypothesis that M, or the holonomy representation of $\overline{\nabla}$, are irreducible and that η is totally skew-symmetric and parallel (the last condition is automatically satisfied for a nearly Kähler manifold by Proposition 1.3). Then M is (i) a homogeneous space or (ii) a manifold with weak holonomy $\mathrm{SU}(3)$ or G_2. The first case in (ii) corresponds exactly to the 6-dimensional irreducible nearly Kähler manifolds while the second is otherwise called *nearly parallel* G_2. Moreover, the geometry of the homogeneous spaces in (i) may be specified. Indeed, the proof consists in showing that the curvature of the intrinsic connection is also parallel. Then $\overline{\nabla}$ is an Ambrose–Singer connection. This reminds us of Theorem 1.8. Finally, irreducible nearly Kähler manifolds are (i) 3-symmetric of type $\mathfrak{B}$ or (ii) 6-dimensional.

c. Eventually, Nagy has proved in [35], Corollary 3.1, that the complete, simply connected, real reducible, nearly Kähler manifolds are 3-symmetric of type $\mathfrak{C}$.

He summarized his results in a partial classification theorem ([35], Theorem 1.1): let M be a complete, simply connected, (strictly) nearly Kähler manifold. Then M is an almost Hermitian product of the following spaces:

- 3-symmetric spaces of type $\mathfrak{A}$, $\mathfrak{B}$ or $\mathfrak{C}$;
- twistor spaces of non locally symmetric, quaternion-Kähler manifolds;

- 6-dimensional irreducible nearly Kähler manifolds.

If we suppose furthermore that the manifold is homogeneous, then there remains only 3-symmetric spaces and 6-dimensional, nearly Kähler, *homogeneous* manifolds. As a consequence, Conjecture 1 is reduced to dimension 6.

Now, the proof of the conjecture in dimension 6 has two parts. First, we must show that the only homogeneous spaces G/H admitting an invariant nearly Kähler structure are those considered in Sections 2, 3, 4. Secondly, for each of these spaces, we must look for all the invariant nearly Kähler structures on it and prove they are all 3-symmetric, as was already done.

Lemma 5.1. *Let $(G/H, g, J)$ be a simply connected 6-dimensional almost Hermitian homogeneous space, such that the almost Hermitian structure (g, J) is nearly Kähler. Then the Lie algebras of G, H and the manifold are given in following table.*

$\dim \mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{g}$	
0	$\{0\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$S^3 \times S^3$
1	$i\mathbb{R}$	$i\mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$S^3 \times S^3$
2	$i\mathbb{R} \oplus i\mathbb{R}$	$i\mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$S^3 \times S^3$
	$i\mathbb{R} \oplus i\mathbb{R}$	$\mathfrak{su}(3)$	$\mathbb{F}^3$
3	$\mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$S^3 \times S^3$
4	$i\mathbb{R} \oplus \mathfrak{su}(2)$	$i\mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$S^3 \times S^3$
	$i\mathbb{R} \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(2)$	$\mathbb{C}P^3$
8	$\mathfrak{su}(3)$	$\mathfrak{g}_2$	S^6

(28)

Proof. By a result of Nagy [36], the Ricci tensor of a nearly Kähler manifold M is positive (in dimension 6, this a consequence of Gray's theorem in [25] that M is Einstein, with positive scalar curvature). Then, by Myer's theorem, M is compact with finite fundamental group and the universal cover $\widetilde{M}$ of M is a nearly Kähler manifold of the same dimension. Moreover, if $M \simeq G/H$ is homogeneous, $\widetilde{M}$ is isomorphic to $\widetilde{G}/\widetilde{H}$ where the groups G, $\widetilde{G}$ and H, $\widetilde{H}$ have the same Lie algebras. Consequently, we shall work with $\widetilde{M}$ instead of M.

For an homogeneous space, we have the following homotopy sequence:

$$\cdots \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to H/H^0 \to 0. \tag{29}$$

If the manifold is simply connected, this provides us with a compact surjective morphism φ from the fundamental group of H to the fundamental group of G. We shall

look only at the consequences at the Lie algebra level (i.e. at the S^1 or $i\mathbb{R}$ factors and not at the finite quotients).

The second feature we use is the natural reduction to SU(3) defined in Section 2. If g and J or ω are invariant for the left action of G, then, so is $\psi = d\omega$. As a consequence, the isotropy group H is a subgroup of SU(3). This leaves the following possibilities for $\mathfrak{h}$: $\{0\}$, $\mathfrak{u}(1)$, $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$, $\mathfrak{su}(2)$, $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and $\mathfrak{su}(3)$. Next, to find $\mathfrak{g}$, we use the fact that the difference between the dimensions of the two algebras is 6, the dimension of the manifold, and the existence of φ above. The latter allows us to eliminate the following: $\mathfrak{g} = \mathfrak{su}(2) \oplus 3\mathfrak{u}(1)$ or $6\mathfrak{u}(1)$ for $\mathfrak{h} = \{0\}$ and $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ for $\mathfrak{h} = \mathfrak{su}(2)$. Table (28) is a list of the remaining cases.

Now, $\mathfrak{h}$ acts as a subgroup of $\mathfrak{su}(3)$ on the 6-dimensional space $\mathfrak{m}$. Using this, we determine the isotropy representation and the embedding of H into G. In particular we show, when $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, that G/H is isomorphic to $S^3 \times S^3$ and G contains $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on the left. So the nearly Kähler structures arising from these cases will always induce a left-invariant nearly Kähler structure on $S^3 \times S^3 \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$. Thus, we need only to consider this more general situation. This was done in Section 2. □

Now, Theorem 1 is a consequence of Lemma 5.1 and Propositions 2.5, 3.2, 3.3 and Corollary 4.3.

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Chapter 12

Nearly pseudo-Kähler and nearly para-Kähler six-manifolds

Lars Schäfer and Fabian Schulte-Hengesbach

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1 Introduction

The notion of a nearly Kähler manifold was introduced and studied in a series of papers by A. Gray in the seventies in the context of weak holonomy. In the last two decades, six-dimensional nearly Kähler manifolds turned out to be of interest in a multitude of different areas including SU(3)-geometries, stable forms, geometries with torsion, existence of Killing spinors, (weak) holonomy, supersymmetric models and compactifications of string theories. For a survey explaining the relations between most of these areas we refer to Chapter 9 of this handbook ([1]).

One observes that most of the literature on nearly Kähler geometry deals with Riemannian signature. To our best knowledge the paper on 3-symmetric spaces [16] is the only article by Gray considering also indefinite nearly Kähler metrics. Killing spinors on pseudo-Riemannian manifolds were studied in [23] where nearly pseudo-Kähler and nearly para-Kähler manifolds appear in a natural way. The subject of nearly para-Kähler manifolds was further developed in [21]. The prefix "para" roughly means

that the anti-involutive complex structure is replaced by an involutive para-complex structure. We refer to Section 2.1 for details on para-complex geometry.

Motivated by a class of solutions of the topological-antitopological fusion equations on the tangent bundle [31], [30] and the similarity to special Kähler geometry, we became interested in Levi-Civita flat nearly Kähler and Levi-Civita flat nearly para-Kähler manifolds. A classification of these manifolds in a constructive manner has been established in [10], [11]. From these results it follows that non-Kählerian examples only exist in pseudo-Riemannian geometry. In other words, nearly Kähler geometry in the pseudo-Riemannian world can be very different from the better-understood Riemannian world.

There is a left-invariant nearly Kähler structure on $S^3 \times S^3$ which arises from a classical construction of 3-symmetric spaces by Ledger and Obata [25]. It is shown in [5] (and also in the extended English version [6] contained in this handbook) that this nearly Kähler structure is the only one on $S^3 \times S^3$ up to homothety. In fact, the proof of this uniqueness result has been the most difficult step in the classification of homogeneous nearly Kähler structures in dimension six. The main tool is the well-known characterisation [27] of a nearly Kähler structure on a six-manifold as an SU(3)-structure (ω, ψ^+, ψ^-) satisfying the exterior system

$$d\omega = 3\psi^+, \tag{1.1}$$

$$d\psi^- = \nu\, \omega \wedge \omega \tag{1.2}$$

for a real constant ν which depends on sign and normalisation conventions.

The starting point of this article is the following observation. The construction of a 3-symmetric space from $G = \mathrm{SL}(2,\mathbb{R})$ instead of $\mathrm{SU}(2)$ defines a left-invariant nearly pseudo-Kähler structure on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We shortly recall this construction explicitly. The group $G \times G \times G$ admits a symmetry of order three given by $(g_1, g_2, g_3) \mapsto (g_2, g_3, g_1)$ which stabilises the diagonal Δ. The tangent space of $M^6 = G \times G \times G/\Delta$ is identified with

$$\mathfrak{p} = \{(X, Y, Z) \in \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \mid X + Y + Z = 0\}.$$

Denote by $K_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$ and define an invariant scalar product on $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ by $g = K_{\mathfrak{g}} \oplus K_{\mathfrak{g}} \oplus K_{\mathfrak{g}}$. This yields a naturally reductive metric on M^6. Using Proposition 5.6 of [16] this metric is nearly pseudo-Kähler. For completeness sake we recall that the complex structure is given by

$$J(X, Y, Z) = \frac{2}{\sqrt{3}}(Z, X, Y) + \frac{1}{\sqrt{3}}(X, Y, Z).$$

Considering Butruille's results, it is natural to ask how many left-invariant nearly pseudo-Kähler structures there are on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Comparing with the results mentioned in the last paragraph, the answer seems a priori hard to guess. The main result of this article is the proof that there is a unique left-invariant nearly pseudo-Kähler structure on all Lie groups with Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. A byproduct of the proof is the result that there are no nearly para-Kähler structures

on these Lie groups. We add the remark that there exist co-compact lattices for these Lie groups. Indeed, the article [28] contains a complete list of the compact quotients of Lie groups with Lie algebras $\mathfrak{sl}(2,\mathbb{R})$, which also give rise to compact quotients on a direct product of such groups.

When dealing with nearly pseudo-Kähler structures, the problem arises that many facts which are well known for the Riemannian signature have never been shown for indefinite metrics. For instance, a hyperbolically nearly Kähler structure is defined in [4] as a $\mathrm{SU}(2,1)$-structure satisfying the same exterior system (1.1), (1.2) as a nearly Kähler structure. It is not obvious whether this definition is equivalent to Gray's classical definition of an (indefinite) nearly pseudo-Kähler manifold which is used in [10], [31]. However, the proof of our main result in Section 3 essentially relies on this exterior system and we have to prove the equivalence.

The close analogy between the pseudo-Hermitian and the para-Hermitian case makes it desirable to give a unified proof dealing with all possible cases at the same time. Therefore, we seize the opportunity and introduce a language that allows us to treat analogous aspects of almost pseudo-Hermitian geometry and almost para-Hermitian geometry simultaneously. This language is consistent with [29] and similar to [24]. In the preliminary section, the necessary basic notions are recalled in this unified language. In particular, we recall some facts about stable forms in dimension six which turn out to be very useful in characterising special almost Hermitian structures and special almost para-Hermitian structures.

Section 3 is devoted to proving the mentioned characterisation of six-dimensional nearly pseudo-Kähler and nearly para-Kähler manifolds by the exterior system. Since we have to generalise many facts from the Riemannian setting, we give a self-contained proof. Although we follow the ideas of the proof in [27], we clarify the structure of the proof by elaborating the role of the Nijenhuis tensor. In particular, we prove that a half-flat structure is additionally nearly half-flat if and only if the Nijenhuis tensor is skew-symmetric. As a first application, we prove some results on the automorphism group of a nearly (para-)Kähler six-manifold in Section 3.3.

In Section 4, we finally obtain the aforementioned structure results on $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. It turns out that the proof is considerably more technical than in the compact case $S^3\times S^3$, cf. [5] or [6]. We also extend the results on $S^3\times S^3$ by proving the non-existence of nearly (para-)Kähler structures of indefinite signature.

The authors wish to thank V. Cortés and P.-A. Nagy for useful discussions. In particular, Section 3.3 has been inspired by P.-A. Nagy.

2 Preliminaries

2.1 Almost pseudo-Hermitian and almost para-Hermitian geometry

We recall that an *almost para-complex structure* on a $2m$-dimensional manifold M is an endomorphism field squaring to the identity such that both eigendistributions (for the eigenvalues ± 1) are m-dimensional. An *almost para-Hermitian structure* consists of a neutral metric and an antiorthogonal almost para-complex structure. For a survey on para-complex geometry we refer to [2] or [12].

In the following, we introduce the unified language describing almost pseudo-Hermitian and almost para-Hermitian geometry simultaneously. The philosophy is to put an "ε" in front of all notions which is to be replaced by "para" for $\varepsilon = 1$ and is to be replaced by "pseudo" or to be omitted for $\varepsilon = -1$. From now on, we always suppose $\varepsilon \in \{\pm 1\}$.

To begin with, we consider the *ε-complex numbers* $\mathbb{C}_\varepsilon = \{x + \mathrm{i}_\varepsilon y,\ x, y \in \mathbb{R}\}$ with $\mathrm{i}_\varepsilon^2 = \varepsilon$. For the para-complex numbers, $\varepsilon = 1$, there are obvious analogues of conjugation, real and imaginary parts and the square of the (not necessarily positive) absolute value given by $|z|^2 = z\bar{z}$.

Moreover, let V be a real vector space of even dimension $n = 2m$. We call an endomorphism J an *ε-complex structure* if $J^2 = \varepsilon\,\mathrm{id}_V$ and if additionally, for $\varepsilon = 1$, the ± 1-eigenspaces $V^\pm$ are m-dimensional. An *ε-Hermitian structure* is an ε-complex structure J together with a pseudo-Euclidean scalar-product g which is *ε-Hermitian* in the sense that it holds

$$g(J\cdot, J\cdot) = -\varepsilon g(\cdot, \cdot).$$

We denote the stabiliser in $\mathrm{GL}(V)$ of an ε-Hermitian structure as the *ε-unitary group*

$$\begin{aligned} \mathrm{U}^\varepsilon(p,q) &= \{L \in \mathrm{GL}(V) \mid [L, J] = 0,\ L^* g = g\} \\ &\cong \begin{cases} \mathrm{U}(p,q),\ p+q=m, & \text{for } \varepsilon = -1, \\ \mathrm{GL}(m,\mathbb{R}), & \text{for } \varepsilon = 1. \end{cases} \end{aligned}$$

Here, the pair $(2p, 2q)$ is the signature[1] of the metric for $\varepsilon = -1$. For $\varepsilon = 1$, the group $\mathrm{GL}(m,\mathbb{R})$ acts reducibly such that $V = V^+ \oplus V^-$ and the signature is always (m, m).

An *almost ε-Hermitian manifold* is a manifold M of dimension $n = 2m$ endowed with a $\mathrm{U}^\varepsilon(p,q)$-structure or, equivalently, with an *almost ε-Hermitian structure* which consists of an almost ε-Hermitian structure J and an ε-Hermitian metric g. The non-degenerate two-form $\omega := g(\cdot, J\cdot)$ is called *fundamental two-form*.

Given an almost ε-Hermitian structure (g, J, ω), there exist pseudo-orthonormal local frames $\{e_1, \ldots, e_{2m}\}$ such that $Je_i = e_{i+m}$ for $i = 1, \ldots, m$ and $\omega = \varepsilon \sum_{i=1}^m \sigma_i e^{i(i+m)}$, where $\sigma_i := g(e_i, e_i)$ for $i = 1, \ldots, m$. Upper indices will always denote dual (not metric dual) one-forms and e^{ij} stands for $e^i \wedge e^j$. We call such a

[1] Please note that in our convention $2p$ refers to the negative directions.

frame *ε-unitary*. If $m \geq 3$, we can always achieve $\sigma_1 = \sigma_2$ by reordering the basis vectors.

For both values of ε, the ε-complexification $TM \otimes \mathbb{C}_\varepsilon$ of the tangent bundle decomposes into the $\pm \mathrm{i}_\varepsilon$-eigenbundles $TM^{1,0}$ and $TM^{0,1}$. This induces the well-known bi-grading of $\mathbb{C}_\varepsilon$-valued exterior forms

$$\Omega^{r,s} = \Gamma(\Lambda^{r,s}) = \Gamma(\Lambda^r(TM^{1,0})^* \otimes \Lambda^s(TM^{0,1})^*).$$

If X is a vector field on M, we use the notation

$$X^{1,0} = \frac{1}{2}(X + \mathrm{i}_\varepsilon \varepsilon JX) \in \Gamma(TM^{1,0}), \quad X^{0,1} = \frac{1}{2}(X - \mathrm{i}_\varepsilon \varepsilon JX) \in \Gamma(TM^{0,1}),$$

for the real isomorphisms from TM to $TM^{1,0}$ respectively $TM^{0,1}$. As usual in almost Hermitian geometry, we define the bundles $[\![\Lambda^{r,s}]\!]$ for $r \neq s$ and $[\Lambda^{r,r}]$ by the property

$$\begin{aligned} [\![\Lambda^{r,s}]\!] \otimes \mathbb{C}_\varepsilon &= [\![\Lambda^{r,s}]\!] \oplus \mathrm{i}_\varepsilon [\![\Lambda^{r,s}]\!] = \Lambda^{r,s} \oplus \Lambda^{s,r}, \\ [\Lambda^{r,r}] \otimes \mathbb{C}_\varepsilon &= [\Lambda^{r,r}] \oplus \mathrm{i}_\varepsilon [\Lambda^{r,r}] = \Lambda^{r,r}. \end{aligned}$$

The sections in these bundles are denoted as *real forms of type* $(r,s)+(s,r)$ respectively *of type* (r,r) and the spaces of sections by $[\![\Omega^{r,s}]\!]$ respectively by $[\Omega^{r,r}]$. For instance, it holds that

$$[\Omega^{1,1}] = \{\alpha \in \Omega^2 M \mid \alpha(X,Y) = \varepsilon\alpha(JX, JY)\},$$

such that the fundamental form is of type $(1,1)$ and similarly

$$[\![\Omega^{3,0}]\!] = \{\alpha \in \Omega^3 M \mid \alpha(X,Y,Z) = \varepsilon\alpha(X, JY, JZ)\}. \tag{2.1}$$

Only in the para-complex case, $\varepsilon = 1$, there is a decomposition of the real tangent bundle $TM = \mathcal{V} \oplus \mathcal{H}$ into the ± 1-eigenbundles of J which also induces a bi-grading of real forms. It is also straightforward to show that

$$[\![\Lambda^{3,0}]\!] \cong \Lambda^3 \mathcal{V}^* \oplus \Lambda^3 \mathcal{H}^*, \tag{2.2}$$

when considering the characterisation (2.1).

Returning to analogies, we recall that the Nijenhuis tensor of the almost ε-complex structure J satisfies

$$\begin{aligned} N(X,Y) &= -\varepsilon[X,Y] - [JX, JY] + J[JX,Y] + J[X, JY] \\ &= -(\nabla_{JX} J)\, Y + (\nabla_{JY} J)\, X + J(\nabla_X J)\, Y - J(\nabla_Y J)\, X \end{aligned} \tag{2.3}$$

for real vector fields X, Y, Z and for any torsion-free connection ∇ on M. For both values of ε, it is well known that the Nijenhuis tensor is the obstruction to the integrability of the almost ε-complex structure.

In the following, let ∇ always denote the Levi-Civita connection of the metric g of an almost ε-Hermitian manifold. Differentiating the almost ε-complex structure,

its square and the fundamental two-form yields for both values of ε the formulas

$$\begin{aligned}(\nabla_X J)\, JY &= -J(\nabla_X J)\, Y,\\ g((\nabla_X J)Y, Z) &= -(\nabla_X \omega)(Y, Z),\end{aligned} \tag{2.4}$$

for all vector fields X, Y, Z. Using these formulas, it is easy to show that for any almost ε-Hermitian manifold, the tensor A defined by

$$A(X, Y, Z) = g((\nabla_X J)Y, Z) = -(\nabla_X \omega)(Y, Z)$$

has the symmetries

$$A(X, Y, Z) = -A(X, Z, Y), \tag{2.5}$$

$$A(X, Y, Z) = \varepsilon A(X, JY, JZ) \tag{2.6}$$

for all vector fields X, Y, Z.

The decomposition of the $\mathrm{U}^{\varepsilon}(p, q)$-representation space of tensors with the same symmetries as A into irreducible components leads to a classification of almost ε-Hermitian manifolds which is classical for $\mathrm{U}(m)$ [18]. The para-complex case for the group $\mathrm{GL}(m, \mathbb{R})$ is completely worked out in [15]. In [24], the Gray–Hervella classes are generalised to almost ε-Hermitian structures, which are denoted by generalised almost Hermitian or $\mathcal{GAH}$ structures there. Analogues of all sixteen Gray–Hervella classes are established. These are invariant under the respective group action, but obviously not irreducible for the para-Hermitian case when compared to the decomposition in [15].

Finally, we mention the useful formula

$$2(\nabla_X \omega)(Y, Z) = d\omega(X, Y, Z) + \varepsilon d\omega(X, JY, JZ) + \varepsilon g(N(Y, Z), JX) \tag{2.7}$$

holding true for all vector fields X, Y, Z on any almost ε-Hermitian manifold. A short direct proof for $\varepsilon = -1$, g Riemannian, is given in the handbook article [26], which also holds literally for pseudo-Riemannian metrics and with sign modifications for $\varepsilon = 1$. Alternatively, we refer to [22] for $\varepsilon = 1$.

2.2 Stable three-forms in dimension six and ε-complex structures

We review a construction given in [19] which associates to a stable three-form ρ on a six-dimensional oriented real vector space V an ε-complex structure on the same vector space. Therefore we recall that a k-form $\rho \in \Lambda^k V^*$ is said to be *stable* if its orbit U under $\mathrm{GL}(V)$ is open. Denote by κ the canonical isomorphism

$$\Lambda^k V^* \cong \Lambda^{6-k} V \otimes \Lambda^6 V^*.$$

For any three-form ρ, one considers $K_\rho \colon V \to V \otimes \Lambda^6 V^*$ defined by

$$K_\rho(v) := \kappa((v \lrcorner \rho) \wedge \rho)$$

and the quartic invariant

$$\lambda(\rho) := \frac{1}{6}\mathrm{tr}\,(K_\rho^2) \in (\Lambda^6 V^*)^{\otimes 2}.$$

This invariant is different from zero if and only if ρ is stable. Since $L = \Lambda^6 V^*$ is of dimension one, there exists a well-defined notion of positivity and norm in $L \otimes L$. Therefore we can, by means of the orientation, associate a volume form $\phi(\rho)$ to a stable three-form ρ by

$$\phi(\rho) := \sqrt{|\lambda(\rho)|}.$$

Using this volume we define an endomorphism

$$J_\rho(v) := \frac{1}{\phi(\rho)} K_\rho(v),$$

which can be proven (cf. [19], [9]) to be an ε-complex structure, where ε is the sign of $\lambda(\rho)$. For both values of ε, a stable three-form is of type $(3,0)+(0,3)$ with respect to its induced ε-complex structure J_ρ or, in other words,

$$\Psi_\rho = \rho + \mathrm{i}_\varepsilon J_\rho^* \rho$$

is a $(3,0)$-form (where $J_\rho^*\rho(X,Y,Z) = \rho(J_\rho X, J_\rho Y, J_\rho Z)$). Moreover, a stable three-form ρ is non-degenerate in the sense that for $v \in V$,

$$v \lrcorner \rho = 0 \implies v = 0 \tag{2.8}$$

and the induced volume form satisfies the formula

$$\phi(\rho) = \frac{1}{2} J_\rho^* \rho \wedge \rho. \tag{2.9}$$

Almost all assertions are straightforward to verify when choosing a basis such that the stable three-form is in the normal form

$$\rho = e^{123} + \varepsilon(e^{156} + e^{426} + e^{453}) \tag{2.10}$$

which satisfies $\lambda(\rho) = 4\varepsilon(e^{1\dots 6})^{\otimes 2}$, $J_\rho^2 = \varepsilon\,\mathrm{id}_V$ and $J(e_i) = \pm e_{i+3}$ for $i = 1,2,3$ where the sign $\pm$ depends on the orientation.

It is worth mentioning that for every stable three-form in the orbit with $\varepsilon = 1$, there is also a basis such that

$$\rho = e^{123} + e^{456} \tag{2.11}$$

where $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ span the ± 1-eigenspaces $V^\pm$ of J_ρ.

2.3 Structure reduction of almost ε-Hermitian six-manifolds

Let (V, g, J, ω) be a $2m$-dimensional ε-Hermitian vector space and $\Psi = \psi^+ + \mathrm{i}_\varepsilon \psi^-$ be an $(m,0)$-form of *non-zero* length. We define the special ε-unitary group $\mathrm{SU}^\varepsilon(p,q)$

as the stabiliser of Ψ in the ε-unitary group $\mathrm{U}^{\varepsilon}(p,q)$ such that

$$\mathrm{SU}^{\varepsilon}(p,q) = \mathrm{Stab}_{\mathrm{GL}(V)}(g,J,\Psi) \cong \begin{cases} \mathrm{SU}(p,q)\,,\ p+q=m, & \text{for } \varepsilon=-1,\\ \mathrm{SL}(m,\mathbb{R}), & \text{for } \varepsilon=1, \end{cases}$$

where $\mathrm{SL}(m,\mathbb{R})$ acts reducibly such that $V = V^+ \oplus V^-$.

With this notation, an $\mathrm{SU}^{\varepsilon}(p,q)$-*structure* on a manifold M^{2m} is an almost ε-Hermitian structure (g,J,ω) together with a global $(m,0)$-form Ψ of non-zero constant length. Locally, there exists an ε-unitary frame

$$\{e_1,\dots,e_m,e_{m+1}=Je_1,\dots,e_{2m}=Je_m\}$$

which is adapted to the $\mathrm{SU}^{\varepsilon}(p,q)$-reduction Ψ in the sense that

$$\Psi = a(e^1+\mathrm{i}_{\varepsilon}e^{(m+1)})\wedge\cdots\wedge(e^m+\mathrm{i}_{\varepsilon}e^{2m}) \tag{2.12}$$

for a constant $a\in\mathbb{R}^*$.

In dimension six, there is a characterisation of $\mathrm{SU}^{\varepsilon}(p,q)$-structures in terms of stable forms. Given a six-dimensional real vector space V, we call a pair (ω,ρ) of a stable $\omega\in\Lambda^2V^*$ and a stable $\rho\in\Lambda^3V^*$ compatible if it holds

$$\omega\wedge\rho = 0. \tag{2.13}$$

We claim that the stabiliser in $\mathrm{GL}(V)$ of a compatible pair is

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\omega,\rho) = \mathrm{SU}^{\varepsilon}(p,q), \quad p+q=3,$$

where $\varepsilon\in\{\pm1\}$ is the sign of $\lambda(\rho)$, that is, $J_\rho^2=\varepsilon\,\mathrm{id}_V$. This can be seen as follows. For the two-form ω, stability is equivalent to non-degeneracy and we choose the orientation on V such that ω^3 is positive. By the previous section, we can associate an ε-complex structure J_ρ to the stable three-form ρ. For instance in an adequate basis, it is easy to verify that $\omega\wedge\rho=0$ is equivalent to the skew-symmetry of J_ρ with respect to ω. Equivalently, the pseudo-Euclidean metric

$$g = \varepsilon\,\omega(\cdot,J_\rho\cdot), \tag{2.14}$$

induced by ω and ρ is ε-Hermitian with respect to J_ρ. Since $\Psi_\rho=\rho+\mathrm{i}_{\varepsilon}J_\rho^*\rho$ is a $(3,0)$-form and the stabiliser of ω and ρ also stabilises the tensors induced by them, the claim follows.

We conclude that an $\mathrm{SU}^{\varepsilon}(p,q)$-structure, $p+q=3$, on a six-manifold is characterised by a pair $(\omega,\psi^+)\in\Omega^2M\times\Omega^3M$ of everywhere stable and compatible forms such that the induced $(3,0)$-form $\Psi=\psi^+ + \mathrm{i}_{\varepsilon}J_{\psi^+}^*\psi^+ = \psi^+ + \mathrm{i}_{\varepsilon}\psi^-$ has constant non-zero length with respect to the induced metric (2.14). In an ε-unitary frame which is adapted to Ψ in the sense of (2.12), the formula

$$\psi^-\wedge\psi^+ = \|\psi^+\|^2\frac{1}{6}\omega^3 \tag{2.15}$$

is easily verified. Thus, given a compatible pair (ω, ψ^+) of stable forms, it can be checked that the induced $(3,0)$-form Ψ has constant non-zero length without explicitly computing the induced metric.

We remark that in the almost Hermitian case, the literature often requires Ψ to be normalised such that $\|\psi^+\|^2 = 4$, for instance in [8]. In the more general almost ε-Hermitian case, we have in an adapted local ε-unitary frame (2.12) with $\sigma_1 = \sigma_2$,

$$\psi^+ = a(e^{123} + \varepsilon(e^{156} + e^{426} + e^{453})) \quad \text{and} \quad \|\psi^+\|^2 = 4a^2\sigma_3 \tag{2.16}$$

for a real constant a. Therefore we have to consider two different normalisations $\|\psi^+\| = \pm 4$ or we multiply the metric by -1 if necessary such that $\|\psi^+\|$ is always positive.

Finally, we remark that SU(3)-structures are classified in [8] and it is shown that the intrinsic torsion is completely determined by the exterior derivatives $d\omega$, $d\psi^+$ and $d\psi^-$.

3 Nearly pseudo-Kähler and nearly para-Kähler manifolds

The main objective of this section is to generalise the characterisation of six-dimensional nearly Kähler manifolds by an exterior differential system to nearly pseudo-Kähler and nearly para-Kähler manifolds. We remark that Riemannian nearly Kähler manifolds are discussed intensively in the articles [26] and [6] of this handbook.

3.1 General properties

Definition 3.1. An almost ε-Hermitian manifold (M^{2m}, g, J, ω) is called *nearly ε-Kähler manifold* if its Levi-Civita connection ∇ satisfies the *nearly ε-Kähler condition*

$$(\nabla_X J)\, X = 0 \quad \text{for all } X \in \Gamma(TM).$$

A nearly ε-Kähler manifold is called *strict* if $\nabla_X J \neq 0$ for all non-trivial vector fields X.

A tensor field $B \in \Gamma((TM^*)^{\otimes 2} \otimes TM)$ is called totally skew-symmetric if the tensor $g(B(X,Y),Z)$ is a three-form. The following characterisation of a nearly ε-Kähler manifold is well known in the Riemannian context.

Proposition 3.2. *An almost ε-Hermitian manifold (M^{2m}, g, J, ω) satisfies the nearly ε-Kähler condition if and only if $d\omega$ is of real type $(3,0) + (0,3)$ and the Nijenhuis tensor is totally skew-symmetric.*

Proof. The nearly ε-Kähler condition is satisfied if and only if the tensor $A = -\nabla\omega$ is a three-form because of the antisymmetry (2.5).

Assume first that (g, J, ω) is a nearly ε-Kähler structure. Comparing the identities (2.1) and (2.6), we see that the real three-form A is of type $(3,0)+(0,3)$. Since $d\omega$ is the alternation of $\nabla\omega$, we have

$$d\omega = 3\nabla\omega = -3A \in [\![\Omega^{3,0}]\!]. \tag{3.1}$$

Furthermore, if we apply the nearly ε-Kähler condition to the expression (2.3), the Nijenhuis tensor of a nearly ε-Kähler structure simplifies to

$$N(X,Y) = 4\,J(\nabla_X J)\,Y. \tag{3.2}$$

We conclude that the Nijenhuis tensor is skew-symmetric since

$$g(N(X,Y),Z) = -4A(X,Y,JZ) \overset{(2.6)}{=} -4\varepsilon J^* A(X,Y,Z). \tag{3.3}$$

The converse follows immediately from the identity (2.7) when considering (2.1). For self-containedness we give a direct proof. Assume that $d\omega \in [\![\Omega^{3,0}]\!]$ and the Nijenhuis tensor is skew-symmetric. To begin with, we observe that

$$(\nabla_Y\omega)(X,X) = 0 = (\nabla_{JY}\omega)(X,JX)$$

by (2.5) and (2.6). With this identity, we have on the one hand

$$\begin{aligned} 0 &= \varepsilon g(N(JX,JY),JX) = g(N(X,Y),JX) \\ &\overset{(2.3)}{=} -g((\nabla_{JX}J)\,Y,JX) + g((\nabla_{JY}J)\,X,JX) \\ &\quad + g(J(\nabla_X J)\,Y,JX) - g(J(\nabla_Y J)\,X,JX) \\ &\overset{(2.4)}{=} (\nabla_{JX}\omega)(Y,JX) + \varepsilon(\nabla_X\omega)(Y,X) \\ &\overset{(2.5)}{=} (\nabla_{JX}\omega)(Y,JX) - \varepsilon(\nabla_X\omega)(X,Y), \end{aligned}$$

and on the other hand

$$\begin{aligned} 0 &= \varepsilon d\omega(X,X,Y) \overset{(2.1)}{=} d\omega(X,JX,JY) \\ &= (\nabla_X\omega)(JX,JY) + (\nabla_{JX}\omega)(JY,X) + (\nabla_{JY}\omega)(X,JX) \\ &\overset{(2.6)}{=} \varepsilon(\nabla_X\omega)(X,Y) + (\nabla_{JX}\omega)(Y,JX). \end{aligned}$$

It follows that $(\nabla_X\omega)(X,Y) = 0$ which is equivalent to the nearly ε-Kähler condition. □

Remark 3.3. The notion of nearly ε-Kähler manifold corresponds to the generalised class $\mathcal{W}_1$ in [24]. However, in the para-Hermitian case, there are two subclasses, see [15]. Indeed, we already observed that

$$A = -\nabla\omega \in [\![\Omega^{3,0}]\!] \overset{(2.2)}{=} \Gamma(\Lambda^3\mathcal{V}^* \oplus \Lambda^3\mathcal{H}^*)$$

for a nearly para-Kähler manifold.

We call a connection $\bar{\nabla}$ on an almost ε-Hermitian manifold (M^{2m}, g, J, ω) *ε-Hermitian* if $\bar{\nabla} g = 0$ and $\bar{\nabla} J = 0$.

Proposition 3.4. *An almost ε-Hermitian manifold (M^{2m}, g, J, ω) admits an ε-Hermitian connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor is totally skew-symmetric. If this is the case, the connection $\overline{\nabla}$ and its torsion T are uniquely defined by*

$$g(\overline{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} g(T(X,Y), Z),$$
$$g(T(X,Y), Z) = \varepsilon g(N(X,Y), Z) - d\omega(JX, JY, JZ),$$

and we call $\overline{\nabla}$ the canonical ε-Hermitian connection (with skew-symmetric torsion).

Proof. The Riemannian case is proved in [14], the para-complex case in [21]. In fact, the sketched proof in [14] holds literally for the almost pseudo-Hermitian case with indefinite signature as well. For completeness, we give a direct proof for all cases simultaneously.

Let $T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = S_X Y - S_Y X$ be the totally skew-symmetric torsion of an ε-Hermitian connection $\overline{\nabla}$ where $S_X Y = \overline{\nabla}_X Y - \nabla_X Y$ is the difference tensor with respect to the Levi-Civita connection ∇ of g. Then, the Nijenhuis tensor is totally skew-symmetric as well, since we have

$$\begin{aligned} g(N(X,Y), Z) = \varepsilon g(T(X,Y), Z) &+ g(T(JX, JY), Z) \\ &+ g(T(JX, Y), JZ) + g(T(X, JY), JZ), \end{aligned} \tag{3.4}$$

using only $\overline{\nabla} J = 0$. Moreover, the difference tensor S_X is skew-symmetric with respect to g, for $\overline{\nabla} g = 0$. Combining this fact with the total skew-symmetry of the torsion, we find that $S_X Y = -S_Y X$ and consequently

$$g(\overline{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} g(T(X,Y), Z).$$

With this identity and $\overline{\nabla}\omega = 0$, the equation

$$2\nabla_{JX}\omega(Y,Z) = g(T(JX,Y), JZ) + g(T(JX, JY), Z) \tag{3.5}$$

follows. Finally, we verify the claimed formula for the torsion:

$$\begin{aligned} d\omega(JX, JY, JZ) &\overset{(2.6)}{=} \varepsilon(\nabla_{JX}\omega(Y,Z) + \nabla_{JY}\omega(Z,X) + \nabla_{JZ}\omega(X,Y)) \\ &\overset{(3.5)}{=} \varepsilon(g(T(JX, JY), Z) + g(T(JX, Y), JZ) \\ &\qquad + g(T(X, JY), JZ)) \\ &\overset{(3.4)}{=} \varepsilon g(N(X,Y), Z) - g(T(X,Y), Z). \end{aligned}$$

Conversely, if the Nijenhuis tensor is skew-symmetric, it is straightforward to verify that the defined connection is ε-Hermitian with skew-symmetric torsion. □

Remark 3.5. An almost Hermitian manifold is said to be of type $\mathcal{G}_1$ if it admits a Hermitian connection with skew-symmetric torsion. Such manifolds are studied in [26] in this handbook. More generally, the proposition justifies to say that an almost

ε-Hermitian manifold is of type $\mathcal{G}_1$ if it admits an ε-Hermitian connection with skew-symmetric torsion.

In particular, the proposition applies to nearly ε-Kähler manifolds. In this case, the skew-symmetric torsion T of the canonical ε-Hermitian connection simplifies to

$$T(X,Y) = \varepsilon J(\nabla_X J)Y = \frac{1}{4}\varepsilon N(X,Y)$$

due to the identities (3.1), (3.2) and (3.3).

Proposition 3.6. *The canonical ε-Hermitian connection $\overline{\nabla}$ of a nearly ε-Kähler manifold (M^{2m}, J, g, ω) satisfies*

$$\overline{\nabla}(\nabla J) = 0 \quad \textit{and} \quad \overline{\nabla}(T) = 0.$$

Proof. The two assertions are equivalent since $\overline{\nabla} J = 0$. A short proof of the first assertion for the Hermitian case is given in [3]. This proof generalises without changes to the pseudo-Hermitian case since it essentially uses the identity

$$2g((\nabla^2_{W,X} J)Y, Z) = -\sigma_{X,Y,Z}\, g((\nabla_W J)X, (\nabla_Y J)JZ),$$

which was proved in [17] for Riemannian metrics and also holds true in the pseudo-Riemannian setting (see [23], Proposition 7.1). The para-Hermitian version is proved in [21], Theorem 5.3. □

Corollary 3.7. *On a nearly ε-Kähler manifold (M^{2m}, J, g, ω), the tensors ∇J and $N = 4\varepsilon T$ have constant length.*

Proof. This is obvious since both tensors are parallel with respect to the connection $\overline{\nabla}$ which preserves in particular the metric. □

Remark 3.8. In dimension six, the fact that ∇J has constant length is usually expressed by the equivalent assertion that a nearly ε-Kähler six-manifold is of constant type, i.e. there is a constant $\kappa \in \mathbb{R}$ such that

$$g((\nabla_X J)\,Y, (\nabla_X J)\,Y) = \kappa\,\{\,g(X,X)g(Y,Y) - g(X,Y)^2 + \varepsilon g(JX,Y)^2\,\}.$$

In fact, the constant is $\kappa = \frac{1}{4}\|\nabla J\|^2$. Furthermore, it is well known in the Riemannian case that strict nearly Kähler six-manifolds are Einstein manifolds with Einstein constant 5κ [17]. The same is true in the para-Hermitian case [21] and in the pseudo-Hermitian case [32].

The case $\|\nabla J\|^2 = 0$ for a strict nearly ε-Kähler six-manifold can only occur in the para-complex world. We give different characterisations of such structures which provide an obvious break in the analogy of nearly para-Kähler and nearly pseudo-Kähler manifolds.

Proposition 3.9. *Let (M^6, g, J, ω) be a six-dimensional strict nearly para-Kähler manifold. Then the following properties are equivalent:*

(i) $\|\nabla J\|^2 = \|A\|^2 = 0$.

(ii) *The three-form $A = -\nabla\omega \in [\![\Omega^{3,0}]\!]$ is either in $\Gamma(\Lambda^3\mathcal{V}^*)$ or in $\Gamma(\Lambda^3\mathcal{H}^*)$.*

(iii) *The three-form $A = -\nabla\omega \in [\![\Omega^{3,0}]\!]$ is not stable.*

(iv) *The metric g is Ricci-flat.*

Proof. We choose a local frame $\{e_1, \dots, e_6\}$ such that $\{e^1, e^2, e^3\}$ spans the $+1$-eigenspace $\mathcal{V}^*$ of J, $\{e^4, e^5, e^6\}$ spans the -1-eigenspace $\mathcal{H}^*$ of J and $g(e_i, e_{i+3}) = 1$ for $i = 1, 2, 3$. According to (2.2), there are local functions a and b such that $A = ae^{123} + be^{456}$. Thus, it holds

$$\|A\|^2 \frac{1}{6}\omega^3 \overset{(2.15)}{=} J^*A \wedge A = (ae^{123} - be^{456}) \wedge (ae^{123} + be^{456}) = 2abe^{123456}.$$

With $\omega = -e^{14} - e^{25} - e^{36}$ and $\omega^3 = 6e^{123456}$, we have $\|A\|^2 = 2ab$. Since A is nowhere zero due to the strictness and considering also (2.9), the first three assertions are equivalent to $a = 0$ or $b = 0$. Finally, assertions (i) and (iv) are equivalent by Theorem 5.5 in [21]. □

Flat strict nearly para-Kähler manifolds (M, g, J, ω) are classified in [11]. It turns out that they always satisfy $\|\nabla J\|^2 = 0$. In [15], almost para-Hermitian structures on tangent bundles TN of real three-dimensional manifolds N^3 are discussed. It is shown that the existence of nearly para-Kähler manifolds satisfying the second condition of Proposition 3.9 is equivalent to the existence of a certain connection on N^3. However, to the authors best knowledge, there exists no reference for an example of a Ricci-flat nearly para-Kähler structure which is not flat.

3.2 Characterisations by exterior differential systems in dimension six

The following lemma explicitly relates the Nijenhuis tensor to the exterior differential. For $\varepsilon = -1$, it gives a characterisation of Bryant's notion of a quasi-integrable $\mathrm{U}(p, q)$-structure, $p + q = 3$, in dimension six [4].

Let (M^6, g, J, ω) be a six-dimensional almost ε-Hermitian manifold. Given a local ε-unitary frame $\{e_1, \dots, e_6 = Je_3\}$, we define a local frame $\{E^1, E^2, E^3\}$ of $(TM^{1,0})^*$ by

$$E^i := (e^i + \mathrm{i}_\varepsilon Je^i) = (e^i + \mathrm{i}_\varepsilon e^{i+m})$$

for $i = 1, 2, 3$ and denote it as a local ε-unitary frame of $(1, 0)$-forms. The dual vector fields of the $(1, 0)$-forms are

$$E_i = e_i^{1,0} = \frac{1}{2}(e_i + \mathrm{i}_\varepsilon Je_i) = \frac{1}{2}(e_i + \mathrm{i}_\varepsilon e_{i+m})$$

such that the $\mathbb{C}_\varepsilon$-bilinearly extended metric satisfies

$$g(E_i, \bar{E}_j) = \frac{1}{2}\sigma_i \delta_{ij} \quad \text{and} \quad g(E_i, E_j) = 0$$

in such a frame.

Lemma 3.10. *The Nijenhuis tensor of an almost ε-Hermitian six-manifold (M^6, g, J, ω) is totally skew-symmetric if and only if for every local ε-unitary frame of $(1,0)$-forms, there exists a local $\mathbb{C}_\varepsilon$-valued function λ such that*

$$(dE^{\tau(1)})^{0,2} = \lambda\, \sigma_{\tau(1)}\, E^{\overline{\tau(2)}\,\overline{\tau(3)}} \tag{3.6}$$

for all even permutations τ of $\{1,2,3\}$.

Proof. First of all, the identities

$$N(\bar{V}, \bar{W}) = -4\varepsilon[\bar{V}, \bar{W}]^{1,0} \quad \text{and} \quad N(V, \bar{W}) = 0$$

for any vector fields $V = V^{1,0}$, $W = W^{1,0}$ in $TM^{1,0}$ follow immediately from the definition of N. Using the first identity, we compute in an arbitrary local ε-unitary frame

$$\begin{aligned} dE^i(\bar{E}_j, \bar{E}_k) &= -E^i([\bar{E}_j, \bar{E}_k]) = -2\sigma_i\, g([\bar{E}_j, \bar{E}_k], \bar{E}_i) \\ &= -2\sigma_i\, g([\bar{E}_j, \bar{E}_k]^{1,0}, \bar{E}_i) = \frac{1}{2}\varepsilon\, \sigma_i\, g(N(\bar{E}_j, \bar{E}_k), \bar{E}_i) \end{aligned}$$

for all possible indices $1 \le i, j, k \le 3$. If the Nijenhuis tensor is totally skew-symmetric, equation (3.6) follows by setting

$$\lambda = \frac{1}{2}\varepsilon\, g(N(\bar{E}_1, \bar{E}_2), \bar{E}_3). \tag{3.7}$$

Conversely, the assumption (3.6) for every local ε-unitary frame implies that the Nijenhuis tensor is everywhere a three-form when considering the same computation and $N(V, \bar{W}) = 0$. □

If there is an $\mathrm{SU}^\varepsilon(p,q)$-reduction with closed real part, this characterisation can be reformulated globally in the following sense.

Proposition 3.11. *Let (ω, ψ^+) be an $\mathrm{SU}^\varepsilon(p,q)$-structure on a six-manifold M such that ψ^+ is closed. Then the Nijenhuis tensor is totally skew-symmetric if and only if*

$$d\psi^- = \nu\, \omega \wedge \omega \tag{3.8}$$

for a global real function ν.

Proof. It suffices to proof this locally. Let $\{E^i\}$ be an ε-unitary frame of $(1,0)$-forms with $\sigma_1 = \sigma_2$ which is adapted to the $\mathrm{SU}^\varepsilon(p,q)$-reduction such that $\Psi =$

$\psi^+ + \mathrm{i}_\varepsilon \psi^- = a E^{123}$ for a real constant a as in (2.12). The fundamental two-form is

$$\omega = -\frac{1}{2}\mathrm{i}_\varepsilon \sum_{k=1}^{m} \sigma_k \, E^{k\bar{k}}$$

in such a frame. Furthermore, as ψ^+ is closed, we have $d\Psi = \mathrm{i}_\varepsilon d\psi^- = -d\bar{\Psi}$, which implies that $d\Psi \in \Lambda^{2,2}$. Considering this, we compute the real 4-form

$$d\psi^- = \varepsilon \mathrm{i}_\varepsilon \, d\Psi = \varepsilon \mathrm{i}_\varepsilon a \, \left((dE^1)^{0,2} \wedge E^{23} + (dE^2)^{0,2} \wedge E^{31} + (dE^3)^{0,2} \wedge E^{12} \right)$$

and compare this expression with

$$\begin{aligned} \omega \wedge \omega &= \frac{1}{2}\varepsilon(\sigma_2\sigma_3 \, E^{2\bar{2}3\bar{3}} + \sigma_1\sigma_3 \, E^{1\bar{1}3\bar{3}} + \sigma_1\sigma_2 \, E^{1\bar{1}2\bar{2}}) \\ &= -\frac{1}{2}\varepsilon\sigma_3(\sigma_1 \, E^{\bar{2}\bar{3}23} + \sigma_2 \, E^{\bar{3}\bar{1}31} + \sigma_3 \, E^{\bar{1}\bar{2}12}). \end{aligned}$$

Hence, by Lemma 3.10, the Nijenhuis tensor is totally skew-symmetric if and only if $d\psi^- = \nu \, \omega \wedge \omega$ holds true for a real function ν. More precisely, the two functions ν and λ are related by the formula

$$\nu = -2\sigma_3 \mathrm{i}_\varepsilon a \lambda. \tag{3.9}$$

□

An $\mathrm{SU}^\varepsilon(p,q)$-structure (ω, ψ) is called *half-flat* if

$$d\psi = 0, \quad d\omega^2 = 0,$$

and *nearly half-flat* if

$$d\psi = \nu \, \omega \wedge \omega$$

for a real constant ν. These notions are defined for the Riemannian signature in [8] respectively [13] and extended to all signatures in [9].

Corollary 3.12. *Let (ω, ψ^+) be a half-flat $\mathrm{SU}^\varepsilon(p,q)$-structure on a six-manifold M. Then, the Nijenhuis tensor is totally skew-symmetric if and only if (ω, ψ^-) is nearly half-flat.*

Proof. If (ω, ψ^-) is nearly half-flat, the equation (3.8) is satisfied by definition and the Nijenhuis tensor is skew-symmetric by the previous proposition. In particular one has $d\omega^2 = 0$. Conversely, if the Nijenhuis tensor is skew, we know that (3.8) holds true for a real function ν, since we have $d\psi^+ = 0$. Differentiating this equation and using $d\omega^2 = 0$, we obtain $d\nu \wedge \omega^2 = 0$. The assertion follows as wedging by ω^2 is injective on one-forms. □

Remark 3.13. An interesting property of $\mathrm{SU}(p,q)^\varepsilon$-structures which are both half-flat and nearly half-flat in the sense of the corollary is the fact that, given that the manifold and the $\mathrm{SU}(p,q)^\varepsilon$-structure are analytic, the structure can be evolved to both a parallel

G_2-structure and a nearly parallel G_2-structure via the Hitchin flow. For details, we refer to [20] and [33] for the compact Riemannian case and [9] for the non-compact case and indefinite signatures.

In [7], six-dimensional nilmanifolds N admitting an invariant half-flat SU(3)-structure (ω, ψ^+) such that (ω, ψ^-) is nearly half-flat are classified. As six nilmanifolds admit such a structure, we conclude that these structures are not as scarce as nearly Kähler manifolds. It is also shown in this reference that these structures induce invariant G_2-structures with torsion on $N \times S^1$.

We give another example of a (normalised) left-invariant SU(3)-structure on $S^3 \times S^3$ which satisfies $d\psi^+ = 0$, $d\psi^- = \omega \wedge \omega$ such that $d\omega$ neither vanishes nor is of type (3,0) + (0,3). We choose a global frame of left-invariant vector fields $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ on $S^3 \times S^3$ such that

$$de^1 = e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12},$$
$$df^1 = f^{23}, \quad df^2 = f^{31}, \quad df^3 = f^{12},$$

and set with $x = 2 + \sqrt{3}$,

$$\begin{aligned}
\omega &= e^1 f^1 + e^2 f^2 + e^3 f^3,\\
\psi^+ &= -\frac{1}{2}x^2 e^{123} + 2x e^{12} f^3 - 2x e^{13} f^2 - 2x e^1 f^{23} + 2x e^{23} f^1\\
&\quad + 2x e^2 f^{13} - 2x e^3 f^{12} + (4x - 8) f^{123},\\
\psi^- &= \frac{1}{2}x e^{123} - 2e^1 f^{23} + 2e^2 f^{13} - 2e^3 f^{12} + 4 f^{123},\\
g &= x(e^1)^2 + x\,(e^2)^2 + x\,(e^3)^2 + 4\,(f^1)^2 + 4\,(f^2)^2 + 4\,(f^3)^2\\
&\quad - 2x\, e^1 \cdot f^1 - 2x\, e^2 \cdot f^2 - 2x e^3 \cdot f^3.
\end{aligned}$$

Finally, we come to the characterisation of six-dimensional nearly ε-Kähler manifolds by an exterior differential system generalising the classical result of [27] which holds for $\varepsilon = -1$ and Riemannian metrics.

Theorem 3.14. *Let (M, g, J, ω) be an almost ε-Hermitian six-manifold. Then M is a strict nearly ε-Kähler manifold with $\|\nabla J\|^2 \neq 0$ if and only if there is a reduction $\Psi = \psi^+ + \mathrm{i}_\varepsilon \psi^-$ to $\mathrm{SU}(p,q)^\varepsilon$ which satisfies*

$$d\omega = 3\psi^+, \tag{3.10}$$
$$d\psi^- = 2\kappa\omega \wedge \omega, \tag{3.11}$$

where $\kappa = \frac{1}{4}\|\nabla J\|^2$ is constant and non-zero.

Remark 3.15. Due to our sign convention $\omega = g(.,J.)$, the constant κ is positive in the Riemannian case and the second equation differs from that of other authors.

Furthermore, we will sometimes use the term nearly ε-Kähler manifold of non-zero type if $\|\nabla J\|^2 \neq 0$.

Proof. By Proposition 3.2, the manifold M is nearly ε-Kähler if and only if $d\omega$ is of type $(3,0)+(0,3)$ and the Nijenhuis tensor is totally skew-symmetric.

Therefore, when (g, J, ω) is a strict nearly ε-Kähler structure such that $\|A\|^2 = \|\nabla J\|^2$ is constant (by Corollary 3.7) and not zero (by assumption), we can define the reduction $\Psi = \psi^+ + \mathrm{i}_\varepsilon \psi^-$ by $\psi^+ = \frac{1}{3} d\omega = -A$ and $\psi^- = J^* \psi^+$ such that the first equation is satisfied. Since ω is of type $(1,1)$ and therefore $d(\omega \wedge \omega) = 2 d\omega \wedge \omega = 0$, this reduction is half-flat. Thus, Corollary 3.12 and the skew-symmetry of N imply that there is a constant $\nu \in \mathbb{R}$ such that $d\psi^- = \nu\, \omega \wedge \omega$.

According to (2.12), we can choose an ε-unitary local frame with $\sigma_1 = \sigma_2$ such that

$$\Psi = -A - \mathrm{i}_\varepsilon J^* A = a E^{123},$$

where a is constant and satisfies $4\kappa = \|\nabla J\|^2 = \|\psi^+\|^2 = 4a^2\sigma_3$ by (2.16). Now, the functions defined in Lemma 3.10 and Proposition 3.11 evaluate as

$$\lambda \overset{(3.7)}{=} \frac{1}{2} \varepsilon g(N(\bar{E}_1, \bar{E}_2), \bar{E}_3) \overset{(3.3)}{=} -2 J^* A(\bar{E}_1, \bar{E}_2, \bar{E}_3) = -\varepsilon\, \mathrm{i}_\varepsilon a$$

and

$$\nu \overset{(3.9)}{=} -2\sigma_3 \mathrm{i}_\varepsilon a \lambda = 2\sigma_3 a^2 = 2\kappa.$$

Conversely, if a given $\mathrm{SU}(p,q)^\varepsilon$-structure satisfies the exterior system, the real three-form ψ^+ is obviously closed and the Nijenhuis tensor is totally skew-symmetric by Corollary 3.12. Considering that $d\omega = 3\nabla\omega$ is of type $(3,0)+(0,3)$ by the first equation, the structure is nearly ε-Kähler. Since $A = -\psi^+$ is stable, the structure is strict nearly ε-Kähler by (2.8) and $\|\nabla J\| = \|A\| \neq 0$ by Proposition 3.9. Now, the computation of the constants in the adapted ε-unitary frame shows that in fact $\|\nabla J\| = 4\kappa$. □

3.3 Consequences for automorphism groups

An automorphism of an $\mathrm{SU}^\varepsilon(p,q)$-structure on a six-manifold M is an automorphism of principal fibre bundles or equivalently, a diffeomorphism of M preserving all tensors defining the $\mathrm{SU}^\varepsilon(p,q)$-structure. By our discussion on stable forms in Section 2.3, an $\mathrm{SU}^\varepsilon(p,q)$-structure is characterised by a pair of compatible stable forms $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$. Since the construction of the remaining tensors J, ψ^- and g is invariant, an diffeomorphism preserving the two stable forms is already an automorphism of the $\mathrm{SU}^\varepsilon(p,q)$-structure and in particular an isometry.

This easy observation has the following consequences when combined with the exterior systems of the previous section and the naturality of the exterior derivative.

Proposition 3.16. *Let* (ω, ψ^+) *be an* $\mathrm{SU}^\varepsilon(p,q)$*-structure on a six-manifold* M*.*

(i) *If the exterior differential equation*

$$d\omega = \mu\, \psi^+$$

is satisfied for a constant $\mu \neq 0$*, then a diffeomorphism* Φ *of* M *preserving* ω *is an automorphism of the* $\mathrm{SU}^\varepsilon(p,q)$*-structure and in particular an isometry.*

(ii) *If the exterior differential equation*

$$d\psi^- = \nu\, \omega \wedge \omega$$

is satisfied for a constant $\nu \neq 0$*, then a diffeomorphism* Φ *of* M *preserving*

(a) *the real volume form and* ψ^+*,*

(b) *or the real volume form and* ψ^-*,*

(c) *or the* ε*-complex volume form* $\Psi = \psi^+ + \mathrm{i}_\varepsilon \psi^-$*,*

is an automorphism of the $\mathrm{SU}^\varepsilon(p,q)$*-structure and in particular an isometry.*

We like to emphasise that both parts of the proposition apply to strict nearly ε-Kähler structures of non-zero type.

Conversely, it is known for *complete* Riemannian nearly Kähler manifolds that orientation-preserving isometries are automorphism of the almost Hermitian structure except for the round sphere S^6, see for instance Proposition 4.1 in [6] in this handbook. However, this is not true if the metric is incomplete. In [13], Theorem 3.6, a nearly Kähler structure is constructed on the incomplete sine-cone over a Sasaki–Einstein five-manifold $(N^5, \eta, \omega_1, \omega_2, \omega_3)$. In fact, the Reeb vector field dual to the one-form η is a Killing vector field which does not preserve ω_2 and ω_3. Thus, by the formulae given in [13], its lift to the nearly Kähler six-manifold is a Killing field for the sine-cone metric which does neither preserve Ψ nor ω nor J.

4 Left-invariant nearly ε-Kähler structures on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

4.1 An algebraic prerequisite

The following lemma is the key to proving the forthcoming structure result, since it considerably reduces the number of algebraic equations on the nearly ε-Kähler candidates.

Lemma 4.1. *Denote by* $(\mathbb{R}^{1,2}, \langle\cdot,\cdot\rangle)$ *the vector space* $\mathbb{R}^3$ *endowed with its standard Minkowskian scalar-product and denote by* $\mathrm{SO}_0(1,2)$ *the connected component of the*

identity of its group of isometries. Consider the action of $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$ *on the space of real* 3×3 *matrices* $\mathrm{Mat}(3,\mathbb{R})$ *given by*

$$\Phi\colon \mathrm{SO}_0(1,2) \times \mathrm{Mat}(3,\mathbb{R}) \times \mathrm{SO}_0(1,2) \longrightarrow \mathrm{Mat}(3,\mathbb{R}),$$
$$(A, C, B) \longmapsto A^t C B.$$

Then any invertible element $C \in \mathrm{Mat}(3,\mathbb{R})$ *lies in the orbit of an element of the form*

$$\begin{pmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & \beta & z \\ \alpha & x & y \\ 0 & 0 & \gamma \end{pmatrix}$$

with $\alpha, \beta, \gamma, x, y, z \in \mathbb{R}$ *and* $\alpha\beta\gamma \neq 0$.

Proof. Let an arbitrary invertible element $C \in \mathrm{Mat}(3,\mathbb{R})$ be given. Denote by $\{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^{1,2}$. There are three different cases:

1.) Suppose that the first column c of C has negative length. We extend c to a Lorentzian basis $\{l_1 = c/\alpha, l_2, l_3\}$ with $\alpha := \sqrt{|\langle c, c\rangle|}$. The linear map L defined by extension of $L(l_i) = e_i$ is by definition a Lorentz transformation. The transformation L can be chosen time-oriented (by replacing l_1 by $\pm\, l_1$) and oriented (by replacing l_3 by $\pm\, l_3$). With this definition we obtain

$$\Phi(L^t, C, \mathbb{1}) = \begin{pmatrix} \alpha & * \\ 0 & C' \end{pmatrix} \quad \text{with an element } C' \in \mathrm{Mat}(2,\mathbb{R}).$$

Using the polar decomposition we can express $C' = O_1 S$ as a product of $O_1 \in \mathrm{SO}(2)$ and a symmetric matrix S in $\mathrm{Mat}(2,\mathbb{R})$ and diagonalise S by $O_2 \in \mathrm{SO}(2)$. If we put

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & O_2^{-1} O_1^{-1} \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 1 & 0 \\ 0 & O_2 \end{pmatrix}$$

we obtain

$$\Phi(L_1^t, \Phi(L^t, C, \mathbb{1}), L_2) = \begin{pmatrix} \alpha & x & y \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

2.) Next suppose that the first column c of C has positive length. Again, we extend c to a Lorentzian basis $\{l_1, l_2 = c/\alpha, l_3\}$ with $\alpha := \sqrt{|\langle c, c\rangle|}$. The linear map L defined by extension of $L(l_i) = e_i$ is by definition a Lorentz transformation. The transformation L can be chosen time-oriented (by replacing l_1 by $\pm\, l_1$) and oriented (by replacing l_3 by $\pm\, l_3$). We get

$$\Phi(L^t, C, \mathbb{1}) = \begin{pmatrix} 0 & * \\ \alpha & C' \\ 0 & \end{pmatrix} \quad \text{with an element } C' \in \mathrm{Mat}(2,\mathbb{R}).$$

The first column of this matrix is stable under the right-operation of

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & O_1 \end{pmatrix} \quad \text{with } O_1 \in \mathrm{SO}(2)$$

and there exists an element $O_1 \in \mathrm{SO}(2)$ such that it holds

$$\Phi(\mathbb{1}, \Phi(L^t, C, \mathbb{1}), L_1) = \begin{pmatrix} 0 & \beta & z \\ \alpha & x & y \\ 0 & 0 & \gamma \end{pmatrix}.$$

3.) Finally suppose that it holds $\langle c, c \rangle = 0$. Then there exists an oriented and time-oriented Lorentz transformation L such that $L(c) = \kappa(e_1 + e_2)$ with $\kappa \neq 0$. Afterwards one finds as in point 2.) an element $O \in \mathrm{SO}(2)$, such that it holds

$$C' := \Phi(L^t, C, O) = \begin{pmatrix} \kappa & c_1 & * \\ \kappa & c_2 & * \\ 0 & 0 & * \end{pmatrix}.$$

Let

$$B(q) := \begin{pmatrix} \cosh(q) & \sinh(q) & 0 \\ \sinh(q) & \cosh(q) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim. There exist $q_1, q_2 \in \mathbb{R}$ such that

$$\Phi\left(B(q_1)^t\,,\, C'\,,\, B(q_2)\right) = \begin{pmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{pmatrix}.$$

To prove this claim let us first consider the right-action of $B(q)$ on $C'' := \Phi\left(B(q_1)^t, C', \mathbb{1}\right)$

$$\Phi\left(\mathbb{1}, C'', B(q)\right) = \begin{pmatrix} c''_{11}\cosh(q) + c''_{12}\sinh(q) & * & * \\ c''_{21}\cosh(q) + c''_{22}\sinh(q) & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{for } q \in \mathbb{R}.$$

We choose q_2 such that $c''_{21}\cosh(q_2) + c''_{22}\sinh(q_2)$ vanishes. This is only possible if $-c''_{22}/c''_{21}$ is in the range of coth, i.e. $|c''_{22}/c''_{21}| > 1$.
In the sequel we show that this can always be achieved by the left-action of an element $B(q_1)$ on C' and that $c''_{21} \neq 0$. In fact, it is

$$\begin{aligned} c''_{22} &= c_1 \sinh(q_1) + c_2 \cosh(q_1), \\ c''_{21} &= \kappa(\sinh(q_1) + \cosh(q_1)) = \kappa e^{q_1}, \\ \frac{c''_{22}}{c''_{21}} &= \frac{c_1 + c_2}{2\kappa} + \frac{c_2 - c_1}{2\kappa} e^{-2q_1}. \end{aligned}$$

We observe that $c_1 \neq c_2$, since the matrix C is invertible. Therefore we can always achieve $|c''_{22}/c''_{21}| > 1$. This proves the claim and finishes the proof of the lemma. □

4.2 Proof of the uniqueness result

Finally, we prove our main result which is the following theorem. By a homothety, we define the rescaling of the metric by a real number which we do not demand to be positive since we are working with all possible signatures.

Theorem 4.2. *Let G be a Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Up to homothety, there is a unique left-invariant nearly ε-Kähler structure with $\|\nabla J\|^2 \neq 0$ on $G \times G$. This is the nearly pseudo-Kähler structure of signature $(4,2)$ constructed as 3-symmetric space in the introduction. In particular, there is no left-invariant nearly para-Kähler structure.*

Remark 4.3. The proof also shows that there is a left-invariant nearly ε-Kähler structure of non-zero type on $G \times H$ with $\mathrm{Lie}(G) = \mathrm{Lie}(H) = \mathfrak{sl}(2,\mathbb{R})$ if $G \neq H$ which is unique up to homothety *and* exchanging the orientation.

Proof. More precisely, we will prove uniqueness up to equivalence of left-invariant almost ε-Hermitian structures and homothety. We will consider the algebraic exterior system

$$d\omega = 3\,\psi^+, \tag{4.1}$$

$$d\psi^- = 2\,\omega \wedge \omega \tag{4.2}$$

on the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. By Theorem 3.14, solutions of this system are in one-to-one correspondence to left-invariant nearly ε-Kähler structure on $G \times G$ with $\|\nabla J\|^2 = 4$. This normalisation can always be achieved by applying a homothety. Furthermore, two solutions which are isomorphic under an inner Lie algebra automorphism from

$$\mathrm{Inn}(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) = \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$$

are equivalent under the corresponding Lie group isomorphism. Since both factors are equal, we can also lift the outer Lie algebra automorphism exchanging the two summands to the group level. In summary, it suffices to show the existence of a solution of the algebraic exterior system (4.1), (4.2) on the Lie algebra which is unique up to inner Lie algebra automorphisms and exchanging the summands.

A further significant simplification is the observation that all tensors defining a nearly ε-Kähler structure of non-zero type can be constructed out of the fundamental two-form ω with the help of the first nearly Kähler equation (4.1) and the stable form formalism described in Section 2.3. We break the main part of the proof into three

lemmas, step by step simplifying ω under Lie algebra automorphisms in a fixed Lie bracket.

We call $\{e_1, e_2, e_3\}$ a *standard basis* of $\mathfrak{so}(1,2)$ if the Lie bracket satisfies

$$de^1 = -e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12}.$$

In this basis, an inner automorphism in $\mathrm{SO}_0(1,2)$ acts by usual matrix multiplication on $\mathfrak{so}(1,2)$.

Lemma 4.4. *Let* $\mathfrak{g} = \mathfrak{h} = \mathfrak{so}(1,2)$ *and let* ω *be a non-degenerate two-form in*

$$\Lambda^2(\mathfrak{g} \oplus \mathfrak{h})^* = \Lambda^2\mathfrak{g}^* \oplus (\mathfrak{g} \otimes \mathfrak{h}) \oplus \Lambda^2\mathfrak{h}^*.$$

Then we have

$$d\omega^2 = 0 \iff \omega \in \mathfrak{g} \otimes \mathfrak{h}.$$

Proof. By inspecting the standard basis, we observe that all two-forms on $\mathfrak{so}(1,2)$ are closed whereas no non-trivial 1-form is closed. Thus, when separately taking the exterior derivative of the components of ω^2 in $\Lambda^4 = (\Lambda^3\mathfrak{g}^* \otimes \mathfrak{h}^*) \oplus (\Lambda^2\mathfrak{g}^* \otimes \Lambda^2\mathfrak{h}^*) \oplus (\mathfrak{g}^* \otimes \Lambda^3\mathfrak{h}^*)$, the equivalence is easily deduced. □

Lemma 4.5. *Let* $\mathfrak{g} = \mathfrak{h} = \mathfrak{so}(1,2)$ *and let* $\{e^1, e^2, e^3\}$ *be a basis of* $\mathfrak{g}^*$ *and* $\{e^4, e^5, e^6\}$ *a basis of* $\mathfrak{h}^*$ *such that the Lie brackets are given by*

$$\begin{aligned} de^1 = -e^{23}, \quad de^2 = e^{31}, \quad de^3 = \tau e^{12}, \quad \text{and} \\ de^4 = -e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45} \end{aligned} \tag{4.3}$$

for some $\tau \in \{\pm 1\}$. *Then, every non-degenerate two-form* ω *on* $\mathfrak{g} \oplus \mathfrak{h}$ *satisfying* $d\omega^2 = 0$ *can be written*

$$\omega \;=\; \alpha\, e^{14} + \beta\, e^{25} + \gamma\, e^{36} + x\, e^{15} + y\, e^{16} + z\, e^{26} \tag{4.4}$$

for $\alpha, \beta, \gamma \in \mathbb{R} - \{0\}$ *and* $x, y, z \in \mathbb{R}$ *modulo an automorphism in* $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$.

Proof. We choose standard bases $\{e^1, e^2, e^3\}$ for $\mathfrak{g}$ and $\{e^4, e^5, e^6\}$ for $\mathfrak{h}$. Using the previous lemma and the assumption $d\omega^2 = 0$, we may write $\omega = \sum_{i,j=1}^{3} c_{ij} e^{i(j+3)}$ for an invertible matrix $C = (c_{ij}) \in \mathrm{Mat}(3, \mathbb{R})$. When a pair $(A, B) \in \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$ acts on the two-form ω, the matrix C is transformed to $A^t C B$. Applying Lemma 4.1, we can achieve by an inner automorphism that C is in one of the normal forms given in that lemma. However, an exchange of the base vectors e_1 and e_2 corresponds exactly to exchanging the first and the second row of C. Therefore, we can always write ω in the claimed normal form by adding the sign τ in the Lie bracket of the first summand $\mathfrak{g}$. □

Lemma 4.6. *Let* $\{e^1,\dots,e^6\}$ *be a basis of* $\mathfrak{so}(1,2)\times\mathfrak{so}(1,2)$ *such that*

$$\begin{aligned} de^1 &= -e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12} \quad \text{and} \\ de^4 &= -e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45}. \end{aligned} \tag{4.5}$$

Then the only $\mathrm{SU}^{\varepsilon}(p,q)$*-structure* (ω,ψ^+) *modulo inner automorphisms and modulo exchanging the summands, which solves the two nearly* ε*-Kähler equations* (4.1) *and* (4.2)*, is determined by*

$$\omega = \frac{\sqrt{3}}{18}(e^{14}+e^{25}+e^{36}). \tag{4.6}$$

Proof. Since $d\omega^2=0$ by the second equation (4.2), we can choose a basis satisfying (4.3) such that ω is in the normal form (4.4). In order to satisfy the first equation (4.1), we have to set

$$\begin{aligned} 3\psi^+ = d\omega = {} & -\alpha\, e^{234} + \alpha\, e^{156} - x\, e^{235} + x\, e^{146} - y\, e^{236} - y\, e^{145} \\ & - \beta\, e^{135} + \beta\, e^{246} - z\, e^{136} - z\, e^{245} + \tau\gamma\, e^{126} - \gamma\, e^{345}. \end{aligned}$$

The compatibility $\omega\wedge\psi^+=0$ is equivalent to $d(\omega^2)=0$. It remains to determine all solutions of the second nearly ε-Kähler equation (4.2) modulo automorphisms.

For the sake of readability, we identify $\Lambda^6(\mathfrak{g}\oplus\mathfrak{h})^*$ with $\mathbb{R}$ by means of e^{123456}. Supported by Maple, we compute

$$\begin{aligned} K_{\psi^+}(e_1) = {} & (x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)e_1 - (2x\beta+2yz)e_2 \\ & - 2\tau\gamma y e_3 + 2\tau\gamma\beta e_4, \\ K_{\psi^+}(e_2) = {} & (2x\beta+2yz)e_1 + (-x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)e_2 \\ & - 2\tau\gamma z e_3 + 2\tau\gamma x e_4 - 2\tau\alpha\gamma e_5, \\ K_{\psi^+}(e_3) = {} & 2y\gamma e_1 - 2z\gamma e_2 + (-x^2-y^2+z^2+\alpha^2+\beta^2-\tau\gamma^2)e_3 \\ & + (2y\beta-2xz)e_4 + 2\alpha z e_5 - 2\alpha\beta e_6, \\ K_{\psi^+}(e_4) = {} & -2\beta\gamma e_1 + 2x\gamma e_2 + (2y\beta-2xz)e_3 \\ & + (x^2+y^2-z^2+\alpha^2-\beta^2-\tau\gamma^2)e_4 - 2\alpha x e_5 - 2\alpha y e_6, \\ K_{\psi^+}(e_5) = {} & 2\alpha\gamma e_2 - 2\alpha z e_3 + 2\alpha x e_4 \\ & + (-x^2+y^2-z^2-\alpha^2+\beta^2-\tau\gamma^2)e_5 + (2\beta z-2xy)e_6, \\ K_{\psi^+}(e_6) = {} & 2\alpha\beta e_3 + 2\alpha y e_4 \\ & + (2\beta z-2xy)e_5 + (x^2-y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)e_6. \end{aligned}$$

We assume that $\lambda(\psi^+)\neq 0$ and check this a posteriori for the solutions we find. Hence, we can set $k := \frac{1}{\pm\sqrt{|\lambda(\psi^+)|}}$ and $J_{\psi^+}=kK_{\psi^+}$. Since $\psi^+ + \mathrm{i}_\varepsilon J^*_{\psi^+}\psi^+$ is a $(3,0)$-form with respect to J_{ψ^+}, we have $\psi^- = J^*_{\psi^+}\psi^+ = \varepsilon\psi^+(J_{\psi^+}.,.,.)$ which

turns out to be

$$\begin{aligned}\varepsilon\frac{27}{k}\,\psi^- = {} & 2\tau\alpha\beta\gamma\; e^{123} + 2\tau y\alpha\gamma\; e^{124} + 2\tau\gamma(xy-\beta z)\; e^{125} - 2(x\beta+yz)\alpha\; e^{134}\\ & + \tau\gamma(-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2)\; e^{126}\\ & - \{\beta(x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xyz\}\; e^{135}\\ & + \{z(x^2-y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)-2xy\beta\}\; e^{136}\\ & - \{y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xz\beta\}\; e^{145}\\ & - \{x(x^2+y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)-2yz\beta\}\; e^{146}\\ & - \alpha(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)\; e^{156}\\ & - \alpha(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)\; e^{234}\\ & - \{x(x^2+y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)-2yz\beta\}\; e^{235}\\ & + \{y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xz\beta\}\; e^{236}\\ & - \{z(x^2-y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)-2xy\beta\}\; e^{245}\\ & - \{\beta(x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xyz\}\; e^{246}\\ & + \gamma(-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2)\; e^{345}\\ & - 2(x\beta+yz)\alpha\; e^{256} - 2\gamma(xy-\beta z)\; e^{346} - 2y\alpha\gamma\; e^{356} + 2\alpha\beta\gamma\; e^{456}.\end{aligned}$$

Furthermore, we compute the exterior derivative

$$\begin{aligned}\varepsilon\frac{27}{k}d\psi^- = {} & -4\tau\gamma\alpha y\; e^{1256} - 4\tau\gamma(xy-\beta z)\; e^{1246} + 4\alpha(x\beta+yz)\; e^{1356}\\ & + 2\tau\gamma(-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2)\; e^{1245}\\ & + 2\{\beta(x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xyz\}\; e^{1346}\\ & + 2\{z(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)-2xy\beta\}\; e^{1345}\\ & + 2\{y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2)+2xz\beta\}\; e^{2345}\\ & + 2\{x(x^2+y^2+z^2-\alpha^2-\beta^2+\tau\gamma^2)-2yz\beta\}\; e^{2346}\\ & + 2\alpha(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2)\; e^{2356}\end{aligned}$$

and

$$\omega^2 = 2((y\beta-xz)\; e^{1256} - \alpha z\; e^{1246} - x\gamma\; e^{1356} - \alpha\beta\; e^{1245} - \alpha\gamma\; e^{1346} - \beta\gamma\; e^{2356}).$$

The second nearly Kähler equation (4.2) is therefore equivalent to the following nine coefficient equations:

$$(\alpha\beta - 27\varepsilon k^{-1}\gamma)\, x = -\alpha\, yz, \qquad (e^{1356})$$

$$(\tau\gamma\alpha - 27\varepsilon k^{-1}\beta)\, y = -27\varepsilon k^{-1} xz, \qquad (e^{1256})$$

$$(\tau\beta\gamma - 27\varepsilon k^{-1}\alpha)\, z = \tau\gamma xy, \qquad (e^{1246})$$

$$x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2 = 54\varepsilon k^{-1}\frac{\beta\gamma}{\alpha}, \qquad (e^{2356})$$
$$z(x^2+y^2+z^2-\alpha^2+\beta^2+\tau\gamma^2) = -2\beta yx, \qquad (e^{1345})$$
$$x^2-y^2-z^2+\alpha^2-\beta^2+\tau\gamma^2 = 54\varepsilon k^{-1}\frac{\alpha\gamma}{\beta}-2\frac{xyz}{\beta}, \qquad (e^{1346})$$
$$y(-x^2-y^2+z^2+\alpha^2-\beta^2+\tau\gamma^2) = -2\beta zx, \qquad (e^{2345})$$
$$-x^2+y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2 = 54\tau\varepsilon k^{-1}\frac{\alpha\beta}{\gamma}, \qquad (e^{1245})$$
$$x(-x^2-y^2-z^2+\alpha^2+\beta^2-\tau\gamma^2) = -2\beta yz. \qquad (e^{2346})$$

Recall that $\alpha, \beta, \gamma \neq 0$ because ω is non-degenerate. We claim that there is no solution if any of x, y or z is different from zero.

On the one hand, assume that one of them is zero. Using one of the first three equations respectively, we find that at least one of the other two has to be zero as well. However, in all three cases, we may easily deduce that the third one has to be zero as well by comparing equations 4 and 5 respectively 6 and 7 respectively 8 and 9.

On the other hand, if we assume that all three of them are different from zero, the bracket in the first equation is necessarily different from zero and we may express x by a multiple of yz. Substituting this expression into equations 2 and 3, yields expressions for y^2 and z^2 in terms of α,β, γ and k. But if we insert all this into equation 4 (or 6 or 8 alternatively), we end up with a contradiction after a slightly tedious calculation.

To conclude, we can set $x = y = z = 0$ without losing any solutions of the second nearly Kähler equation which simplifies to the equations

$$\alpha^3-\alpha\beta^2-\tau\alpha\gamma^2-54\varepsilon k^{-1}\beta\gamma = 0,$$
$$\beta^3-\tau\beta\gamma^2-\beta\alpha^2-54\varepsilon k^{-1}\gamma\alpha = 0,$$
$$\gamma^3-\tau\gamma\alpha^2-\tau\gamma\beta^2-54\varepsilon k^{-1}\alpha\beta = 0.$$

Setting $c_1 = \alpha^2+\beta^2+\tau\gamma^2$ and $c_2 = 54\varepsilon k^{-1}\alpha\beta\gamma$, these are equivalent to

$$\begin{aligned} 2\alpha^4-c_1\alpha^2-c_2 &= 0,\\ 2\beta^4-c_1\beta^2-c_2 &= 0,\\ 2\gamma^4-c_1\tau\gamma^2-c_2 &= 0. \end{aligned} \qquad (4.7)$$

To finish the proof, we have to show that all real solutions of the system (4.7) are isomorphic under $\mathrm{SO}_0(1,2)\times\mathrm{SO}_0(1,2)$ to

$$\alpha = \beta = \gamma = \frac{\sqrt{3}}{18}, \quad \tau = 1.$$

Since α^2, β^2 and $\tau\gamma^2$ satisfy the same quadratic equation, at least two of them have to be identical, say $\alpha^2 = \beta^2$. However, if $\tau\gamma^2$ was the other root of the quadratic equation, we would have $\alpha^2+\tau\gamma^2 = \frac{1}{2}c_1$ and by definition of c_1 at the same time

$2\alpha^2 + \tau\gamma^2 = c_1$. This would only be possible if γ was zero, a contradiction to the non-degeneracy of ω. Therefore τ has to be $+1$ and α,β and γ have to be identical up to sign. By applying one of the proper and orthochronous Lorentz transformations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

on, say, the second summand, it is always possible to achieve that the signs of α, β and γ are identical.

So far, we found a basis satisfying (4.5) such that

$$\omega = \alpha(e^{14} + e^{25} + e^{36}).$$

It is straightforward to check that the quartic invariant in this basis is

$$\lambda(\frac{1}{3}d\omega) = -\frac{1}{27}\alpha^4. \tag{4.8}$$

Therefore, there cannot exist a nearly para-Kähler structure and we can set $\varepsilon = -1$. Inserting $k = \pm\frac{1}{\sqrt{-\lambda}} = \pm 3\sqrt{3}\alpha^{-2}$ into equations (4.7) yields

$$2\alpha^4 - 3\alpha^4 \pm \frac{54}{3\sqrt{3}}\alpha^5 = 0 \iff \alpha = \pm\frac{1}{18}\sqrt{3}.$$

Finally, we can achieve that α is positive by applying the Lie algebra automorphism exchanging the two summands, i.e. $e_i \mapsto e_{i+3 \bmod 6}$ and the lemma is proven. □

In fact, the uniqueness, existence and non-existence statements claimed in the theorem follow directly from this lemma and formula (4.8).

As explained in the introduction, we know that there is a left-invariant nearly pseudo-Kähler structure of indefinite signature on all the groups in question. After applying a homothety, we can achieve $\|\nabla J\|^2 = 4$ and this structure has to coincide with the unique structure we just constructed. Therefore, the indefinite metric has to be of signature (4,2) by our sign conventions.

We summarise the data of the unique nearly pseudo-Kähler structure in the basis (4.5) and can easily double-check the signature of the metric explicitly:

$$\omega = \frac{1}{18}\sqrt{3}(e^{14} + e^{25} + e^{36}),$$

$$\psi^+ = \frac{1}{54}\sqrt{3}(e^{126} - e^{135} + e^{156} - e^{234} + e^{246} - e^{345}),$$

$$\psi^- = -\frac{1}{54}(2\,e^{123} + e^{126} - e^{135} - e^{156} - e^{234} - e^{246} + e^{345} + 2e^{456}),$$

$$J(e_1) = -\frac{1}{3}\sqrt{3}e_1 - \frac{2}{3}\sqrt{3}e_4, \quad J(e_4) = \frac{2}{3}\sqrt{3}e_1 + \frac{1}{3}\sqrt{3}e_4,$$

$$J(e_2) = -\frac{1}{3}\sqrt{3}e_2 + \frac{2}{3}\sqrt{3}e_5, \quad J(e_5) = -\frac{2}{3}\sqrt{3}e_2 + \frac{1}{3}\sqrt{3}e_5,$$

$$J(e_3) = -\frac{1}{3}\sqrt{3}e_3 + \frac{2}{3}\sqrt{3}e_6, \quad J(e_6) = -\frac{2}{3}\sqrt{3}e_3 + \frac{1}{3}\sqrt{3}e_6,$$

$$g = \frac{1}{9}((e^1)^2 - (e^2)^2 - (e^3)^2 + (e^4)^2 - (e^5)^2 - (e^6)^2 - e^1 \cdot e^4 - e^2 \cdot e^5 - e^3 \cdot e^6).$$

□

Observing that in [5] very similar arguments have been applied to the Lie group $S^3 \times S^3$, we find the following non-existence result.

Proposition 4.7. *On the Lie groups $G \times H$ with $\mathrm{Lie}(G) = \mathrm{Lie}(H) = \mathfrak{so}(3)$, there is neither a left-invariant nearly para-Kähler structure of non-zero type nor a left-invariant nearly pseudo-Kähler structure with an indefinite metric.*

Proof. The unicity of the left-invariant nearly Kähler structure $S^3 \times S^3$ is proved in [5], Section 3, with a strategy analogous to the proof of Theorem 4.2. In the following, we will refer to the English version [6]. There, it is shown in the proof of Proposition 2.5 that for any solution of the exterior system

$$d\omega = 3\psi^+$$
$$d\psi^+ = -2\mu\omega^2$$

there is a basis of the Lie algebra of $S^3 \times S^3$ and a real constant α such that

$$de^1 = e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12}$$

and

$$de^4 = e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45},$$

$$\omega = \alpha(e^{14} + e^{25} + e^{36}).$$

In this basis, a direct computation or formula (18) in [6] show that the quartic invariant that we denote by λ is

$$\lambda = -\frac{1}{27}\alpha^4$$

with respect to the volume form e^{123456}. Therefore, a nearly para-Kähler structure cannot exist on all the Lie groups with the same Lie algebra as $S^3 \times S^3$ by Theorem 3.14. A nearly pseudo-Kähler structure with an indefinite metric cannot exist either, since the induced metric is always definite as computed in the second part of Lemma 2.3 in [6]. □

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Chapter 13

Quaternionic geometries from superconformal symmetry

Andrew Swann

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1 Introduction

Although often presented as abstract algebraic objects, quaternions have a long association with physics. They were introduced by Hamilton [29] who wrote extensively on applications, and the unit quaternions **i**, **j** and **k** appear as standard notation in elementary discussion of three-dimensional geometry and mechanics.

Quaternions reappear in at least two guises in modern theoretical physics. One is through the geometry of four-dimensions, where the close links between the orthogonal group, groups of unit quaternions and SU(2) arise in the conformal geometry and self-duality conditions. This is particularly relevant for the Yang–Mills and Seiberg–Witten theories, and the quaternions often occur again in descriptions of the geometries of related moduli spaces.

Another situation where quaternions arise is in the discussion of supersymmetry. Bagger and Witten [3] demonstrated how hyper-Kähler and quaternionic Kähler

geometry are necessary in certain models. More recently, extensive discussion of one-dimensional quantum mechanics with supersymmetry has shown the relevance of geometries with torsion. This article will discuss certain aspects of these geometries mostly from a mathematical viewpoint.

It will concentrate on the so-called hyper-Kähler manifolds with torsion, taking in discussion of both weak and strong geometries. Following the work of Michelson and Strominger [47], I will describe the specialisation to the geometries arising from models with superconformal symmetry. Although quantum mechanically the main interest is in positive definite metrics one finds that there are direct relations with corresponding geometries in mixed signature and so we will discuss indefinite geometries from the start. It turns out that there are intimate links with another type of quaternion-based geometry, namely quaternionic geometry with torsion, that arises in certain two-dimensional field theories. Much of the discussion is based on my paper [56] with Yat Sun Poon, but with inclusion of newer developments. I will not attempt to give a comprehensive guide to the literature, which particularly in the physics directions, is vast. In particular, while allowing indefinite metrics, we will not discuss geometries based on the split quaternions, cf. work of Stefan Ivanov in this volume, nor non-linear realisations of the supersymmetry algebra as discussed in [34].

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2 HKT geometry

Let M be a manifold and suppose I, J and K are endomorphisms of the tangent bundle satisfying

$$I^2 = -1 = J^2 = K^2, \quad IJ = K = -JI. \tag{2.1}$$

We say that (I, J, K) is an *almost hypercomplex* structure on M. A metric g, possibly indefinite, is said to be *compatible* if $g(AX, AY) = g(X, Y)$ for all $X, Y \in TM$ and each $A = I, J, K$. In this situation the tensors $F_A(\cdot, \cdot) = g(A\cdot, \cdot)$, $A = I, J, K$, are two-forms and (M, g, I, J, K) is called an almost hyper-Hermitian manifold.

Definition 2.1. A manifold M is (*weak*) *hyper-Kähler with torsion* HKT if it is almost hyper-Hermitian and

$$d_I F_I = d_J F_J = d_K F_K.$$

Here d_A is given on r-forms by $d_A = (-1)^r A d A$ and A acts on an r-tensor α by

$$A\alpha(X_1, \ldots, X_r) = (-1)^r \alpha(AX_1, \ldots, AX_r), \quad A = I, J, K.$$

On an HKT manifold M, consider the three-form

$$c = -I d F_I;$$

this is independent of I. Since c is totally skew, the connection

$$\nabla = \nabla^{\mathrm{LC}} + \tfrac{1}{2}c$$

is metric $\nabla g = 0$ with torsion $c \in \Lambda^3 T^*M$. The HKT condition ensures that ∇ also preserves the almost complex structures:

$$\nabla I = 0 = \nabla J = \nabla K.$$

An almost complex structure I is said to be *integrable* if there are local complex coordinates $z^1, \ldots, z^{2n}$ such that $I\,\partial/\partial z^k = i\,\partial/\partial z^k$. The obstruction to integrability is expressed by the Nijenhuis tensor

$$N_I(X, Y) = [IX, IY] - I[IX, Y] - I[X, IY] - [X, Y].$$

Proposition 2.2 ([45]). *On an almost hyper-Hermitian manifold the Nijenhuis tensors satisfy*

$$2N_I = J_{(23)}\big(I_{(23)} - I_{(12)} - I_{(13)} - 1\big)(d_J F_J - d_K F_K),$$

etc. In particular, on an HKT *manifold, the almost complex structures I, J and K are integrable.*

Here $I_{(abc\ldots)}\beta$ is $-I$ applied in slots $a, b, c, \ldots$ of β.

A manifold with integrable complex structures I, J and K satisfying (2.1) is called *hypercomplex*. It is important to note that such manifolds do not in general admit local quaternionic coordinates. Indeed a hypercomplex manifold carries a unique torsion-free connection ∇^{Ob}, the *Obata connection* [49], such that

$$\nabla^{\mathrm{Ob}} I = 0 = \nabla^{\mathrm{Ob}} J = \nabla^{\mathrm{Ob}} K.$$

The manifold admits local quaternionic coordinates if and only if the curvature tensor R^{Ob} of ∇^{Ob} is zero.

In the HKT case, as I, J and K are integrable we have a *hyper-Hermitian* structure. The pair (g, I) is a Hermitian structure on M. Gauduchon [23] showed that a Hermitian manifold (M, g, I) admits a unique connection with skew-symmetric torsion satisfying

$$\nabla^{\mathrm{B}}_I g = 0 \quad \text{and} \quad \nabla^{\mathrm{B}}_I I = 0.$$

Due to its applications in [7], ∇^{B}_I is known as the *Bismut connection*. The HKT condition therefore says that the Bismut connections ∇^{B}_I, ∇^{B}_J and ∇^{B}_K coincide and so equal ∇.

Definition 2.3. An HKT structure is *strong* if $dc = 0$, i.e., the torsion three-form is closed.

When $c = 0$ we have a *hyper-Kähler* structure.

Strong, as opposed to weak, HKT structures are important in some physical applications, but not all. Hyper-Kähler metrics are historically much better understood

than general HKT structrues: hyper-Kähler metrics are Ricci-flat, so Calabi–Yau, and appear in the classification of Riemannian holonomies.

3 Examples

3.1 Four dimensions

In four dimensions any hyper-Hermitian manifold is HKT and the metric is necessarily definite. Indeed a hypercomplex structure in dimension four determines a unique conformal class $[g]$ of compatible metrics: if X is a non-zero tangent vector then $\{X, IX, JX, KX\}$ is an oriented conformal basis for TM. The Obata connection ∇^{Ob} preserves the conformal structure, so $\nabla^{\mathrm{Ob}} g = \omega \otimes g$ for a representative metric $g \in [g]$. One now checks that (M, g, I, J, K) is HKT with torsion-form $c = (I\omega \wedge J\omega \wedge K\omega)/\|\omega\|^2 = *\omega$.

The first example here is $M = \mathbb{R}^4 \cong \mathbb{H}$, the quaternions, with standard basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. As M is a vector space we have $T_p M = \mathbb{H}$ canonically, and one takes I, J and K to be given by right multiplication by $-\mathbf{i}$, $-\mathbf{j}$ and $-\mathbf{k}$ respectively; the signs being necessary to ensure $IJ = K$. The Obata connection is flat differentiation and the standard inner product on $\mathbb{R}^4 \cong \mathbb{H}$ given by $\langle p, q\rangle = \mathrm{Re}(p\bar{q})$ gives the metric $g_0 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 = \mathrm{Re}(dp\, d\bar{p})$ that is hyper-Hermitian. Indeed $\nabla^{\mathrm{Ob}} g_0 = 0$, so g_0 is hyper-Kähler. If $g = f g_0$, we have $\omega = d \log f$ and $c = *_0 df$. Thus g is a strong HKT metric if and only if f is harmonic. The important case here is if $f = f(r^2)$, where $r^2 = p\bar{p}$ is the radius squared. The resulting non-flat strong metrics are $g = \mathrm{Re}((a + b/p\bar{p}) dp\, d\bar{p})$, with a, b constant. When $a = 0$, this descends to the standard product metric on $S^3 \times S^1$.

Four-dimensional compact hypercomplex manifolds have been classified by Boyer [9]. They are the four-torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4 = \mathbb{H}/\mathbb{Z}^4$, K3 surfaces (simply-connected complex surfaces with $c_1 = 0$) and $S^3 \times S^1$. The first two admit hyper-Kähler metrics, and we have seen that the last manifold has a strong HKT structure locally conformal to the flat hyper-Kähler metric. We thus have in dimension four that a compact HKT manifold is (i) conformal to a strong HKT structure, (ii) locally conformal to a hyper-Kähler metric, (iii) if not conformally flat, the space is a K3 surface.

Passing to the non-compact realm, there are many examples of four-dimensional hyper-Kähler structures known. Two important series are the multi-instanton metrics of Gibbons and Hawking [24], the first of which is the Eguchi–Hanson metric [19] on the cotangent bundle $T^*\mathbb{C}\mathrm{P}(1)$ (see equation (4.2) below with $a = 0$), and the multi-Taub-NUT metrics [30]. Both series live on the same topological manifolds, which have the homotopy type of a chain of two-spheres, and each metric admits a tri-holomorphic circle symmetry. As such one may apply a version of T-duality to construct HKT structures in a new conformal class, as we will now explain.

For a little more generality consider a four-dimensional HKT manifold (M, g) admitting a circle symmetry $X = \partial/\partial\tau$ preserving both the metric and the complex structures, and such that the one-form $X^\flat = d\tau + \omega$ metric dual to X satisfies $AdX^\flat = dX^\flat$ for $A = I, J$ and K. (If g is hyper-Kähler, the last condition is automatically satisfied.) Let

$$V = 1/\|X\|^2, \quad \varphi = VX^\flat.$$

Then $L_X\varphi = 0, \varphi(X) = 1$ and

$$g = \frac{1}{V}\varphi^2 + V\pi^* h = \frac{1}{V}(d\tau + \omega)^2 + Vh_{ij}dy^i dy^j,$$

where h is a metric on the three-dimensional manifold $Y = M/\langle X\rangle$ and ω is the connection one-form on the circle bundle $\pi : M \to Y$. The function V descends to Y and the HKT condition forces V and h to satisfy a generalised monopole equation. (In the hyper-Kähler case $dV = *_h d\omega$.) Write

$$c = \varphi \wedge \gamma_2 + \gamma_3,$$

where $X \lrcorner \gamma_i = 0$ and $\gamma_i \in \Omega^i(Y)$. Then one can show that the four-dimensional manifold $W = Y \times \mathbb{R}$, or $Y \times S^1$, carries an HKT structure with metric and torsion

$$g_W = V(dt^2 + h), \quad c_W = \gamma_3 + dt \wedge (V\gamma_2 - d\varphi).$$

As Callan, Harvey and Strominger [12] found in particular cases, starting with a hyper-Kähler metric on M, $c = 0$, we always obtain a strong HKT structure on W, cf. [25]. One can check that in general the metrics g_W are not conformal to hyper-Kähler [8], [52], unlike the compact case.

3.2 Higher dimensions

In dimensions greater than four, an HKT metric will in general have signature $(4p, 4q)$ and these structures will be the focus of much of the rest of this article. First examples may be constructed from products of four-dimensional examples and from totally geodesic submanifolds (this concept is the same for ∇^{LC} and ∇) as successfully exploited in [25].

Beyond that there is a wide range of constructions of hypercomplex manifolds, and it may be shown that many of these admit compatible positive-definite HKT metrics, see [18], [20], [26], [28], [32], [61]. One particularly interesting class here is given as follows. Let G be any compact Lie group. Then Joyce [38] showed that there is a torus T^r, $0 \leqslant r \leqslant \operatorname{rank} G$, such that $G \times T^r$ admits a left-invariant hypercomplex structure. One may adapt Joyce's construction so that it is compatible with a bi-invariant metric g and in this way obtain strong HKT structures on $G \times T^r$ with $c(X, Y, Z) = B([X, Y], Z)$, where B is the Killing form. Note that some extra factor such as T^r is necessary in general in order that the dimension of the manifold be

divisible by 4. However, in some cases, for example when $G = \mathrm{SU}(3)$, we can take $r = 0$ and obtain a strong HKT structure on a compact simply-connected manifold.

Locally any hypercomplex structure admits a compatible positive definite HKT metric, as we will see below. However, Grantcharov and Fino [21] showed that there exist compact 8-dimensional hypercomplex manifolds for which this is impossible globally. On the other hand it is interesting to note that their concrete example admits an HKT structure of signature $(4,4)$.

4 Potentials

A function μ on M is an HKT *potential* if

$$F_I = \tfrac{1}{2}(d d_I + d_J d_K)\mu = \tfrac{1}{2}(1-J)d I d\mu. \tag{4.1}$$

In complex coordinates for I, this says

$$2\mathbf{i}\,(F_I)_{i\bar{\jmath}} = \frac{\partial^2\mu}{\partial z^i \partial \bar{z}^j} - J_i^{\bar{q}} J_{\bar{\jmath}}^{p} \frac{\partial^2 \mu}{\partial z^p \partial \bar{z}^q}.$$

Equation (4.1) implies similar equations for F_J and F_K and that

$$g = \tfrac{1}{4}(1 + I + J + K)(\nabla^{\mathrm{Ob}})^2\mu.$$

Given a hypercomplex manifold we can always find a local μ so that $F_I = \frac{1}{2}(d d_I + d_J d_K)\mu$ is non-degenerate and hence a local HKT metric; we may even ensure that this is positive definite.

Example 4.1. Let g be a hyper-Kähler metric and choose a local Kähler potential μ for F_I, i.e., $F_I = \frac{1}{2}\mathbf{i}\,\partial_I \bar{\partial}_I \mu$. Then μ is a local HKT potential for g.

Many hyper-Kähler metrics can be presented in this way globally by specifying a single Kähler potential. These include Biquard and Gauduchon's descriptions [5], [6] of hyper-Kähler metrics on $T^*(G/K)$, where G/K is a Hermitian symmetric space, local versions in [48], and families of G-invariant hyper-Kähler metrics on nilpotent orbits $\mathcal{O} = G^{\mathbb{C}}.X \subset \mathfrak{g}^{\mathbb{C}}$, where G is compact and $\mathrm{ad}_X = [X,\cdot]\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is nilpotent [40], [42], [43], [41], [62], [63]. Indeed Calabi's original construction [11] of a non-flat hyper-Kähler metric is essentially obtained this way. The prototype for most of these constructions is the family of Eguchi–Hanson metrics

$$g = \frac{1}{W} dr^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2 + W\sigma_3^2) \tag{4.2}$$

with $W = (1 - a^2 r^{-4})$, where σ_i is a basis of left-invariant one-forms on $S^3 = \mathrm{SU}(2)$ with $d\sigma_1 = \sigma_2 \wedge \sigma_3$, etc. This has potential

$$\mu = \frac{1}{2}\left(r^2 - a \tan^{-1}\left(\frac{r^2}{a}\right)\right).$$

In general, if μ is an HKT potential, then the torsion tensor is given by

$$c = -\tfrac{1}{2} d_I d_J d_K \mu = \tfrac{1}{2} I dK d I d \mu$$

so the structure is strong if and only if $d d_I d_J d_K \mu = 0$. For the standard hypercomplex structure on $\mathbb{R}^4$, we have that μ is the potential for a non-degenerate HKT structure if $\Delta\mu \neq 0$, and that the structure is strong if $\Delta^2\mu = 0$, i.e., if μ is bi-harmonic. Taking $\mu = \mu(r^2)$, the bi-harmonic condition is $R^2 \frac{d^4\mu}{dR^4} + 6R\frac{d^3\mu}{dR^3} + 6\frac{d^2\mu}{dR^2} = 0$, for $R = r^2$, which has elementary solutions R, $\log R$, R^{-1} and 1; the first gives the flat metric, the second the strong structure of $S^3 \times S^1$, whereas the last two are harmonic and so do not give non-degenerate metrics.

The question of which HKT structures are determined by a potential is answered by

Theorem 4.2 (Banos–Swann [4]). *Every* HKT *structure locally admits a potential.*

Note that in the positive definite case on a compact manifold this potential can not be global due to the maximum principle [28].

The local existence is obtained as follows. Salamon [57] introduced a differential complex for hypercomplex manifolds. There is a two-dimensional family of complex structures on M given by $\mathcal{I} = aI + bJ + cK$, where $a^2 + b^2 + c^2 = 1$. For each $\mathcal{I}$ we may decompose the complex forms in to types: $T^*_{\mathbb{C}}M = \Lambda^{1,0}_{\mathcal{I}} \oplus \Lambda^{0,1}_{\mathcal{I}}$ is the eigenspace decomposition under $\mathcal{I}$; and $\Lambda^k T^*_{\mathbb{C}}M = \Lambda^{k,0}_{\mathcal{I}} \oplus \Lambda^{k-1,1}_{\mathcal{I}} \oplus \cdots \oplus \Lambda^{0,k}_{\mathcal{I}}$, with $\Lambda^{p,q}_{\mathcal{I}} = \Lambda^p(\Lambda^{1,0}_{\mathcal{I}}) \otimes \Lambda^q(\Lambda^{0,1}_{\mathcal{I}})$. Let $\Lambda^{\{p,k-p\}}_{\mathcal{I}} = \Lambda^k T^*M \cap (\Lambda^{p,k-p}_{\mathcal{I}} \oplus \Lambda^{k-p,p}_{\mathcal{I}})$ and define

$$A^k = \bigcup_{\mathcal{I}} \Lambda^{\{k,0\}}_{\mathcal{I}}, \qquad B^k = \bigcap_{\mathcal{I}} \sum_{1 \leqslant p \leqslant k/2} \Lambda^{\{p,k-p\}}_{\mathcal{I}},$$

so $\Lambda^k T^*M = A^k \oplus B^k$. The Salamon operator

$$D^{\mathrm{Sal}} \colon A^k \to A^{k+1}$$

is then exterior differentiation $d \colon A^k \to \Lambda^{k+1} T^*M$ followed by projection. That I, J, K form a hypercomplex structure implies that (A^*, D^{Sal}) is a complex, i.e., $D^{\mathrm{Sal}} \circ D^{\mathrm{Sal}} = 0$.

Using results of Verbitsky [60], one finds that the HKT condition on a hypercomplex manifold is equivalent to

$$D^{\mathrm{Sal}} F_I = 0$$

for a non-degenerate $F_I \in \Lambda^{1,1}_I$ with $JF_I = -F_I$. This is equivalent to the relations

$$(1 - J_{(12)} - J_{(13)} - J_{(23)}) d F_I = 0$$

where I, J anti-commute. There is a local potential if and only if

$$F_I = D^{\mathrm{Sal}} \theta = \tfrac{1}{4}(3 - I - J - K) d\theta$$

for some one-form θ: when there is a potential μ one can take $\theta = I d\mu$, conversely when $F_I = D^{\mathrm{Sal}}\theta$ one finds μ locally via $d\theta = d d_I \mu$, i.e., the usual $\partial\bar{\partial}$-lemma.

On a hypercomplex manifold the cohomology in degree $k \leqslant \frac{1}{2}\dim M$ is equal to the $\bar{\partial}$-cohomology of the twistor space Z by results of Mamone Capria and Salamon [44]. In particular, $H^2(U, D^{\mathrm{Sal}}) = 0$ locally, so the HKT condition $D^{\mathrm{Sal}} F_I = 0$ implies F_I is D^{Sal}-exact.

Alternatively, Michelson and Strominger [47] showed that any HKT structure on the flat hypercomplex manifold $\mathbb{H}^n = \mathbb{R}^{4n}$ admits a potential via considerations in quaternionic coordinates. However, to first order any HKT structure may be identified with the flat model, and the equation $F_I = D^{\mathrm{Sal}}\theta$ only uses first order information, so is solved by the Michelson and Strominger proof. This argument, avoiding the twistor space, was used in [36] for analogous split signature structures.

4.1 Change of potential and metric

As we noted above, the HKT property is not conformally invariant in dimensions greater than 4. However, using potentials there is a related type of transformation as described by Grantcharov and Poon [28]. Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth, then $f(\mu)$ is an HKT potential for the metric

$$g_f = f'(\mu)g + \tfrac{1}{2}f''(\mu)(d^{\mathbb{H}}\mu)^2, \tag{4.3}$$

where $(d^{\mathbb{H}}\mu)^2 = (1 + I + J + K)(d\mu)^2$, whenever g_f is non-degenerate. Note these transformations may change the signature.

5 Some physical motivation

In the work of Michelson and Strominger [47], [10], [46] the principal motivation is quantum mechanics of certain one-dimensional systems. One considers N particles moving on a line with the dynamics governed by a potential function $V(x) = V(x^1, \dots, x^N)$ defined on some open subset of $\mathbb{R}^N$. The Hamiltonian takes the form

$$H = \tfrac{1}{2}g^{ab}P_aP_b + V(x),$$

for some symmetric tensor g^{ab}, which we may take to be non-degenerate, where P_a are the generalised momenta. One quantises this system by replacing the coordinates x^a by the multiplication operator $x^a\cdot$ and the generalised momenta P_a by $-i\frac{\partial}{\partial x^a}$, for $a = 1, \dots, N$. The Hamiltonian now becomes the operator

$$H = \tfrac{1}{2}P_a^\dagger g^{ab}P_b + V(x), \tag{5.1}$$

where $P_a^\dagger = \sqrt{\det g}\, P_a \frac{1}{\sqrt{\det g}}$ and the operators are applied in the given order. Acting on functions one has $H = \frac{1}{2}\Delta + V$, where Δ is the Laplacian of g.

The system is said to have conformal symmetry if it scales homogeneously under dilations. One looks for mechanical systems that reflect this symmetry, meaning that

the quantisation contains a copy of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) = \left\{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a, b, c \in \mathbb{R}\right\}$. In the physics literature a standard basis for this algebra is

$$iD = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iH = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad iK = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.2}$$

which satisfies

$$[D, H] = 2iH, \quad [D, K] = -2iK, \quad [H, K] = -iD.$$

Michelson and Strominger find

Proposition 5.1 ([47]). *Let H be the operator given in equation* (5.1).

(1) *There exists D with $[D, H] = 2iH$ if and only if there exists a vector field X with*

$$L_X g = 2g \quad \text{and} \quad L_X V = -2V.$$

(2) *There exists K completing the $\mathfrak{sl}_2$-triple $\{H, D, K\}$ if and only if $X^\flat = g(X, \cdot)$ is closed.*
(3) *In this situation, $K = \frac{1}{2} g(X, X)$.*

On the other hand introducing supersymmetry, one puts $V = 0$ and adds super-commuting odd-operators Q_i, $i = 1, \ldots, N$, such that $Q_i^2 = H$. The actions of Q_i and Q_j are related via some endomorphism A_{ij} of the tangent bundle

$$Q_i \psi = A_{ij} Q_j \psi.$$

The algebra relations imply immediately that for $i \neq j$, A_{ij} are anti-commuting almost complex structures and a result of Coles and Papadopoulos [13] is that each A_{ij} is integrable with the mixed Nijenhuis tensors $N(A_{ij}, A_{k\ell})$ also zero. Thus when $N = 2$ we have a complex manifold, and hypercomplex geometry can arise in the case $N = 4$ if the algebra of A_{ij}'s is closed. In addition, the operators Q_i are required to preserve a superspace action determined not only by g but also by an additional three-form c on M. In the $N = 1$ case the action is usually given as

$$-i \int (\tfrac{1}{2} g_{ij} \mathcal{D}\phi^i \partial_t \phi^j + \tfrac{1}{6} c_{ijk} \mathcal{D}\phi^i \mathcal{D}\phi^j \mathcal{D}\phi^k)\, dt\, d\theta$$

where (t, θ) are coordinates on $\mathbb{R}^{1|1}$, ϕ is the superfield thought of as map from $\mathbb{R}^{1|1}$ to the target space M, and $\mathcal{D} = \partial/\partial\theta + i\theta\partial/\partial t$. The geometric consequences of this were determined in [22], [13], [25] who found in the $N = 4$ case (strictly $N = 4B$) that g is Hermitian with respect to each complex structure and HKT manifolds are amongst the possible target spaces. Examples that are not HKT may be found in [25] and further discussion of this case is given by Hull [33]. However, Michelson and Strominger [47] showed that if the endomorphisms A_{ij} satisfy the quaternion algebra, then HKT is the only possible geometry.

Superconformal symmetry is present when there is an action of a superalgebra containing both an even subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and supercharges Q_i, as above. In the next section, we will look at the essentially the smallest possible such superalgebra.

6 A family of simple superalgebras

A Lie superalgebra is a $\mathbb{Z}/2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ with Lie bracket satisfying $[X,Y] = (-1)^{pq+1}[Y,X]$ for $X \in \mathfrak{g}_p$, $Y \in \mathfrak{g}_q$, and a super Jacobi identity. The simple Lie superalgebras were classified by Kac [39]. One notable difference with the classification of complex simple Lie algebras is the occurrence of one continuous family $D(2,1;\alpha)$ with a real parameter α. For $\mathfrak{g}$ in this family one has

$$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})_+ \oplus \mathfrak{sl}(2,\mathbb{C})_-,$$
$$\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_-.$$

In particular, the even part $\mathfrak{g}_0$ contains the ideal $\mathfrak{sl}(2,\mathbb{C})$ which is the complexification of the $\mathfrak{sl}(2,\mathbb{R})$ of conformal symmetry. The algebra $\mathfrak{r} = \mathfrak{sl}(2,\mathbb{C})_+ \oplus \mathfrak{sl}(2,\mathbb{C})_- \cong \mathfrak{so}(4,\mathbb{C})$ is often referred to as the algebra of R-symmetries. The action of $\mathfrak{g}_0$ on the odd part $\mathfrak{g}_1$ is as the tensor product of the standard two-dimensional representations. To specify the Lie superalgebra it remains to give the map $[\cdot,\cdot]\colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. This map is symmetric and $\mathfrak{g}_0$ equivariant, so by Schur's Lemma there is one free-parameter for each $\mathfrak{sl}(2,\mathbb{C})$-factor of $\mathfrak{g}_0$. However, the Jacobi identity on elements of $\mathfrak{g}_1$ imposes a homogeneous linear constraint on these parameters and an overall scaling leads to isomorphic algebras. There is thus one degree of freedom. The customary description of this via the parameter α is as follows.

Let D, H, K be the basis of $\mathfrak{sl}(2,\mathbb{C})$ given in (5.2). Write $\mathfrak{g}_1 = \mathbb{C}^4_Q + \mathbb{C}^4_S$ corresponding to the $-i$ and $+i$ eigenspaces of D. Let $Q^1,\dots,Q^4$ be a standard basis of $\mathbb{C}^4_Q$ under the action of $\mathfrak{r} = \mathfrak{so}(4,\mathbb{C})$, and put $S^a = -i[K,Q^a]$, which gives a corresponding standard basis for $\mathbb{C}^4_S$. The Jacobi identity then gives $[Q^a,S^a] = \lambda_1 D$ for some λ_1. One also finds $[Q^1,S^2] = \lambda_2 D_+ + \lambda_3 D_- \in \mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})_+ \oplus \mathfrak{sl}(2,\mathbb{C})_-$. The Jacobi identity implies $\lambda_1+\lambda_2+\lambda_3 = 0$. When $\lambda_1 \neq 0$ one rescales to $\lambda_1 = 1$ and writes $\lambda_2 = -4\alpha/(1+\alpha)$ and $\lambda_3 = -4/(1+\alpha)$. Note that $\alpha = -1$ corresponds to $\lambda_1 = 0$, in that case one may take $\lambda_2 = 1$, $\lambda_3 = -1$.

The algebras $D(2,1;\alpha)_{\mathbb{C}}$ are simple for $\alpha \neq -1, 0, \infty$. Over $\mathbb{C}$, there are isomorphisms between the cases $\alpha^{\pm 1}$, $-(1+\alpha)^{\pm 1}$, $-(\alpha/(1+\alpha))^{\pm 1}$, corresponding to permuting the labelling of the $\mathfrak{sl}(2,\mathbb{C})$-factors in $\mathfrak{g}_0$.

To obtain real forms of this algebra one chooses real structures on each $\mathfrak{sl}(2,\mathbb{C})$-factor. These induce either a real or quaternionic structure on the corresponding two-dimensional representation $\mathbb{C}^2$ depending on whether one has a non-compact or compact form of $\mathfrak{sl}(2,\mathbb{C})$. Thus in order to have a real structure on $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_-$ one needs an odd number of non-compact real forms in the factors of $\mathfrak{g}_0$.

For superconformal symmetry related to quaternionic geometry the appropriate real form is $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) + \mathfrak{su}(2)_+ + \mathfrak{su}(2)_-$. These are the algebras we will refer to as $D(2,1;\alpha)$. Note that only the algebras corresponding to $\alpha^{\pm 1}$ are isomorphic.

7 Superconformal symmetry and special homotheties

The conditions on the target space of $N = 4B$ models with superconformal symmetry $D(2,1;\alpha)$ were determined by Michelson and Strominger [47] in the HKT case, see also Papadopoulos [53]. The extra conformal ingredient comes from the generator D of the $\mathfrak{sl}(2,\mathbb{R})$-factor which gives rise to a vector field X on M.

Let (M, g, I, J, K) be an HKT manifold. A vector field X on M is called a *special homothety of type* (a,b) if (i) $L_X g = ag$, (ii) $L_X A = 0$, for $A = I, J, K$, and (iii) $L_{AX} A = 0$ and $L_{AX} B = bC$, whenever (A, B, C) is a cyclic permutation of (I, J, K). Note that scaling X by a constant does the same for the pair (a,b), so we make speak of the type $[a,b] \in \mathbb{R}\mathrm{P}(1)$ if IX is not tri-holomorphic. In the standard flat model $M = \mathbb{H}^n$, one has that $X = \partial/\partial r$ is a special homothety of type $(2,-2)$. For $a \neq 0$ one may wish to describe the geometry as an HKT cone. When $a = 0$, we will sometimes talk of a *special isometry* since X now preserves the metric g; we now have an HKT cylinder. This is the case in the $S^1 \times S^3$-model thought of as $\mathbb{H}^*/\mathbb{Z}$ where the vector field $X = \partial/\partial r$ above gives a special homothety of type $(0,-2)$.

Theorem 7.1 ([47], [53]). *$D(2,1;\alpha)$-symmetry corresponds to the existence of a special homothety of type $(\alpha+1, 1)$.*

The existence of a special homothety usually leads directly to an HKT potential.

Proposition 7.2 ([56], [55]). *Suppose X is a special homothety of type $(\alpha+1,1)$.*

(1) *For $\alpha \neq -1$, we have $dX^\flat = 0$ and $X \lrcorner c = 0$.*
(2) *For $\alpha \neq -1, 0$, the function*

$$\mu = \frac{2}{\alpha(\alpha+1)} g(X,X)$$

is an HKT potential.

In the cases $\alpha \neq -1, 0$, we may now consider the related HKT metrics g_f with potential $f(\mu)$ of (4.3). To do this it is necessary to work away from the (possibly empty) hypersurface where X is null. Then the cases for which g_f still has X as a special homothety are

(1) for $f(\mu) = |\mu|^k$, g_f has X of type $(k(\alpha+1),1) \neq (1,1)$, and
(2) for $f(\mu) = \log|\mu|$, g_f has X of type $(0,1)$, i.e., $\alpha = -1$

The isomorphic algebras $D(2,1;\alpha)$ and $D(2,1;1/\alpha)$ are obtained by taking $f(\mu) = |\mu|^{1/\alpha}$.

The signature of g_f may differ from that of g. First note that by replacing g by $-g$ we may assume $\mu > 0$ in the neighbourhood of a given point. Suppose g has signature $(4p, 4q)$. Then g_{μ^k} has signature $(4p, 4q)$ precisely when the parameters α for the two structures have the same sign; if the signs are different then the new signature is $(4p-4, 4q+4)$. The metric $g_{\log\mu}$ has signature $(4p-4, 4q+4)$ if $\alpha > 0$, otherwise it is signature $(4p, 4q)$.

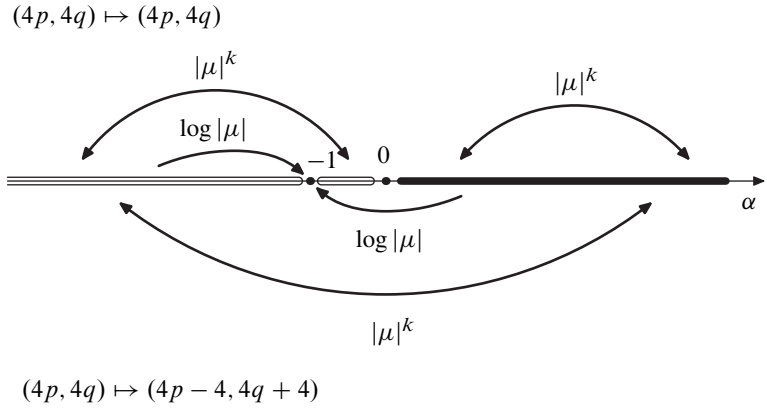

Figure 7.1. Signature changes for $D(2,1;\alpha)$ symmetry.

Locally, structures with $\alpha = -1$ may be converted to other values provided X is not null and $X \lrcorner c = 0$. This is done by solving $d\log\mu = X^\flat$ and considering g_{μ^k}, with $k \neq 1$.

In conclusion, working in the category of indefinite HKT structures we have:

Proposition 7.3 ([56]). *The existence of an* HKT *structure with a nowhere null special homothety X of type $(\alpha+1, 1)$, $\alpha \neq -1, 0$, is equivalent to the existence of such a structure for any other such value α.*

Remark 7.4. This equivalence extends to the case $\alpha = -1$ locally when $X \lrcorner c = 0$, cf. [51]. But other cases can arise for $\alpha = -1$ via T-duality. This requires the existence of a closed invariant 2-form F_θ that is of type $(1,1)$ for each I and has $X \lrcorner F_\theta = 0$, where X is a special homothety of type $(0,1)$. The cohomology class $[F_\theta]$, when integral, then determines a circle bundle P over M to which one may lift the action of X. The quotient $W = P/\langle X\rangle$ carries an HKT structure with $D(2,1;-1)$-symmetry generated by X_W such that $X_W \lrcorner c_W$ is a constant times F_θ when $X \lrcorner c = 0$. Locally, every special isometry with $X \lrcorner c$ of type $(1,1)$ for all I may be obtained in this way [59].

8 Relation to QKT geometry

Let us continue to consider an HKT manifold M with a special homothety X of type $(\alpha+1,1)$, $\alpha\neq 0$, and assume that X is nowhere null. In this situation the vector fields IX, JX and KX generate a local action of the compact group SU(2), that is isometric.

When $\alpha=-1$, the action of X is also isometric and one may consider the manifold $N=M/\langle X, IX, JX, KX\rangle$ as a Riemannian submersion. For $\alpha\neq -1$, X scales the metric so this procedure is no longer possible. However, the level sets of the potential μ are transverse to the homothety X and we consider the quotient

$$N=\frac{\mu^{-1}(1)}{\langle IX, JX, KX\rangle}. \tag{8.1}$$

We will concentrate on the case when N is a manifold, but the discussion carries over easily to the general case, when N is an orbifold, and generates a wide range of examples.

For all $\alpha\neq 0$, the space N inherits a metric g^N and local almost complex structures $\mathcal{I},\mathcal{J},\mathcal{K}$, induced by I,J,K, defining an integrable quaternionic structure, cf. [54]; when $\dim N=4$ this means that N has a self-dual conformal structure. For $\alpha\neq -1$ there is also a torsion three-form c^N, induced by restriction of c to $\mu^{-1}(1)$, which is of type $\Lambda^{\{1,2\}}$ for each $\mathcal{I}$. For $\alpha=-1$, we get a torsion three-form when $X\lrcorner c=0$, cf. Remark 7.4.

Using c^N, we may define first a connection ∇ with skew-symmetric torsion via

$$\nabla=\nabla^{\mathrm{LC}}+\tfrac{1}{2}c^N$$

that preserves the metric g^N and the quaternionic structure. This gives N the structure of a quaternionic Kähler manifold with torsion.

Definition 8.1. A manifold N is *quaternionic Kähler with torsion* (QKT) if there is a metric g, a connection ∇^N and a subbundle $\mathcal{G}$ of $\mathrm{End}\,TN$ satisfying the following: (i) there are local bases $\{I_N, J_N, K_N\}$ of $\mathcal{G}$ satisfying the quaternion relations (2.1) and compatible with g; (ii) the connection ∇^N is metric, $\nabla^N g=0$, and quaternionic, meaning $\nabla^N I_N$ is a linear combination of J_N and K_N, etc.; (iii) the torsion tensor c^N of ∇^N is totally skew-symmetric and of type $(2,1)+(1,2)$ with respect to each I_N.

Ivanov [35] showed that c^N is uniquely determined by g^N and the local almost complex structures when $\dim N>4$.

In [56] it was shown that there is a another connection ∇^{Q} on N, that is torsion-free and preserves the quaternionic structure, given via the formula

$$\nabla^{\mathrm{Q}}=\nabla+\xi$$

with

$$12\,\xi_Y Z = -6c^N(Y,Z) + 2\sum_{A=\mathcal{I},\mathcal{J},\mathcal{K}} Ac^N(Y,AZ) - Ac^N(AY,Z)$$
$$- \mathop{\mathfrak{S}}_{\substack{(A,B,C)\\=(\mathcal{I},\mathcal{J},\mathcal{K})}} Ac^N(BY,CZ) - Ac^N(BY,CZ).$$

This connection is characterised by the property that ∇^{Q} is torsion-free and agrees with ∇ on $\mathcal{G} = \operatorname{Span}\{\mathcal{I},\mathcal{J},\mathcal{K}\}$.

It turns out that N arising in (8.1) is not a general QKT manifold, but rather that it satisfies two stronger conditions. The first is that of being of *instanton type*, meaning that the curvature of the quaternionic connection ∇^{Q} on

$$\kappa^{\mathbb{R}} = \Lambda^{\mathrm{top}} T^*N$$

is of type $(1,1)$ for each $A = \mathcal{I},\mathcal{J},\mathcal{K}$. Equivalently,

$$d\tau \in \bigcap_A \Lambda^{1,1}_A,$$

where τ is the torsion 1-form given by

$$2\tau(X) = \sum_{\substack{A=\mathcal{I},\mathcal{J},\mathcal{K}\\ i=1,\dots,4n}} c^N(AX, e_i, Ae_i).$$

Note that conformally rescaling g^N and using Ivanov's construction [35] of a QKT connection preserves the condition of being of instanton type.

The second condition satisfied by N in (8.1) is a compatibility between the curvature R^∇ of ∇ and the metric g^N. The bundle $\mathcal{G} = \operatorname{Span}\{\mathcal{I},\mathcal{J},\mathcal{K}\} \subset \operatorname{End} TN$ is preserved by ∇ and has curvature

$$R^\nabla \mathcal{I} = R^{\mathrm{Q}}\mathcal{I} = -\beta_K \mathcal{J} + \beta_J \mathcal{K}, \quad \text{etc.}$$

When N is of instanton type, one may show that

$$\sigma = \tfrac{1}{2}\big(\beta_I(\cdot,\mathcal{I}\cdot) + \beta_I(\mathcal{J}\cdot,\mathcal{K}\cdot)\big) = \tfrac{1}{2}(1-\mathcal{J})\beta_I(\cdot,\mathcal{I}\cdot)$$

is symmetric, independent of $\mathcal{I}$, and Hermitian with respect to each $\mathcal{I}$. We call σ the *curvature metric* of N. Note that conformally rescaling we can locally ensure that σ is non-degenerate on any given QKT manifold of instanton type.

Theorem 8.2. *Suppose M^{4n+4} is an* HKT *manifold with a nowhere null special homothety of type $(\alpha+1,1)$, $\alpha \neq -1, 0$. Then*

$$N^{4n} = \frac{\mu^{-1}(1)}{\langle IX, JX, KX\rangle}$$

is QKT *of instanton type, with $g^N = \sigma$ the curvature metric.*

This result extends to the case $\alpha = 0$ by replacing μ with $g(X, X)$. It also applies locally to the case $\alpha = -1$ if $X \lrcorner c = 0$.

For a converse, consider the following two bundles over a quaternionic manifold N:

$$\mathcal{S}(N) := \{(\mathcal{I}, \mathcal{J}, \mathcal{K}) \in \mathcal{G}_x \text{ satisfying the quaternion relations (2.1)} : x \in N\},$$
$$\mathcal{U}(N) := \mathcal{S}(N) \times \mathbb{R}_{>0}.$$

The bundle $\mathcal{S}(N)$ has fibres $\mathrm{SO}(3) = \mathbb{R}\mathrm{P}(3)$; the second bundle $\mathcal{U}(N)$ was called the associated bundle in [58], but it is also known as the Swann bundle.

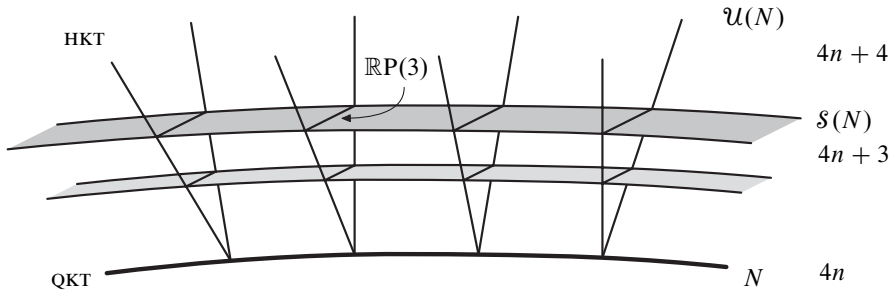

Figure 8.1. Bundles over the QKT base N.

Theorem 8.3. *Suppose N^{4n} is* QKT *of instanton type, then using the curvature metric horizontally, the Swann bundle $M^{4n+4} = \mathcal{U}(N)$ admits* HKT *structures with special homotheties of type $(\alpha + 1, 1)$ for all $\alpha \neq 0$. The structures with $\alpha = -1$ descend to* HKT *structures with special isometry on the discrete quotient $M/\mathbb{Z} = \mathcal{S}(N) \times S^1$.*

This result is proved by constructing a potential for a special homothety of type $(2, -2)$ and then applying the results on type change in §7.

9 Examples

9.1 Quaternionic Kähler bases

A quaternionic Kähler manifold of dimension $4n$ is a (pseudo-) Riemannian manifold (N^{4n}, g^N) such that either

(1) $n > 1$ and for some $0 \leqslant p \leqslant n$ the Levi-Civita connection has holonomy contained in $\mathrm{Sp}(p, n-p)\,\mathrm{Sp}(1)$, or

(2) $n = 1$ and g^N is definite, self-dual and Einstein.

In both cases, the Levi-Civita connection is also a QKT and a quaternionic connection and the metric g^N is Einstein with constant scalar curvature s^N. The curvature metric σ is a positive constant times $s^N g^N$, and so is non-degenerate precisely when $s^N \neq 0$.

The constructions of Swann [58] are a special case of Theorem 8.3 and provide $\mathcal{U}(N)$ with a pseudo-hyper-Kähler metric g and a special homothety of type $(2,-2)$, $\alpha=-2$. This metric g has signature $(4p+4,4n-4p)$ if $s^N>0$, or $(4n-4p+4,4p)$ if $s^N<0$. In particular, it is definite if g^N is positive definite with $s^N>0$. Compact examples of such N are provided by the Wolf spaces

$$\mathbb{H}\mathrm{P}(n)=\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\,\mathrm{Sp}(1)},\quad \mathrm{Gr}_2(\mathbb{C}^{n+2})=\frac{\mathrm{U}(n+2)}{\mathrm{U}(n)\,\mathrm{U}(2)},\quad \widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+4})=\frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\,\mathrm{SO}(4)},$$
$$\frac{\mathrm{G}_2}{\mathrm{SO}(4)},\quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\,\mathrm{Sp}(1)},\quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)\,\mathrm{Sp}(1)},\quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\,\mathrm{Sp}(1)},\quad \frac{\mathrm{E}_8}{\mathrm{E}_7\,\mathrm{Sp}(1)}.$$

These are exactly the compact symmetric spaces of the form $G/K\,\mathrm{Sp}(1)$. A full list of the quaternionic Kähler symmetric spaces in all signatures may be found in [2]. Other homogeneous examples have been found in [15]. When these metrics are definite they are the metrics found in [1], [16], [14] and have $s^N<0$. The hyper-Kähler metric on $\mathcal{U}(N)$ is then of signature $(4,4n)$, but using the type change $\mu\mapsto\mu^{-2}$ of §7 one obtains positive definite HKT structures with special homotheties of type $(-4,-2)$, $\alpha=1$.

Note that the question of when $\mathcal{U}(N)$ might be hyper-Kähler was addressed in [31]. The authors gave conditions that are equivalent to the requirement that the curvature metric on N be quaternionic Kähler.

9.2 Group manifolds

As discussed in §3.2, for a compact group G we can always find a torus T^r such that $M=G\times T^r$ admits a strong HKT structure. Opfermann and Papadopoulos have described in [50] when such manifolds admit a QKT quotient N. In practice, their quotients arise by considering a special isometry on $G\times T^r$ leading to an action of $\mathrm{U}(2)$, so $N=G\times T^r/\mathrm{U}(2)$ and these are instances of the $\alpha=-1$ variation of Theorem 8.2. For G simple, the corresponding M are

$$\mathrm{SU}(2n+1),\quad \mathrm{SU}(2n)\times T^1,\quad \mathrm{SO}(2n+1)\times T^n,\quad \mathrm{Sp}(n)\times T^n,$$
$$\mathrm{SO}(4n)\times T^{2n},\quad \mathrm{SO}(4n+2)\times T^{2n-1},\quad \mathrm{G}_2\times T^2,\quad \mathrm{F}_4,\quad \mathrm{E}_6\times T^2,$$
$$\mathrm{E}_7\times T^7,\quad \mathrm{E}_8\times T^8.$$

The examples where $M=G$ has no torus factor can not arise from the inverse construction of Theorem 8.3. For such M the homotopy exact sequence shows that $b_2(N)=1$. Pulling a 2-form on N representing a non-trivial integral cohomology class back to $\mathcal{S}(N)\times S^1$, we may now recover the HKT structure on M via T-duality as in Remark 7.4 up to finite quotients. A particular case is $M=G=\mathrm{SU}(3)$, $N=\mathbb{C}\mathrm{P}(2)$, where one obtains $\mathrm{SU}(3)/(\mathbb{Z}/2)$; see [54] for the hypercomplex geometry.

9.3 Dimension four and connected sums

If the base manifold N has dimension four, then the condition to be QKT of instanton type is no more than g^N being self-dual. Choosing any one-form t, one may construct a QKT connection with $c^N = *t$, this is of instanton type if and only if dt is self-dual. For N^4 compact this forces t to be closed. Theorem 8.3 now implies that $\mathcal{S}(N^4) \times S^1$ carries an HKT structure (g, c) provided the curvature metric is non-degenerate. Although not strong in general, one has that dc is the pull-back of a four-form on N. If $b_2(N)_+ \neq 0$ we may now find a T-dual M^8 of $\mathcal{S}(N^4) \times S^1$ that is HKT and has finite fundamental group. Examples of this are obtained by taking $N^4 = n\,\mathbb{C}\mathrm{P}(2)$, the connected sum of n-copies of $\mathbb{C}\mathrm{P}(2)$, which is self-dual by [17]. In this case, an alternative to checking the non-degeneracy of the curvature metric is provided by Joyce's construction [37] of $n\,\mathbb{C}\mathrm{P}(2)$ as a quaternionic quotient of $\mathbb{H}\mathrm{P}(n)$ by $\mathrm{U}(1)^{n-1}$. At the level of Swann bundles this may be viewed as a construction of $\mathcal{S}(n\,\mathbb{C}\mathrm{P}(2)) \times S^1$ as a hypercomplex quotient of $\mathbb{H}^{n+1}$, which in turn may be regarded as an HKT quotient using the results of [27]. Thus there are compact simply-connected HKT manifolds M^8_n fibering over $n\,\mathbb{C}\mathrm{P}(2)$.

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Part D

Para-geometries

Chapter 14

Twistor and reflector spaces of almost para-quaternionic manifolds

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1 Introduction and statement of the results

We study the geometry of structures on a differentiable manifold related to the algebra of paraquaternions. This structure leads to the notion of (almost) hyper-paracomplex and almost paraquaternionic manifolds in dimensions divisible by four. These structures are also attractive in theoretical physic since they play a role in string theory [36], [24], [8], [25], [15], [16] and integrable systems [19]. For example, hyper-paracomplex geometry arises in connection with different versions of the c-map [16]. New versions of the c-map are constructed in [16] which allow the authors to obtain the target manifolds of hypermultiplets in Euclidean theories with rigid $N = 2$ supersymmetry. The authors show that the resulting hypermultiplet target spaces are para-hyper-Kähler manifolds. One might also consult the contribution of Th. Mohaupt [35] in this volume where the role of paraquaternions in theories with Euclidean supersymmetry is discussed as well as the exposition written by M. Rocek, C. Vafa and S. Vandoren [37] and the work of A. Swann [40] (booth in this book) concerning the implement of quaternions in string theories.

Both quaternions H and paraquaternions $\widetilde{H}$ are real Clifford algebras, $H = C(2,0)$, $\widetilde{H} = C(1,1) \cong C(0,2)$. In other words, the algebra $\widetilde{H}$ of paraquaternions is generated by the unity 1 and the generators J_1^0, J_2^0, J_3^0 satisfying the *paraquaternionic identities*,

$$(J_1^0)^2 = (J_2^0)^2 = -(J_3^0)^2 = 1, \quad J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0. \tag{1.1}$$

We recall the notion of almost hyper-paracomplex manifold introduced by Libermann [34]. An *almost quaternionic structure of the second kind* on a smooth manifold consists of two almost product structures J_1, J_2 and an almost complex structure J_3 which mutually anti-commute, i.e., these structures satisfy the paraquaternionic identities (1.1). Such a structure is also called *complex product structure* [5], [4].

We recall that an almost product structure P on M is a $(1,1)$ tensor field such that $P^2 = -\mathrm{id}_M$. If moreover the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of P, respectively, have the same rank then the structure P is said to be an almost paracomplex structure (see [17] for a survey of paracomplex geometry). Sometimes these structures are also called almost bilagrangian [30].

An *almost hyper-paracomplex structure* on a $4n$-dimensional manifold M is a triple $\widetilde{H} = (J_\alpha)$, $\alpha = 1,2,3$, where J_1 and J_2 are almost paracomplex structures $J_1, J_2 \colon TM \to TM$, and $J_3 \colon TM \to TM$ is an almost complex structure, satisfying the para-quaternionic identities (1.1). We note that on an almost hyper-paracomplex manifold there is actually a 2-sheeted hyperboloid worth of almost complex structures:

$$S_1^2(-1) = \{c_1 J_1 + c_2 J_2 + c_3 J_3 \mid c_1^2 + c_2^2 - c_3^2 = -1\}$$

and a 1-sheeted hyperboloid worth of almost paracomplex structures:

$$S_1^2(1) = \{b_1 J_1 + b_2 J_2 + b_3 J_3 \mid b_1^2 + b_2^2 - b_3^2 = 1\}.$$

When each J_α, $\alpha = 1,2,3$ is an integrable structure, $\widetilde{H}$ is said to be a *hyper-paracomplex structure* on M. Such a structure is also called sometimes *pseudo-hypercomplex* [19].

It is well known that the structure J_α is integrable if and only if the corresponding Nijenhuis tensor $N_\alpha = [J_\alpha, J_\alpha] + J_\alpha^2[\,,\,] - J_\alpha[J_\alpha,\,] - J_\alpha[\,, J_\alpha]$ vanishes, $N_\alpha = 0$. In fact an almost hyper-paracomplex structure is hyper-paracomplex if and only if any two of the three structures J_α, $\alpha = 1,2,3$ are integrable due to the existence of a linear identity between the three Nijenhuis tensors [29], [14]. In this case all almost complex structures of the two-sheeted hyperboloid $S_1^2(-1)$ as well as all para-complex structures of the one-sheeted hyperboloid $S_1^2(1)$ are integrable. Examples of hyper-paracomplex structures on the simple Lie groups $\mathrm{SL}(2n+1,\mathbb{R})$, $\mathrm{SU}(n,n+1)$ are constructed in [27].

A *hyper-paraHermitian metric* is a pseudo Riemannian metric which is compatible with the (almost) hyper-paracomplex structure $\widetilde{H} = (J_\alpha)$, $\alpha = 1,2,3$ in the sense that the metric is skew-symmetric with respect to each J_α, $\alpha = 1,2,3$. Such a metric is necessarily of neutral signature $(2n,2n)$. Such a structure is called (*almost*) *hyper-paraHermitian structure*.

An *almost para-quaternionic structure* on M is a rank-3 subbundle $\mathcal{P} \subset \mathrm{End}(TM)$ which is locally spanned by an almost hyper-paracomplex structure $\widetilde{H} = (J_\alpha)$; such a locally defined triple $\widetilde{H}$ will be called admissible basis of $\mathcal{P}$. A linear connection ∇ on TM is called *para-quaternionic connection* if ∇ preserves $\mathcal{P}$. We denote the space all para-quaternionic connections on an almost para-quaternionic manifold by $\Delta(\mathcal{P})$.

An almost para-quaternionic structure is said to be a *para-quaternionic* if there is a torsion-free para-quaternionic connection.

An almost para-quaternionic (resp. para-quaternionic) manifold with hyper-para-Hermitian metric is called an *almost para-quaternionic Hermitian* (resp. *para-quaternionic Hermitian*) manifold. If the Levi-Civita connection of a para-quaternionic Hermitian manifold is para-quaternionic connection, then the manifold is said to be *para-quaternionic Kähler* manifold. This condition is equivalent to the statement that the holonomy group of g is contained in $\mathrm{Sp}(n,\mathbb{R})\mathrm{Sp}(1,\mathbb{R})$ for $n \geq 2$ [22]. A typical example is the para-quaternionic projective space endowed with the standard para-quaternionic Kähler structure [13]. The para-quaternionic analogue of the quaternionic geometry is developed in [22] where it is also shown that any para-quaternionic Kähler manifold of dimension $4n \geq 8$ is Einstein. If on a para-quaternionic Kähler manifold there exists an admissible basis $(\widetilde{H})$ such that each J_α, $\alpha = 1,2,3$ is parallel with respect to the Levi-Civita connection, then the manifold is said to be (locally) *hyper-paraKähler*. Such manifolds are also called *hypersymplectic* [23], *neutral hyper-Kähler* [31], [21]. The equivalent characterization is that the holonomy group of g is contained in $\mathrm{Sp}(n,\mathbb{R})$ if $n \geq 2$ [41].

For $n = 1$ an almost para-quaternionic structure is the same as an oriented neutral conformal structure [19], [22], [41], [14] and turns out to be always quaternionic. The existence of a (local) hyper-paracomplex structure is a strong condition since the integrability of the (local) almost hyper-paracomplex structure implies that the corresponding neutral conformal structure is anti-self-dual [1], [24], [29].

When $n \geq 2$, the para-quaternionic condition, i.e., the existence of torsion-free para-quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $\mathrm{GL}(n,\widetilde{H})\,\mathrm{Sp}(1,\mathbb{R}) \cong \mathrm{GL}(2n,\mathbb{R})\,\mathrm{Sp}(1,\mathbb{R})$- structure [4], [5]. The paraquaternionic condition controls the Nijenhuis tensors in the sense that $NJ_\alpha := N_\alpha$ preserves the subbundle $\mathcal{P}$. An invariant first order differential operator D is defined on any almost paraquaternionic manifolds which is two-step nilpotent i.e., $D^2 = 0$ exactly when the structure is paraquaternionic [28]. Para-quaternionic structure is a type of a para-conformal structure [7] as well as a type of generalized hypercomplex structure [10].

Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold. The vector bundle $\mathcal{P}$ carries a Lorentz structure of signature $(+,+,-)$ (note that this is the opposite to the naturally induced signature $(-,-,+)$ as discussed in [12]) such that (J_1,J_2,J_3) forms an orthonormal local basis of P. There are two kinds of "unit sphere" bundles according to the existence of the 1-sheeted hyperboloid $S_1^2(1)$ and the 2-sheeted hyperboloid $S_1^2(-1)$. The *twistor space* $Z^-(M)$ is the unit pseudo-sphere bundle with fibre (the one sheet of) $S_1^2(-1)$. The *reflector space* $Z^+(M)$ is the unit pseudo-sphere bundle with fibre $S_1^2(1)$. In other words, the fibre of $Z^-(M)$ consists of all almost complex structures compatible with the given paraquaternionic structure while the fibre of $Z^+(M)$ consists of all almost paracomplex structures compatible with the given paraquaternionic structure. Reflector spaces where introduced in [30] in the study of neutral surfaces in 4-dimensional neutral pseudo-

Riemannian manifolds. The construction of the reflector space in [30] is based on the splitting of the non-semi-simple structure group $SO(2,2)$, (on the Lie algebra level $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1) \cong \mathfrak{sp}(1,\mathbb{R}) \oplus \mathfrak{sp}(1,\mathbb{R})$) and the reflector spaces are the neutral space analogs of the twistor spaces of Riemannian geometry [30].

Keeping in mind the formal similarity with the quaternionic geometry where there are two natural almost complex structures on the corresponding twistor space [6], [20], one observes the existence of two naturally arising almost complex structures I_1^∇, I_2^∇ on $Z^-(M)$ and two almost paracomplex structures P_1^∇, P_2^∇ on $Z^+(M)$ defined with the help of the horizontal spaces of an arbitrary para-quaternionic connection $\nabla \in \Delta(\mathcal{P})$. The almost paracomplex structures on the reflector space of a 4-dimensional manifold with neutral signature metric are defined using the horizontal spaces of the Levi-Civita connection in [30]. The authors show that one of the almost paracomplex structure is never integrable while the other almost paracomplex structure is integrable if and only if the neutral metric is anti-self-dual. The almost complex structures on the twistor space of a para-quaternionic Kähler manifold are defined and investigated in [12] using the horizontal spaces of the Levi-Civita connection (see also [2]). The authors show that one of the almost complex structure is never integrable while the other almost complex structure is always integrable. These construction are generalized in the case of twistor and reflector space of a para-quaternionic manifold [11] (see also [29]).

In the present chapter we investigate the dependence on the para-quaternionic connection of these structures on the twistor and reflector spaces over an almost para-quaternionic manifold. We obtain conditions on the torsion of the para-quaternionic connection which imply the coincidence of the corresponding structures (Theorem 3.4, Corollary 3.3). We show that the existence of an integrable almost complex structure on the twistor space (resp. the existence of an integrable almost paracomplex structure on the reflector space) does not depend on the para-quaternionic connection and it is equivalent to the condition that the almost para-quaternionic manifold is quaternionic provided the dimension is bigger than four (Theorem 3.9, Theorem 3.12).

In dimension four we find new relations between the Riemannian Ricci forms,i.e., the 2-forms which determine the $Sp(1,\mathbb{R})$-component of the Riemannian curvature, which are equivalent to the ant-self-duality of the oriented neutral conformal structure corresponding to a given para-quaternionic structure (Theorem 3.7).

In the last section we apply our considerations to the paraquaternionic Kähler manifold with torsion recently described by the third author in [42].

2 Preliminaries

Let $\widetilde{\mathbb{H}}$ be the para-quaternions and identify $\widetilde{\mathbb{H}}^n = \mathbb{R}^{4n}$. To fix notation we assume that $\widetilde{\mathbb{H}}$ acts on $\widetilde{\mathbb{H}}^n$ by right multiplication. This defines an antihomomorphism

$$\lambda\colon \{\text{unit para-quaternions}\} = \{x + j_1 y + j_2 z + j_3 w \mid x^2 - y^2 - z^2 + w^2 = \pm 1\}$$
$$\longrightarrow \mathrm{SO}(2n, 2n) \subset \mathrm{GL}(4n, \mathbb{R}),$$

where our convention is that $\mathrm{SO}(2n, 2n)$ acts on $\widetilde{\mathbb{H}}^n$ on the left. Denote the image by $\mathrm{Sp}(1, \mathbb{R})$ and let $J_1^0 = -\lambda(j_1)$, $J_2^0 = -\lambda(j_2)$, $J_3^0 = -\lambda(j_3)$. The Lie algebra of $\mathrm{Sp}(1, \mathbb{R})$ is $\mathfrak{sp}(1, \mathbb{R}) = \mathrm{span}\{J_1^0, J_2^0, J_3^0\}$ and we have

$$J_1^{0\,2} = J_2^{0\,2} = -J_3^{0\,2} = 1, \quad J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0.$$

Define $\mathrm{GL}(n, \widetilde{H}) = \{A \in \mathrm{GL}(4n, \mathbb{R}) \mid A(\mathfrak{sp}(1, \mathbb{R}))A^{-1} = \mathfrak{sp}(1, \mathbb{R})\}$. The Lie algebra of $\mathrm{GL}(n, \widetilde{H})$ is $\mathfrak{gl}(n, \widetilde{H}) = \{A \in \mathfrak{gl}(4n, \mathbb{R}) \mid AB = BA \text{ for all } B \in \mathfrak{sp}(1, \mathbb{R})\}$.

Let $(M, \mathcal{P})$ be an almost paraquaternionic manifold and $\widetilde{H} = (J_\alpha)$, $\alpha = 1, 2, 3$ be an admissible local basis. We shall use the notation $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$.

Let $\mathcal{B} \in \Gamma(\Lambda^2(TM) \otimes TM)$. We say that $\mathcal{B}$ is of type $(0, 2)_{J_\alpha}$ with respect to J_α if

$$\mathcal{B}(J_\alpha X, Y) = -J_\alpha \mathcal{B}(X, Y)$$

and denote this space by $\Lambda^{0,2}_{J_\alpha}$. The projection $\mathcal{B}^{0,2}_{J_\alpha}$ is given by

$$\mathcal{B}^{0,2}_{J_\alpha}(X, Y) = \frac{1}{4}(\mathcal{B}(X, Y) + \epsilon_\alpha \mathcal{B}(J_\alpha X, J_\alpha Y) - \epsilon_\alpha J_\alpha \mathcal{B}(J_\alpha X, Y) - \epsilon_\alpha J_\alpha \mathcal{B}(X, J_\alpha Y)).$$

For example, the Nijenhuis tensor N_α is an element of $\Lambda^{0,2}_{J_\alpha}$.

Let $\nabla \in \Delta(\mathcal{P})$ be a para-quaternionic connection on an almost paraquaternionic manifold $(M, \mathcal{P})$. This means that there exist locally defined 1-forms ω_α, $\alpha = 1, 2, 3$ such that

$$\begin{aligned} \nabla J_1 &= -\omega_3 \otimes J_2 + \omega_2 \otimes J_3, \\ \nabla J_2 &= \omega_3 \otimes J_1 + \omega_1 \otimes J_3, \\ \nabla J_3 &= \omega_2 \otimes J_1 + \omega_1 \otimes J_2. \end{aligned} \tag{2.1}$$

An easy consequence of (2.1) is that the curvature R^∇ of any para-quaternionic connection $\nabla \in \Delta(\mathcal{P})$ satisfies the relations

$$\begin{aligned} [R^\nabla, J_1] &= -A_3 \otimes J_2 + A_2 \otimes J_3, & A_1 &= d\omega_1 + \omega_2 \wedge \omega_3, \\ [R^\nabla, J_2] &= A_3 \otimes J_1 + A_1 \otimes J_3, & A_2 &= d\omega_2 + \omega_3 \wedge \omega_1, \\ [R^\nabla, J_3] &= A_2 \otimes J_1 + A \otimes J_2, & A_3 &= d\omega_3 - \omega_1 \wedge \omega_2. \end{aligned} \tag{2.2}$$

The Ricci 2-forms of a para-quaternionic connection are defined by

$$\rho^{\nabla}_{\alpha}(X,Y)=\frac{1}{2}\mathrm{tr}(Z\longrightarrow J_{\alpha}R^{\nabla}(X,Y)Z),\quad \alpha=1,2,$$
$$\rho^{\nabla}_{3}(X,Y)=-\frac{1}{2}\mathrm{tr}(Z\longrightarrow J_{3}R^{\nabla}(X,Y)Z).$$

It is easy to see using (2.2) that the Ricci forms are given by

$$\rho^{\nabla}_{1}=d\omega_1+\omega_2\wedge\omega_3,\quad \rho^{\nabla}_{2}=-d\omega_2-\omega_3\wedge\omega_1,\quad \rho^{\nabla}_{3}=d\omega_3-\omega_1\wedge\omega_2. \quad (2.3)$$

We split the curvature of ∇ into $\mathfrak{gl}(n,\widetilde{H})$-valued part $(R^{\nabla})'$ and $\mathfrak{sp}(1,\mathbb{R})$-valued part $(R^{\nabla})''$ following the classical scheme (see e.g. [3], [26], [9])

Proposition 2.1. *The curvature of an almost para-quaternionic connection on M splits as follows*

$$R^{\nabla}(X,Y)=(R^{\nabla})'(X,Y)+\frac{1}{2n}(\rho^{\nabla}_{1}(X,Y)J_1+\rho^{\nabla}_{2}(X,Y)J_2+\rho^{\nabla}_{3}(X,Y)J_3),$$
$$[(R^{\nabla})'(X,Y),J_{\alpha}]=0,\quad \alpha=1,2,3,$$

Let Ω, Θ be the curvature 2-form and the torsion 2-form of ∇ on the principle $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})$-bundle $P(M)$, respectively (see e.g. [32]). We denote the splitting of the $\mathfrak{gl}(n,\widetilde{H})\oplus\mathfrak{sp}(1,\mathbb{R})$-valued curvature 2-form Ω on $P(M)$ according to Proposition 2.1, by $\Omega=\Omega'+\Omega''$, where Ω' is a $\mathfrak{gl}(n,\widetilde{H})$-valued 2-form and Ω'' is a $\mathfrak{sp}(1,\mathbb{R})$-valued form. Explicitly,

$$\Omega''=\Omega''_1J^0_1+\Omega''_2J^0_2+\Omega''_3J^0_3,$$

where $\Omega''_{\alpha},\alpha=1,2,3$, are 2-forms. If $\xi,\eta,\zeta\in\mathbb{R}^{4n}$, then the 2-forms $\Omega''_{\alpha},\alpha=1,2,3$, are given by

$$\Omega''_{\alpha}(B(\xi),B(\eta))=\frac{1}{2n}\rho_{\alpha}(X,Y)\quad\text{and}\quad X=u(\xi),Y=u(\eta).$$

3 Twistor and reflector spaces of almost para-quaternionic manifolds

Consider the space $\widetilde{H}_1$ of imaginary para-quaternions. It is isomorphic to the Lorentz space $\mathbb{R}^2_1$ with a Lorentz metric of signature $(+,+,-)$ defined by $\langle q,q'\rangle=-\mathrm{Re}(q\bar{q}')$, where $\bar{q}=-q$ is the conjugate imaginary para-quaternion. In $\mathbb{R}^2_1$ there are two kinds of 'unit spheres', namely the pseudo-sphere $S^2_1(1)$ of radius 1 (the 1-sheeted hyperboloid) which consists of all imaginary para-quaternions of norm 1 and the pseudo-sphere $S^2_1(-1)$ of radius (-1) (the 2-sheeted hyperboloid) which contains all imaginary para-quaternions of norm (-1). The 1-sheeted hyperboloid $S^2_1(1)$ carries a natural paracomplex structure while the 2-sheeted hyperboloid $S^2_1(-1)$ carries a

natural complex structure, both induced by the cross-product on $\widetilde{H}_1 \cong \mathbb{R}^2_1$ defined by

$$X \times Y = \sum_{i \neq k} x^i y^k J_i J_k$$

for vectors $X = x^i J_i$, $Y = y^k J_k$. Namely, for a tangent vector $X = x^i J_i$ to the 1-sheeted hyperboloid $S^2_1(1)$ at a point $q_+ = q^k_+ J_k$ (resp. tangent vector $Y = y^k_- J_k$ to the 2-sheeted hyperboloid $S^2_1(-1)$ at a point $q_- = q^k_- J_k$) we define $PX := q_+ \times X$ (resp. $JY = q_- \times Y$). It is easy to check that PX is again tangent vector to $S^2_1(1)$ and $P^2X = X$ (resp. JY is tangent vector to $S^2_1(-1)$ and $J^2Y = -Y$).

Let M be a $4n$-dimensional manifold endowed with an almost para-quaternionic structure $\mathcal{P}$. Let J_1, J_2, J_3 be an admissible basis of $\mathcal{P}$ defined in some neighborhood of a given point $p \in M$. Any linear frame u of T_pM can be considered as an isomorphism $u\colon \mathbb{R}^{4n} \to T_pM$. If we pick such a frame u we can define a subspace of the space of the all endomorphisms of T_pM by $u(\mathfrak{sp}(1,\mathbb{R}))u^{-1}$. Clearly, this subset is a para-quaternionic structure at the point p and in general this para-quaternionic structure is different from $\mathcal{P}_p$. We define $P(M)$ to be the set of all linear frames u which satisfy $u(\mathfrak{sp}(1,\mathbb{R}))u^{-1} = \mathcal{P}$. It is easy to see that $P(M)$ is a principal frame bundle of M with structure group $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})$, it is also called a $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})$-structure on M.

Let $\pi\colon P(M) \to M$ be the natural projection. For each $u \in P(M)$ we consider two linear isomorphisms $j^+(u)$ and $j^-(u)$ on $T_{\pi(u)}M$ defined by $j^+(u) = uJ^0_1u^{-1}$ and $j^-(u) = uJ^0_3u^{-1}$. It is easy to see that $(j^+(u))^2 = \mathrm{id}$ and $(j^-(u))^2 = -\mathrm{id}$. For each point $p \in M$ we define $Z^+_p(M) = \{j^+(u) \mid u \in P(M), \pi(u) = p\}$ and $Z^-_p(M) = \{j^-(u) \mid u \in P(M), \pi(u) = p\}$. In other words, $Z^-_p(M)$ is the connected component of J_3 of the space of all complex structures (resp. $Z^+_p(M)$ is the space of all paracomplex structures) in the tangent space T_pM which are compatible with the almost para-quaternionic structure on M.

We define the twistor space Z^- of M, by setting $Z^- = \bigcup_{p\in M} Z^-_p(M)$. Let H_3 be the stabilizer of J^0_3 in the group $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})$. There is a bijective correspondence between the symmetric space $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})/H_3 \cong S^2_1(-1)^+ = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1,\ z > 0\}$ and $Z^-_p(M)$ for each $p \in M$. So we can consider Z^- as the associated fibre bundle of $P(M)$ with standard fibre $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})/H_3$. Hence, $P(M)$ is a principal fibre bundle over Z^- with structure group H_3 and projection j^-. We consider the symmetric spaces $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})/H_3$. We have the following Cartan decomposition $\mathfrak{gl}(n,\widetilde{H}) \oplus \mathfrak{sp}(1,\mathbb{R}) = h_3 \oplus m_3$ where

$$h_3 = \{A \in \mathfrak{gl}(n,\widetilde{H}) \oplus \mathfrak{sp}(1,\mathbb{R}) \mid AJ^0_3 = J^0_3A\}$$

is the Lie algebra of H_3 and $m_3 = \{A \in \mathfrak{gl}(n,\widetilde{H}) \oplus \mathfrak{sp}(1,\mathbb{R}) \mid AJ^0_3 = -J^0_3A\}$. It is clear that m_3 is generated by J^0_1, J^0_2, i.e., $m_3 = \mathrm{span}\{J^0_1, J^0_2\}$. Hence, if $A \in m_3$ then $J^0_3A \in m_3$.

We proceed with defining the reflector space Z^+ of M. Put $Z^+ = \bigcup_{p\in M} Z_p^+(M)$. Let H_1 be the stabilizer of J_1^0 in the group $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})$. There is a bijective correspondence between the symmetric space $\mathrm{GL}(n,\widetilde{H})\ \mathrm{Sp}(1,\mathbb{R})/H_1 \cong S_1^2(1) = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-z^2=1\}$ and $Z_p^+(M)$ for each $p\in M$. So we can consider Z^+ as the associated fibre bundle of $P(M)$ with standard fibre $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})/H_1$. Hence, $P(M)$ is a principal fibre bundle over Z^+ with structure group H_1 and projection j^+. We consider the symmetric spaces $\mathrm{GL}(n,\widetilde{H})\mathrm{Sp}(1,\mathbb{R})/H_1$. We have the following Cartan decomposition $\mathfrak{gl}(n,\widetilde{H})\oplus\mathfrak{sl}(1,\mathbb{R}) = h_1\oplus m_1$ where

$$h_1 = \{A\in\mathfrak{gl}(n,\widetilde{H})\oplus\mathfrak{sl}(1,\mathbb{R}) \mid AJ_1^0 = J_1^0A\}$$

is the Lie algebra of H_1 and $m_1 = \{A\in\mathfrak{gl}(n,\widetilde{H})\oplus\mathfrak{sl}(1,\mathbb{R}) \mid AJ_1^0 = -J_1^0A\}$. It is clear that m_1 is generated by J_2^0, J_3^0, i.e., $m_1 = \mathrm{span}\{J_2^0, J_3^0\}$. Hence, if $A\in m_1$ then $J_1^0A\in m_1$.

Let ∇ be a para-quaternionic connection on M, i.e., ∇ is a linear connection in the principal bundle $P(M)$ (see e.g. [32]). Note that we make no assumptions on the torsion or on the curvature of ∇. Keeping in mind the formal similarity with the quaternionic geometry where one uses a quaternionic connection to define two almost complex structures on the corresponding twistor space [6], [20], [38], [39], we use ∇ to define two almost complex structures I_1^∇ and I_2^∇ on the twistor space Z^- and two almost paracomplex structures P_1^∇ and P_2^∇ on the reflector space Z^+.

We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the standard horizontal vector field) on $P(M)$ corresponding to $A\in\mathfrak{gl}(n,\widetilde{H})\oplus\mathfrak{sl}(1,\mathbb{R})$ (resp. $\xi\in\mathbb{R}^{4n}$).

Let $u\in P(M)$ and Q_u be the horizontal subspace of the tangent space $T_uP(M)$ induced by ∇ (see e.g. [32]). The vertical space i.e., the vector space tangent to a fibre is isomorphic to

$$\left(\mathfrak{gl}(n,\widetilde{H})\oplus\mathfrak{sl}(1,\mathbb{R})\right)_u^* = (h_3)_u^*\oplus(m_3)_u^* = (h_1)_u^*\oplus(m_1)_u^*,$$

where $(h_i)_u^* = \{A_u^* \mid A\in h_i\}$ and $(m_i)_u^* = \{A_u^* \mid A\in m_i\}$, $i = 1,3$. Hence, $T_uP(M) = (h_i)_u^*\oplus(m_i)_u^*\oplus Q_u$.

For each $u\in P(M)$ we put

$$V^-_{j^-(u)} = j^-_{*u}((m_3)_u^*),\ H^-_{j^-(u)} = j^-_{*u}Q_u,\quad V^+_{j^+(u)} = j^+_{*u}((m_1)_u^*),\ H^+_{j(u)} = j^+_{*u}Q_u.$$

Thus we obtain vertical and horizontal distributions V^- and H^- on Z^-(resp. V^+ and H^+ on Z^+). Since $P(M)$ is a principal fibre bundle over Z^- (resp. Z^+) with structure group H_3 (resp. H_1) we have $\mathrm{Ker}\, j^-_{*u} = (h_3)_u^*$ (resp. $\mathrm{Ker}\, j^+_{*u} = (h_1)_u^*$).

Hence $V^-_{j^-(u)} = j^-_{*u}(m_3)_u^*$ and $j^-_{*u|(m_3)_u^*\oplus Q_u}\colon (m_3)_u^*\oplus Q_u \to T_{j^-(u)}Z^-$ is an isomorphism (resp. $V^+_{j^+(u)} = j^+_{*u}(m_1)_u^*$ and $j^+_{*u|(m_1)_u^*\oplus Q_u}\colon (m_1)_u^*\oplus Q_u \to T_{j^+(u)}Z^+$ is an isomorphism).

We define two almost complex structures I_1^∇ and I_2^∇ on Z^- by

$$\begin{gathered} I_1^\nabla(j_{*u}^- A^*) = j_{*u}^-(J_3^0 A)^*, \quad I_2^\nabla(j_{*u}^- A^*) = -j_{*u}^-(J_3^0 A)^* \\ [4pt] I_i^\nabla(j_{*u}^- B(\xi)) = j_{*u}^- B(J_3^0 \xi), \quad i = 1, 2, \end{gathered} \tag{3.1}$$

for $A \in m_3, \xi \in \mathbb{R}^{4n}$.

Similarly, we define two almost paracomplex structures P_1^∇ and P_2^∇ on Z^+ by

$$\begin{gathered} P_1^\nabla(j_{*u}^+ A^*) = j_{*u}^+(J_1^0 A)^*, \quad P_2^\nabla(j_{*u}^+ A^*) = -j_{*u}^+(J_1^0 A)^* \\ P_i^\nabla(j_{*u}^+ B(\xi)) = j_{*u}^+ B(J_1^0 \xi), \quad i = 1, 2, \end{gathered} \tag{3.2}$$

for $A \in m_1, \xi \in \mathbb{R}^{4n}$. Clearly, the construction of these structures depends on the choice of the para-quaternionic connection ∇.

The almost paracomplex structures (3.2) on the reflector space of a 4-dimensional manifold with neutral signature metric are defined using the horizontal spaces of the Levi-Civita connection ∇^g in [30]. The authors show that the almost paracomplex structure $P_2^{\nabla^g}$ is never integrable while the almost paracomplex structure $P_1^{\nabla^g}$ is integrable if and only if the neutral metric is anti-self-dual. The almost complex structures (3.1) on the twistor space of a para-quaternionic Kähler manifold are defined and investigated in [12] with the help of the horizontal spaces of the Levi-Civita connection. The authors show that the almost complex structure $I_2^{\nabla^g}$ is never integrable while the almost complex structure $I_1^{\nabla^g}$ is always integrable. Both construction are generalized in the case of twistor and reflector space of a para-quaternionic manifold in [11], [29]. Twistor space of para-quaternionic Kähler manifold is investigated also in [18] where the LeBrun's inverse twistor construction for quaternionic Kähler manifolds [33] has been adapted to the case of para-quaternionic Kähler manifolds.

We finish this section with the next useful

Lemma 3.1. *Let $J_- \in Z^-$ be an almost complex structure or $J_+ \in Z^+$ be an almost paracomplex structure and $\mathcal{B} \in \Gamma(\Lambda^2(TM) \otimes TM)$.*

If $\mathcal{B}^{0,2}_{J_-} = 0$ for all $J_- \in Z^-$ then $\mathcal{B}^{0,2}_{J_+} = 0$ for all $J_+ \in Z^+$ and vice versa.

Proof. Let $J_t = \sinh t J_1 + \cosh t J_3, t \in \mathbb{R}$ be an almost complex structure in Z^-. Using the conditions $\mathcal{B}^{0,2}_{J_t} = 0 = \mathcal{B}^{0,2}_{J_3}$, we calculate

$$\frac{1}{2}(1 + \cosh 2t)\mathcal{B}^{0,2}_{J_1} + \frac{1}{2}\sinh 2t\,[\mathcal{B}] = 0.$$

The latter leads to $\mathcal{B}^{0,2}_{J_1} = 0$. Similarly, $\mathcal{B}^{0,2}_{J_2} = 0$ and the lemma follows. □

3.1 Dependence on the para-quaternionic connection

In this section we investigate when different almost para-quaternionic connections induce the same structure on the twistor or reflector space over an almost para-quaternionic manifold.

Let ∇ and ∇' be two different almost para-quaternionic connections on an almost para-quaternionic manifold $(M, \mathcal{P})$. Then we have

$$\nabla'_X = \nabla_X + S_X, \quad X \in \Gamma(TM),$$

where S_X is a $(1,1)$ tensor on M and $u^{-1}(S_X)u$ belongs to $\mathfrak{gl}(n, \widetilde{H}) \oplus \mathfrak{sp}(1, \mathbb{R})$ for any $u \in P(M)$. Thus we have the splitting

$$S_X(Y) = S^0_X(Y) + s^1(X)J_1Y + s^2(X)J_2Y + s^3(X)J_3Y, \tag{3.3}$$

where $X, Y \in \Gamma(TM)$, s^i are 1-forms and $[S^0_X, J_i] = 0$, $i = 1, 2, 3$.

Proposition 3.2. *Let ∇ and ∇' be two different para-quaternionic connections on an almost para-quaternionic manifold $(M, \mathcal{P})$. The following conditions are equivalent.*

(i) *The two almost complex structures I_1^{∇} and $I_1^{\nabla'}$ on the twistor space Z^- coincide.*

(ii) *The 1-forms s^1, s^2, s^3 are related as follows:*

$$s^1(J_1X) = s^2(J_2X) = s^3(J_3X), \quad X \in \Gamma(TM).$$

(iii) *The two almost paracomplex structures P_1^{∇} and $P_1^{\nabla'}$ on the reflector space Z^+ coincide.*

Proof. We fix a point J of the twistor space Z^-. We have $J = a_1J_1 + a_2J_2 + a_3J_3$ with $a_1^2 + a_2^2 - a_3^2 = -1$. Let $\pi\colon Z^- \to M$ be the natural projection and $x = \pi(J)$. The connection ∇ induces a splitting of the tangent space of Z^- into vertical and horizontal components: $T_JZ^- = V_J^- \oplus H_J^-$. Let v and h be the vertical and horizontal projections corresponding to this splitting. Let $T_JZ^- = V'^-_J \oplus H'^-_J$ be the splitting induced by ∇' with the projections v' and h', respectively. It is easy to observe the following identities:

$$\begin{aligned} v + h &= 1, \\ v' + h' &= 1, \\ vv' &= v', \\ v' + vh' &= v. \end{aligned} \tag{3.4}$$

In fact, $V_J^- = V'^-_J$ and we may regard this space as a subspace of $\mathcal{P}_x$. We have that

$$\begin{aligned} V_J^- &= \{W \in \mathcal{P}_x \mid WJ + JW = 0\} \\ &= \{w_1J_1 + w_2J_2 + w_3J_3 \mid w_1a_1 + w_2a_2 - w_3a_3 = 0\}, \end{aligned}$$

where $J = a_1J_1 + a_2J_2 + a_3J_3$. It follows that for any $W \in V_J^-$, $I_1^{\nabla}(W) = I_1^{\nabla'}(W) = JW$. In general, for any $W \in T_JZ^-$, we have

$$\begin{aligned} I_1^{\nabla}(W) &= J(vW) + (J\pi(W))^h, \\ I_1^{\nabla'}(W) &= J(v'W) + (J\pi(W))^{h'}, \end{aligned}$$

where $(\cdot)^h$ (resp. $(\cdot)^{h'}$) denotes the horizontal lift on Z^- of the corresponding vector field on M with respect to ∇ (resp. ∇'). Using (3.4), we calculate that

$$\begin{aligned} v(I_1^{\nabla'}W) &= J(v'W) + v(J\pi(W))^{h'} = J((v - vh')W) + v(J\pi(W))^{h'} \\ &= v(I_1^{\nabla}W) - J(vh'W) + v(J\pi(W))^{h'}. \end{aligned} \tag{3.5}$$

We investigate the equality

$$J(vh'W) = v(J\pi(W))^{h'}, \quad W \in T_J Z^-. \tag{3.6}$$

Take $W = Y^{h'}$, $Y \in \Gamma(TM)$ in (3.6) to get

$$J(vY^{h'}) = v(JY)^{h'}, \quad Y \in T_x M. \tag{3.7}$$

Hence, (3.7) is equivalent to $I_1^{\nabla} = I_1^{\nabla'}$ because of (3.5).

Let $(U, x_1, \dots, x_{4n})$ be a local coordinate system on M and let $Y = \sum Y^i \frac{\partial}{\partial x^i}$. The horizontal lift of Y with respect to ∇' at the point $J \in Z^-$ is given by

$$Y_J^{h'} = \sum_{i=1}^{4n} (Y^i \circ \pi) \frac{\partial}{\partial x^i} - \sum_{s=1}^{3} a_s \nabla'_{Y} J_s.$$

We calculate

$$\begin{aligned} v(JY)^{h'} &= (JY)^{h'} - h(JY)^{h'} = (JY)^{h'} - (JY)^h \\ &= \sum_{s=1}^{3} a_s(-\nabla'_{JY} J_s + \nabla_{JY} J_s) = -[S_{JY}, J]. \end{aligned} \tag{3.8}$$

On the other hand, we have

$$J(vY^{h'}) = J(Y^{h'} - Y^h) = J \sum_{s=1}^{3} a_s(-\nabla'_Y J_s + \nabla_Y J_s) = -J[S_Y, J]. \tag{3.9}$$

Substitute (3.8) and (3.9) into (3.7) to get that $I_1^{\nabla} = I_1^{\nabla'}$ is equivalent to the condition

$$J[S_Y, J] = [S_{JY}, J], \quad Y \in \Gamma(TM),\ J \in Z^-. \tag{3.10}$$

Now, (3.10) and (3.3) easily lead to the equivalence of (i) and (ii).

Similarly, we obtain that $P_1^{\nabla} = P_1^{\nabla'}$ if and only if

$$P[S_Y, J] = [S_{PY}, J] \tag{3.11}$$

for any choice of $P \in Z^+$ and $Y \in TM$.

The equality (3.11) together with (3.3) implies the equivalence of (ii) and (iii). □

Corollary 3.3. *Let ∇ and ∇' be two different para-quaternionic connections on an almost para-quaternionic manifold $(M, \mathcal{P})$. The following conditions are equivalent:*

(i) *The two almost complex structures I_2^{∇} and $I_2^{\nabla'}$ on the twistor space Z^- coincide.*

(ii) *The* 1*-forms* s^1, s^2, s^3 *vanish,* $s^1 = s^2 = s^3 = 0$.

(iii) *The two almost paracomplex structures* P_2^{∇} *and* $P_2^{\nabla'}$ *on the reflector space* Z^+ *coincide.*

Proof. It is sufficient to observe from the proof of Proposition 3.2 that $I_2^{\nabla} = I_2^{\nabla'}$ is equivalent to $J[S_Y, J] = -[S_{JY}, J]$, $Y \in \Gamma(TM)$, $J \in Z^-$ while $P_1^{\nabla} = P_1^{\nabla'}$ if and only if $P[S_Y, J] = -[S_{PY}, J]$ for any choice of $P \in Z^+$ and $Y \in TM$. Each one of the last two conditions imply $s^1 = s^2 = s^3 = 0$. □

Theorem 3.4. *Let* ∇ *and* ∇' *be two different para-quaternionic connections with torsion tensors* $T^{\nabla'}$ *and* T^{∇}*, respectively, on an almost para-quaternionic manifold* $(M, \mathcal{P})$*. The following conditions are equivalent:*

(i) *The two almost complex structures* I_1^{∇} *and* $I_1^{\nabla'}$ *on the twistor space* Z^- *coincide.*

(ii) *The* $(0,2)_J$ *parts with respect to all* $J \in \mathcal{P}$ *of the torsion* T^{∇} *and* $T^{\nabla'}$ *coincide,* $(T^{\nabla})^{0,2}_J = (T^{\nabla'})^{0,2}_J$.

(iii) *The two almost paracomplex structures* P_1^{∇} *and* $P_1^{\nabla'}$ *on the reflector space* Z^+ *coincide.*

Proof. The equivalence of (i) and (iii) has been proved in Proposition 3.2.

Let $S = \nabla' - \nabla$. Then we have

$$T^{\nabla'}(X,Y) = T^{\nabla}(X,Y) + S_X(Y) - S_Y(X). \tag{3.12}$$

The $(0,2)_J$-part with respect to J of (3.12) gives

$$(T^{\nabla'})^{0,2}_J - (T^{\nabla})^{0,2}_J = [S_{JX}, J]Y - J[S_X, J]Y - [S_{JY}, J]X + J[S_Y, J]X. \tag{3.13}$$

Suppose (iii) holds. Substitute (3.11) into the right hand side of (3.13) and use Lemma 3.1 to get $(T^{\nabla'})^{0,2}_J = (T^{\nabla})^{0,2}_J$, i.e., (ii) is true.

For the converse, put $J = J_2$ in (3.13) and use the splitting (3.3) to obtain

$$\begin{aligned} &\frac{1}{2}\Big((T^{\nabla'})^{0,2}_{J_2} - (T^{\nabla})^{0,2}_{J_2}\Big) \\ &\quad = \big[s^1(X) + s^3(J_2X)\big] J_1Y + \big[s^1(J_2X) + s^3(X)\big] J_3Y \\ &\qquad - \big[s^1(Y) + s^3(J_2Y)\big] J_1X - \big[s^1(J_2Y) + s^3(Y)\big] J_3X. \end{aligned} \tag{3.14}$$

Hence, $s^1(J_1X) = s^3(J_3X)$ is equivalent to $(T^{\nabla'})^{0,2}_{J_2} = (T^{\nabla})^{0,2}_{J_2}$ Substitute $J = J_1$ in (3.13) and use the splitting (3.3) to get $s^1(J_2X) = s^3(J_3X)$ is equivalent to $(T^{\nabla'})^{0,2}_{J_1} = (T^{\nabla})^{0,2}_{J_1}$. Now, Lemma 3.1 together with Proposition 3.2 completes the proof. □

3.2 Integrability

In this section we investigate conditions on the para- quaternionic connection ∇ which imply the integrability of the almost complex structure I_1^∇ on Z^- and almost para-complex structure P_1^∇ on Z^+. We also show that I_2^∇ and P_2^∇ are never integrable, i.e., for any choice of the para-quaternionic connection ∇ each of these two structures has non-vanishing Nijenhuis tensor.

We denote by IN_i, PN_i, $i = 1, 2$ the Nijenhuis tensors of I_i and P_i, respectively and recall that

$$\begin{aligned} IN_i(U,W) &= [I_iU, I_iW] - [U,W] \\ &\quad - I_i[I_iU, W] - I_i[U, I_iW], \qquad U, W \in \Gamma(TZ^-), \\ PN_i(U,W) &= [P_iU, P_iW] + [U,W] \\ &\quad - P_i[P_iU, W] - P_i[U, P_iW], \quad U, W \in \Gamma(TZ^+). \end{aligned}$$

Proposition 3.5. *Let ∇ be a para-quaternionic connection on an almost para-quaternionic manifold $(M, \mathcal{P})$ with torsion tensor T^∇. The following conditions are equivalent:*

(i) *The almost complex structure I_1^∇ on the twistor space Z^- of $(M, \mathcal{P})$ is integrable.*

(ii) *The $(0,2)_J$-parts $(T^\nabla)_J^{0,2}$ of the torsion with respect to all $J \in \mathcal{P}$ vanish,*

$$(T^\nabla)_J^{0,2} = 0, J \in \mathcal{P} \tag{3.15}$$

and the $(2,0) + (0,2)$ parts of the Ricci 2-forms with respect to an admissible basis J_1, J_2, J_3 of $\mathcal{P}$ coincide in the sense that the following identities hold

$$\epsilon_\beta \rho_\alpha(J_\beta X, J_\beta Y) + \rho_\alpha(X,Y) - \epsilon_\gamma \rho_\gamma(J_\beta X, Y) - \epsilon_\gamma \rho_\gamma(X, J_\beta Y) = 0, \tag{3.16}$$

where $\{\alpha, \beta, \gamma\}$ is a cyclic permutation of $\{1,2,3\}$ and $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$.

(iii) *The almost paracomplex structure P_1^∇ on the reflector space Z^+ of $(M, \mathcal{P})$ is integrable.*

Proof. Let J_1, J_2, J_3 be an admissible basis of the almost para-quaternionic structure $\mathcal{P}$.

Let hor be the natural projection $T_uP \to (m_3)_u^* \oplus Q_u$, with $\ker(\mathrm{hor}) = (h_3)_u^*$. We define a tensor field I_1' on $P(M)$ by

$$\begin{aligned} I_1'(U) &\in (m_3)_u^* \oplus Q_u, \\ (j^-)_{*u}(I_1'(U)) &= I_1((j^-)_{*u}U), \quad U \in T_uP. \end{aligned}$$

For any $U, W \in \Gamma(TP(M))$ we define

$$IN_1'(U,W) = \mathrm{hor}[I_1'U, I_1'W] - \mathrm{hor}[\mathrm{hor}\, U, \mathrm{hor}\, W] - I_1'[I_1'U, \mathrm{hor}\, W] - I_1'[\mathrm{hor}\, U, I_1'W].$$

It is easy to check that IN_1' is a tensor field on $P(M)$. We also observe that

$$j_{*u}^-(IN_1'(U,W)) = IN_1(j_{*u}^-U, j_{*u}^-W), \quad U, W \in T_uP(M). \tag{3.17}$$

Let $A, B \in m_3$ and $\xi, \eta \in \mathbb{R}^{4n}$. Using the well-known general commutation relations among the fundamental vector fields and standard horizontal vector fields on the principal bundle $P(M)$ (see e.g. [32]), we calculate taking into account (3.17) that

$$
\begin{aligned}
&IN_1(j^-_{*u}(A^*_u), j^-_{*u}(B^*_u)) = 0,\\
&IN_1(j^-_{*u}(A^*_u), j^-_{*u}(B(\xi)_u)) = 0,\\
&[IN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{H^-}\\
&\quad = j^-_{*u}(B(-\Theta(B(J^0_3\xi), B(J^0_3\eta)) + \Theta(B(\xi), B(\eta))\\
&\qquad + J^0_3\Theta(B(J^0_3\xi), B(\eta)) + J^0_3\Theta(B(\xi), B(J^0_3\eta)))_u),
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&[IN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{V^-}\\
&\quad = \{-\rho_1(B(J^0_3\xi), B(J^0_3\eta)) + \rho_1(B(\xi), B(\eta))\\
&\qquad + \rho_2(B(J^0_3\xi), B(\eta)) + \rho_2(B(\xi), B(J^0_3\eta))\}j^-_{*u}(J^0_1)\\
&\qquad + \{-\rho_2(B(J^0_3\xi), B(J^0_3\eta)) + \rho_2(B(\xi), B(\eta))\\
&\qquad - \rho_1(B(J^0_3\xi), B(\eta)) - \rho_1(B(\xi), B(J^0_3\eta))\}j^-_{*u}(J^0_2),
\end{aligned}
\tag{3.19}
$$

$$
IN_2(j^-_{*u}(A^*_u), j^-_{*u}(B(\xi)_u)) = -4j^-_{*u}(B(A\xi)_u) \neq 0. \tag{3.20}
$$

Concerning the reflector space, let horr be the natural projection $T_uP \to (m_1)^*_u \oplus Q_u$, with $\ker(\mathrm{horr}) = (h_1)^*_u$. In a very similar way as above, we calculate

$$
\begin{aligned}
&PN_1(j^-_{*u}(A^*_u), j^-_{*u}(B^*_u)) = 0,\\
&PN_1(j^-_{*u}(A^*_u), j^-_{*u}(B(\xi)_u)) = 0,\\
&[PN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{H^+}\\
&\quad = j^-_{*u}(B(-\Theta(B(J^0_1\xi), B(J^0_1\eta)) - \Theta(B(\xi), B(\eta))\\
&\qquad + J^0_1\Theta(B(J^0_1\xi), B(\eta)) + J^0_1\Theta(B(\xi), B(J^0_1\eta)))_u),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&[PN_1(j^-_{*u}(B(\xi)_u)), j^-_{*u}(B(\eta)_u))]_{V^+}\\
&\quad = \{-\rho_2(B(J^0_1\xi), B(J^0_1\eta)) - \rho_2(B(\xi), B(\eta))\\
&\qquad + \rho_3(B(J^0_1\xi), B(\eta)) + \rho_3(B(\xi), B(J^0_1\eta))\}j^-_{*u}(J^0_2)\\
&\qquad + \{-\rho_3(B(J^0_1\xi), B(J^0_1\eta)) - \rho_3(B(\xi), B(\eta))\\
&\qquad + \rho_2(B(J^0_1\xi), B(\eta)) + \rho_2(B(\xi), B(J^0_1\eta))\}j^-_{*u}(J^0_3),
\end{aligned}
\tag{3.22}
$$

$$
PN_2(j^+_{*u}(A^*_u), j^+_{*u}(B(\xi)_u)) = 4j^+_{*u}(B(A\xi)_u) \neq 0. \tag{3.23}
$$

Take $X = u(\xi), Y = u(\eta)$ we see that (3.18) and (3.19) are equivalent to

$$
(T^\nabla)^{0,2}_{J_3} = T^\nabla(X,Y) - T^\nabla(J_3X, J_3Y) + J_3T^\nabla(J_3X, Y) + J_3T^\nabla(X, J_3Y) = 0, \tag{3.24}
$$

$$
\rho^\nabla_1(J_3X, J_3Y) - \rho^\nabla_1(X,Y) - \rho^\nabla_2(J_3X, Y) - \rho^\nabla_2(X, J_3Y) = 0, \tag{3.25}
$$

for any admissible basis J_1, J_2, J_3 of $\mathcal{P}$ which proves the equivalence of (i) and (ii).

Similarly, (3.21) and (3.22) are equivalent to

$$(T^{\nabla})^{0,2}_{J_1} = T^{\nabla}(J_1X, J_1Y) + T^{\nabla}(X,Y) - J_1T^{\nabla}(J_1X,Y) - J_1T^{\nabla}(X,J_1Y) = 0, \tag{3.26}$$

$$\rho^{\nabla}_3(J_1X, J_1Y) + \rho^{\nabla}_3(X,Y) - \rho^{\nabla}_2(J_1X,Y) - \rho^{\nabla}_2(X,J_1Y) = 0, \tag{3.27}$$

for any admissible basis J_1, J_2, J_3 of $\mathcal{P}$ which proves the equivalence of (iii) and (ii). □

The equations (3.20) and (3.23) in the proof of Proposition 3.5 yield

Corollary 3.6. *Let ∇ be a para-quaternionic connection on an almost para-quaternionic manifold $(M, \mathcal{P})$ with torsion tensor T^{∇}.*

(1) *The almost complex structure I_2^{∇} on the twistor space Z^- of $(M, \mathcal{P})$ is never integrable.*

(2) *The almost paracomplex structure P_2^{∇} on the reflector space Z^+ of $(M, \mathcal{P})$ is never integrable.*

In the 4-dimensional case we derive

Theorem 3.7. *Let (M^4, g) be a 4-dimensional pseudo-Riemannian manifold with neutral metric g and let $\mathcal{P}$ be the para-quaternionic structure corresponding to the conformal class generated by g with a local basis J_1, J_2, J_3. Then the following conditions are equivalent*

(i) *The neutral metric g is anti-self-dual.*

(ii) *The Ricci forms ρ^g_α of the Levi-Civita connection ∇^g satisfy (3.16), i.e.,*

$$\epsilon_\beta \rho^g_\alpha(J_\beta X, J_\beta Y) + \rho^g_\alpha(X,Y) - \epsilon_\gamma \rho^g_\gamma(J_\beta X, Y) - \epsilon_\gamma \rho^g_\gamma(X, J_\beta Y) = 0.$$

(iii) *The torsion condition (3.15) for a linear connection ∇ always implies the curvature condition (3.16).*

Proof. The proof is a direct consequence of Proposition 3.5, Theorem 3.4 and the result in [30] (resp. [12]) which states that the almost paracomplex structure $P_1^{\nabla^g}$ (resp. the almost complex structure $I_1^{\nabla^g}$) is integrable exactly when the neutral conformal structure generated by g is anti-self-dual. □

Remark 3.8. We note that for any compatible with the orientation integrable almost (para) complex structure, say $J = J_3$, the condition (ii) of Theorem 3.7 holds for $\alpha = 1$, $\beta = 2$, $\gamma = 3$ but the oriented neutral conformal structure may not be anti-self-dual. Indeed, if J_α is a (para)complex structure then the connection 1-forms $\omega^g_\beta, \omega^g_\gamma$ of ∇^g satisfy the equality [42]

$$\omega^g_\beta(X) = -\epsilon_\alpha \omega^g_\gamma(J_\alpha X). \tag{3.28}$$

An easy calculations, using (2.1) and (3.28), give

$$\begin{aligned} d\omega^g_\beta(X,Y) = &-\epsilon_\alpha(\nabla^g{}_X\omega^g_\gamma)J_\alpha Y + \epsilon_\alpha(\nabla^g{}_Y\omega^g_\gamma)J_\alpha X \\ &- \epsilon_\alpha\omega^g_\beta(X)\omega^g_\gamma(J_\gamma Y) + \epsilon_\beta\omega^g_\gamma(X)\omega^g_\gamma(J_\beta Y) \\ &+ \epsilon_\alpha\omega^g_\beta(Y)\omega^g_\gamma(J_\gamma X) - \epsilon_\beta\omega^g_\gamma(Y)\omega^g_\gamma(J_\beta X). \end{aligned} \tag{3.29}$$

The Ricci 2-forms $\rho^{\nabla^g}_\alpha$ of ∇^g with respect to $(J_1, J_2, J_3 = J)$ are given by (2.3). Insert (2.3) and (3.29) into (3.16) taken for $\alpha = 1, \beta = 2, \gamma = 3$ to check that (3.16) holds.

In higher dimensions, the curvature condition (3.16) is always a consequence of the torsion condition (3.15) in the sense of the next

Theorem 3.9. *Let ∇ be a para-quaternionic connection on an almost para-quaternionic $4n$-dimensional $n \geq 2$ manifold $(M, \mathcal{P})$ with torsion tensor T^∇. Then the following conditions are equivalent:*

(i) *The almost complex structure I_1^∇ on the twistor space Z^- of $(M, \mathcal{P})$ is integrable.*

(ii) *The $(0,2)_J$-part $(T^\nabla)^{0,2}_J$ of the torsion with respect to all $J \in \mathcal{P}$ vanishes,*

$$(T^\nabla)^{0,2}_J = 0, \quad J \in \mathcal{P}.$$

(iii) *The almost paracomplex structure P_1^∇ on the reflector space Z^+ of $(M, \mathcal{P})$ is integrable.*

Proof. We use Proposition 3.5. Since the connection ∇ is a para-quaternionic connection, $\nabla \in \Delta(\mathcal{P})$, the condition (3.15) yields the next expression of the Nijenhuis tensor N_J of any local $J \in \mathcal{P}$,

$$N_J(X,Y) \in \operatorname{span}\{J_1X, J_1Y, J_2X, J_2Y, J_3X, J_3Y\}, \tag{3.30}$$

where J_1, J_2, J_3 is an admissible local basis of $\mathcal{P}$.

To prove that (ii) implies the integrability of I_1^∇ and P_1^∇, we apply the result of Zamkovoy [42] which states that an almost para-quaternionic $4n$-manifold $(n \geq 2)$ is para-quaternionic if and only if the three Nijenhuis tensors N_1, N_2, N_3 satisfy the condition

$$(N_1(X,Y) + N_2(X,Y) - N_3(X,Y)) \in \operatorname{span}\{J_1X, J_1Y, J_2X, J_2Y, J_3X, J_3Y\}. \tag{3.31}$$

Clearly, (3.31) follows from (3.30) which shows that the almost para-quaternionic $4n$-manifold $(n \geq 2)$ $(M, \mathcal{P})$ is a para-quaternionic manifold. Let ∇^0 be a torsion-free para-quaternionic connection on $(M, \mathcal{P})$. Then the almost complex structure $I_1^{\nabla^0}$ on the twistor space Z^- as well as the almost paracomplex structure $P_1^{\nabla^0}$ on the reflector space Z^+ are integrable [29] and $I_1^\nabla = I_1^{\nabla^0}$, $P_1^\nabla = P_1^{\nabla^0}$ due to Theorem 3.4.

Hence, the equivalence between (i), (ii) and (iii) is established, which completes the proof. □

From the proof of Proposition 3.5 and Theorem 3.9, we easily derive

Corollary 3.10. *Let ∇ be a para-quaternionic connection on a $4n$-dimensional $(n \geq 2)$ almost para-quaternionic manifold $(M, \mathcal{P})$ with torsion tensor T^{∇}. Then the torsion condition (3.15) implies the curvature condition (3.16).*

We note that Corollary 3.10 generalizes the same statement proved in the case of PQKT-connection (see below) in [42] where the first Bianchi identity is used.

Theorem 3.9 and Theorem 3.4 imply

Corollary 3.11. *Let $(M, \mathcal{P})$ be an almost para-quaternionic manifold.*

(i) *Among the all almost complex structures I_1^{∇}, $\nabla \in \Delta(\mathcal{P})$ on the twistor space Z^- at most one is integrable.*

(ii) *Among the all almost paracomplex structures P_1^{∇}, $\nabla \in \Delta(\mathcal{P})$ on the reflector space Z^+ at most one is integrable.*

A direct consequence of the proof of Theorem 3.9, Theorem 3.7 and Corollary 3.11 is the following

Theorem 3.12. *Let $(M, \mathcal{P})$ be an almost para-quaternionic $4n$-manifold. The next three conditions are equivalent:*

(1) *Either $(M, \mathcal{P})$ is a para-quaternionic manifold (if $n \geq 2$) or $(M, \mathcal{P} = [g])$ is anti-self dual for $n = 1$.*

(2) *There exists an integrable almost complex structure I_1^{∇} on the twistor space Z^- which does not depend on the para-quaternionic connection ∇.*

(3) *There exists an integrable almost paracomplex structure P_1^{∇} on the reflector space Z^+ which does not depend on the para-quaternionic connection ∇.*

4 Para-quaternionic Kähler manifolds with torsion

An almost para-quaternionic Hermitian manifold $(M, \mathcal{P}, g)$ is called para-quaternionic Kähler with torsion (PQKT) if there exists an almost para-quaternionic Hermitian connection $\nabla^T \in \Delta(\mathcal{P})$ whose torsion tensor T is a 3-form which is $(1,2)+(2,1)$ with respect to each J_a, i.e., the tensor $T(X,Y,Z) := g(T(X,Y),Z)$ is totally skew-symmetric and satisfies the conditions

$$T(X,Y,Z) = -T(J_\alpha X, J_\alpha Y, Z) - T(J_\alpha X, Y, J_\alpha Z) - T(X, J_\alpha Y, J_\alpha Z), \quad \alpha = 1,2,$$

$$T(X,Y,Z) = T(J_3 X, J_3 Y, Z) + T(J_3 X, Y, J_3 Z) + T(X, J_3 Y, J_3 Z).$$

We recall that each PQKT is a para-quaternionic manifold [42]. The condition on the torsion implies that the $(0,2)$-part of the torsion of a PQKT connection vanishes. Applying Theorem 3.9, we obtain

Theorem 4.1. *Let $(M, \mathcal{P}, \nabla^T)$ be a PQKT and $\nabla^0 \in \Delta(\mathcal{P})$ be a torsion-free para-quaternionic connection. Then*

(i) *The almost complex structure $I_1^{\nabla^T}$ on the twistor space Z^- is integrable and therefore it coincides with $I_1^{\nabla^0}$.*

(ii) *The almost paracomplex structure $P_1^{\nabla^T}$ on the reflector space Z^+ is integrable and therefore it coincides with $P_1^{\nabla^0}$.*

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Chapter 15

Para-pluriharmonic maps and twistor spaces

Matthias Krahe

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Introduction

Twistor theory goes back to the 1960s, where twistor methods were introduced by R. Penrose to provide an approach to quantum gravity. The general idea is to translate a problem on a differentiable manifold M endowed with a certain geometric structure to a problem on a complex manifold Z, the *twistor space*. Usually, Z is a fibre bundle over M, with the fibres over $p \in M$ being complex structures on T_pM which are compatible with the geometric structure on M. Z carries a canonical almost complex structure, whose integrability can be derived from the properties of M.

An important application of twistor theory is the construction of minimal and pluriminimal submanifolds: In 1982, Bryant used the fibration $\mathbb{C}\mathrm{P}^3 \to \mathbb{H}\mathrm{P}^1$ to construct minimal surfaces in $\mathbb{H}\mathrm{P}^1 = \mathrm{S}^4$ [Br]. This has been generalized to the construction of minimal surfaces in various Grassmannians and symmetric spaces by considering horizontal maps into certain flag manifolds and flag domains [BR], [BS].

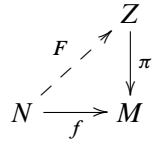
The reinterpretation of Bryant's work by Lawson [L] using a birational equivalence $\mathbb{C}P^3 \sim F_{1,1,1}(\mathbb{C}^3) := SU(3)/S(U(1) \times U(1) \times U(1))$ led to the remarkable observation that the twistor spaces of any two symmetric quaternionic Kähler spaces are birationally equivalent [Bu].

The purpose of this work is to transfer some of these methods to *para-complex geometry*:

A para-complex (or hyperbolic complex) manifold is a smooth manifold with a field of involutions $J_p : T_pM \to T_pM$, $J^2 = 1$, such that the ± 1-eigendistributions are integrable and of the same dimension. The obvious example is a product of two manifolds $M_+ \times M_-$ of the same dimension; in fact, every para-complex manifold is locally of this form. Para-complex and para-quaternionic manifolds have recently been studied in mathematics as well as in physics, where they are related to supersymmetric field theories with Euclidean (rather than Lorentzian) space-times [CMMS1], [CMMS2].

The first section of this chapter contains the definitions of para-complex manifolds, para-Kähler manifolds and para-quaternionic Kähler manifolds. We give a proof of the para-complex Dolbeault Lemma, which states that in a para-complex manifold, the $\bar{\partial}$-operator is locally exact (Theorem 1.2.8 and Corollary 1.2.11).

Extending earlier results by Blair et al. [BDM], it has been shown by Alekseevsky and Cortés [AC2], that for any para-quaternionic Kähler manifold M there is a twistor bundle $\pi : Z \to M$ of para-complex structures (actually there is also a twistor bundle $Z^- \to M$ of *complex* structures, which we neglect here). Z has a canonical para-Kähler structure, and the horizontal distribution is a para-holomorphic contact structure. A submanifold $N \subset M$ such that there exists a parallel section of $Z|_N$ is called *para-complex submanifold*. If the pseudo-Riemannian metric g_M is nondegenerate on TN, then N is a *para-Kähler submanifold*, and it follows that N is a minimal submanifold (i.e. the trace of the shape operator is zero). Any para-complex submanifold can be lifted to a para-holomorphic horizontal submanifold of Z, in other words, para-complex immersions $f : N \to M$ are precisely the projections of para-holomorphic horizontal immersions $F : N \to Z$.



It follows that N has at most half the dimension of M, and the para-complex submanifolds with maximal dimension correspond to the Legendrian submanifolds of Z.

Our aim in Section 2 is to show that these Legendrian submanifolds and therefore the maximal para-complex submanifolds of M are locally given by a para-holomorphic function.

The main step will be the proof of the para-complex equivalent of the Darboux theorem, which states that on each para-complex contact manifold, there exist local co-

ordinate charts which identify the contact structure with the standard contact structure on $\boldsymbol{C}^{2n+1}$, where $\boldsymbol{C}$ denotes the algebra of *para-complex numbers* (Theorem 2.1.11).

The notion of twistors can be generalized to the setting where M a pseudo-Riemannian symmetric space (not necessarily para-quaternionic Kähler). There is a twistor construction analogous to the one in Section 2, where the role of para-complex immersions is played by *isotropic para-pluriharmonic* maps (in fact, in the case of para-quaternionic Kähler symmetric spaces, these constructions coincide). A map $f\colon N \to M$ is called para-pluriharmonic, if it is harmonic along every para-complex curve in N. Under certain conditions, these maps have a one-parameter family $(f_\theta)_{\theta\in\mathbb{R}}$ of deformations, the *associated family*. A special case occurs when this family is constant in θ; such maps f are called *isotropic*. In Section 3 we prove the following: Each full isotropic para-pluriharmonic map $f\colon N \to M$ from a Kähler manifold N into a symmetric space $M = G/G^\sigma$ defines a twistor fibration $Z \to M$ and a holomorphic superhorizontal lift $F\colon N \to Z$; vice versa, given such a map F, its projection $f = \pi \circ F$ is isotropic para-pluriharmonic (Theorem 3.3.6).

The twistor space is an open subset of the para-complex coset space $\overline{Z} = (G/P)\times(G/P)$, where the para-complex structure is just multiplication by $+1$ on the first factor and by -1 on the second. If G is a real linear algebraic group, then P is a parabolic subgroup, and thus G/P a real projective variety. This variety has a cell decomposition (Bruhat decomposition) with exactly one cell N in the largest dimension, which is an open (even Zariski-open) and dense subset.

Now, if Z is the twistor space associated to a symmetric para-quaternionic Kähler space, then it turns out that there is a contact isomorphism between the product $N \times N \subset \overline{Z}$ of the big cells and the para-complex Heisenberg group. It follows that, given the twistor spaces of two symmetric para-quaternionic Kähler spaces of the same dimension, then there is a biholomorphism defined on Zariski-open subsets preserving the contact distribution; moreover, this biholomorphism extends to a birational equivalence. This can be used to transfer the construction of para-pluriharmonic maps from one para-quaternionic Kähler symmetric space to another. Among these spaces is the para-complex Grassmannian $\mathrm{Gr}_2(\boldsymbol{C}^{n+2})$ with corresponding twistor fibration $\mathrm{F}_{1,n,1}(\boldsymbol{C}^{n+2}) \to \mathrm{Gr}_2(\boldsymbol{C}^{n+2})$, where this construction is fairly easy.

In the last section we consider tt*-bundles. (Para-)pluriharmonic maps are closely related to a geometric structure in topological field theories called *topological-anti-topological fusion,* or *tt*-geometry*: There is a bijection between tt*-bundles and (para-)pluriharmonic maps into the symmetric space $\mathrm{GL}(p+q)/\mathrm{O}(p,q)$ which admit an associated family (called *admissible* in [S2]). Here we relate this to the construction of associated families and give a criterion for the (para-)pluriharmonic map to be isotropic.

Differences to the complex theory

For those who are familiar with the twistor theory in the complex case, we point out a few differences to the para-complex case:

- *Para-complex manifolds:* A para-complex structure defines a splitting of the tangent bundle into eigenbundles – there is no need to consider the para-complexified tangent bundle (at times we do so for convenience, but this yields no new information).
- *Regularity:* Para-holomorphic and para-pluriharmonic maps are not necessarily analytic (any pair of differentiable functions $f_\pm$ defines a para-holomorphic map $f_+ \times f_- \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$).
- *Associated families:* All pluriharmonic maps into Riemannian symmetric spaces have associated families, provided that the symmetric space has nonpositive or nonnegative curvature. In the para-complex context, this is different: Unless the symmetric space is flat, there exist para-pluriharmonic maps which do not admit an associated family.

  Moreover, the parameter range of the associated family is $\mathbb{R}$ rather than S^1. Consequently, in our context the loop group of G will be defined as the set of smooth curves $\mathbb{R} \to G$.
- *Topology:* In contrast to hermitian symmetric spaces, para-hermitian spaces are not necessarily simply connected, an example being the para-complex projective space

  $$\boldsymbol{C}\mathrm{P}^n = \mathrm{SL}(n+1, \mathbb{R})/\mathrm{S}(\mathrm{GL}(1, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})).$$

  This example also shows that the twistor fibration $G/H \to G/K$ is not defined globally, if $G_0^\sigma \subset K \subsetneq G^\sigma$ is "too small" (where H is the centralizer of a canonical element and G^σ denotes the fixed-point set of the Cartan involution).

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1 Preliminaries

1.1 Para-complex vector spaces

Definition 1.1.1. Let V be a finite-dimensional real vector space. A *para-complex structure* on V is an involution $J \colon V \to V$, $J^2 = \mathrm{id}_V$, such that the eigenspaces $V^\pm := \ker(\mathrm{id} \mp J)$ have the same dimension. The pair (V, J) is then called a *para-complex vector space.* A *homomorphism* between para-complex vector spaces (V, J) and (V', J') is a linear map $\phi \colon V \to V'$ satisfying $\phi \circ J = J' \circ \phi$.

Just like complex vector spaces (i.e. (V, I) with $I^2 = -1$) are vector spaces over the field $\mathbb{C}$, para-complex vector spaces are free modules over the *algebra of para-complex numbers*:

Definition 1.1.2. The algebra $\boldsymbol{C} = \mathbb{R} \oplus j\mathbb{R}$ of *para-complex numbers* (also called *split complex* or *hyperbolic complex* numbers) is the real algebra generated by 1 and j, with the relation $j^2 = 1$ (in fact, $\boldsymbol{C}$ is the Clifford algebra $\mathrm{Cl}(0, 1)$).

The *conjugation map* $\bar{\cdot} : \boldsymbol{C} \to \boldsymbol{C}$ is defined by $\overline{x + jy} = x - jy$ for $x, y \in \mathbb{R}$. The real numbers $x = \mathrm{Re}(z) := (z + \bar{z})/2$ and $y = \mathrm{Im}(z) := (z - \bar{z})/2j$ are called the *real* and *imaginary* part of $z = x + jy$, respectively. Note that $\bar{z}z = x^2 - y^2 \in \mathbb{R}$. Thus, the group of units is $\boldsymbol{C}^* = \{x + jy : x \neq \pm y\} = \{z \in \boldsymbol{C} : z \notin (1 \pm j)\mathbb{R}\}$. It has four connected components; in fact, each $z \in \boldsymbol{C}^*$ can be written uniquely as $\pm re^{j\theta}$ or $\pm jre^{j\theta}$, where $r \in \mathbb{R}^+$, $\theta \in \mathbb{R}$, and $e^{j\theta} = \cosh(\theta) + j \sinh(\theta)$. This is the para-complex analogue of the polar decomposition; the unit circle is replaced by the four hyperbolas $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = \pm 1\}$.

The free module $\boldsymbol{C}^n$ is a para-complex vector space, with the multiplication by j as para-complex structure. Conversely, any para-complex vector space (V, J) can be regarded as a $\boldsymbol{C}$-module via $(x + jy)v := xv + yJv$. In order to see that this module is free, let $\{v_1^+, \dots, v_n^+, v_1^-, \dots, v_n^-\}$ be a basis of V, such that $v_a^\pm \in V^\pm$. Then the equation $\sum (x^a + jy^a)(v_a^+ + v_a^-) = \frac{1+j}{2} \sum (x^a + y^a)v_a^+ + \frac{1-j}{2} \sum (x^a - y^a)v_a^-$ for coefficients $x^a + jy^a \in \boldsymbol{C}$ shows that $\{w_1, \dots, w_n\}$, $w_a := v_a^+ + v_a^-$ is a free basis of the $\boldsymbol{C}$-module V. Homomorphisms between para-complex vector spaces correspond to $\boldsymbol{C}$-linear maps.

Remark 1.1.3. Since $\boldsymbol{C}$ is not a field, some basic facts from linear algebra must be treated with caution:

(1) Let $\phi : V \to V'$ be $\boldsymbol{C}$-linear. Then the image and the kernel of ϕ are J-invariant real subspaces of V' and V, respectively. They are not necessarily para-complex subspaces, because the dimensions of $\phi(V^\pm)$, resp. $\ker(\phi)^\pm$ may not be the same. – Of course, it still holds that

$$\phi \text{ injective} \iff \ker(\phi) = \{0\}.$$

(2) Let $\phi : V \to V$ be a $\boldsymbol{C}$-linear endomorphism and let $z = x + jy \in \boldsymbol{C}$. Then the eigenspace $V_z := \ker(\phi - z \cdot \mathrm{id})$ is J-invariant, and therefore we have $V_z = V_z^+ \oplus V_z^-$. The $V_z^\pm$ are eigenspaces of ϕ with *real* eigenvalue $(x \pm y)$. In particular, eigenspaces to different para-complex eigenvalues need not be disjoint.

(3) The $\boldsymbol{C}$-span of a single vector $v \in V$ is not necessarily a para-complex vector space; this motivates the following definition:

Definition 1.1.4. Let (V, J) be a para-complex vector space. A vector $v \in V$ is called *regular*, if it fulfills the following equivalent conditions:

(1) $v \notin V^+$ and $v \notin V^-$.

(2) $\boldsymbol{C} \cdot v$ is a one-dimensional (over $\boldsymbol{C}$) para-complex vector space.

(3) v and Jv are linearly independent over $\mathbb{R}$.

(4) For $z \in \boldsymbol{C}$ it holds that $(zv = 0 \iff z = 0)$.

Proof of the equivalence. (1) $\iff$ (2): We write $v = v^+ + v^- \in V^+ \oplus V^-$. Then $(\boldsymbol{C} \cdot v)^\pm = \mathbb{R} \cdot v^\pm$. So $\boldsymbol{C} \cdot v$ is a one-dimensional para-complex vector space, if and only if $v^- \neq 0$ and $v^+ \neq 0$. (1) $\implies$ (3): We may assume that $v \neq 0$. If $Jv = rv$, $r \in \mathbb{R}$, then $v = J^2 v = r^2 v$, so $r = \pm 1$. (3) $\implies$ (1) is trivial. (3) $\iff$ (4) follows immediately from $(x + jy)v = xv + yJv$. □

The following example is fundamental since it illustrates the principle that *para-complex objects can be described either by using para-complex numbers or by using real numbers*:

Example 1.1.5. The map

$$\begin{aligned}
(\boldsymbol{C}^n, j) &\xrightarrow{\cong} (\mathbb{R}^n \times \mathbb{R}^n, (\mathrm{id})\times(-\mathrm{id})),\\
(x^a + jy^a)_{a=1,\dots,n} &\longmapsto (x^a + y^a, x^a - y^a)_{a=1,\dots,n},\\
\left(\frac{1+j}{2} z_+^a + \frac{1-j}{2} z_-^a\right)_{a=1,\dots,n} &\longleftarrow (z_+^a, z_-^a)_{a=1,\dots,n},
\end{aligned}$$

is an isomorphism of para-complex vector spaces. It induces an isomorphism of groups

$$\begin{aligned}
\mathrm{GL}(n, \boldsymbol{C}) &\xrightarrow{\cong} \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}),\\
B + jC &\longmapsto \begin{pmatrix} B + C & 0 \\ 0 & B - C \end{pmatrix},\\
\frac{1+j}{2} A_+ + \frac{1-j}{2} A_- &\longleftarrow \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}.
\end{aligned}$$

Lemma 1.1.6. *Let $A_+, A_- \in \mathrm{GL}(n, \mathbb{R})$. Then*

$$\det_{\boldsymbol{C}} \left(\frac{1+j}{2} A_+ + \frac{1-j}{2} A_-\right) = \frac{1+j}{2} \det(A_+) + \frac{1-j}{2} \det(A_-).$$

Proof. This follows immediately from the Leibniz formula and from $\left(\frac{1+j}{2}\right)^2 = \frac{1+j}{2}$, $\left(\frac{1-j}{2}\right)^2 = \frac{1-j}{2}$, $\frac{1+j}{2}\frac{1-j}{2} = 0$. □

Corollary 1.1.7.

$$\det_{\boldsymbol{C}} \left(\frac{1+j}{2} A_+ + \frac{1-j}{2} A_-\right) \in \boldsymbol{C}^* \iff \det(A_+) \neq 0 \wedge \det(A_-) \neq 0.$$

Corollary 1.1.8.

$$\mathrm{SL}(n, \boldsymbol{C}) \cong \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}).$$ □

1.2 Para-complex manifolds

Definition 1.2.1. An *almost para-complex structure* on a manifold M is a smooth endomorphism field J on TM, such that for every $p \in M$, J_p is a para-complex structure on T_pM. An almost para-complex structure is said to be *integrable,* if the eigendistributions $T^{\pm}M := \ker(\mathrm{id} \mp J)$ are integrable. In this case, (M, J) is called a *para-complex manifold.*

As a consequence of the Frobenius theorem – see for example [CMMS1, Chapter 2.1] – we have

Proposition 1.2.2. *Let (M, J) be an almost para-complex manifold. Then the following conditions are equivalent:*

(1) *J is integrable.*

(2) *$[T^+M, T^+M] \subset T^+M$ and $[T^-M, T^-M] \subset T^-M$.*

(3) *$N_J = 0$, where N_J is the* Nijenhuis tensor

$$N_J(X, Y) := [X, Y] + [JX, JY] - J[X, JY] - J[JX, Y].$$

(4) *For any point $p \in M$, there exist local real coordinates $(z^1_+, \dots, z^n_+, z^1_-, \dots, z^n_-)$: $U \xrightarrow{\sim} \widetilde{U}_+ \times \widetilde{U}_- \subset \mathbb{R}^n \times \mathbb{R}^n$ defined on an open neighbourhood U of p, such that $dz^a_{\pm} \circ J = \pm\, dz^a_{\pm}$.*

(4′) *For any point $p \in M$, there exist local para-complex coordinates $(z^1, \dots, z^n)$: $U \xrightarrow{\sim} \widetilde{U} \subset \boldsymbol{C}^n$ defined on an open neighbourhood U of p, such that $dz^a \circ J = j\, dz^a$.*

These coordinates are *para-holomorphic* in the following sense:

Definition 1.2.3. A smooth map $\phi\colon (M, J_M) \to (M', J_{M'})$ between para-complex manifolds is called *para-holomorphic* (or just *holomorphic*, if no confusion is possible), if

$$d\phi \circ J_M = J_{M'} \circ d\phi.$$

Para-holomorphic maps $(M, J) \to (\boldsymbol{C}, j)$ or $(M, J) \to (\mathbb{R} \times \mathbb{R}, (\mathrm{id}) \times (-\mathrm{id}))$ are called *para-holomorphic functions.*

Definition 1.2.4. Let M be a para-complex manifold and let $z^a = \frac{1+j}{2} z^a_+ + \frac{1-j}{2} z^a_-$, $a = 1 \dots n$, be a system of local holomorphic coordinates. Then $(dz^a, d\bar{z}^a)_{a=1\dots n}$ is local frame of the complexified cotangent bundle $T^*M \otimes \boldsymbol{C}$. The dual frame of $TM \otimes \boldsymbol{C}$ is given by

$$\frac{\partial}{\partial z^a} := \frac{1+j}{2} \frac{\partial}{\partial z^a_+} + \frac{1-j}{2} \frac{\partial}{\partial z^a_-},$$
$$\frac{\partial}{\partial \bar{z}^a} := \frac{1-j}{2} \frac{\partial}{\partial z^a_+} + \frac{1+j}{2} \frac{\partial}{\partial z^a_-}.$$

Proposition 1.2.5. *Let M be a para-complex manifold and let $z^a = \frac{1+j}{2} z^a_+ + \frac{1-j}{2} z^a_-$, $a = 1 \dots n$, be a system of local holomorphic coordinates. Then a smooth function $f : M \to \boldsymbol{C}$ is para-holomorphic, if and only if it fulfills the* ***Cauchy–Riemann equations***

$$\frac{\partial f}{\partial \bar{z}^a} = 0.$$

Proof. We write $f = \frac{1+j}{2} f_+ + \frac{1-j}{2} f_-$, where $f_\pm := \mathrm{Re}(f) \pm \mathrm{Im}(f)$. Then it is easy to see that the Cauchy–Riemann equations are equivalent to

$$\frac{\partial f_+}{\partial z^a_-} = \frac{\partial f_-}{\partial z^a_+} = 0.$$

But this in turn is just a rewriting of $df \circ J_M = J_{\mathbb{R}\times\mathbb{R}} \circ df$. □

Lemma 1.2.6. *Let $U \subset \boldsymbol{C}$ be a simply connected domain and let $f : U \to \boldsymbol{C}$ be a para-holomorphic function. Then f has an antiderivative, i.e. a para-holomorphic function F, such that $\frac{\partial F}{\partial z} = f$.*

Proof. Since $d(f\,dz) = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = 0$, the $\boldsymbol{C}$-valued 1-form $f\,dz$ is closed, hence exact, i.e. $f\,dz = dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z}$. It follows that $\frac{\partial F}{\partial z} = f$ and $\frac{\partial F}{\partial \bar{z}} = 0$. □

1.2.1 Bigrading of differential forms, Dolbeault sequences

Real version. Let (M, J) be an almost para-complex manifold. The decomposition $T^*M = (T^*M)^+ \oplus (T^*M)^-$ into the ± 1-eigenspaces of J^* induces a bigrading on exterior and differential forms:

$$\bigwedge^k T^*M = \bigoplus_{p+q=k} \bigwedge^{(+p,-q)} T^*M,$$
$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{(+p,-q)}(M).$$

Definition 1.2.7. The elements of $\Omega^{(+p,-q)}(M)$ are called *differential forms of degree* $(+p, -q)$.

In the case that J is integrable, then we can use holomorphic coordinates to show that $d\Omega^{(+p,-q)}(M) \subset \Omega^{(+(p+1),-q)}(M) \oplus \Omega^{(+p,-(q+1))}(M)$. Hence the exterior derivative splits as $d = \partial_+ + \partial_-$, where

$$\partial_+ \colon \Omega^{(+p,-q)}(M) \longrightarrow \Omega^{(+(p+1),-q)}(M),$$
$$\partial_- \colon \Omega^{(+p,-q)}(M) \longrightarrow \Omega^{(+p,-(q+1))}(M).$$

From $d^2 = 0$ we get $\partial_+^2 = \partial_-^2 = \partial_+\partial_- + \partial_-\partial_+ = 0$, so for any para-complex manifold the operators $\partial_\pm$ can be used to define real versions of the *Dolbeault sequence*. This sequence is locally exact:

Theorem 1.2.8. *Let $U = U_+ \times U_-$ be a para-complex manifold (with the para-complex structure on $TU_+ \times TU_-$ given by* $\mathrm{id} \times -\mathrm{id}$*), such that both U_+ and U_- are contractible.*

Then any ∂_+-closed form $\omega \in \Omega^{(+p,-q)}(U)$, $p \geq 1$, is ∂_+-exact. Likewise, any ∂_--closed form $\omega \in \Omega^{(+p,-q)}(U)$, $q \geq 1$, is ∂_--exact.

Remark. (1) From Proposition 1.2.2 (4) it follows that each point in an arbitrary para-complex manifold has a neighbourhood U of this form.

(2) The condition that U be contractible is not sufficient, as shown by the following counterexample: Let

$$U := \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \setminus (\{0\} \times \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R})$$

and $\omega(z_+, w_+, z_-, w_-) := f(z_-)\frac{z_+ dw_+ - w_+ dz_+}{z_+^2 + w_+^2}$, where $f(z_-)$ is a smooth nonzero function that vanishes on $\mathbb{R}_{>0}$. Then ω is a smooth $(1,0)$-form, which is ∂_+-closed, but not ∂_+-exact.

Proof of the theorem. For the proof of the first statement, let $\omega \in \Omega^{(+p,-q)}(U)$, $p \geq 1$, and $\partial_+\omega = 0$. Let the contraction maps on U_+ and U_- be denoted by h_+ and h_- respectively. We can define a para-holomorphic homotopy map $h\colon [0,1]^2 \times U \to U$ by $h_{(s_+,s_-)}(u_+, u_-) = (h_{s_+}(u_+), h_{s_-}(u_-))$. Then $h_{(1,1)} = \mathrm{id}_U$ and $dh_{(0,1)} \cdot T^+U = 0$ and therefore, since $p \geq 1$,

$$\begin{aligned} \omega - 0 &= h_{(1,1)}^*\omega - h_{(0,1)}^*\omega \\ &= \int_0^1 \frac{d}{ds_+}(h_{(s_+,1)}^*\omega)ds_+. \end{aligned} \tag{1}$$

In order to compute the integrand, we write $h_{(s_+,1)} = h \circ \Psi_{s_+} \circ \iota$, where ι is the embedding $U \to \{0\} \times \{1\} \times U \subset [0,1]^2 \times U$ and Ψ is the flow of the vector field $\frac{\partial}{\partial s_+}$ on $[0,1]^2 \times U$. Then by Cartan's formula we have

$$\begin{aligned} \frac{d}{ds_+}h_{(s_+,1)}^*\omega &= \iota^*\Psi_{s_+}^*\left(\mathcal{L}_{s_+}(h^*\omega)\right) \\ &= \iota^*\Psi_{s_+}^*\left(\frac{\partial}{\partial s_+} \lrcorner\, d(h^*\omega) + d\left(\frac{\partial}{\partial s_+} \lrcorner\, h^*\omega\right)\right) \\ &= \iota^*\Psi_{s_+}^*\left(\frac{\partial}{\partial s_+} \lrcorner\, \partial_+(h^*\omega) + \partial_+\left(\frac{\partial}{\partial s_+} \lrcorner\, h^*\omega\right)\right. \\ &\qquad \left. + \frac{\partial}{\partial s_+} \lrcorner\, \partial_-(h^*\omega) + \partial_-\left(\frac{\partial}{\partial s_+} \lrcorner\, h^*\omega\right)\right). \end{aligned}$$

Because the maps ι, Ψ_{s_+}, h are para-holomorphic, the respective pull-back maps commute with $\partial_\pm$ and preserve the types of differential forms. Hence the sum of the terms containing ∂_- must be zero, since these terms are of degree $(+(p-1), -(q+1))$ and the rest is of degree $(+p, -q)$. Further, by assumption we have $\partial_+(h^*\omega) = h^*\partial_+\omega = 0$ and thus the expression reduces to

$$\frac{d}{ds_+} h^*_{(s_+,1)}\omega = \partial_+\Big(\iota^*\Psi^*_{s_+}\Big(\frac{\partial}{\partial s_+} \lrcorner\, h^*\omega\Big)\Big).$$

Inserting this in (1) yields

$$\omega = \partial_+\Big(\int_0^1 \iota^*\Psi^*_{s_+}\Big(\frac{\partial}{\partial s_+} \lrcorner\, h^*\omega\Big) ds_+\Big).$$

The proof of the second statement is completely analogous. □

Para-complex version. There is also a para-complex version of the Dolbeault sequence defined on the para-complexified cotangent bundle $T^*M^{\boldsymbol{c}} := T^*M \otimes_{\mathbb{R}} \boldsymbol{C}$: The $\boldsymbol{C}$-linear extension of $J^*\colon T^*M \to T^*M$ to an endomorphism field on $T^*M^{\boldsymbol{c}}$ yields a decomposition $T^*M^{\boldsymbol{c}} = \bigwedge^{(1,0)} T^*M \oplus \bigwedge^{(0,1)} T^*M$ into the $\pm j$-eigenbundles

$$\bigwedge^{(1,0)} T^*M = \{\alpha + jJ\alpha : \alpha \in T^*M\},$$
$$\bigwedge^{(0,1)} T^*M = \{\alpha - jJ\alpha : \alpha \in T^*M\}.$$

This decomposition extends to a bigrading of $\boldsymbol{C}$-valued exterior and differential forms:

$$\bigwedge^k T^*M^{\boldsymbol{c}} = \bigoplus_{p+q=k} \bigwedge^{(p,q)} T^*M,$$
$$\Omega^k(M)^{\boldsymbol{c}} = \bigoplus_{p+q=k} \Omega^{(p,q)}(M).$$

If J is integrable, then the $\boldsymbol{C}$-linear extension of the exterior derivative splits as $d^{\boldsymbol{c}} = \partial + \bar\partial$, where

$$\partial\colon \Omega^{(p,q)}(M) \longrightarrow \Omega^{(p+1,q)}(M),$$
$$\bar\partial\colon \Omega^{(p,q)}(M) \longrightarrow \Omega^{(p,q+1)}(M).$$

Definition 1.2.9. The elements of $\Omega^{(p,q)}(M)$ are called para-complex *differential forms of degree* (p,q). A $\bar\partial$-closed $(p,0)$-form is called *para-holomorphic*.

From $d^2 = 0$ we get $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$, so for any para-complex manifold the operators $\bar\partial$, ∂ can be used to define para-complex versions of the Dolbeault sequence. The real and para-complex versions are related in the following way:

Proposition 1.2.10. *For a para-complex manifold M, there is an ($\mathbb{R}$-linear) isomorphism*

$$\begin{aligned}\textstyle\bigwedge^{(+p,-q)} T^*M \times \bigwedge^{(+q,-p)} T^*M &\xrightarrow{\varphi} \textstyle\bigwedge^{(p,q)} T^*M,\\ (\eta,\eta') &\longmapsto \frac{1+j}{2}\eta + \frac{1-j}{2}\eta',\end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccc}\Omega^{(+p,-q)}(M)\times\Omega^{(+q,-p)}(M) & \xrightarrow{\quad\varphi\quad} & \Omega^{(p,q)}(M)\\ \downarrow{\scriptstyle \partial_-\times\partial_+} & & \downarrow{\scriptstyle \bar\partial}\\ \Omega^{(+p,-(q+1))}(M)\times\Omega^{(+(q+1),-p)}(M) & \xrightarrow[\varphi]{} & \Omega^{(p,q+1)}(M).\end{array}$$

Proof. The fact that the image of φ is contained in $\bigwedge^{(p,q)} T^*M$ is an easy consequence of $\left(\frac{1\pm j}{2}\right)^2 = \frac{1\pm j}{2}$, $\frac{1+j}{2}\frac{1-j}{2} = 0$ and of the equations

$$\begin{aligned}\tfrac{1+j}{2}dz^a_+ &= \frac{1+j}{2}dz^a, \quad \tfrac{1-j}{2}dz^a_- = \frac{1-j}{2}dz^a,\\ \tfrac{1+j}{2}dz^a_- &= \frac{1+j}{2}d\bar z^a, \quad \tfrac{1-j}{2}dz^a_+ = \frac{1-j}{2}d\bar z^a\end{aligned}$$

for local holomorphic coordinates $(z^a_\pm)_{k=1\dots n}$ and $z^a = \frac{1+j}{2}z^a_+ + \frac{1-j}{2}z^a_-$. The map φ is clearly a bijection, with the inverse given by

$$\omega \mapsto (\mathrm{Re}(\omega) + \mathrm{Im}(\omega), \mathrm{Re}(\omega) - \mathrm{Im}(\omega))\,.$$

For the commutativity of the diagram, it is sufficient to consider functions, i.e. the case that $f, f' \in \Omega^{(+0,-0)}$. Then we calculate $\bar\partial\left(\frac{1+j}{2}f\right) = \left(\frac{1-jJ^*}{2}d\right)\left(\frac{1+j}{2}f\right) = \frac{1+j}{2}\left(\frac{1-J^*}{2}df\right) = \frac{1+j}{2}\partial_- f$ and in the same way $\bar\partial\frac{1-j}{2}f' = \frac{1-j}{2}\partial_+ f'$. □

Corollary 1.2.11. *For $U = U_+ \times U_-$ with U_+ and U_- contractible, any $\bar\partial$-closed form $\omega \in \Omega^{(p,q)}(U)$, $q \geq 1$, is $\bar\partial$-exact.* □

1.3 Para-hermitian metrics and para-Kähler manifolds

We recall that a pseudo-Euclidean vector space is a real vector space V equipped with a nondegenerate symmetric bilinear form $g\colon V \times V \to \mathbb{R}$.

If $W \subset V$ is a subspace with orthogonal complement $W^\perp := \{v \in V : g(w,v) = 0 \text{ for all } w \in W\}$, then W is called

- *nondegenerate*, if $W^\perp \cap W = \{0\}$,
- *degenerate*, if $W^\perp \cap W \neq \{0\}$,
- *totally degenerate*, if $W^\perp \supset W$, i.e. if $g|_{W\times W} = 0$.

Fact 1.3.1. *A subspace W is totally degenerate if and only if it consists only of degenerate vectors.*

Proof. This follows from the polarization identity $4g(w,w') = g(w+w', w+w') - g(w-w', w-w')$. □

Definition 1.3.2. Let (V,J) be a para-complex vector space. A pseudo-Euclidean scalar product g on V is said to be *compatible* with J or *para-hermitian*, if J is g-skew, i.e. if $g(J\cdot,\cdot) + g(\cdot, J\cdot) = 0$.

Since $J^2 = 1$, this is equivalent to J being an anti-isometry, i.e. $g(J\cdot, J\cdot) = -g$. In particular, g has real signature (n,n) (where $n = \frac{1}{2}\dim_{\mathbb{R}} V$), and the subspaces $V^{\pm}$ are totally degenerate.

On $V = \boldsymbol{C}^n$ the standard para-hermitian scalar product is given by

$$g\left((z^1,\dots,z^n),(w^1,\dots,w^n)\right) = \mathrm{Re}\Big(\sum_a \overline{z^a} w^a\Big).$$

The group of $\boldsymbol{C}$-linear automorphisms of $\boldsymbol{C}^n$ preserving g is called the *para-unitary group*

$$\mathrm{U}^{\pi}(n) := \{A \in \mathrm{GL}(n,\boldsymbol{C}) : A^*A = \mathbf{1}_n\} \quad (\text{where } A^* := \bar{A}^t).$$

It is easy to see that

$$\mathrm{U}^{\pi}(n) = \left\{\tfrac{1+j}{2}B + \tfrac{1-j}{2}B^{-t} : B \in \mathrm{GL}(n,\mathbb{R})\right\} \quad (\text{where } B^{-t} := (B^{-1})^t).$$

By Lemma 1.1.6 we have $\det_{\boldsymbol{C}}\left(\frac{1+j}{2}B + \frac{1-j}{2}B^{-t}\right) = \frac{1+j}{2}\det(B) + \frac{1-j}{2}\det(B)^{-1}$; this proves

$$\mathrm{U}^{\pi}(n) \cong \mathrm{GL}(n,\mathbb{R}),$$
$$\mathrm{SU}^{\pi}(n) \cong \mathrm{SL}(n,\mathbb{R}).$$

Under the identification $\boldsymbol{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, the standard para-hermitian scalar product corresponds to the symmetric bilinear form given by the matrix

$$E := \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}.$$

The isometry group is $\left\{X \in \mathrm{GL}(2n,\mathbb{R}) : X^t E X = E\right\} \cong \mathrm{O}(n,n)$, and $\mathrm{U}^{\pi}(n)$ is identified with the subgroup

$$\left\{\begin{pmatrix} B & \mathbf{0}_n \\ \mathbf{0}_n & B^{-t} \end{pmatrix} : B \in \mathrm{GL}(n,\mathbb{R})\right\}.$$

Proposition 1.3.3. *Every para-hermitian vector space has a unitary basis, i.e. a free $\boldsymbol{C}$-basis $(e_a)_{a=1\dots n}$ in which the metric is given by $g(e_a,e_b) = \delta_{ab}$.*

Proof. Let V be a para-hermitian vector space, $\dim_{\boldsymbol{C}}(V) = n$. V is nondegenerate with respect to g, so there is a vector e_1, such that $g(e_1,e_1) \neq 0$. After multiplying

e_1 by $\sqrt{g(e_1,e_1)}$ (if $g(e_1,e_1)>0$) or by $j\sqrt{-g(e_1,e_1)}$ (if $g(e_1,e_1)<0$), we may assume that $g(e_1,e_1)=1$. It follows that $e_1\notin V^{\pm}$, so e_1 is regular and $\boldsymbol{C}\cdot e_1$ is a 1-dimensional para-complex subspace. The orthogonal complement $(\boldsymbol{C}\cdot e_1)^{\perp}$ is an $(n-1)$-dimensional para-complex subspace, which is nondegenerate, so we can proceed by induction. □

Definition 1.3.4. A *para-Kähler manifold* is an almost para-complex manifold (M,J) equipped with a compatible pseudo-Riemannian metric g, such that $DJ=0$, where D denotes the Levi-Civita connection.

Equivalently, a para-Kähler manifold is a pseudo-Riemannian manifold with holonomy group $\mathrm{GL}(n,\mathbb{R})\subset \mathrm{O}(n,n)$.

From the parallelity of J it follows that $N_J=0$ (in particular, (M,J) is a para-complex manifold) and that the $(+1,-1)$-form $\omega:=g(\cdot,J\cdot)$ is also parallel, and therefore closed. The converse is also true: If $N_J=0$ and $d\omega=0$, then $DJ=0$ (see [CMMS1, Theorem 1]).

1.4 Para-quaternionic Kähler manifolds

Definition 1.4.1. A *para-quaternionic structure* on a finite-dimensional vector space V is a three-dimensional subalgebra $Q\subset \mathfrak{gl}(V)$ with a basis I,J,K that satisfies the para-quaternionic relations

$$I^2=-1,\quad J^2=K^2=+1,\quad -IJ=JI=K.$$

Remark. From $-IJ=JI$ we see that I maps the $+1$-eigenspace of J isomorphically to the -1-eigenspace, so these eigenspaces have the same dimension. In other words, J (as well as K) is a para-complex structure in the sense of Definition 1.1.1.

Para-quaternionic vector spaces are identified with free right-modules over $\boldsymbol{H}$, where $\boldsymbol{H}=\mathbb{R}\oplus i\mathbb{R}\oplus j\mathbb{R}\oplus k\mathbb{R}$, $-i^2=j^2=k^2=1$, $ij=-ji=k$, is the algebra of *para-quaternions* (also called *split quaternions*) and I,J,K correspond to the multiplication by i,j,k from the right. Accordingly, $\boldsymbol{H}^n$ is the space of para-quaternionic column vectors with scalar multiplication from the right. Multiplication by para-quaternionic matrices from the left is $\boldsymbol{H}$-linear.

The identification $\boldsymbol{H}=(\mathbb{R}\oplus i\mathbb{R})\oplus j(\mathbb{R}\oplus i\mathbb{R})=\mathbb{C}\oplus j\mathbb{C}$ shows that any para-quaternionic vector space is a complex vector space of the double dimension. A map is $\boldsymbol{H}$-linear if and only if it is $\mathbb{C}$-linear and commutes with the right multiplication by j (which is a $\mathbb{C}$-antilinear involution).

In the same way, $\boldsymbol{H}=(\mathbb{R}\oplus j\mathbb{R})\oplus i(\mathbb{R}\oplus j\mathbb{R})=\boldsymbol{C}\oplus i\boldsymbol{C}$ shows that any para-quaternionic vector space is a para-complex vector space of the double dimension. A map is $\boldsymbol{H}$-linear if and only if it is $\boldsymbol{C}$-linear and commutes with the right multiplication by i (which is a $\boldsymbol{C}$-antilinear complex structure).

Definition 1.4.2. Let (V, Q) be a para-quaternionic vector space. A pseudo-Euclidean metric g on V is said to be *compatible* with Q or *para-quaternion hermitian*, if the elements of Q are g-skew.

The standard para-quaternion hermitian metric on $\boldsymbol{H}^n$ is given by

$$g((h_1,\dots,h_n),(h'_1,\dots,h'_n)) := \mathrm{Re}\Big(\sum \overline{h_a} h'_a\Big).$$

The isometry group

$$\mathrm{Sp}^\pi(n) := \{A \in \mathrm{GL}(n,\boldsymbol{H}) : A^* A = \mathrm{id}_n\}$$

is called the *para-symplectic group*.

By the same argument as in Proposition 1.3.3, we see that any two para-quaternion hermitian vector spaces of the same dimension are isometric.

Proposition 1.4.3. $\mathrm{Sp}^\pi(n) \cong \mathrm{Sp}(2n,\mathbb{R})$.

Proof. Let $\boldsymbol{\Omega}_{2n} = \left(\begin{smallmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right)$ be the standard symplectic structure on $\mathbb{R}^{2n}$, then $\boldsymbol{H}^n \cong (\mathbb{R}^{4n}, I, J, K, g)$, where

$$I = \begin{pmatrix} 0 & -\mathbf{1}_{2n} \\ \mathbf{1}_{2n} & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \mathbf{1}_{2n} & 0 \\ 0 & -\mathbf{1}_{2n} \end{pmatrix}, \quad K = JI,$$

$$g = -\omega_I(\cdot, I\cdot), \quad \omega_I = \begin{pmatrix} \boldsymbol{\Omega}_{2n} & 0 \\ 0 & \boldsymbol{\Omega}_{2n} \end{pmatrix}.$$

Thus, $\mathrm{Sp}^\pi(n)$ is identified with the endomorphism of $\mathbb{R}^{4n}$ that commute with I and J and preserve ω_I. This shows that

$$\mathrm{Sp}^\pi(n) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \mathrm{Sp}(2n,\mathbb{R}) \right\}. \qquad \square$$

Definition 1.4.4. A *para-quaternionic Kähler manifold* (M, g, Q) is a pseudo-Riemannian manifold (M, g) endowed with a g-compatible para-quaternionic structure $Q \subset \mathrm{End}(TM)$ which is parallel with respect to the Levi-Civita connection.

It is not difficult to see (cf. [AC1, Proposition 1], [AC2, Proposition 9]) that the normalizer of $Q_{\mathbb{R}^{4n}} = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ in $\mathrm{O}(2n,2n)$ is $G_0 \cup J \cdot G_0$, where $G_0 = \mathrm{Sp}^\pi(1) \cdot \mathrm{Sp}^\pi(n)$ is the subgroup of $\mathrm{GL}(4n,\mathbb{R}) = \mathrm{GL}_{\mathbb{R}}(\boldsymbol{H}^n)$ generated by left multiplication by elements of $\mathrm{Sp}^\pi(n)$ and right multiplication by elements of $\mathrm{Sp}^\pi(1)$ (which is just the group of unit para-quaternions), and where J denotes the right multiplication by j (which is an *anti*-unit, i.e. $\bar{j}j = -1$). So we have:

Proposition 1.4.5. *A pseudo-Riemannian manifold is para-quaternionic Kähler if and only if*

$$\mathrm{Hol}(M) \subset (\mathrm{Sp}^\pi(1) \cdot \mathrm{Sp}^\pi(n)) \cup J \cdot (\mathrm{Sp}^\pi(1) \cdot \mathrm{Sp}^\pi(n)).$$

Note. In the definition in [AC1], it is also required that M be *strongly oriented*, i.e. that $\mathrm{Hol}(M) \subset \mathrm{SO}^+(2n, 2n)$. Of course, this is always fulfilled, if M is simply connected.

It can be shown [AC1, Theorem 3] that any para-quaternionic Kähler manifold is Einstein and that its curvature tensor admits a decomposition

$$R = \frac{\mathrm{scal}}{4n(n+2)} R_0 + W,$$

where R_0 is the curvature tensor of $\boldsymbol{H}\mathrm{P}^n$ and W is an algebraic curvature tensor of type $\mathfrak{sp}^\pi(n)$. It follows that, if scal $\neq 0$, then $\mathfrak{sp}^\pi(1) \subset \mathfrak{hol}$ (see [AC1, Corollary 1]).

2 Para-Kähler submanifolds of para-quaternionic Kähler manifolds

2.1 Para-complex symplectic and contact structures

The classical Darboux theorem states that all $2n$-dimensional symplectic manifolds are locally isomorphic to $T^*\mathbb{R}^n$ equipped with the canonical symplectic form $dp_a \wedge dq^a$. As shown in this paragraph, the analogous statement in the para-holomorphic setting also holds.

2.1.1 Para-complex symplectic vector spaces

Definition 2.1.1. Let (V, J) be a para-complex vector space. A *para-complex symplectic form* is an anti-symmetric $\boldsymbol{C}$-bilinear map $\omega\colon V \times V \to \boldsymbol{C}$ which is nondegenerate, i.e. $(\omega(X, \cdot) = 0 \iff X = 0)$.

Equivalently, a para-complex symplectic form is an element $\omega \in \bigwedge^{(2,0)} V^*$, such that the map $\omega\colon V^{(1,0)} \to \bigwedge^{(1,0)} V^*$ is an isomorphism.

Proposition 2.1.2. *Let (V, J) be a para-complex vector space and ω a para-complex symplectic form. Then there is a basis $\{Z^a, W_b\}_{a,b=1,\dots,n}$ of the $\boldsymbol{C}$-module V such that $\omega(Z^a, W_b) = \delta^a_b$, $\omega(Z^a, Z^b) = 0 = \omega(W_a, W_b)$ for $a, b \in \{1, \dots, n\}$.*

Proof. Choose a regular vector $Z^1 \in V$. We claim that there exists $W_1 \in V$ such that $\omega(Z^1, W_1) = 1$: Otherwise the image of $W \mapsto \omega(Z^1, W)$ would be contained in one of the maximal ideals $(1 \pm j)\mathbb{R} \subset \boldsymbol{C}$. But then $\omega((1 \mp j)Z^1, \cdot) \equiv 0$, and therefore $(1 \mp j)Z^1 = 0$ by the nondegeneracy of ω, in contradiction to the regularity of Z^1. From $\omega(Z^1, W_1) = 1$ it now follows that W_1 is regular and that Z^1 and W_1 are $\boldsymbol{C}$-linearly independent. Therefore, $\mathrm{span}_{\boldsymbol{C}}(Z^1, W_1)$ is a para-complex subspace, as well as the symplectic complement $(\mathrm{span}_{\boldsymbol{C}}(Z^1, W_1))^{\perp_\omega} :=$

$\{X \in V : \omega(Z^1, X) = \omega(W_1, X) = 0\}$. The latter is again nondegenerate with respect to ω, and the proposition follows by induction. □

Corollary 2.1.3. *An antisymmetric $\boldsymbol{C}$-bilinear 2-form ω is nondegenerate if and only if $\omega^n \in \bigwedge^{(2n,0)} V^*$ is regular (where $n = \dim_{\boldsymbol{C}}(V)/2$).*

Proof. If the form ω is nondegenerate and $\{Z^a, W_b\}_{a,b=1,\ldots,n}$ a basis as above, then $\omega^n(Z^1, W_1, \ldots, Z^n, W_n) = n!$, in particular $(1 \pm j)\omega^n \neq 0$.

Conversely, suppose that ω^n is regular. Let $\{B_1, \ldots, B_{2n}\}$ be an arbitrary $\boldsymbol{C}$-basis of V and $X = \sum_a x^a B_a \in \ker(\omega)$, $x^a \in \boldsymbol{C}$. We have to show that $X = 0$. By the skew-symmetry of ω^n we have

$$0 = \omega^n(B_1, \ldots, B_{a-1}, X, B_{a+1}, \ldots, B_{2n}) = x^a \cdot \omega^n(B_1, \ldots, B_{2n}).$$

But by assumption we have $\omega^n(B_1, \ldots, B_{2n}) \in \boldsymbol{C}^*$, so $x^a = 0$. □

2.1.2 Para-holomorphic symplectic structures on manifolds

Definition 2.1.4. Let (M, J) be a para-complex manifold. A *para-holomorphic symplectic structure* is a nondegenerate closed form $\omega \in \Omega^{(2,0)}M$.

Lemma 2.1.5. *Let (M, J) be a para-complex manifold and ω_0, ω_1 two para-holomorphic symplectic forms on M, which coincide in a point $p \in M$, $\omega_0(p) = \omega_1(p)$. Then there is a para-holomorphic diffeomorphism Φ defined on a neighbourhood of p, such that $\Phi(p) = p$ and $\Phi^*\omega_1 = \omega_0$.*

We want to apply Moser's method of constructing Φ as time-1-map of a para-holomorphic flow, i.e. a flow $\Phi^X : (-\epsilon, \epsilon) \times U \to U$ of a (time-dependent) vector field X_s, such that $(\Phi_s^X)^* J = J$ for any $s \in (-\epsilon, \epsilon)$. The vector field X_s then needs to be an infinitesimal automorphism of the para-complex structure, i.e. $\mathcal{L}_{X_s} J = 0$.

Remark (on notation). If X_s is a time-dependent vector field, then $\Phi^X_{s,\cdot}$ denotes the flow of X starting at time s, i.e. $\frac{d}{dt}\Phi^X_{s,t}(p) = X(\Phi^X_{s,t}(p))$, $\Phi^X_{s,s}(p) = p$. Instead of $\Phi^X_{0,s}$ we will also write Φ^X_s. The Lie-derivative of a (possibly also time-dependent) tensor T_s is defined as

$$\mathcal{L}_{X_s} T_s := \frac{d}{dt} \mid_{t=0} \Phi^*_{s,s+t} T_{s+t}.$$

For functions, differential forms and vector fields this yields

$$\begin{aligned}
\mathcal{L}_{X_s} f_s &= X_s(f_s) + \frac{\partial}{\partial s} f_s, \\
\mathcal{L}_{X_s} \alpha_s &= d(X_s \lrcorner \alpha_s) + X_s \lrcorner d\alpha_s + \frac{\partial}{\partial s}\alpha_s, \\
\mathcal{L}_{X_s} Y_s &= [X_s, Y_s] + \frac{\partial}{\partial s} Y_s.
\end{aligned}$$

Lemma 2.1.6. *Let (M, J) be a para-complex manifold. Then $\mathcal{L}_{X_s} J = 0$ if and only if the $(1,0)$-vector field $\frac{1}{2}(X_s + jJX_s)$ is para-holomorphic.*

Proof. Using para-holomorphic coordinates $(z^a = x^a + jy^a)_{a=1\ldots n}$ we write $X_s = \sum_a \xi_s^a \frac{\partial}{\partial x^a} + \sum_a \eta_s^a \frac{\partial}{\partial y^a}$, $\frac{1}{2}(X + jJX) = \sum(\xi_s^a + j\eta_s^a)\frac{\partial}{\partial z^a}$. The equations $J\frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^a}$ and $J\frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^a}$ yield

$$(\mathcal{L}_{X_s} J)\Big(\frac{\partial}{\partial x^a}\Big) = [X_s, J\frac{\partial}{\partial x^a}] - J\Big[X_s, \frac{\partial}{\partial x^a}\Big]$$
$$= \sum_b \left(\frac{\partial \eta_s^b}{\partial x^a} - \frac{\partial \xi_s^b}{\partial y^a}\right)\frac{\partial}{\partial x^b} + \left(\frac{\partial \xi_s^b}{\partial x^a} - \frac{\partial \eta_s^b}{\partial y^a}\right)\frac{\partial}{\partial y^b}.$$

The vanishing of this expression is equivalent to the para-Cauchy–Riemann equations for the function $(\xi_s^b + j\eta_s^b)$. The calculation for $(\mathcal{L}_X J)(\partial_{y_k})$ is analogous. □

Remark 2.1.7. $\mathcal{L}_X J = 0$ implies $\mathcal{L}_{JX} J = 0$ and $[X, JX] = 0$. If X is time-independent, the flows Φ^X and Φ^{JX} commute, so Φ^X can be extended to a flow of $Z = \frac{1}{2}(X + jJX)$, depending on a para-complex parameter $s + jt \in \boldsymbol{C}$, by $\Phi^Z_{s+jt}(p) := \Phi^X_s(\Phi^{JX}_t(p))$, fulfilling $\frac{d}{dz}\Phi^Z_z(p) = Z\left(\Phi^Z_z(p)\right)$.

Proof of Lemma 2.1.5. Let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$, $0 \le t \le 1$. We have $\omega_t(p) = \omega_0(p) = \omega_1(p)$ for all t, so by the compactness of $[0, 1]$ there is a neighbourhood of p, such that in this neighbourhood ω_t is nondegenerate for all $t \in [0, 1]$. We will now construct a 1-parameter family Φ_t of diffeomorphisms (defined on a neighbourhood of p), such that

$$\Phi_t(p) = p,$$
$$\Phi_t^* \omega_t = \omega_0. \tag{2}$$

Since $\omega' := \frac{d}{dt}\omega_t = \omega_1 - \omega_0$ is closed by assumption, it locally is the differential of a 1-form, $\omega' = d\alpha$. Further, we may assume that $\alpha(p) = 0$ (by adding the differential of a function). Since $\bar\partial(\alpha^{(0,1)}) = (\omega')^{(0,2)} = 0$, we can apply Corollary 1.2.11 and find a (locally defined) function β, such that $\alpha^{(0,1)} = \bar\partial\beta$. Replacing α with $\alpha - d\beta$, we obtain $\alpha \in \Omega^{(1,0)}M$. Further, by construction, $\bar\partial\alpha = \omega'^{(1,1)} = 0$, i.e. α is para-holomorphic. Thus the equation $Z_t \lrcorner \omega_t = -\alpha$ defines a time-dependent vector field Z_t, which is para-holomorphic since α and ω_t^{-1} are, the latter by Cramer's rule. According to Remark 2.1.7, the flow of $X_t = 2\mathrm{Re}(Z_t) = Z_t + \bar Z_t$ is a 1-parameter family Φ_t of para-holomorphic diffeomorphisms. For all $t \in [0, 1]$ we have $X_t(p) = 0$, thus $\Phi_t(p) = p$, and Φ_t is defined on a neighbourhood of p. By writing $\mathcal{L}_X(U + jV) = [X, U + jV]$ and extending the definition of the Lie bracket and Lie derivative $\boldsymbol{C}$-linearly to the tensor algebra on $T^{\boldsymbol{C}}M$, we calculate (using

Cartan's formula and $\bar{Z}_t \lrcorner \omega_t = 0$):

$$\begin{aligned}\frac{d}{dt}\Phi_t^*\omega_t &= \Phi_t^*\left(d(X_t \lrcorner \omega_t) + X_t \lrcorner d\omega_t + \frac{d}{dt}\omega_t\right)\\ &= \Phi_t^*\left(d(Z_t \lrcorner \omega_t) + 0 + \frac{d}{dt}\omega_t\right)\\ &= \Phi_t^*(-d\alpha + \omega') = 0.\end{aligned}$$

Integration then yields (2). □

Theorem 2.1.8. *Let (M, J) be a $2n$-dimensional para-complex manifold and let ω be a para-holomorphic symplectic form. Then in the neighbourhood of any $p \in M$, there is a para-holomorphic coordinate system $(z^a, w_b)_{a,b=1,\dots,n}$ such that $\omega = \sum_a dz^a \wedge dw_a$.*

Proof. In a sufficiently small neighbourhood of $p \in M$, choose an arbitrary para-holomorphic coordinate system $(\tilde{z}^a, \tilde{w}_b)$. From Proposition 2.1.2 it follows that by changing the coordinates by a $\boldsymbol{C}$-linear map $\boldsymbol{C}^n \to \boldsymbol{C}^n$, we can arrange that $\omega(p) = \sum d\tilde{z}^a(p) \wedge d\tilde{w}_a(p)$. Now apply the above lemma to $\omega_0 = \omega$ and $\omega_1 = \sum d\tilde{z}^a \wedge d\tilde{w}_a$. Then $(z^a := \tilde{z}^a \circ \Phi,\ w_b := \tilde{w}_b \circ \Phi)$ is the desired coordinate system. □

2.1.3 Para-holomorphic contact forms

Definition. Let M be a $(2n+1)$-dimensional para-complex manifold. A *para-holomorphic contact form* is a para-holomorphic 1-form $\alpha \in \Omega^{(1,0)}(M)$, such that $(\alpha \wedge (d\alpha)^n) \in \Omega^{(2n+1,0)}(M)$ is regular.

Remark. This condition is equivalent to $\ker(\alpha)$ being a $2n$-dimensional para-holomorphic distribution in the sense of [AC2], Chapter 2.2 (see Proposition 3) and $d\alpha|_{\ker(\alpha)}$ being nondegenerate (cf. Corollary 2.1.3).

Lemma 2.1.9. *There is a unique vector field R_α, called* Reeb vector field, *such that*

$$\begin{aligned}R_\alpha \lrcorner d\alpha &= 0,\\ R_\alpha \lrcorner \alpha &= 1.\end{aligned}$$

Proof. At each $p \in M$, $d\alpha(p)$ has rank $2n$, so $\ker(d\alpha(p))$ is 1-dimensional. With similar reasoning as in the proof of Proposition 2.1.2, there is a (unique) vector $R(p) \in \ker(d\alpha(p))$ which normalizes $\alpha(p)$: Otherwise, either $(1+j)\alpha(p)$ or $(1-j)\alpha(p)$ would be zero on $\ker(d\alpha(p))$, and therefore $(1 \pm j)\alpha \wedge (d\alpha)^n = 0$. □

Lemma 2.1.10. *Let M be a para-complex manifold and let α_0, α_1 be two para-holomorphic contact forms on M. Let $p \in M$ and $\alpha_0(p) = \alpha_1(p)$, $d\alpha_0(p) = d\alpha_1(p)$. Then there is a para-holomorphic diffeomorphism Φ defined on a neighbourhood of p, such that $\Phi(p) = p$ and $\Phi^*\alpha_1 = \alpha_0$.*

Proof. As in the proof of Lemma 2.1.5, let $\alpha_t = \alpha_0 + t(\alpha_1 - \alpha_0)$, $0 \leq t \leq 1$. α_t is a contact form for all $t \in [0,1]$ in a neighbourhood of p. We will construct a 1-parameter family Φ_t of diffeomorphisms, such that

$$\begin{aligned} \Phi_t(p) &= p, \\ \Phi_t^* \alpha_t &= \alpha_0. \end{aligned} \tag{3}$$

As before, if Φ_t is defined as the flow of a time-dependent vector field $X_t = 2\mathrm{Re}(Z_t)$, with Z_t a para-holomorphic vectorfield to be specified below, then (3) follows from

$$0 = \frac{d}{dt}\Phi_t^* \alpha_t = Z_t \lrcorner d\alpha_t + d(Z_t \lrcorner \alpha_t) + \frac{d}{dt}\alpha_t.$$

We decompose $Z_t = r_t R_{\alpha_t} + W_t$ with W_t tangential to $\ker(\alpha_t)$ and obtain

$$W_t \lrcorner d\alpha_t + dr_t + \frac{d}{dt}\alpha_t = 0. \tag{4}$$

Now $d\alpha_t$ is nondegenerate on $\ker(\alpha_t)$ and zero on $\mathrm{span}(R_{\alpha_t})$, so for given r_t there is a vector field W_t which solves (4), if and only if r_t satisfies

$$R_{\alpha_t} \lrcorner \left(dr_t + \frac{d}{dt}\alpha_t \right) = 0. \tag{5}$$

This equation can be solved as follows: Let $\Psi_{t,z}$ denote the flow of R_{α_t} (see Remark 2.1.7) and let N be a $2n$-dimensional para-complex submanifold of M such that $p \in N$ and such that N is transversal to R_{α_t} in a neighbourhood of p (note that $R_{\alpha_t}(p) = R_{\alpha_0}(p)$, so we can take the same N for all t). Then $(z,q) \mapsto \Psi_{t,z}(q)$ is an isomorphism from a neighbourhood of $(0,p)$ in $\boldsymbol{C} \times N$ onto a neighbourhood of p in M, and the equation (5) reads:

$$\frac{d}{dz} r_t(\Psi_{t,z}(q)) = -R_{\alpha_t}(\Psi_{t,z}(q)) \lrcorner \left(\frac{d}{dt}\alpha_t \right)$$

The right-hand side is para-holomorphic in z, so a solution for r_t exists by Lemma 1.2.6. Moreover, by construction and because of $\frac{d}{dt}\alpha_t(p) = 0$, we have $dr_t(p) = 0 = r_t(p)$, so that $Z_t(p) = 0$, hence $\Phi_t(p) = p$, and Φ_t is well defined for all $t \in [0,1]$ in a neighbourhood of p. □

Theorem 2.1.11. *Let M be a $(2n+1)$-dimensional para-complex manifold and let α be a para-holomorphic contact form. Then in the neighbourhood of any $p \in M$, there is a para-holomorphic coordinate system $(w_0, z^a, w_b)_{a,b=1,\ldots,n}$ such that $\alpha = dw_0 - \sum w_a dz^a$.*

Proof. Let $(\tilde{w}_0, \tilde{z}^a, \tilde{w}_b)_{a,b=1,\ldots,n}$ be a coordinate system such that $\frac{\partial}{\partial \tilde{w}_0}(p) = R_\alpha(p)$ and such that $(\frac{\partial}{\partial \tilde{z}^a}(p), \frac{\partial}{\partial \tilde{w}_b}(p))$ is a symplectic basis for $d\alpha(p)|_{\ker(\alpha(p))}$. By adding a constant, we can assume that p is mapped to zero. Then it follows that $\alpha(p) = (d\tilde{w}_0 - \sum \tilde{w}_a d\tilde{z}^a)(p)$ and $d\alpha(p) = \sum d\tilde{z}^a(p) \wedge d\tilde{w}_a(p)$. Now apply Lemma 2.1.10

to $\alpha_0 = \alpha$, $\alpha_1 = d\tilde{w}_0 - \sum \tilde{w}_a d\tilde{z}^a$ and set $z = \tilde{z} \circ \Phi$, $v^a = \tilde{v}^a \circ \Phi$, $w_b = \tilde{w}_b \circ \Phi$. □

2.2 Para-Kähler submanifolds and twistor lift

Let (M, Q, g) be a para-quaternionic Kähler manifold.

Definition 2.2.1. The *twistor bundle* $\pi\colon Z \to M$ is the bundle of para-complex structures in $Q \subset \mathrm{End}(TM)$:

$$Z := \{q \in Q : q^2 = \mathrm{id}_{TM}\}.$$

The fibres of Z are one-sheeted hyperboloids: If I_p, J_p, K_p is a basis for Q_p as in Definition 1.4.1, then $Z_p = \{rI_p + sJ_p + tK_p : -r^2 + s^2 + t^2 = 1\}$. This can be identified with the homogenous space $\mathrm{SO}(2,1)/\mathrm{SO}(1,1) = \mathrm{Sp}^\pi(1)/\mathrm{U}^\pi(1)$.

The Levi-Civita connection on TM induces a splitting of TZ into a vertical and a horizontal distribution $TZ = \mathcal{V} \oplus \mathcal{H} = \ker(d\pi) \oplus \mathcal{H}^D$. With respect to this splitting, we can define an almost complex structure $J_Z := J_\mathcal{V} + J_\mathcal{H}$, where at each point $z \in Z$, $J_\mathcal{H}(z)$ is the para-complex structure given by $z \in Q_{\pi(z)} \subset \mathrm{End}(T_{\pi(z)}M) \cong \mathrm{End}(\mathcal{H}_z)$, and where $J_\mathcal{V}(z)$ is a $\mathrm{SO}(2,1)$-invariant para-complex structure on the homogenous space $Z_{\pi(z)}$ (see [AC2, Proposition 6]).

Likewise, we can define a one-parameter family $g_Z^t = t g_\mathcal{V} + g_\mathcal{H}$, $t \in \mathbb{R} \setminus \{0\}$, of para-hermitian metrics, such that the map $\pi\colon Z \to M$ is a pseudo-Riemannian submersion (see [AC2, Proposition 7]), and where $-2g_\mathcal{V}$ is the Killing form on $\mathfrak{so}(2,1)$.

Theorem 2.2.2 ([AC2, Theorem 2 and 3]). *Let Z be the twistor space of a para-quaternionic Kähler manifold with nonzero scalar curvature. Then*

(1) *The almost para-complex structure J_Z is integrable.*
(2) *$g_Z^{-4n(n+2)/\mathrm{scal}}$ is a para-Kähler-Einstein metric.*
(3) *The horizontal distribution $\mathcal{H} \subset TZ$ is a para-holomorphic contact structure.*

Definition 2.2.3. Let (M, Q, g) be a para-quaternionic Kähler manifold. A *para-complex submanifold* of M is a pair (N, J), where $N \subset M$ is a smooth submanifold and J a parallel section of $Z|_N = \{q \in Q|_N : q^2 = 1\}$, such that $J(TN) = TN$, and such that the ± 1-eigenspaces $TN^\pm$ have the same dimension. A para-complex submanifold is called *para-Kähler submanifold*, if the induced pseudo-Riemannian metric $g_N := g|_{TN}$ is nondegenerate.

Remark. It follows that the almost complex structure $J_N := J|_{TN}$ is integrable, since it is parallel with respect to a torsion-free connection.

Remark. Let $N \subset M$ be a para-Kähler submanifold. Then the map $f\colon (N, g_N) \to (M, g)$ is an isometric immersion which fulfills

$$df \circ J_N = J \circ df, \tag{6}$$

where J is a parallel section of $f^*Q \subset f^*\mathrm{End}(TM)$. Thus, in the language of Section 3.2, the map f is *isotropic para-pluriharmonic,* see equation (10) (with $\Phi_\theta := e^{\theta J}$). Note that in Section 3.2, M is a symmetric space, but even without that assumption, equation (6) implies that f is para-pluriharmonic: Because J_N and J are parallel, we have $(Ddf)J_N = D(df \cdot J_N) = D(J \cdot df) = J(Ddf)$, and therefore

$$(Ddf)(X, J_N Y) - (Ddf)(Y, J_N X) = J\left((Ddf)(X, Y) - (Ddf)(Y, X)\right) = 0.$$

It follows that $(Ddf)^{(+1,-1)} = 0$, and that f is para-pluriharmonic by Proposition 3.1.3.

The following theorem by Alekseevsky and Cortés relates the para-complex submanifolds of a para-quaternionic Kähler manifold to the horizontal submanifolds of its twistor space:

Theorem ([AC2, Theorem 4]). *Let (M, Q, g) be a para-quaternionic Kähler manifold with nonzero scalar curvature and let $\pi : Z \to M$ be its twistor bundle.*

(1) *If (N, J) is a para-complex submanifold of M, then $F : N \to Z$, $F(p) := J_p$ defines an embedding of N into Z such that $\pi \circ F = \mathrm{id}_N$. The image $\tilde{N} = F(N)$ is a horizontal para-complex submanifold of Z (called the **canonical lift** of N).*

(2) *Conversely, if $\tilde{N} \subset Z$ is a para-complex horizontal submanifold such that $\pi : \tilde{N} \to \pi(\tilde{N}) \subset M$ is a diffeomorphism, then its projection $(N := \pi(\tilde{N}), J := d\pi \circ J_Z \circ d\pi^{-1})$ is a para-complex submanifold of M.*

(3) *Moreover, N is a para-Kähler submanifold, if and only if the one-parameter family of metrics g^t_Z is nondegenerate on $\tilde{N} \subset Z$.*

Because the horizontal distribution $\mathcal{H} \subset TZ$ is a contact distribution, it follows that the dimension of N cannot be greater than half the dimension of M. The maximal (i.e. of maximal dimension) para-complex submanifolds of M correspond to *Legendrian* submanifolds (i.e. submanifolds tangent to the contact distribution with maximal possible dimension) of Z.

Combining this with the results of the preceding section, we can show that the maximal para-complex submanifolds are locally given by para-holomorphic functions:

Theorem 2.2.4. *Let M be a para-quaternionic Kähler manifold of real dimension $4n$, and let $\pi : Z \to M$ be its twistor bundle.*

Then at any point $q \in Z$ there exist local coordinates

$$(z^1, \dots z^n, w_0, \dots, w_n) : U \xrightarrow{\sim} U_z \times U_w \subset \boldsymbol{C}^{2n+1}$$

defined on a neighbourhood U of q, such that the horizontal distribution is $\mathcal{H} = \ker(dw_0 - \sum w_a dz^a)$.

For any para-holomorphic function $h\colon U_z \to \boldsymbol{C}$*, the equations*

$$w_0 = h(z^1, \dots, z^n),$$
$$w_a = \partial h/\partial z^a$$

define a horizontal submanifold $\widetilde{N} \subset Z$ *(provided that* $(w_0, \dots, w_n) \in U_w$*), and the projection* $\pi(\widetilde{N})$ *is a para-complex submanifold of* M*.*

Any maximal para-complex submanifold $N \subset M$ *is locally of this form with respect to a locally finite covering of* Z *by such coordinate domains.*

Proof. Since $\mathcal{H}$ is a para-holomorphic contact distribution, the existence of local coordinates as above follows from Theorem 2.1.11. Since Z is paracompact, it is locally finitely covered by such coordinates. Clearly, with respect to the contact form $dw_0 - \sum w_a dz^a$, a section $U_z \to U_z \times U_w$ is horizontal, if and only if $w_a = \partial w_0/\partial z^a$. It remains to show that, given a horizontal n-dimensional submanifold $\widetilde{N} \subset U$, we can write it as such a section. Let $(z^1, \dots, z^n, w_0, \dots, w_n)$ be a system of local coordinates as above. We may assume that $U_z \times U_w$ is a sufficiently small neighbourhood of $0 \in \boldsymbol{C}^{2n+1}$, so that the $2n$ differential forms $dz^1, \dots dz^n, dw_1, \dots, dw_n$ are pointwise $\boldsymbol{C}$-linearly independent on $\mathcal{H}|_U$. In other words, the $2 \cdot 2n$ real-valued differential forms $dz^1_\pm, \dots, dz^n_\pm, dw^\pm_1, \dots, dw^\pm_n$, are $\mathbb{R}$-linearly independent on $\mathcal{H}^\pm \subset TZ^\pm$. Therefore a subset of $2n$ of these forms (n for each sign) is $\mathbb{R}$-linearly independent on $T\widetilde{N}^\pm$. After exchanging some of the $z_\pm$- and $w^\pm$-coordinates (note that $dw^\pm_0 - w^\pm dz_\pm = d(w^\pm_0 - w^\pm z_\pm) + z_\pm dw^\pm$), we can assume that $dz^1_\pm, \dots, dz^n_\pm$ are linearly independent on $T\widetilde{N}^\pm$, so by the inverse function theorem, the z-coordinates form a local coordinate system on $\widetilde{N}$. □

2.2.1 Example: The twistor space $\boldsymbol{C}\mathrm{P}^{2n+1} \to \boldsymbol{H}\mathrm{P}^n$. Let $\boldsymbol{C}\mathrm{P}^n$ be the space of nondegenerate para-complex lines in $\boldsymbol{C}^{n+1}$,

$$\boldsymbol{C}\mathrm{P}^n = \left\{(z_o, \dots, z_n) \in \boldsymbol{C}^{n+1} : \textstyle\sum_{a=0}^n \bar{z}_a z_a \neq 0\right\}/\boldsymbol{C}^*.$$

Accordingly, let

$$\boldsymbol{H}\mathrm{P}^n = \left\{(h_0, \dots, h_n) \in \boldsymbol{H}^{n+1} : \textstyle\sum_{a=0}^n \bar{h}_a h_a \neq 0\right\}/\boldsymbol{H}^*.$$

In each para-quaternionic line there is a unit element, which can be completed to a unitary basis of $\boldsymbol{H}^{n+1}$ (see the proof of Proposition 1.3.3), so $\mathrm{Sp}^\pi(n+1)$ acts transitively on $\boldsymbol{H}\mathrm{P}^n$; therefore we can identify the latter with the coset space $\mathrm{Sp}^\pi(n+1)/\mathrm{Sp}^\pi(n) \times \mathrm{Sp}^\pi(1)$.

Consider the identification $\boldsymbol{H}^{n+1} = \boldsymbol{C}^{n+1} \oplus i\boldsymbol{C}^{n+1} \cong \boldsymbol{C}^{2n+2}$ and note that $\overline{(z + iw)}(z + iw) = \bar{z}z + \bar{w}w$ for $z, w \in \boldsymbol{C}$, so the scalar products (and the notions of nondegeneracy) on $\boldsymbol{H}^{n+1}$ and $\boldsymbol{C}^{2n+2}$ agree. In particular, there is a fibration $\boldsymbol{C}\mathrm{P}^{2n+1} \to \boldsymbol{H}\mathrm{P}^n$, which is the fibration of homogenous spaces

$$\frac{\mathrm{Sp}^\pi(n+1)}{\mathrm{Sp}^\pi(n) \times \mathrm{U}^\pi(1)} \longrightarrow \frac{\mathrm{Sp}^\pi(n+1)}{\mathrm{Sp}^\pi(n) \times \mathrm{Sp}^\pi(1)}.$$

We want to give an explicit expression for the contact form: Let $\omega = \sum_{a=0}^{n} dz^a \wedge dw_a$ be the standard symplectic form on $\boldsymbol{C}^{2n+2}$ and let $X_{\mathrm{rad}} = \sum_{a=0}^{n}(z_a \frac{\partial}{\partial z^a} + w_a \frac{\partial}{\partial w_a})$ be the radial vector field. The 1-form $\alpha' := X_{\mathrm{rad}} \lrcorner\, \omega = \sum_{a=0}^{n}(z^a dw_a - w_a dz^a)$ is homogenous of degree one and vanishes in the radial direction, so it yields a holomorphic form α on $\boldsymbol{C}\mathrm{P}^{2n+1}$ which is well-defined up to multiplication by a non-vanishing function. The kernel of α' in $\boldsymbol{C}^{2n+2}$ is the orthogonal complement $(X_{\mathrm{rad}} i)^{\perp}$ of the vertical vector field $X_{\mathrm{rad}} i = i \sum_{a=0}^{n}(\bar{z}^a \frac{\partial}{\partial w_a} - \bar{w}_a \frac{\partial}{\partial z^a})$, and thus $\ker(\alpha)$ is indeed the horizontal distribution in $\boldsymbol{C}\mathrm{P}^{2n+1}$.

On the subset $U_0 := \{[z^0, w_0, \dots z^n, w_n] : z_0 \in \boldsymbol{C}^*\} \subset \boldsymbol{C}\mathrm{P}^{2n+1}$ we introduce affine coordinates $[z^0, \dots, w_n] \mapsto (z^0)^{-1}(w_0, z^1, \dots, w_n)$, thereby identifying U_0 with the hyperplane $\{z^0 \equiv 1\} \subset \boldsymbol{C}^{2n+2}$. In these coordinates we have

$$\alpha|_{z^0 \equiv 1} = dw_0 + \sum_{a=1}^{n}(z^a dw_a - w_a dz^a).$$

Coordinates $(\tilde{z}^a, \tilde{w}_b)$ as in Theorem 2.1.11 are given by

$$\begin{aligned} \tilde{w}_0 &= w_0 + \sum z^a w_a, \\ \tilde{z}^a &= z^a, \\ \tilde{w}_a &= 2w_a. \end{aligned}$$

Using these coordinates, we can construct the following horizontal immersions $\boldsymbol{C}^n \supset U \to \boldsymbol{C}\mathrm{P}^{2n+1}$:

$$(z^1, \dots, z^n) \longmapsto \left[1 : h - \sum z^a \frac{\partial h/2}{\partial z^a} : z^1 : \frac{\partial h/2}{\partial z^1} : \cdots : z^n : \frac{\partial h/2}{\partial z^n}\right],$$

where $h\colon U \to \boldsymbol{C}$ is a holomorphic function.

Accordingly, the para-complex immersions $U \to \boldsymbol{H}\mathrm{P}^n$ are

$$(z^1, \dots, z^n) \longmapsto \left[1 + i(h - \sum z^a \frac{\partial h/2}{\partial z^a}) : z^1 + i\frac{\partial h/2}{\partial z^1} : \cdots : z^n + i\frac{\partial h/2}{\partial z^n}\right].$$

3 Para-pluriharmonic maps into symmetric spaces

3.1 Para-pluriharmonic maps

3.1.1 Definitions

Definition 3.1.1. Let (N, g_N), (M, g_M) be pseudo-Riemannian manifolds. A map $f : N \to M$ is called *harmonic* if it is a critical point of the *energy functional* under variations with compact support

$$E(f) := \frac{1}{2}\int_N \mathrm{tr}_{g_N}(f^* g_M)\, d\mathrm{vol}_N.$$

The first variation of the energy is given by

$$\frac{d}{dt}E(f_t) = \int_N g_M(\frac{\partial f_t}{\partial t}, \mathrm{tr}_{g_N}(Ddf)) + \mathrm{div}X\, d\mathrm{vol}_N,$$

where D is the connection on $\mathrm{End}(TN, f^*TM)$ induced by the Levi-Civita connections on TN and TM and where X is the vector field on N defined by $g_N(X,\cdot) = g_M(\frac{\partial f_t}{\partial t}, df\cdot)$. Therefore, f is harmonic if and only if

$$\mathrm{tr}_{g_N}(Ddf) = 0.$$

If N is 2-dimensional, then the energy functional is invariant under conformal changes of the metric g_N, so the definition depends only on the conformal class of g_N. But in two dimensions, the conformal classes of definite metrics correspond to the complex structures (up to sign), and the conformal classes of indefinite metrics correspond to the para-complex structures (any indefinite scalar product in two dimensions has two null directions, which we take as the ± 1-eigenspaces of the para-complex structure). The former case leads to the definition of *pluriharmonic* maps, which have already been studied (see [BR], [ET2]), the latter case to *para-pluriharmonic* maps:

Definition 3.1.2. Let (N, J, g) be a para-Kähler manifold and let (M, g_M) be a pseudo-Riemannian manifold. A smooth map $f\colon N \to M$ is called *para-pluriharmonic* if it is harmonic along every 1-dimensional para-complex submanifold.

Proposition 3.1.3. *For a map $f\colon N \to M$ with N, M as above the following conditions are equivalent:*

(1) *f is para-pluriharmonic.*

(2) *$(Ddf)^{(+1,-1)} = 0$, i.e. Ddf vanishes on $T^+N \otimes T^-N$.*

(2') *$(Ddf)^{(1,1)} = 0$, i.e. $(Ddf)^{\boldsymbol{c}}$ vanishes on $T^{(1,0)}N \otimes T^{(0,1)}N$.*

Proof. (2) $\Longrightarrow$ (1): Let $L \subset N$ be a para-complex curve and $z_\pm = x \pm y$ a pair of holomorphic coordinates on L. The Levi Civita connection on L differs from the one on N only by the (traceless) second fundamental form, and at any point $p \in L$ the pseudo-Riemannian metric $g|_{T_pL}$ is proportional to $dz_+ \vee dz_- = dx \otimes dx - dy \otimes dy$. So we have $\mathrm{tr}_g(Ddf) \sim (Ddf)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) - (Ddf)(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = (Ddf)(\frac{\partial}{\partial z_+}, \frac{\partial}{\partial z_-}) = 0$.

(1) $\Longrightarrow$ (2): Any pair of vectors $Z_\pm \in T_p^\pm N$ is tangent to a para-complex curve through p. The same calculation as before shows that $(Ddf)(Z_+, Z_-) \sim \mathrm{tr}_g(Ddf) = 0$.

(2) $\Longleftrightarrow$ (2') is immediate from $\vee^{(1,1)}(TN) = \vee^{(+1,-1)}(TN) \otimes \boldsymbol{C}$ (cf. Proposition 1.2.10). □

3.1.2 A remark on complexification. We observe that if I is a complex structure on V, then $I^{\mathbb{C}}$ is a $\mathbb{C}$-linear complex structure on $V \otimes \mathbb{C}$, and $J^{\mathbb{C}} = i \cdot I^{\mathbb{C}}$ is a para-complex structure. This leads to the idea to construct para-pluriharmonic maps

in the following way: Start with a pluriharmonic map $f\colon N \to M$, extend this to map $f^{\mathbb{C}}\colon N^{\mathbb{C}} \to M^{\mathbb{C}}$ and then restrict it to an alternate real form $\widetilde{N}$ of $N^{\mathbb{C}}$, such that $iI_N^{\mathbb{C}}$ is a para-complex structure of $T\widetilde{N}$. – The problem with this is that one cannot expect the image $f(\widetilde{N})$ to be contained in a totally real submanifold $\widetilde{M}$ of $M^{\mathbb{C}}$.

To illustrate this, we consider $N = \mathbb{R}^2$ with the standard complex structure $I = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $M = \mathbb{R}$: A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic, if and only if

$$d(df \circ I) = 0.$$

Then the complex analytic continuation $f^{\mathbb{C}}\colon \mathbb{C}^2 \to \mathbb{C}$ of f is harmonic *and* para-harmonic,

$$d(df^{\mathbb{C}} \circ I^{\mathbb{C}}) = d(df^{\mathbb{C}} \circ iI^{\mathbb{C}}) = 0.$$

Restricting this to $(\boldsymbol{C}, j) \cong (\mathbb{R}e_1 \oplus i\mathbb{R}e_2, iI)$, we obtain a para-harmonic function

$$f^{\mathbb{C}}|_{\mathbb{R}e_1 \oplus i\mathbb{R}e_2}\colon \mathbb{R}e_1 \oplus i\mathbb{R}e_2 \to \mathbb{C}.$$

However, the image of $f^{\mathbb{C}}(\mathbb{R}e_1 \oplus i\mathbb{R}e_2)$ is usually not contained in a one-dimensional real subspace, e.g. if $f(x, y) = x + y$, then $f(x, iy) = \mathbb{C}$.

3.1.3 Pseudo-Riemannian symmetric spaces. Recall that a pseudo-Riemannian symmetric space is a pseudo-Riemannian manifold M, such that for each point $p \in M$ there exists an isometry $s_p\colon M \xrightarrow{\sim} M$ (the *geodesic reflection*) whose differential at p is $ds_p(p) = -\mathrm{id}_{T_pM}$. We will always assume that M is connected. It follows that the identity component of the isometry group $G := \mathrm{Iso}_0(M)$ acts transitively on M. Therefore M can be identified with the coset space G/K, where $K = \{g \in G : g \cdot o = o\}$ is the stabilizer subgroup of a chosen base point $o \in M$. The conjugation by s_o is an involutive automorphism $\sigma\colon G \to G$, the *Cartan involution.* It holds that $G_0^\sigma \subset K \subset G^\sigma$, where G^σ denotes the fixed-point set of σ. The differential $d\sigma_e\colon \mathfrak{g} \to \mathfrak{g}$ defines the Ad_K-invariant *Cartan decomposition* $\mathfrak{g} = \ker(d\sigma_e - \mathrm{id}) \oplus \ker(d\sigma_e + \mathrm{id}) =: \mathfrak{k} \oplus \mathfrak{m}$. In fact, $\mathfrak{k}$ is the Lie algebra of K, so the differential of $G \to M$ annihilates $\mathfrak{k}$ and maps $\mathfrak{m}$ isomorphically to T_oM.

Conversely, let G be a Lie group, σ an involution of G, $K \subset G$ a closed subgroup such that $G_0^\sigma \subset K \subset G^\sigma$ and suppose that there exists an Ad_K-invariant pseudo-Euclidean scalar product on $\mathfrak{m} := \ker(d\sigma_e + \mathrm{id}_{\mathfrak{g}})$ (e.g. the Killing form, if it is nondegenerate) for which $d\sigma$ is an isometry. Then $M = G/K$ has the structure of a pseudo-Riemannian symmetric space with the geodesic reflection given by $s_{gK}(hK) = g\sigma(g^{-1}hK)$. The group G acts by isometries.

By saying that $M = G/K$ is a symmetric space, we will implicitly assume that the image of $G \to \mathrm{Iso}(M)$ contains $\mathrm{Iso}_0(M)$. (This is automatically the case if G is semisimple.) Usually, we will also demand that the kernel of $G \to \mathrm{Iso}(M)$ be discrete, in other words, that the G-action on M be *almost effective.*

The following facts hold (see e.g. [H], [O'N], [E])

Fact 3.1.4. *Let G/K be a pseudo-Riemannian symmetric space with almost effective G-action. Then*

(1) $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k},\mathfrak{m}] \subset \mathfrak{m},\ [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$.

(2) If $\mathfrak{g}$ is semisimple then $[\mathfrak{m},\mathfrak{m}] = \mathfrak{k}$.

(3) $[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}] \subset \mathfrak{m}$. The isomorphism $\mathfrak{m} \xrightarrow{\sim} T_oM$ identifies the Lie triple product on $\mathfrak{m}$ with the Riemannian curvature tensor (possibly up to sign, depending on the definition of the latter).

(4) The group gKg^{-1} acts on $T_{gK}M$ by linear isometries preserving R. Moreover, if G is semisimple then the map $gK_0g^{-1} \to \mathrm{Aut}_0(T_{gK}M, g, R)$ is surjective. The same holds if M is simply connected and the image of G in $\mathrm{Iso}(M)$ contains $\mathrm{Iso}_0(M)$ ([O'N, Theorem 8.14]).

3.1.4 Existence of associated families. Pluriharmonic maps into *Riemannian* symmetric spaces of non-positive or non-negative curvature are characterized by the fact that they have *associated families*. In our setting this is slightly different:

Theorem 3.1.5. *Let N be a connected and simply connected para-complex manifold and M a pseudo-Riemannian symmetric space. The following conditions for a smooth map $f: N \to M$ are equivalent:*

(1) *f has an associated family, i.e. there exists a 1-parameter family $(f_\theta, \Phi_\theta)_{\theta\in\mathbb{R}}$ of maps $f_\theta: N \to M$ and parallel bundle isomorphisms $\Phi_\theta: f_\theta^*TM \to f^*TM$ preserving the pseudo-Riemannian metric g^M and the curvature R^M, such that*

$$df \circ e^{\theta J} = \Phi_\theta \circ df_\theta. \tag{7}$$

(2) *f is para-pluriharmonic and*

$$R^M(df(T^+N), df(T^+N)) = 0,$$
$$R^M(df(T^-N), df(T^-N)) = 0.$$

(2') *f is para-pluriharmonic and*

$$R^M(df(T^{(1,0)}N), df(T^{(1,0)}N)) = 0.$$

Proof. The equivalence (1) $\Longleftrightarrow$ (2) follows from the answer to a more general question, which is, roughly speaking:

Given a vector bundle $E \to N$, such that the fibres are isomorphic to those of TM, when is a bundle homomorphism $F: TN \to E$ the differential of a function $f: N \to M$?

The answer is given in [ET1, Theorem 1]:

Theorem (Eschenburg, Tribuzy). *Let N be a simply connected smooth manifold and let M be a pseudo-Riemannian symmetric space with Levi-Civita connection D^M and curvature tensor R^M. Let E be a vector bundle over N, equipped with a connection D, a D-parallel fibre metric g and a D-parallel triple product $R^E: \bigwedge^2 E \to \mathrm{End}(E)$,*

such that there exists a linear isometry $\Phi_0 \colon T_oM \xrightarrow{\sim} E_p$ *for some fixed points* $o \in M$, $p \in N$, *such that* $\Phi_0^* R^E_p = R^M_o$. *Let* $F \colon TN \to E$ *be a vector bundle homomorphism. Then there exists a smooth map* $f \colon N \to M$ *and a parallel bundle isomorphism* $\Phi \colon f^*TM \xrightarrow{\sim} E$ *preserving* R *and* g *such that*

$$F = \Phi \circ df$$

if and only if

$$(D_X F)(Y) - (D_Y F)(X) = 0, \tag{8}$$

$$D_X D_Y \sigma - D_Y D_X \sigma - D_{[X,Y]}\sigma = R(FX, FY)\sigma \tag{9}$$

for all vector fields $X, Y \in \mathrm{X}(N)$ *and sections* σ *of* E, *where in the first equation* D *denotes the connection on* $\mathrm{Hom}(TN, E)$ *induced by* D *and a torsion-free connection on* TN.

Applying this theorem to the bundle $E = f^*TM$ with the pull-back connection, metric and curvature tensor and to the map $F_\theta = df \circ e^{\theta J}$, we see that a solution (Φ_θ, f_θ) of the equation $F_\theta = \Phi_\theta \circ df_\theta$ exists if and only if the conditions (8) and (9) are fulfilled. We already know that they are fulfilled for $F_0 = df$. We consider the first equation: Since $DJ = 0$, we have $DF_\theta = DF \cdot e^{\theta J}$. Now, if X and Y are of the same type, say $X, Y \in T^+N$, then $e^{\theta J}$ is just multiplication by e^θ, so equation (8) remains valid for all $\theta \in \mathbb{R}$. Thus, it has only be checked for vectors of *different* types; in this case it holds for all $\theta \in \mathbb{R}$, if and only if both terms are zero – but this is precisely the pluriharmonicity. A similar consideration can be done for the second equation: This time, if X, Y are of different types, the factors e^θ and $e^{-\theta}$ cancel, so we have to test equation (9) with vectors X, Y of the same type: Then the right-hand side picks up the factor $e^{\pm 2\theta}$, so equation (9) can only be fulfilled for all $\theta \in \mathbb{R}$ if both sides are zero. But for $F_0 = df$, this is precisely the condition for the images of $T^\pm N$ given in (2).

The equivalence (2) $\Longleftrightarrow$ (2′) follows immediately from $T^{(1,0)}N = (1+j)T^+N \oplus (1-j)T^-N$. □

Remark 3.1.6. If M is simply connected, then the solution (Φ_θ, f_θ) is unique up to isometries of M ([ET1, Theorem 1]). Moreover, if f is *full* (i.e. if the image of f is not contained in a totally geodesic subspace of M), then Φ_θ is determined by f_θ; this follows from [ET1, Theorem 2].

Remark 3.1.7. In the Riemannian case, the condition

$$R^M(df(T^{(1,0)}N), df(T^{(1,0)}N)) = 0$$

actually follows from the pluriharmonicity, provided that the curvature tensor of M (viewed as a symmetric bilinear form on $\bigwedge^{(1,1)} TM$) is semi-definite; therefore, if M is a Riemannian symmetric space of nonpositive or nonnegative curvature, *all* pluriharmonic maps have associated families. In our case this is not true, as shown by the following counterexample:

Let M be any symmetric space with nonzero curvature and let $U \subset M$ be an open subset diffeomorphic to $\mathbb{R}^n$. Let $N := U \times U$, $J_N = \mathrm{id}_{TU} \times (-\mathrm{id}_{TU})$ and let g be a compatible metric. The projection on the first factor $f : N \xrightarrow{\pi_1} U \hookrightarrow M$ is obviously para-pluriharmonic since it depends only on the +-coordinates. But since $R^M(df(T^+N), df(T^+N)) \neq 0$ by assumption, f does not have an associated family.

Remark 3.1.8. If $M = G/K$ is a *para-Kähler* symmetric space, then the conditions for the existence of an associated family (Theorem 3.1.5 (2)) can be expressed in a very nice way; this has been discovered recently by Eschenburg and Quast [EQ]:

Suppose that G is semisimple. We consider the standard immersion $J : M \to \mathfrak{g}$, which assigns to each point $p \in M$ the para-complex structure J_p, which is a derivation of the Lie triple algebra (T_pM, R_p) and therefore given by an element of $\mathfrak{g}$ acting by the adjoint representation. From the parallelity of the endomorphism field J we conclude that the image $J(M) \subset \mathfrak{g}$ is the adjoint orbit $\mathrm{Ad}_G(\xi)$, $\mathrm{ad}_\xi := J_{eK}$ (see the proof of Proposition 3.2.2). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$. By definition, ad_ξ is a para-complex structure on $\mathfrak{m}$ and zero on $\mathfrak{k}$. In particular, $\mathfrak{k}$ coincides with the stabilizer subalgebra of ξ. It follows that the map $M \to \mathrm{Ad}_G(\xi) \subset \mathfrak{g}$ is a G-equivariant covering. The tangent and normal bundles of $J(M) \subset \mathfrak{g}$ are $\mathrm{Ad}_G(\mathfrak{m})$ and $\mathrm{Ad}_G(\mathfrak{k})$, respectively. At the point $\eta \in \mathrm{Ad}_G(\xi)$, the para-complex structure on $T_\eta J(M)$ is ad_η.

Now let $f : N \to M$ be a map from a para-Kähler manifold N into M. The differential of $\tilde{f} : N \to M \to \mathfrak{g}$ can be considered as a $\mathfrak{g}$-valued differential form on N. Let

$$\gamma := J_M \cdot d\tilde{f} \cdot J_N = \mathrm{ad}_{\tilde{f}}(d\tilde{f} \cdot J_N) \in \Omega^1(N; \mathfrak{g}).$$

We claim that *f has an associated family if and only if γ is closed* [EQ, Theorem 5.1].

Proof. A short calculation shows that

$$\begin{aligned} d\gamma(X, J_N Y) = {} & [\tilde{f}, \nabla d\tilde{f}(X, Y) - \nabla d\tilde{f}(J_N X, J_N Y)] \\ & + [d\tilde{f}(X), d\tilde{f}(Y)] + [d\tilde{f}(J_N X), d\tilde{f}(J_N Y)], \end{aligned}$$

where ∇ denotes the connection on $T^*N \otimes \mathfrak{g}$ defined by the Levi-Civita connection on TN. Since $\mathrm{ad}_{\tilde{f}}$ vanishes on the normal bundle, the expression in the first line takes values in the tangent bundle. It is zero for all X, Y, if and only if

$$\begin{aligned} 0 &= (\nabla d\tilde{f}(X, Y) - \nabla d\tilde{f}(J_N X, J_N Y))^T \\ &= Dd\tilde{f}(X, Y) - Dd\tilde{f}(J_N X, J_N Y), \end{aligned}$$

which is precisely the definition of $\tilde{f}$ being para-pluriharmonic.

The expression in the second line takes values in the normal bundle. It is zero for all X, Y, if and only if

$$\begin{aligned} 0 &= [d\tilde{f}(X), d\tilde{f}(Y)] + [d\tilde{f}(J_N X), d\tilde{f}(J_N Y)] \\ &= R^{J(M)}(d\tilde{f}(X), d\tilde{f}(Y)) + R^{J(M)}(d\tilde{f}(J_N X), d\tilde{f}(J_N Y)), \end{aligned}$$

which is equivalent the condition $R^M(df(T^{\pm}N), df(T^{\pm}N)) = 0$. □

In [EQ] this is used to generalize the Sym–Bobenko construction of surfaces in $\mathbb{R}^3$ with prescribed constant mean curvature from their (harmonic) Gauß map.

3.1.5 The Maurer–Cartan form. If G is a matrix group, i.e. a closed subgroup of $\mathrm{GL}(\cdot, \mathbb{R})$, then there is an alternative approach to associated families using the Maurer–Cartan form. The following is completely analogous to the Riemannian case, see [DE] (note that our Φ_θ is Φ_θ^{-1} there).

Let N_0 be a contractible open subset of N. Then any map $f: N_0 \to M = G/K$ can be lifted to a map $F: N_0 \to G$ such that $f = \pi \circ F$; this is called a *framing* of f. This framing is unique up to right multiplication by K-valued maps. Without loss of generality, we may assume that $p_o \in N_0$ is mapped to $o = eK \in M$, and that $F(p_o) = e$.

To the framing F we assign the $\mathfrak{g}$-valued differential form

$$\alpha := F^{-1}dF \in \Omega^1(N_0; \mathfrak{g}),$$

the *Maurer–Cartan form*. It holds that $d\alpha + \alpha \wedge \alpha = 0$.

Suppose that $f: N \to M = G/K$ fulfills the conditions of Theorem 3.1.5. Let (Φ_θ, f_θ) be the associated family of f. For any $p \in N$, $\Phi_\theta(p)$ is a linear isometry $T_{f_\theta(p)}M \to T_{f(p)}M$ which preserves R, therefore under the requirements of Fact 3.1.4 (4), $\Phi_\theta(p)$ is given by an element of G.

We can easily check that

$$F_\theta := \Phi_\theta^{-1} F$$

is a framing of f_θ with corresponding Maurer–Cartan form

$$\alpha_\theta = F_\theta^{-1} dF_\theta = \alpha + F^{-1}\Phi_\theta d\Phi_\theta^{-1} F.$$

From the parallelity of Φ_θ it follows (cf. [DE, Lemma 1]) that the last term on the right-hand side takes values in $\mathfrak{m} \subset \mathfrak{g}$ and that $(\alpha_\theta)_{\mathfrak{k}} = \alpha_{\mathfrak{k}}$ for any $\theta \in \mathbb{R}$ (where $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ due to the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$). By definition, the component $(\alpha_\theta)_{\mathfrak{m}}$ is the horizontal lift of df_θ, up to left multiplication by elements of G, therefore from equation (7) we obtain

$$(\alpha_\theta)_{\mathfrak{m}} = \alpha_{\mathfrak{m}} \circ e^{\theta J} = e^{\theta}\alpha_{\mathfrak{m}}^{(+1,0)} + e^{-\theta}\alpha_{\mathfrak{m}}^{(0,-1)}.$$

For fixed $p \in N_0$, the map $\theta \mapsto F_\theta(p)$ can be considered as an element of the *loop group*

$$\Lambda := \{\gamma: \mathbb{R} \to G;\ \gamma \text{ smooth}\}.$$

Thus, we have defined a map $\mathcal{F}: N_0 \to \Lambda$, $\mathcal{F}(p)(\theta) := F_\theta(p)$ which satisfies

$$\mathcal{F}^{-1} d\mathcal{F} = \alpha_{\mathfrak{k}} + e^{\theta}\alpha_{\mathfrak{m}}^{(+1,0)} + e^{-\theta}\alpha_{\mathfrak{m}}^{(0,-1)};$$

this is called an *extended framing*. By construction, $\mathcal{F}$ is unique up to constant loops with values in K, hence the map $\widetilde{\mathcal{F}} := \pi \circ \mathcal{F}: N \to \Lambda/K$ is uniquely determined by f and globally defined on N.

Conversely we can reconstruct the associated family from $\mathcal{F}$: Let $F_\theta = \mathcal{F}(\theta)$ and $f_\theta = \pi \circ F_\theta$. Then a short calculation shows that

$$df_\theta \circ e^{\theta J} = (F_0 F_\theta^{-1}) \circ df_0$$

and that $\Phi_\theta := F_0 F_\theta^{-1}$ is parallel.

3.2 Isotropic para-pluriharmonic maps

We now want to focus our attention to the case where the family (f_θ) is constant in θ. Such maps are called *isotropic* para-pluriharmonic maps. Equation 7 then reads

$$df \circ e^{\theta J} = \Phi_\theta \circ df, \tag{10}$$

where Φ_θ is a parallel bundle automorphism of f^*TM preserving R and g.

We now assume that f is *full*, i.e. that the image is not contained in a totally geodesic subspace of M.

Proposition 3.2.1. *Let $f: N \to M$ be a full isotropic para-pluriharmonic map. Then we have the following:*

(1) $(\Phi)_{\theta\in\mathbb{R}}$ *is a one-parameter group of automorphisms of f^*TM, i.e. $\Phi_\theta \circ \Phi_{\theta'} = \Phi_{\theta+\theta'}$;*

(2) f^*TM *has a decomposition into parallel eigenbundles: $f^*TM = \bigoplus_{\lambda \text{ odd}} E_\lambda$, where $\Phi_\theta|_{E_\lambda} = e^{\theta\lambda}$;*

(3) *The image of df is contained in $E_{-1} \oplus E_{+1}$, and $E_{-1} \oplus E_{+1}$ generates f^*TM as Lie triple algebra;*

(4) $g(E_\lambda, E_\mu) = 0$ *if* $\lambda + \mu \neq 0$ *and* $\dim E_\lambda = \dim E_{-\lambda}$.

Proof. Let $E_\lambda := \bigcap_{\theta\in\mathbb{R}} \ker(\Phi_\theta - e^{\theta\lambda})$. From equation (10) we see immediately that $df(T^{\pm}N) \subset E_{\pm 1}$. Let $E'_{\pm 1}$ be the smallest parallel subbundle of f^*TM containing $df(T^{\pm}N)$, then $E'_{\pm 1} \subset E_{\pm 1}$, because Φ_θ commutes with the parallel translation on f^*TM. Now let $E \subset f^*TM$ be the bundle generated by $(E'_{-1} \oplus E'_{+1})$ as a Lie triple algebra. By definition we have $[[E_\lambda, E_\mu], E_\nu] \subset E_{\lambda+\mu+\nu}$ and thus $E \subset \bigoplus_{\lambda \text{ odd}} E_\lambda$. We claim that $E = f^*TM$: In fact, because E is parallel and R-invariant and because f is full, this is the consequence of [ET1, Theorem 2]. This proves (2) and (3). (1) is an immediate consequence. Now (4) follows from the fact that Φ_θ is an isometry of f^*TM: We have $g|_{E_\lambda \times E_\mu} = \Phi_\theta^* g|_{E_\lambda \times E_\mu} = e^{(\lambda+\mu)\theta} g|_{E_\lambda \times E_\mu}$, which obviously implies the first statement. But then, g is an isomorphism $E_\lambda \xrightarrow{\sim} E_{-\lambda}^*$, which proves the equality of dimensions. □

Assume now that G is semisimple or that M is simply connected (and that the image of G in $\mathrm{Iso}(M)$ contains $\mathrm{Iso}_0(M)$).

For each $p \in N$, $\Phi_\theta(p)$ is a one-parameter group of automorphisms of the Lie triple algebra $(T_{f(p)}M, R)$, so by Fact 3.1.4(4), it can be considered as a one-parameter subgroup of gKg^{-1} (where $gK = f(p)$). Its generator $\frac{d}{d\theta}\Phi_\theta(p)$ is a derivation of the $(T_{gK}M, R)$ and therefore given by the adjoint representation of an element of $\mathrm{Ad}_g(\mathfrak{k})$. This element is uniquely determined, if the action of G on M is almost effective.

Let $p_o \in N$ and choose $o = f(p_o)$ as the base point in M.

Proposition 3.2.2. *Under the identification* $\mathrm{ad}\colon \mathrm{Ad}_g(\mathfrak{k}) \xrightarrow{\sim} \mathrm{Der}(T_{gK}M, g, R)$*, the map*

$$F = \frac{d}{d\theta}|_{\theta=0}\Phi_\theta$$

takes values in the adjoint orbit $\mathrm{Ad}_G(\xi) \subset \mathfrak{g}$*, where* $\mathrm{ad}_\xi := F(p_0)$.

Proof. Let $p \in N$ and let $\gamma\colon [0,1] \to N$ be a smooth path from p_o to p. Recall that the left invariant distribution $g\mathfrak{m} \subset T_gG$ yields a connection on the K-principal bundle $G \to M$ which agrees with the Levi-Civita connection on $TM = G \times_K \mathfrak{m}$. Therefore the parallel translation along $f \circ \gamma$ is given by a horizontal curve $g(t)$ in G, such that $gK = f \circ \gamma$, $g(0) = e$ and $g' \in g\mathfrak{m}$. It follows that $F(\gamma(t)) = \mathrm{Ad}_{g(t)}\xi$, because both sides are parallel endomorphism fields along $f \circ \gamma$, which coincide at $t = 0$. □

From Proposition 3.2.1 we see immediately that ad_ξ has odd integer eigenvalues on $\mathfrak{m} \cong T_oM$. Therefore, if $\mathfrak{g}$ is semisimple, ad_ξ has even integer eigenvalues on $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$, and the sum of the ± 1-eigenspaces generates all of $\mathfrak{g}$ as Lie algebra. Elements $\xi \in \mathfrak{g}$ with this property are called *canonical:*

3.3 Canonical elements and twistor spaces

Definition 3.3.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra. An element $\xi \in \mathfrak{g}$ is called *canonical,* if

(1) $\mathfrak{g}$ is the sum $\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{Z}} \mathfrak{g}_\lambda$ of ad_ξ-eigenspaces $\mathfrak{g}_\lambda := \ker(\mathrm{ad}_\xi - \lambda\,\mathrm{id}_\mathfrak{g})$ with integer eigenvalues,

(2) $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$ as Lie algebra.

Some authors replace the condition (2) by

(2′) the Lie algebra generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ contains $\bigoplus_{\lambda\neq 0} \mathfrak{g}_\lambda$.

To distinguish these notions, we call an element ξ *semi-canonical,* if (1) and (2′) hold.

Lemma 3.3.2. *If* $\mathfrak{g}$ *is simple, then any semi-canonical element of* $\mathfrak{g}$ *is either canonical or zero.*

More generally, let $\mathfrak{g}$ *be semisimple and let* $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'' \oplus \cdots$ *be the decomposition into simple summands. Then a semi-canonical element* $\xi = \xi' + \xi'' + \cdots$ *is canonical, if and only if all the* $\xi', \xi'', \ldots$ *are nonzero.*

Proof. Let $\mathfrak{g}$ be simple and ξ semi-canonical. From (2') and $[\mathfrak{g}_0, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 1}$ and the Jacobi identity it follows that the Lie algebra generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is an ideal in $\mathfrak{g}$ and therefore either all of $\mathfrak{g}$ or trivial. This proves the first statement. The rest is straightforward. □

Definition 3.3.3. Let $\xi \in \mathfrak{g}$ be semi-canonical. The adjoint orbit $Z(\xi) := \mathrm{Ad}_G(\xi)$ is called the *twistor space* associated to ξ.

Fact. *Z is the homogenous space G/H, where $H = \{g \in G : \mathrm{Ad}_g(\xi) = \xi\}$ is the centralizer of ξ with Lie algebra $\mathfrak{h} = \mathfrak{g}_0$.*

Proposition 3.3.4. *Let G be a connected real form of a complex semisimple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and $\xi \in \mathfrak{g}$ semi-canonical.*

(1) *The map $\sigma\colon g \mapsto e^{i\pi\xi} g e^{-i\pi\xi}$ ($i^2 = -1$) is an involutive automorphism of G with corresponding Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \big(\bigoplus\nolimits_{\lambda \text{ even}} \mathfrak{g}_\lambda\big) \oplus \big(\bigoplus\nolimits_{\lambda \text{ odd}} \mathfrak{g}_\lambda\big).$$

(2) *It holds that $H \subset G^\sigma$, and $H \cdot G_0^\sigma$ is a subgroup of G. Let K be a subgroup of G such that $H \cdot G_0^\sigma \subset K \subset G^\sigma$. Then there is a G-equivariant projection map $\pi\colon Z \to M$ onto the symmetric space $M = G/K$, called the* ***twistor fibration****.*

(3) *The G-action on M is almost effective (i.e. the kernel of the map $G \to \mathrm{Iso}(M)$ is discrete) if and only if ξ is canonical.*

Proof. (1) The stated Cartan decomposition follows from the fact that $d\sigma = e^{i\pi \mathrm{ad}_\xi}$ is $+1$ on $\mathfrak{g}_\lambda$ if λ is even, and -1 if λ is odd. This also shows that $d\sigma(\mathfrak{g}) = \mathfrak{g}$ and $d\sigma^2 = \mathrm{id}$, which implies $\sigma(G) = G$ and $\sigma^2 = \mathrm{id}$, since G is connected.

(2) If $g \in H$, then $e^{i\pi\xi} = e^{i\pi \mathrm{Ad}_g \xi} = g e^{i\pi\xi} g^{-1}$, so $g \in G^\sigma$. This shows that $H \subset G^\sigma$. It now follows that $h G_0^\sigma h^{-1} = G_0^\sigma$ for any $h \in H$. Therefore $H \cdot G_0^\sigma = G_0^\sigma \cdot H$, which implies that $H \cdot G_0^\sigma$ is a subgroup of G. The rest is straightforward.

(3) Let ξ be canonical. We have to show that the adjoint action $\mathfrak{k} \to \mathfrak{gl}(\mathfrak{m})$ is effective. Suppose that $X \in \mathfrak{k}$ and $\mathrm{ad}_X|_{\mathfrak{m}} = 0$. Since $\mathfrak{m} \supset \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$, it follows by the Jacobi identity that $\mathrm{ad}_X|_{\mathfrak{g}} = 0$, hence $X = 0$. Vice versa, let ξ be semi-canonical, but not canonical. Then the orthogonal complement (with respect to the Killing form) of the subalgebra generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is an ideal contained in $\mathfrak{g}_0 \subset \mathfrak{k}$ and therefore in the kernel of $\mathrm{ad}\colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{m})$. □

At $\xi \in Z$, the tangent space of Z is $T_\xi Z = \mathrm{ad}_{\mathfrak{g}}(\xi) = \bigoplus_{\lambda \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_\lambda =: \mathfrak{z}$. This eigenspace decomposition is Ad_H-invariant and thus defines a decomposition of TZ into Ad_G-invariant distributions $\mathrm{Ad}_G(\mathfrak{g}_\lambda)$, $\lambda \in \mathbb{Z} \setminus \{0\}$.

Proposition 3.3.5. *The twistor space* $Z = \mathrm{Ad}_G(\xi)$ *has a canonical para-complex structure which is given by multiplication by* $\mathrm{sgn}(\lambda)$ *on* $\mathrm{Ad}_G(\mathfrak{g}_\lambda) \subset TZ$.

Proof. The proposition follows from the fact that Z can be written as a quotient of para-complex groups (see below). However, there is a direct proof: From the fact that e^{ad_ξ} is an isometry with respect to the (nondegenerate) Killing form it follows that $\dim(\mathfrak{g}_\lambda) = \dim(\mathfrak{g}_{-\lambda})$, cf. the proof of 3.2.1 (4). Moreover, the distributions $T^\pm Z = \mathrm{Ad}_G(\bigoplus_{\lambda \gtrless 0} \mathfrak{g}_\lambda)$ are integrable, because they are tangent to the leaves $\mathrm{Ad}_{gN_\pm}(\xi) \subset Z$, where $N_\pm := \exp(\bigoplus_{\lambda \gtrless 0} \mathfrak{g}_\lambda)$. □

Theorem 3.3.6. *Let* N *be a connected para-Kähler manifold and let* G *be a connected real form of a semisimple complex Lie group.*

(1) *Let* $M = G/K$ *be a symmetric space with* $K = G^\sigma$ *and suppose that* G *acts almost effectively. Then any full isotropic para-pluriharmonic map* $f : N \to M$ *defines a canonical element* $\xi \in \mathfrak{g}$ *and a twistor fibration* $\pi : Z \to M$. *The map* f *has a para-holomorphic lift* $F : N \to Z$, $f = \pi \circ F$, *such that the differential* dF *takes values in the subbundle* $\mathcal{H}_1 := \mathrm{Ad}_G(\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}) \subset TZ$ (*called* ***superhorizontal*** *bundle*)

(2) *Let* $\xi \in \mathfrak{g}$ *be canonical and let* $\pi : Z \to M = G/K$ *a twistor fibration as in Proposition 3.3.4. For any para-holomorphic superhorizontal map* $F : N \to Z$ *the map* $f := \pi \circ F$ *is isotropic para-pluriharmonic.*

Proof. For the proof of (1), there is only left to show that $F = \frac{d}{d\theta}|_{\theta=0}\Phi_\theta$ is para-holomorphic and superhorizontal, i.e. that dF maps $T^\pm N$ to $\mathrm{Ad}_G(\mathfrak{g}_{\pm 1})$: In the proof of Proposition 3.2.2 we have already seen that $F \circ \gamma(t) = \mathrm{Ad}_{g(t)}\xi$ with $g^{-1}g' \in \mathfrak{m}$ and therefore $dF \cdot \gamma' \in \mathrm{Ad}_g[\mathfrak{m}, \xi] = \mathrm{Ad}_g(\bigoplus_{\lambda \text{ odd}} \mathfrak{g}_\lambda)$. The map $d\pi : TZ \to TM$ takes $\mathrm{Ad}_G(\mathfrak{g}_\lambda)$ to E_λ, and from equation (10) it follows that $d\pi \circ dF(T^\pm N) = df(T^\pm N) \subset E_{\pm 1}$, hence $dF(T^\pm N) \subset \mathrm{Ad}_G(\mathfrak{g}_{\pm 1})$.

In order to prove (2), recall that the para-complex structure on the subspace $\mathrm{Ad}_g(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1) \subset T_{\xi'}Z$, $\xi' = \mathrm{Ad}_g\xi$, is just $\mathrm{ad}(\xi')$. The fact that F is holomorphic and the image of dF is contained in $\mathrm{Ad}_G(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)$, yields $dF \circ e^{\theta J} = e^{\theta \mathrm{ad}_F} \circ dF$. Composing this equation with $d\pi : TZ \to TM$ (which is multiplication by (-1) on $\mathrm{Ad}_g(\mathfrak{g}_\lambda)$, λ odd), and recalling that the isomorphism $\mathrm{Der}(\mathrm{Ad}_g(\mathfrak{m}), [[,],]) \xrightarrow{\sim} \mathrm{Der}(T_{gK}M, g, R)$ identifies ad_F with F, we obtain $df \circ e^{\theta J} = e^{\theta F} \circ df$, where $e^{\theta F} =: \Phi_\theta$ is an automorphism field of f^*TM, which preserves the curvature tensor and any Ad_K-invariant metric. F and Φ_θ are parallel, since dF has values in the horizontal bundle $\mathrm{Ad}_G(\mathfrak{m})$ (see the proof of Proposition 3.2.2). Thus, $f = \pi \circ F$ is isotropic para-pluriharmonic by the definition. □

Corollary 3.3.7. *A necessary condition for the existence of full isotropic para-pluriharmonic maps* $N \to M = G/G^\sigma$ *is that* σ *is of the form* $\sigma(g) = e^{i\pi\xi} g e^{-i\pi\xi}$ *for* $\xi \in \mathfrak{g}$.

Example 3.3.8 (Para-holomorphic maps). The easiest example of all this is as follows: Assume that $M = G/G^\sigma$ is a para-Kähler symmetric space with para-complex structure J_M. Any para-holomorphic map $f\colon N \to M$ is isotropic para-pluriharmonic, since $df \circ e^{\theta J_N} = e^{\theta J_M} \circ df$. The canonical element $\xi \in \mathfrak{g}$ for this map is exactly the para-complex structure J_M; it defines a depth-1 grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. By definition, ξ is invariant under G^σ, so $G^\sigma \subset H$. We have already seen that $H \subset G^\sigma$, therefore $Z = M$, and the twistor fibration is just the identity map.

3.3.1 Para-complex description of the twistor space. We want to give a description of Z as a quotient of para-complex groups:

Definition 3.3.9. Let G be a (not necessarily semisimple) Lie group. The *para-complexification* of G is the product $G^{\boldsymbol{c}} := G \times G$. This is a para-complex manifold, with the para-complex structure given by $J = \mathrm{id}_{TG} \times (-\mathrm{id}_{TG})$. Its Lie algebra is $\mathfrak{g}^{\boldsymbol{c}} = \mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g} \otimes \boldsymbol{C}$, $(X, Y) \cong \frac{1+j}{2}X + \frac{1-j}{2}Y$. (cf. Example 1.1.5). We can identify G with the diagonal subgroup $\{(g, g) : g \in G\}$, being the fixed-point set of the anti-holomorphic involution $(g, h) \mapsto (h, g)$ of $G^{\boldsymbol{c}}$.

Let G be semisimple and $\xi \in \mathfrak{g}$ semi-canonical. There is a decomposition of $\mathfrak{g}^{\boldsymbol{c}}$ into $\mathrm{ad}_{(\xi,\xi)}$-eigenspaces with *imaginary* eigenvalues (see Remark 1.1.3): For $\lambda \in \mathbb{Z}$, let $\mathfrak{g}^{\boldsymbol{c}}_\lambda := \mathfrak{g}_\lambda \times \mathfrak{g}_{-\lambda}$. Then $\mathfrak{g}^{\boldsymbol{c}}_\lambda$ is the λj-eigenspace of $\mathrm{ad}_{(\xi,\xi)}$ (j equals $+1$ on $\mathfrak{g} \times \{0\}$ and -1 on $\{0\} \times \mathfrak{g}$), and we have $\mathfrak{g}^{\boldsymbol{c}} = \bigoplus_{\lambda\in\mathbb{Z}} \mathfrak{g}^{\boldsymbol{c}}_\lambda$. We define the following subalgebras of $\mathfrak{g}$ and $\mathfrak{g}^{\boldsymbol{c}}$:

$$\mathfrak{n}_\pm := \bigoplus_{\lambda \gtrless 0} \mathfrak{g}_\lambda, \quad \mathfrak{n}^{\boldsymbol{c}}_\pm := \mathfrak{n}_\pm \times \mathfrak{n}_\mp,$$
$$\mathfrak{p}_\pm := \mathfrak{g}_0 \oplus \mathfrak{n}_\pm, \quad \mathfrak{p}^{\boldsymbol{c}}_\pm := \mathfrak{p}_\pm \times \mathfrak{p}_\mp.$$

The $\mathfrak{p}_\pm$ are *parabolic* subalgebras (see below) with nilradical (maximal nilpotent ideal) $\mathfrak{n}_\pm$. Let

$$N_\pm := \exp(\mathfrak{n}_\pm), \quad P_\pm := \{g \in G : \mathrm{Ad}_g \xi \in \xi + \mathfrak{n}_\pm\}.$$

These are closed subgroups with respective Lie algebras $\mathfrak{n}_\pm$ and $\mathfrak{p}_\pm$ (see [ET2]). Let $P^{\boldsymbol{c}}_\pm := P_\pm \times P_\mp$ and $N^{\boldsymbol{c}}_\pm := N_+ \times N_-$.

Proposition 3.3.10. *$Z = G/H$ is an open subset of $\overline{Z} := G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_- = (G/P_-) \times (G/P_+)$, called the **closed twistor** space. The para-complex structure on Z inherited from $\overline{Z}$ coincides with the one given in Proposition 3.3.5.*

Proof. The map $G \xrightarrow{\Delta} G^{\boldsymbol{c}} \to G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_-$ has kernel $P_- \cap P_+ = H$, so we have a well-defined embedding $G/H \to G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_-$. The differential of this map is the isomorphism $\mathfrak{g}/\mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{g}^{\boldsymbol{c}}/\mathfrak{p}^{\boldsymbol{c}}_- \cong \mathfrak{n}_+ \times \mathfrak{n}_-$ which sends $\mathfrak{g}_\lambda$ to $\mathfrak{g}_\lambda \times \{0\}$ or to $\{0\} \times \mathfrak{g}_\lambda$, depending on the sign of λ. □

3.3.2 Parabolic subgroups. In order to describe the coset spaces $G/P_{\mp}$, we need some facts concerning (real) parabolic subgroups. We start with some basic definitions from algebraic geometry. (All of the following can be found in [Bo].)

An *affine algebraic variety* is a subset $V \subset \mathbb{C}^n$ which is the set of zeroes of a family of polynomials. It can be endowed with the *Zariski topology*, in which the closed sets are by definition the subvarieties.

Let $I(V) \subset \mathbb{C}[\mathbb{C}^n]$ be the ideal of polynomials vanishing on V. Then $\mathbb{C}[V] := \mathbb{C}[\mathbb{C}^n]/I(V)$ is the ring of *regular functions.* A *morphism* of varieties is a map $\phi\colon V \to V'$, such that $\phi^*(\mathbb{C}[V']) \subset \mathbb{C}[V]$. V is said to be *irreducible*, if V is not the proper union of two nonempty varieties, which is equivalent to $I(V)$ being a prime ideal. If V is irreducible, then the quotient field of $\mathbb{C}[V]$ is called the field of *rational functions,* denoted by $\mathbb{C}(V)$.

A variety V is said to be *defined over* $\mathbb{R}$, or an $\mathbb{R}$-*variety*, if $I(V)$ is generated by polynomials with real coefficients. In this case, the set $V(\mathbb{R})$ of *real points* of V (i.e. the points of V with real coordinates) is a *real variety.* A morphism $\phi\colon V \to V'$ of $\mathbb{R}$-varieties is *defined over* $\mathbb{R}$ (or an $\mathbb{R}$-*morphism*), if $\phi^*(\mathbb{R}[V']) \subset \mathbb{R}[V]$.

A group is called *linear algebraic,* if it is both a subgroup and a subvariety of $\mathrm{GL}_n(\mathbb{C})$ (where $\mathrm{GL}_n(\mathbb{C})$ is the variety $\{(g, \det(g)^{-1}) : g \in \mathrm{GL}_n(\mathbb{C})\} \subset \mathbb{C}^{n^2+1}$). It follows that any linear algebraic group over $\mathbb{C}$ is a complex Lie group. If G is an $\mathbb{R}$-*group* (i.e. a linear algebraic group defined over $\mathbb{R}$), then $G(\mathbb{R})$ is a real Lie group and a real form of G. Important examples of linear algebraic groups are the classical (real and complex) matrix groups. In fact, any connected semisimple complex Lie group is linear algebraic, as well as the adjoint form of any connected semisimple real Lie group. (In the complex case, the notions of connectivity in the ordinary (Hausdorff) topology and the Zariski topology coincide).

Let G be a connected semisimple $\mathbb{R}$-group.

A subgroup $P \subset G$ is called *parabolic*, if it contains a *Borel subgroup*, i.e. a maximal solvable connected (in fact, the connectivity follows) subgroup. In this case, G/P is compact and a projective algebraic variety (i.e. a subset of $\mathbb{C}\mathrm{P}^n$ defined by homogenous polynomials).

We assume that $G(\mathbb{R})$ is non-compact (otherwise the following would be trivial). Let S be a maximal $\mathbb{R}$-*split* torus, i.e. a maximal algebraic torus S, such that the adjoint action of $S(\mathbb{R})$ on $\mathfrak{g}$ has only real eigenvalues. The dimension r of S is called $\mathbb{R}$-*rank* of G (this is well-defined, since all maximal $\mathbb{R}$-split tori are conjugate, [Bo, 20.9]). Then $\mathfrak{g}$ is the sum of *relative root spaces* $\mathfrak{g} = \bigoplus_{\alpha\in\Phi_{\mathbb{R}}} \mathfrak{g}_\alpha$ (in general not one-dimensional), where $\Phi_{\mathbb{R}} \subset \mathfrak{s}^*$ is the *relative root system*. The group generated by the orthogonal reflections through the hyperplanes $\{\alpha^\perp : \alpha \in \Phi_{\mathbb{R}}\}$ is the *relative Weyl group* $\mathcal{W}_{\mathbb{R}} = \mathcal{N}(S)/\mathcal{Z}(S)$, which can be represented by elements of $G(\mathbb{R})$ [Bo, 21.2]. The connected components of $\mathfrak{s} \setminus \bigcup_{\alpha\in\Phi_{\mathbb{R}}} \alpha^\perp$ are called *relative Weyl chambers.* The choice of a relative Weyl chamber determines a partition $\Phi_{\mathbb{R}} = \Phi^+_{\mathbb{R}} \cup \Phi^-_{\mathbb{R}}$ of the set of roots into positive and negative ones. Let $\Delta_{\mathbb{R}} \subset \Phi^+_{\mathbb{R}}$ denote the set of *simple* roots, i.e. the positive roots that are not the sum of two other positive

roots. Clearly, the simple roots $(\alpha_1, \ldots, \alpha_r)$, $r = \operatorname{rank}_{\mathbb{R}}(G)$ form a basis of $\mathfrak{s}^*$. Let $(\xi_1, \ldots, \xi_r)$ be the basis of $\mathfrak{s}$ dual to this, i.e. $\alpha_a(\xi_b) = \delta_{ab}$.

The following proposition relates subsets of $\Delta_{\mathbb{R}}$ to semi-canonical elements and to parabolic $\mathbb{R}$-subgroups (cf. [Bo, 21.12]):

Proposition 3.3.11. *There is a bijection between*

- *subsets of* $\Delta_{\mathbb{R}}$,
- *conjugacy classes of semi-canonical elements, and*
- *conjugacy classes of parabolic* $\mathbb{R}$*-subgroups.*

More explicitly:

(1) *For each subset* $I \subset \Delta_{\mathbb{R}}$, *the element* $\xi_I := \sum_{\alpha_a \in \Delta_{\mathbb{R}} \setminus I} \xi_a$ *is semi-canonical.*

(2) *Each semi-canonical element is* $G(\mathbb{R})$*-conjugate to exactly one of the* ξ_I.

(3) *If* ξ *is semi-canonical, then the connected* $\mathbb{R}$*-subgroup* $P_\xi = \{g \in G : \operatorname{Ad}_g \xi \in \xi + \bigoplus_{\alpha(\xi)>0} \mathfrak{g}_\alpha\}$ *with Lie algebra* $\mathfrak{p}_\xi = \bigoplus_{\alpha(\xi)\geq 0} \mathfrak{g}_\alpha$ *is parabolic.*

(4) *For each parabolic* $\mathbb{R}$*-subgroup* P *there exists a unique semi-canonical element* ξ, *such that* $P = P_\xi$.

Proof. (1) By definition, the ad_ξ-eigenspaces are $\mathfrak{g}_\lambda = \bigoplus_{\alpha(\xi)=\lambda} \mathfrak{g}_\alpha$. Let ξ be as described. Then we have $\alpha(\xi) \in \{0, 1\}$ for each simple root α. Since each positive root is the sum of simple ones, it follows that $\alpha(\xi) \in \mathbb{N}_0$ for each positive root α (and of course $-\alpha(\xi) \in -\mathbb{N}_0$ for each negative root $-\alpha$). Because $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ contains all simple root spaces, it generates the subalgebra $\bigoplus_{\lambda\geq 0} \mathfrak{g}_\lambda$. From $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$ it then follows that $\mathfrak{g}_1$ generates $\bigoplus_{\lambda>0} \mathfrak{g}_\lambda$. Analogously for $\mathfrak{g}_{-1}$.

(3) By definition, ξ is semisimple, so it is contained in a maximal torus $\mathfrak{t} \subset \mathfrak{g}_0$. Choose a Weyl chamber in $\mathfrak{t}^*$ whose closure contains ξ. This defines a positive direction in $\mathfrak{t}$, such that $\mathfrak{p}_\xi$ contains the sum of $\mathfrak{t}$ and all positive root spaces. The latter is easily seen to be a Borel subalgebra, therefore $\mathfrak{p}_\xi$ is parabolic.

Now (2) and (4) follow from the fact that *each parabolic* $\mathbb{R}$*-group is* $G(\mathbb{R})$*-conjugate to exactly one of the* P_{ξ_I} [Bo, 21.12], and that $P_\xi = P_{\xi'}$ only if $\xi = \xi'$: This follows from the observation that the ad_ξ-eigenspaces $\mathfrak{g}_\lambda$ (and therefore ξ, by semisimplicity) are determined by $\mathfrak{p}_\xi$ and $\mathfrak{p}_{-\xi}$: Let $\mathfrak{n}_\pm$ be the nilradical of $\mathfrak{p}_{\pm\xi}$. Then $\mathfrak{D}_a\mathfrak{n}_\pm = \bigoplus_{\lambda\geq a+1} \mathfrak{g}_{\pm\lambda}$ (where $\mathfrak{D}_0\mathfrak{n} := \mathfrak{n}$, $\mathfrak{D}_{a+1}\mathfrak{n} := [\mathfrak{n}, \mathfrak{D}_a\mathfrak{n}]$). Because $\mathfrak{g}_\lambda \perp \mathfrak{g}_\mu$ for $\lambda \neq -\mu$ with respect to the Killing form, it follows that $\mathfrak{g}_0 = \mathfrak{p}_+ \cap \mathfrak{n}_-^\perp$ and $\mathfrak{g}_{\pm a} = \mathfrak{D}_{a-1}\mathfrak{n}_\pm \cap (\mathfrak{D}_a\mathfrak{n}_\mp)^\perp$ for $a \in \mathbb{N}$. □

There are two special cases of this construction: For $I = \Delta_{\mathbb{R}}$ we obtain $\xi = 0$ and $P_{\Delta_{\mathbb{R}}} = G$. – For $I = \emptyset$ we get the *minimal* parabolic $\mathbb{R}$-subgroups, which play the same role in the real Bruhat decomposition as the Borel subgroups in the complex case:

Theorem ([Bo, 21.15, 21.16]). *Let G be a connected semisimple $\mathbb{R}$-group, S a maximal $\mathbb{R}$-split torus and $P_\emptyset$ a minimal parabolic $\mathbb{R}$-subgroup. Then*

$$G(\mathbb{R}) = \bigcup_{w \in \mathcal{W}_{\mathbb{R}}} P_\emptyset(\mathbb{R}) w P_\emptyset(\mathbb{R}) \quad \textit{(disjoint union)}.$$

More generally, let $P_I \supset P_\emptyset$ be a parabolic $\mathbb{R}$-subgroup of G and let $\mathcal{W}_I \subset \mathcal{W}_{\mathbb{R}}$ be the subgroup generated by the reflections through the $\{\alpha^\perp : \alpha \in I\}$, then

$$G(\mathbb{R}) = \bigcup_{w \in \mathcal{W}_{\mathbb{R}}/\mathcal{W}_I} P_\emptyset(\mathbb{R}) w P_I(\mathbb{R}).$$

Therefore, $G(\mathbb{R})/P_I(\mathbb{R})$ is a union of the *Bruhat cells*

$$G(\mathbb{R})/P_I(\mathbb{R}) = \bigcup_{w \in \mathcal{W}_{\mathbb{R}}/\mathcal{W}_I} P_\emptyset(\mathbb{R}) w P_I(\mathbb{R})/P_I(\mathbb{R}).$$

Let w be the element of *shortest length* in its class $w \cdot \mathcal{W}_I$; then the cell $P_\emptyset w P_I / P_I$ is $\mathbb{R}$-isomorphic to the affine space $N_w := N_\emptyset \cap w N_{\emptyset -} w^{-1}$, where $N_\emptyset$, $N_{\emptyset -}$ are the unipotent radicals of $P_\emptyset$ and of its opposite parabolic subgroup $P_{\emptyset -} := P_{-\xi_\emptyset}$, respectively [Bo, 21.20, 21.29]. Note that $\exp\colon \mathfrak{n}_w \mapsto N_w$ is an $\mathbb{R}$-isomorphism, because $\mathfrak{n}_w$ is nilpotent and the exponential series and its inverse are finite.

There is one *big cell* in this decomposition, belonging to the element w_0 which maps $\Delta_{\mathbb{R}} \setminus I$ to $\Phi^-_{\mathbb{R}}$: The image of $N_{w_0} \cong N_{I-}$ is $\mathbb{R}$-Zariski-open and -dense in G/P_I. Moreover, the map $N_{I-} \to G/P_I$ is a *birational equivalence* (meaning that it induces an isomorphism of the ring of rational functions $\mathbb{C}(G/P_I) \xrightarrow{\sim} \mathbb{C}(N_{I-})$) [Bo, 21.20].

Summarizing this we conclude (back to our original notation, where the real groups are denoted $G, P_\pm, \dots$ rather than $G(\mathbb{R}), P_\pm(\mathbb{R}), \dots$):

Theorem 3.3.12. *Let G be a noncompact real form of a connected semisimple linear algebraic group. Let $\xi \in \mathfrak{g}$ be semi-canonical and $G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_-$ the closed twistor space associated to ξ. Then the following facts hold:*

(1) *The quotient $G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_-$ is a product $(G/P_-) \times (G/P_+)$ of real projective varieties.*

(2) *$(G/P_-) \times (G/P_+)$ is a disjoint union of affine spaces, namely the products of the Bruhat cells*

$$\bigcup_{(w_+, w_-) \in \mathcal{W} \times \mathcal{W}} (P_{\emptyset -} w_+ P_- / P_-) \times (P_\emptyset w_- P_+ / P_+).$$

(3) *The map $\mathfrak{n}_+ \times \mathfrak{n}_- \xrightarrow{\exp} N_+ \times N_- \to (G/P_-) \times (G/P_+)$ is an embedding onto a Zariski-open and -dense subset (namely the product of the respective big Bruhat cells) and induces a birational equivalence $\mathfrak{n}_+ \times \mathfrak{n}_- \sim (G/P_-) \times (G/P_+)$.*

3.3.3 Twistor fibrations of $\mathbb{R}$-split simple Lie groups. By Proposition 3.3.4, in order to find the twistor fibrations of a given Lie group G and therefore the isotropic

para-pluriharmonic maps into G-symmetric spaces (Theorem 3.3.6), we have to find the canonical elements in $\mathfrak{g}$. A direct consequence of Proposition 3.3.11 is the following

Corollary 3.3.13. *Let G be a simple Lie group of $\mathbb{R}$-rank r. Then there exist, up to covering or isomorphism, $2^r - 1$ nontrivial G-twistor fibrations, each of which corresponds to a proper subset of $\Delta_{\mathbb{R}}$.*

This construction is fairly straightforward, if G has an $\mathbb{R}$-split maximal torus (i.e. one that is maximal among all tori). These groups are called $\mathbb{R}$-*split*.

As an example, we will carry this out for $\mathrm{Sp}(2n,\mathbb{R})$:

The group $\mathrm{Sp}(2n,\mathbb{R})$ is defined as the group of automorphisms of $\mathbb{R}^{2n}$ which preserve the standard symplectic form given by $\mathbf{\Omega}_{2n} = \left(\begin{smallmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right)$. Its Lie algebra is

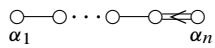
$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{ \begin{pmatrix} A & C \\ B & -A^t \end{pmatrix} : A, B, C \in \mathfrak{gl}(n,\mathbb{R}),\ B, C \text{ symmetric} \right\}.$$

The group $\mathrm{Sp}(2n,\mathbb{R})$ has $\mathbb{R}$-rank n, and a system of simple coroots spanning the maximal torus is given by

$$\xi_a := \begin{pmatrix} D_a & \\ & -D_a \end{pmatrix}, \quad D_a := \begin{pmatrix} \mathbf{0}_a & \\ & \mathbf{1}_{n-a} \end{pmatrix} \text{ for } 1 \le a \le n-1$$

and

$$\xi_n := \begin{pmatrix} \frac{1}{2}\mathbf{1}_n & \\ & -\frac{1}{2}\mathbf{1}_n \end{pmatrix}$$

(α_n is the *long* root of the Dynkin diagram 

).

There are two types of canonical elements:

Case 1: $\alpha_n \in I$, i.e. ξ does not contain the summand ξ_n. In this case we have

$$\xi = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \quad D = \begin{pmatrix} 0\cdot\mathbf{1}_{m_1} & & & \\ & 1\cdot\mathbf{1}_{m_2} & & \\ & & 2\cdot\mathbf{1}_{m_3} & \\ & & & \ddots \end{pmatrix}.$$

The centralizer of ξ is

$$H = \mathrm{Sp}(2m_1,\mathbb{R}) \times \mathrm{GL}(m_2,\mathbb{R}) \times \mathrm{GL}(m_3,\mathbb{R}) \times \cdots$$

(where $\mathrm{GL}(n,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & (A^{-1})^t \end{smallmatrix}\right) : A \in \mathrm{GL}(n,\mathbb{R}) \right\} \subset \mathrm{Sp}(2n,\mathbb{R})$),

and the centralizer of $e^{i\pi\xi}$ is

$$G^\sigma = \mathrm{Sp}(2m_{\mathrm{odd}},\mathbb{R}) \times \mathrm{Sp}(2m_{\mathrm{ev}},\mathbb{R}),$$

where $m_{\mathrm{odd}} := \sum_{a \text{ odd}} m_a$, $m_{\mathrm{ev}} := \sum_{a \text{ even}} m_a$.

The resulting twistor fibration $G/H \to G/K$ (with $K := H \cdot G_0^\sigma = G^\sigma$) is

$$\begin{array}{c} \mathrm{Sp}(2n,\mathbb{R}) \,/\, \mathrm{Sp}(2m_1,\mathbb{R}) \times \mathrm{GL}(m_2,\mathbb{R}) \times \mathrm{GL}(m_3,\mathbb{R}) \times \cdots \\ \downarrow \\ \mathrm{Sp}(2n,\mathbb{R}) \,/\, \mathrm{Sp}(2m_{\mathrm{ev}},\mathbb{R}) \times \mathrm{Sp}(2m_{\mathrm{odd}},\mathbb{R}). \end{array}$$

Case 2: $\alpha_n \notin I$, i.e. ξ contains the summand ξ_n. In this case we have

$$\xi = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{2}\mathbf{1}_{m_1} & & & \\ & \frac{3}{2}\mathbf{1}_{m_2} & & \\ & & \frac{5}{2}\mathbf{1}_{m_3} & \\ & & & \ddots \end{pmatrix}.$$

In order to compute the twistor fibration, it is practical to switch to a different Weyl chamber and change ξ into

$$\xi' = \begin{pmatrix} D' & 0 \\ 0 & -D' \end{pmatrix}, \quad D' = \begin{pmatrix} \frac{1}{2}\mathbf{1}_{m_1} & & & \\ & -\frac{3}{2}\mathbf{1}_{m_2} & & \\ & & \frac{5}{2}\mathbf{1}_{m_3} & \\ & & & \mp\ddots \end{pmatrix}.$$

Then the corresponding twistor fibration $G/H \to G/K$ (with $K = H \cdot G_0^\sigma = G^\sigma$) is

$$\begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})/\mathrm{GL}(m_1,\mathbb{R}) \times \mathrm{GL}(m_2,\mathbb{R}) \times \cdots \\ \downarrow \\ \mathrm{Sp}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R}). \end{array}$$

(The original ξ would lead to a subgroup conjugate to this.)

Note that $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})$ is the adjoint orbit $\mathrm{Ad}_G(\xi_n)$ and therefore has a para-complex structure. However, the map $\mathrm{Ad}_{\mathrm{Sp}(2n,\mathbb{R})}(\xi') \to \mathrm{Ad}_{\mathrm{Sp}(2n,\mathbb{R})}(\xi_n)$ is *not* holomorphic, except in the trivial case $\xi' = \xi_n$.

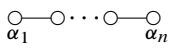
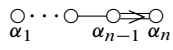
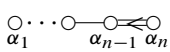
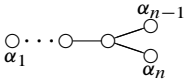
In the same way we can compute the twistor fibrations $G/H \to G/(H \cdot G_0^\sigma)$ of all classical $\mathbb{R}$-split simple Lie groups (see Table 1).

Remark. The reader may have observed that $\mathrm{SO}(n,n)$ and $\mathrm{SO}(n+1,n)$ are not connected. However, the only place where this assumption was needed, was the proof of $\sigma(G) = G$ (Proposition 3.3.4). Since this holds nonetheless, we stick with these groups rather than use their (non-algebraic) identity components.

3.4 Isotropic para-pluriharmonic maps into Grassmannians

The fibrations in the first row of this table are in fact the fibrations of para-complex flag manifolds over para-complex Grassmannians, which we will now describe more in detail, giving an explicit construction of para-pluriharmonic maps.

Table 1. Twistor fibrations of classical $\mathbb{R}$-split simple groups.

A_n	$\mathrm{SL}(n,\mathbb{R})$ 		$\mathrm{SL}(n,\mathbb{R})/\mathrm{S}(\mathrm{GL}(m_1,\mathbb{R})\times\mathrm{GL}(m_2,\mathbb{R})\cdots)$ $\downarrow$ $\mathrm{SL}(n,\mathbb{R})/\mathrm{S}(\mathrm{GL}(m_{\mathrm{odd}},\mathbb{R})\times\mathrm{GL}(m_{\mathrm{ev}},\mathbb{R}))$
B_n	$\mathrm{SO}(n+1,n)$ 	$\alpha_n \notin I$	$\mathrm{SO}(n+1,n)/\mathrm{GL}(m_1,\mathbb{R})\times\mathrm{GL}(m_2,\mathbb{R})\times\mathrm{GL}(m_3,\mathbb{R})\times\cdots$ $\downarrow$ $\mathrm{SO}(n+1,n)/\mathrm{SO}(m_{\mathrm{odd}},m_{\mathrm{odd}})\times\mathrm{SO}(m_{\mathrm{ev}}+1,m_{\mathrm{ev}})$
		$\alpha_n \in I$	$\mathrm{SO}(n+1,n)/\mathrm{SO}(m_1+1,m_1)\times\mathrm{GL}(m_2,\mathbb{R})\times\mathrm{GL}(m_3,\mathbb{R})\times\cdots$ $\downarrow$ $\mathrm{SO}(n+1,n)/\mathrm{SO}(m_{\mathrm{odd}}+1,m_{\mathrm{odd}})\times\mathrm{SO}(m_{\mathrm{ev}},m_{\mathrm{ev}})$
C_n	$\mathrm{Sp}(2n,\mathbb{R})$ 	$\alpha_n \notin I$	$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{GL}(m_1,\mathbb{R})\times\mathrm{GL}(m_2,\mathbb{R})\times\cdots$ $\downarrow$ $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})$
		$\alpha_n \in I$	$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{Sp}(2m_1,\mathbb{R})\times\mathrm{GL}(m_2,\mathbb{R})\times\mathrm{GL}(m_3,\mathbb{R})\times\cdots$ $\downarrow$ $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{Sp}(2m_{\mathrm{odd}},\mathbb{R})\times\mathrm{Sp}(2m_{\mathrm{ev}},\mathbb{R})$
D_n	$\mathrm{SO}(n,n)$ 	either $\alpha_n \in I$ or $\alpha_{n-1} \in I$	$\mathrm{SO}(n,n)/\mathrm{GL}(m_1,\mathbb{R})\times\mathrm{GL}(m_2,\mathbb{R})\times\mathrm{GL}(m_3,\mathbb{R})\times\cdots$ $\downarrow$ $\mathrm{SO}(n,n)/\mathrm{GL}(n,\mathbb{R})$
		both or neither	$\mathrm{SO}(n,n)/\mathrm{SO}(m_1,m_1)\times\mathrm{GL}(m_2,\mathbb{R})\times\mathrm{GL}(m_3,\mathbb{R})\times\cdots$ $\downarrow$ $\mathrm{SO}(n,n)/\mathrm{SO}(m_{\mathrm{odd}},m_{\mathrm{odd}})\times\mathrm{SO}(m_{\mathrm{ev}},m_{\mathrm{ev}})$

Definition 3.4.1. Let $m_1,\dots,m_l \in \mathbb{N}$ and $n := \sum m_a$. The *para-complex flag manifold* $\mathrm{F}_{m_1,\dots,m_l}(\boldsymbol{C}^n)$ is the space of nondegenerate $(m_1,\dots,m_l)$-flags in $\boldsymbol{C}^n$; such a flag is an orthogonal decomposition of $\boldsymbol{C}^n$ into nondegenerate (with respect to the standard para-hermitian scalar product) subspaces $\boldsymbol{C}^n = E_1 \oplus\cdots\oplus E_l$, $\dim_{\boldsymbol{C}}(E_a) = m_a$. Equivalently, a flag is a sequence of nondegenerate subspaces $\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_l = \boldsymbol{C}^n$, $\dim(W_a) = m_1 + \cdots + m_a$, where $W_a := E_1 \oplus\cdots\oplus E_a$. (To see the equivalence, we use the fact that the orthogonal complement of a nondegenerate subspace is again nondegenerate.)

The flag manifolds $\mathrm{F}_{m,n-m}(\boldsymbol{C}^n) =: \mathrm{Gr}_m(\boldsymbol{C}^n)$ are called *para-complex Grassmannians.*

The standard action of $\mathrm{U}^\pi(n)$ on $\boldsymbol{C}^n$ induces a transitive (by Proposition 1.3.3) $\mathrm{U}^\pi(n)$-action on $\mathrm{F}_{m_1,\dots,m_l}(\boldsymbol{C}^n)$. There is an equivariant projection map $\mathrm{U}^\pi(n) \to \mathrm{F}_{m_1,\dots,m_l}(\boldsymbol{C}^n)$ which sends a matrix $g \in \mathrm{U}^\pi(n)$ to the flag $(E_1,\dots,E_l)$, where E_1 is the span of the first m_1 columns of g, E_2 the span of the next m_2 columns, and so on. In particular, the identity matrix is mapped to the standard flag $\boldsymbol{C}^n = \boldsymbol{C}^{m_1} \oplus\cdots\oplus \boldsymbol{C}^{m_l}$.

The restriction of this map to $\mathrm{SU}^\pi(n)$ is still surjective, since we can multiply the first column of a given matrix $g \in \mathrm{U}^\pi(n)$ by $\det(g)^{-1} \in \{z \in \boldsymbol{C} : \bar{z}z = 1\}$ without changing its span.

This yields the identification

$$\mathrm{F}_{(m_1,\dots,m_l)}(\boldsymbol{C}^n) \cong \mathrm{SU}^\pi(n) \,/\, \mathrm{S}\left(\mathrm{U}^\pi(m_1) \times \cdots \times \mathrm{U}^\pi(m_l)\right).$$

Now recall that $\mathrm{SU}^\pi(n) \cong \mathrm{SL}(n,\mathbb{R})$. So the fibration

$$\begin{array}{c} \mathrm{SL}(n,\mathbb{R}) \,/\, \mathrm{S}\left(\mathrm{GL}(m_1,\mathbb{R}) \times \cdots \times \mathrm{GL}(m_l,\mathbb{R})\right) \\ \downarrow \\ \mathrm{SL}(n,\mathbb{R}) \,/\, \mathrm{S}\left(\mathrm{GL}(m_{\mathrm{odd}},\mathbb{R}) \times \mathrm{GL}(m_{\mathrm{ev}},\mathbb{R})\right) \end{array}$$

is the map $\mathrm{F}_{(m_1,\dots,m_l)}(\boldsymbol{C}^n) \to \mathrm{Gr}_{m_{\mathrm{odd}}}(\boldsymbol{C}^n)$, which sends the flag $(E_1,\dots,E_l)$ to the subspace $\bigoplus_{a\ \mathrm{odd}} E_a \subset \boldsymbol{C}^n$ (cf. Table 1). The corresponding canonical element is

$$\xi = \begin{pmatrix} 0\cdot\mathbf{1}_{m_1} & & \\ & \ddots & \\ & & (l-1)\mathbf{1}_{m_l} \end{pmatrix} + \lambda\mathbf{1}_n \in \mathfrak{sl}(n,\mathbb{R})$$

(where λ is the unique real number, such that ξ is trace free). By the identification $\mathfrak{sl}(n,\mathbb{R}) \xrightarrow{\sim} \mathfrak{su}^\pi(n)$, $X \mapsto \frac{1+j}{2}X - \frac{1-j}{2}X^t$, this corresponds to

$$\xi = \begin{pmatrix} 0\cdot j\cdot\mathbf{1}_{m_1} & & \\ & \ddots & \\ & & (l-1)j\cdot\mathbf{1}_{m_l} \end{pmatrix} + \lambda j\,\mathbf{1}_n \in \mathfrak{u}^\pi(n).$$

Lemma.

$$T_{(E_1,\dots,E_l)}Z = \left\{(X_{ab}) \in \textstyle\bigoplus_{a\neq b}\mathrm{Hom}(E_a,E_b) : X_{ab} + X_{ba}^* = 0\right\},$$
$$T^{\boldsymbol{c}}_{(E_1,\dots,E_l)}Z = \bigoplus_{a\neq b}\mathrm{Hom}(E_a,E_b).$$

Proof. The second equation follows immediately from the first. So let π_a denote the orthogonal projection on E_a. From $\pi_a = \pi_a^*$, $\pi_a^2 = \pi_a$ and $\pi_a\pi_b = 0$, $a \neq b$ it follows that a tangent vector to Z at $E = (E_a)$ is given by a collection of maps $\pi_1',\dots\pi_l' \colon \boldsymbol{C}^n \to \boldsymbol{C}^n$ satisfying

$$\pi_a' = \pi_a'^*, \quad \pi_a'\pi_a + \pi_a\pi_a' = \pi_a', \quad \pi_a'\pi_b + \pi_a\pi_b' = 0 \quad \text{for } a \neq b.$$

Now it is easy to see that $X_{ab} := \pi_b\pi_a' = -\pi_b'\pi_a \in \mathrm{Hom}(E_a,E_b)$ for $a \neq b$ satisfies $X_{ab} + X_{ba}^* = 0$ and that any such collection (X_{ab}) defines maps π_a' with the above properties. □

By definition, ad_ξ has eigenvalue $(b-a)j$ on $\mathrm{Hom}(\boldsymbol{C}^{m_a},\boldsymbol{C}^{m_b})$, therefore the superhorizontal bundle is $\mathrm{Ad}_g(\mathfrak{g}_1^{\boldsymbol{c}}) = \bigoplus_a \mathrm{Hom}(E_a,E_{a+1}) \subset T^{\boldsymbol{c}}_{(E_a)}Z$. A holomorphic map $f : N \to Z$ from a para-Kähler manifold N to $Z = \mathrm{F}_{(m_1,\dots,m_l)}(\boldsymbol{C}^n)$ is superhor-

izontal, if and only if $\partial f = df^{(1,0)}$ takes values in $\bigoplus_a \mathrm{Hom}(E_a, E_{a+1})$, or, equivalently (as follows by induction), if ∂f maps $W_a = E_1 \oplus \cdots \oplus E_a$ to W_{a+1}. Thus we can construct para-pluriharmonic maps in the following way: Let $f_1, \ldots, f_{m_1} : N \to \boldsymbol{C}^n$ be holomorphic maps such that for $n \in N$ the images $f_a(n)$ are $\boldsymbol{C}$-linearly independent vectors with nondegenerate span. Let W_1 be the span of the f_a, W_2 the span of the f_a and their derivatives and possibly further $g_1, g_2, \ldots : N \to \boldsymbol{C}^n$ and so on, as long as all these maps are (pointwise) linearly independent and span nondegenerate subspaces. This defines a superhorizontal map $F : N \to \mathrm{F}_{(m_1,\ldots,m_l)}(\boldsymbol{C}^n)$, and the composition $f = \pi \circ F : N \to \mathrm{Gr}_{m_{\mathrm{odd}}}(\boldsymbol{C}^n)$ is para-pluriharmonic.

A special case of this is the fibration $\mathrm{F}_{(1,n,1)}(\boldsymbol{C}^{n+2}) \to \mathrm{Gr}_2(\boldsymbol{C}^{n+2})$, which is the twistor fibration associated to the para-quaternionic Kähler structure on $\mathrm{Gr}_2(\boldsymbol{C}^{n+2})$ (cf. Section 3.6.1). Para-pluriharmonic maps $f : N \to \mathrm{Gr}_2(\boldsymbol{C}^{n+2})$, $\dim_{\boldsymbol{C}} N = n$, are obtained by (para-)holomorphic functions $h : N \to \boldsymbol{C}^{n+2}$ via $f = W_1 \oplus W_2^{\perp}$, where $W_1(n) = \boldsymbol{C} \cdot h(n)$ and $W_2(n) = \mathrm{span}_{\boldsymbol{C}}(h(n), \mathrm{grad}(h(n)))$. In the next section we will prove that, given two para-quaternionic Kähler symmetric spaces of the same dimension, there is a biholomorphism between open and dense subsets (the big Bruhat cells, see above) of their twistor spaces preserving the horizontal structure. This can be used to transfer the preceding construction to any para-quaternionic Kähler symmetric space.

3.5 Para-quaternionic Kähler symmetric spaces

In this section, we examine the twistor fibrations $Z \to M$ associated to a para-quaternionic Kähler structure on M. We will see that these are the twistor spaces which belong to a depth-two grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\dim(\mathfrak{g}_{\pm 2}) = 1$. The main result is the birational equivalence between all these twistor spaces of the same dimension.

Proposition 3.5.1. *Let $M = G/K$ be a pseudo-Riemannian symmetric space such that G acts effectively on M.*

(1) *If M is quaternionic Kähler with nonzero scalar curvature (see Definition 1.4.4), then there exists a semi-canonical element $\xi \in \mathfrak{k}$ which defines a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\xi \in [\mathfrak{g}_{-2}, \mathfrak{g}_2]$, $\mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 = \mathfrak{k}$, $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1 = \mathfrak{m}$, $\dim(\mathfrak{g}_{-2}) = \dim(\mathfrak{g}_2) = 1$ and $[\,,\,] : \mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 2}$ is nondegenerate.*

(2) *Vice versa, if there exist a semi-canonical $\xi \in \mathfrak{k}$ which defines a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as above and if $K \subset H \cdot G_0^{\sigma}$ (where H is the centralizer of ξ in G), then M has a G-invariant para-quaternionic Kähler structure.*

(3) *If $K = H \cdot G_0^{\sigma}$, the twistor space $\mathrm{Ad}_G(\xi) = G/H$ is the twistor space of para-complex structures (Definition 2.2.1).*

Remark. Of course, there may be other twistor bundles over the same base space M. A simple example is the trivial fibration $\mathrm{Gr}_2(\boldsymbol{C}^{n+2}) \to \mathrm{Gr}_2(\boldsymbol{C}^{n+2})$.

Proof. (1) Suppose M is a para-quaternionic Kähler manifold. Then $\mathfrak{k} = \mathfrak{hol} \subset \mathfrak{sp}^\pi(1) \oplus \mathfrak{sp}^\pi(n)$. If $\mathrm{scal}(M) \neq 0$ then the curvature decomposition shows that $\mathfrak{k}$ contains the $\mathfrak{sp}^\pi(1)$-factor (see [AC1, Lemma 1]), so $\mathfrak{k} = \mathfrak{sp}^\pi(1) \oplus \mathfrak{k}'$. The isomorphism $T_oM \cong \mathfrak{m}$ identifies the quaternionic structure $Q_o \subset \mathrm{End}(T_oM)$ with $\mathrm{ad}_{\mathfrak{sp}^\pi(1)}|_{\mathfrak{m}}$. Now choose a para complex structure $J_o \in Q_o$. This corresponds to an element $\xi \in \mathfrak{sp}^\pi(1) \subset \mathfrak{k}$. By [AC1, Lemma 3 and 4], $\mathfrak{g}$ is simple and ξ corresponds to a long root μ with respect to a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}^{\mathbb{C}}$ in the sense that $\alpha(\xi) = 2\frac{\langle\alpha,\mu\rangle}{\langle\mu,\mu\rangle}$ for all $\alpha \in \mathfrak{a}^*$. In particular, $\alpha(\xi) \in \{0, \pm 1, \pm 2\}$, so ξ defines a depth two grading of $\mathfrak{g}$, and $\mathfrak{g}_{\pm 2}$ are the one-dimensional root spaces $\mathfrak{g}_{\pm\mu}$. To see that $[,]\colon \mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 2}$ is nondegenerate, we remark that for each root $\alpha \in \mathfrak{a}^*$ with $\alpha(\xi) = \pm 1$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-w_\mu(\alpha)}] = \mathfrak{g}_{\pm\mu} \neq 0$, where $w_\mu \in \mathcal{W}$ denotes the element of the Weyl group that is the reflection through the hyperplane $\{\mu = 0\}$. Of course, the nondegeneracy implies that $\mathfrak{g}_{\pm 1}$ generates $\bigoplus_{\lambda \gtrless 0} \mathfrak{g}_\lambda$, so ξ is semi-canonical.

(2) Let $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then $\mathfrak{g}_{-2} \oplus \mathbb{R}\xi \oplus \mathfrak{g}_2$ is a subalgebra isomorphic to $\mathfrak{sp}^\pi(1)$, and we have $\mathfrak{g}_0 = \mathbb{R}\xi \oplus Z_{\mathfrak{g}}(\mathfrak{sp}^\pi(1))$. It follows that $\mathfrak{k} = \mathfrak{sp}^\pi(1) \oplus \mathfrak{k}'$ with $\mathfrak{k}' = Z_{\mathfrak{g}}(\mathfrak{sp}^\pi(1))$. This implies that $Q_0 := \mathrm{ad}_{\mathfrak{sp}^\pi(1)}|_{\mathfrak{m}}$ is a para-quaternionic structure on $\mathfrak{m} \cong T_oM$, which is invariant under the adjoint action of $\mathfrak{k}$ and therefore under $K_0 = G_0^\sigma$. By definition, it is also invariant under H, hence under $K \subset H \cdot G_0^\sigma$. Therefore Q_0 extends to a G-invariant para-quaternionic structure on TM.

(3) The twistor space Z^Q of para-complex structures is the G-fibre bundle $G \times_K Z_o^Q$, where $Z_o^Q = \{q \in Q_o : q^2 = 1\}$ is the $\mathrm{Sp}^\pi(1)$-orbit of ξ in Q_o. Since K contains the $\mathrm{Sp}^\pi(1)$ factor, it acts transitively on Z_o^Q. It follows that G acts transitively on Z^Q, so Z^Q is the homogenous space G/H', where $H' = K \cap H$ is the centralizer of ξ in K. But since $H \subset K$ by assumption, we have $H' = H$ and therefore $Z^Q = \mathrm{Ad}_G(\xi)$. □

Birational equivalence of twistor spaces. Let G be a linear algebraic group. We have seen that the twistor space Z is an open subset of $\bar{Z} = G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$, which is a real variety, and that $\exp(\mathfrak{n}_+^{\boldsymbol{c}}) = N_+^{\boldsymbol{c}} \to G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$ is an embedding onto a Zariski-open and -dense subset, which induces a birational equivalence $N_+^{\boldsymbol{c}} \sim G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$.

If Z is the twistor space of a para-quaternionic Kähler symmetric space, then $N_+^{\boldsymbol{c}}$ is isomorphic to the (generalized) para-complex Heisenberg group:

Definition 3.5.2. The (real, complex, para-complex) *Heisenberg group* $\mathrm{H}(n, \mathrm{K})$ ($\mathrm{K} = \mathbb{R}, \mathbb{C}, \boldsymbol{C}$) is the group of $(n+2) \times (n+2)$-matrices of the form

$$\begin{pmatrix} 1 & & & & \\ z^1 & 1 & & & \\ \vdots & & \ddots & & \\ z^n & & & 1 & \\ w_0 & w_1 & \cdots & w_n & 1 \end{pmatrix} \in \mathrm{GL}(n+2, \mathrm{K}).$$

Its Lie algebra is the *Heisenberg algebra* $\mathfrak{h}(n, \mathrm{K})$ and consists of the matrices

$$\begin{pmatrix} 0 & & & & \\ u^1 & 0 & & & \\ \vdots & & \ddots & & \\ u^n & & & 0 & \\ v_0 & v_1 & \cdots & v_n & 0 \end{pmatrix} \in \mathfrak{gl}(n+2, \mathrm{K}).$$

Remark. Note that $\mathrm{H}(n, \boldsymbol{C}) \cong \mathrm{H}(n, \mathbb{R}) \times \mathrm{H}(n, \mathbb{R})$.

Lemma 3.5.3. *A Lie algebra is isomorphic to the Heisenberg algebra if and only if it is two-step nilpotent with one-dimensional kernel.*

(If $\mathrm{K} = \boldsymbol{C}$ this condition includes that the kernel *is* a para-complex subspace.)

Proof. Let $(u_1, \ldots, u_n, v^0, \ldots, v^n)$ be the obvious basis of $\mathfrak{h}(n, \mathrm{K})$ (i.e. the one which is dual to the above coefficients). Then $[v^a, u_b] = \delta^a_b v^0$ for $1 \leq a, b \leq n$, and all other brackets are zero, so obviously $\mathfrak{h}(n, \mathrm{K})$ is two-step nilpotent with kernel $\mathrm{K} \cdot v^0$.

Conversely, let $\mathfrak{g}$ be a two-step nilpotent Lie algebra and let v^0 be a nonzero (resp. regular, if $\mathrm{K} = \boldsymbol{C}$) element of its one-dimensional kernel. Let $\mathfrak{g}'$ a complementary subspace, i.e. $\mathfrak{g} = \mathrm{K} \cdot v^0 \oplus \mathfrak{g}'$. By assumption we have $[\mathfrak{g}', \mathfrak{g}'] = \mathrm{K} \cdot v^0$, so the Lie bracket defines a symplectic form ω on $\mathfrak{g}'$ via $[\cdot, \cdot] = \omega(\cdot, \cdot) v^0$. Let $(v^a, u_b)_{a,b=1\ldots n}$ be a symplectic basis of $(\mathfrak{g}', \omega)$ (see Proposition 2.1.2), then $[v^a, u_b] = \delta^a_b \cdot v^0$, which defines an isomorphism $\mathfrak{g} \cong \mathfrak{h}(n, \mathrm{K})$. □

Proposition 3.5.4. $\mathrm{H}(n, \mathrm{K})$ *has a left-invariant contact form given by*

$$dw_0 - \sum_{a=1}^{n} w_a dz^a.$$

Let $\bar{Z} = G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_-$ be the closed twistor space associated to a para-quaternionic Kähler structure. Let $N^{\boldsymbol{c}}_+$ be the unipotent radical of $P^{\boldsymbol{c}}_+$. Then $N^{\boldsymbol{c}}_+ \cong \mathrm{H}(n, \boldsymbol{C})$ (as real linear algebraic groups). This isomorphism is biholomorphic and identifies the contact distribution on $\mathrm{H}(n, \boldsymbol{C})$ with the horizontal distribution on the orbit of $N^{\boldsymbol{c}}_+$ in $\bar{Z}$. Moreover, it extends to a birational equivalence $N^{\boldsymbol{c}}_+ \sim \mathrm{H}(n, \boldsymbol{C})$.

Proof. By Proposition 3.5.1 we have $\mathfrak{n}^{\boldsymbol{c}}_+ = \mathfrak{g}^{\boldsymbol{c}}_1 \oplus \mathfrak{g}^{\boldsymbol{c}}_2$, and that the Lie bracket $[,]\colon \mathfrak{g}^{\boldsymbol{c}}_1 \times \mathfrak{g}^{\boldsymbol{c}}_1 \to \mathfrak{g}^{\boldsymbol{c}}_2$ is nondegenerate. So $\mathfrak{n}^{\boldsymbol{c}}_+$ is two-step nilpotent with one-dimensional kernel $\mathfrak{g}^{\boldsymbol{c}}_2$. This shows that $\mathfrak{n}^{\boldsymbol{c}}_+ \cong \mathfrak{h}(n, \boldsymbol{C})$, or equivalently $\mathfrak{n}_+ \times \mathfrak{n}_- \cong \mathfrak{h}(n, \mathbb{R}) \times \mathfrak{h}(n, \mathbb{R})$. Exponentiating this, we get an isomorphism of the groups $N_+ \times N_- \cong H(n, \mathbb{R}) \times H(n, \mathbb{R})$. This isomorphism is algebraic (i.e. has polynomial components) because both the exponential and logarithm series are finite. Obviously, it is a biholomorphism. To show that the contact distribution on $N^{\boldsymbol{c}}_+$ obtained through this isomorphism is mapped to the horizontal distribution by the map $\pi\colon N^{\boldsymbol{c}}_+ \to G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_-$,

it suffices to remark that both distributions are $N_+^{\boldsymbol{c}}$-invariant and coincide at the point $e \in N_+^{\boldsymbol{c}}$, where they are given by the linear subspace $\mathfrak{g}_1^{\boldsymbol{c}} \subset \mathfrak{n}_+^{\boldsymbol{c}}$. The statement about the birational equivalence has already been made in Theorem 3.3.12. □

Thus we have shown a refinement of Theorem 2.1.11: The Darboux coordinates can be constructed explicitly by writing down the isomorphism $N_+^{\boldsymbol{c}} \cong H(n, \boldsymbol{C})$. They are defined on the open dense subset $\pi(N_+^{\boldsymbol{c}}) \cap Z$ of Z.

Theorem 3.5.5. *Let Z be the twistor space of a para-quaternionic Kähler symmetric space. Then there is an open dense subset $U \subset Z$ and para-holomorphic coordinates $(z^1, \dots, z^n, w_0, \dots, w_n)\colon U \to \boldsymbol{C}^{2n+1}$ such that the horizontal distribution is given by $\ker(dw_0 - \sum_{a=1}^n w_a dz^a)$.*

Now let Z' be the twistor space of another para-quaternionic Kähler symmetric space of the same dimension, then we can put the birational maps together:

$$G'^{\boldsymbol{c}}/P_-'^{\boldsymbol{c}} \leftarrow N_+'^{\boldsymbol{c}} \cong H(n, \boldsymbol{C}) \cong N_+^{\boldsymbol{c}} \to G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}.$$

This proves:

Theorem 3.5.6. *Let $\bar{Z} = G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$ and $\bar{Z}' = G'^{\boldsymbol{c}}/P_-'^{\boldsymbol{c}}$ be two closed twistor spaces associated to para-quaternionic Kähler symmetric spaces of the same dimension. Then there is a biholomorphic map between Zariski-open and -dense subsets of $\bar{Z}$ and $\bar{Z}'$, which preserves the horizontal distribution. This morphism induces a birational equivalence $\bar{Z} \sim \bar{Z}'$.*

This allows us to construct para-pluriharmonic maps into *any* para-quaternionic Kähler symmetric space by transferring the construction from $\mathrm{Gr}_2(\boldsymbol{C}^{n+2})$ (see Section 3.4).

3.6 The classical para-quaternionic Kähler symmetric spaces

The simply connected para-quaternionic Kähler symmetric spaces have been classified by Alekseevsky and Cortés [AC1], see also [DJS]:

Theorem (Alekseevsky, Cortés). *Let $M = G/K$ be a simply connected para-quaternionic Kähler symmetric space. Then M is one of the following spaces:*

A) $\frac{\mathrm{SL}(n+2,\mathbb{R})}{\mathrm{S}(\mathrm{GL}^+(2,\mathbb{R})\times\mathrm{GL}^+(n,\mathbb{R}))}$, $\frac{\mathrm{SU}(p+1,q+1)}{\mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(p,q))}$,

BD) $\frac{\mathrm{SO}^+(p+2,q+2)}{\mathrm{SO}^+(2,2)\times\mathrm{SO}^+(p,q)}$, $\frac{\mathrm{SO}^*(2n+4)}{\mathrm{SO}^*(4)\times\mathrm{SO}^*(2n)}$,

C) $\frac{\mathrm{Sp}(2n+2,\mathbb{R})}{\mathrm{Sp}(2,\mathbb{R})\times\mathrm{Sp}(2n,\mathbb{R})}$,

E6) $\frac{\mathrm{E}_{6(6)}}{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(6,\mathbb{R})}$, $\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(3,3)\times\mathrm{SU}(1,1)}$, $\frac{\mathrm{E}_{6(-14)}}{\mathrm{SU}(5,1)\times\mathrm{SU}(1,1)}$,

E7) $\frac{\mathrm{E}_{7(7)}}{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}_0(6,6)}$, $\frac{\mathrm{E}_{7(-5)}}{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}^*(12)}$, $\frac{\mathrm{E}_{7(-25)}}{\mathrm{SL}(2,\mathbb{R})\times\mathrm{Spin}_0(10,2)}$,

E8) $\frac{E_{8(8)}}{SL(2,\mathbb{R})\times E_{7(7)}}$, $\frac{E_{8(-24)}}{SL(2,\mathbb{R})\times E_{7(-25)}}$,

F4) $\frac{F_{4(4)}}{SL(2,\mathbb{R})\times Sp(6,\mathbb{R})}$,

G2) $\frac{G_{2(2)}}{SO(2,2)}$.

We want to describe the spaces belonging to the classical groups more in detail. However, to apply the results of this section, it is useful to consider spaces of the form $G/(H\cdot G_0^\sigma)$ rather than their universal covers. We will find that these are the following Grassmannians:

($\mathrm{K}^{p,q}$ ($\mathrm{K}=\mathbb{R},\mathbb{C}$) denotes the vector space K^{p+q}, endowed with an indefinite pseudo-Euclidean or -hermitian scalar product of signature (p,q). A subspace of signature (p',q') is called (p',q')-subspace.)

(1) $\frac{SL(n+2,\mathbb{R})}{S(GL(2,\mathbb{R})\times GL(n,\mathbb{R}))} \cong \mathrm{Gr}_2(\boldsymbol{C}^{n+2})$, the set of nondegenerate 2-dimensional para-complex subspaces of $\boldsymbol{C}^{n+2}$.

(2) $\frac{SU(p+1,q+1)}{S(U(1,1)\times U(p,q))} \cong \mathrm{Gr}_{1,1}(\mathbb{C}^{p+1,q+1})$, the set of complex (1,1)-subspaces of $\mathbb{C}^{p+1,q+1}$.

(3) $\frac{SO(p+2,q+2)}{SO(2,2)\times SO(p,q)} \cong \mathrm{Gr}^{\mathrm{or}}_{2,2}(\mathbb{R}^{p+2,q+2})$, the set of oriented real (2,2)-subspaces of $\mathbb{R}^{p+2,q+2}$.

(4) $\frac{SO^*(2n+4)}{SO^*(4)\times SO^*(2n)} \cong \mathrm{Gr}^{\mathbb{H}}_{2,2}(\mathbb{C}^{n+2,n+2})$, the set of those complex (2,2)-subspaces of $\mathbb{C}^{n+2,n+2}$ which are invariant under a certain quaternionic structure.

(5) $\frac{Sp(2n+2,\mathbb{R})}{Sp(2,\mathbb{R})\times Sp(2n,\mathbb{R})} \cong \boldsymbol{H}\mathrm{P}^n$, the para-quaternionic projective space.

Following the procedure described in the preceding section, we will construct Darboux coordinates on their twistor spaces, which in turn will give us a description of the isotropic para-pluriharmonic maps. (In the cases (1) and (5) it is natural to use $\boldsymbol{C}$-valued coordinates, whereas we will use $(\mathbb{R}\times\mathbb{R})$-valued coordinates in the cases (2)–(4).)

3.6.1 $\mathbf{SL(n+2,\mathbb{R})/S(GL(2,\mathbb{R})\times GL(n,\mathbb{R}))}$. Let $G=\mathrm{SU}^\pi(n+2)\cong \mathrm{SL}(n+2,\mathbb{R})$. The Lie algebra of G is

$$\mathfrak{g}=\{X\in\mathfrak{gl}(n+2,\boldsymbol{C}) : X^*+X=0,\ \mathrm{tr}(X)=0\}.$$

In Section 3.4 we have seen that the canonical element

$$\xi := \begin{pmatrix} -j & & \\ & \mathbf{0}_n & \\ & & j \end{pmatrix}$$

leads to the twistor fibration $G/H\to G/K$ (with $K=G^\sigma=H\cdot G_0^\sigma$)

$$\frac{\mathrm{SU}^\pi(n+2)}{\mathrm{S}(\mathrm{U}^\pi(1)\times\mathrm{U}^\pi(n)\times\mathrm{U}^\pi(1))} \longrightarrow \frac{\mathrm{SU}^\pi(n+2)}{\mathrm{S}(\mathrm{U}^\pi(2)\times\mathrm{U}^\pi(n))}.$$

From Section 3.4 we also know that the fibration $G/H \to G/K$ can be identified with

$$\mathrm{F}_{1,n,1}(\boldsymbol{C}^{n+2}) \longrightarrow \mathrm{Gr}_2(\boldsymbol{C}^{n+2}),$$

where $\mathrm{Gr}_2(\boldsymbol{C}^{n+2})$ is a para-complex Grassmannian and $Z = \mathrm{F}_{1,n,1}(\boldsymbol{C}^{n+2})$ the space of nondegenerate $(1, n, 1)$-flags in $\boldsymbol{C}^{n+2}$.

We have also proven that Z is an open subset of $\overline{Z} = G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$. In the present case, $G^{\boldsymbol{c}} = \mathrm{SL}(n+2, \boldsymbol{C})$, and $P_-^{\boldsymbol{c}}$ is the group of upper block diagonal matrices of the form

$$\left(\begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \\ \hline 0 & 0 & \bullet \end{array}\right) \in \mathrm{SL}(1+n+1, \boldsymbol{C}).$$

Thus, $\overline{Z}$ is the space of *all* $(1, n, 1)$-flags in $\boldsymbol{C}^{n+2}$: Such a flag is a sequence $0 \subsetneq W_1 \subsetneq W_2 \subsetneq \boldsymbol{C}^{n+2}$, where W_1 and W_2 are para-complex subspaces (possibly with degenerate scalar product) of dimension 1 and $1+n$, respectively. The map $G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}} \xrightarrow{\sim} \overline{Z}$ is defined as follows: Given an element of $G^{\boldsymbol{c}}$, let W_1 be the $\boldsymbol{C}$-span of its first column and W_2 the $\boldsymbol{C}$-span of the first $1+n$ columns. This map is easily seen to be surjective and invariant under multiplication by elements of $P_-^{\boldsymbol{c}}$ from the right.

The unipotent radical $N_+^{\boldsymbol{c}}$ of $P_+^{\boldsymbol{c}}$ is the para-complex Heisenberg group $\mathrm{H}(n, \boldsymbol{C})$. We want to determine the orbit of $N_+^{\boldsymbol{c}}$ in $G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$: By definition it consists of those $g \cdot P_-^{\boldsymbol{c}}$, such that $g \in N_+^{\boldsymbol{c}} \cdot P_-^{\boldsymbol{c}}$, i.e.

$$\begin{pmatrix} g_{1,1} & \cdots & g_{1,n+2} \\ \vdots & \ddots & \vdots \\ g_{n+2,1} & \cdots & g_{n+2,n+2} \end{pmatrix} = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \bullet & \mathbf{1}_n & 0 \\ \hline \bullet & \bullet & 1 \end{array}\right) \cdot \left(\begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \\ \hline 0 & 0 & \bullet \end{array}\right).$$

A necessary condition is that the square blocks on the diagonal of the right factor are invertible. From this it follows after short calculation that

$$g_{1,1} \in \boldsymbol{C}^*, \tag{11}$$

$$\det \begin{pmatrix} g_{1,1} & \cdots & g_{1,n+1} \\ \vdots & \ddots & \vdots \\ g_{n+1,1} & \cdots & g_{n+1,n+1} \end{pmatrix} \in \boldsymbol{C}^*. \tag{12}$$

The Gaussian elimination algorithm shows that this condition is also sufficient:

Since $g_{1,1}$ is invertible, we can annihilate the rest of the first column of g by adding multiples of the first row to the other rows. This is achieved by left multiplication of a suitable matrix, so we obtain

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \bullet & \mathbf{1}_n & 0 \\ \hline \bullet & 0 & 1 \end{array}\right) \cdot g = \left(\begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \end{array}\right)$$

This does not change the determinant of the upper left $(n+1)\times(n+1)$ submatrix, which is invertible by equation (12). It follows that the middle $n\times n$-block on the right-hand side is invertible, so we can eliminate the middle block in the last row:

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \mathbf{1}_n & 0 \\ \hline 0 & \bullet & 1 \end{array}\right)\cdot\left(\begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \end{array}\right)=\left(\begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \hline 0 & \bullet & \bullet \\ \hline 0 & 0 & \bullet \end{array}\right).$$

This yields the desired decomposition of g.

From Corollary 1.1.7 it follows that the equations (11) and (12) define an $\mathbb{R}$-Zariski-open subset of $\overline{Z}$, as claimed in Theorem 3.3.12.

It remains to give a description of the isotropic para-pluriharmonic maps:

The coordinates on $N_+^{\boldsymbol{c}}$,

$$\left(\begin{array}{c|c|c} 1 & & \\ \hline \boldsymbol{z} & \mathbf{1}_n & \\ \hline w_0 & \boldsymbol{w}^t & 1 \end{array}\right),$$

$\boldsymbol{z}=(z^1,\dots,z^n)^t$, $\boldsymbol{w}=(w_1,\dots,w_n)^t$, yield holomorphic Darboux coordinates on $\pi(N_+^{\boldsymbol{c}})\subset G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}}$. With respect to these coordinates, a holomorphic map

$$\begin{aligned} F\colon \boldsymbol{C}^n\supset U_z &\longrightarrow G^{\boldsymbol{c}}/P_-^{\boldsymbol{c}},\\ (z^1,\dots,z^n) &\longmapsto (z^1,\dots,z^n,w_0,\dots,w_n),\end{aligned}$$

is horizontal if there is a holomorphic function $h\colon U_z\to\boldsymbol{C}$, such that

$$\begin{aligned} w_0&=h(z^1,\dots,z^n),\\ w_a&=\partial h/\partial z^a,\ a=1\dots n.\end{aligned}$$

This holomorphic horizontal map projects onto the isotropic para-pluriharmonic map

$$\begin{aligned} f=\pi\circ F\colon U_z &\longrightarrow \mathrm{Gr}_2(\boldsymbol{C}^{n+2}),\\ (z^1,\dots,z^n) &\longmapsto W_1\oplus W_2^{\perp},\end{aligned}$$

where

$$W_1=\boldsymbol{C}\cdot\begin{pmatrix}1\\ z^1\\ \vdots\\ z^n\\ h\end{pmatrix},\quad W_2=\mathrm{span}_{\boldsymbol{C}}\left\{\begin{pmatrix}1\\ z^1\\ \vdots\\ z^n\\ h\end{pmatrix},\begin{pmatrix}0\\ 1\\ \vdots\\ 0\\ \frac{\partial h}{\partial z^1}\end{pmatrix},\dots,\begin{pmatrix}0\\ 0\\ \vdots\\ 1\\ \frac{\partial h}{\partial z^n}\end{pmatrix}\right\}.$$

This is what we have seen in Section 3.4 (neglecting that we can change this map by a biholomorphism $\boldsymbol{C}^n\supset U_z\xrightarrow{\sim}\widetilde{U}_z\subset\boldsymbol{C}^n$).

3.6.2 $\mathbf{SU(p+1,q+1)\,/\,S\,(U(1,1)\times U(p,q))}$. Let $G=\mathrm{SU}(p+1,q+1)$ be the group of endomorphisms of $\mathbb{C}^{n+2}$, $n=p+q$, with unit determinant which preserve

the hermitian scalar product given by the matrix

$$E := \left(\begin{array}{c|c|c} & & 1 \\ \hline & \mathbf{1}_p & \\ & & -\mathbf{1}_q \\ \hline 1 & & \end{array}\right).$$

Its Lie algebra is $\mathfrak{g} = \mathfrak{su}(p+1, q+1) = \{X \in \mathfrak{sl}(n+2, \mathbb{C}) : X^* E + EX = 0\}$ and consists of traceless matrices of the form

$$\left(\begin{array}{c|cc|c} a & -C^* & D^* & c \\ \hline A & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathfrak{u}(p,q)$}} & C \\ B & \multicolumn{2}{c|}{} & D \\ \hline b & -A^* & B^* & -\bar{a} \end{array}\right) \in \mathfrak{sl}(1+n+1, \mathbb{C}), \quad b, c \in i\mathbb{R}.$$

The canonical element

$$\xi := \left(\begin{array}{c|c|c} -1 & & \\ \hline & \mathbf{0}_{p+q} & \\ \hline & & 1 \end{array}\right)$$

leads to the twistor fibration

$$\frac{\mathrm{SU}(p+1, q+1)}{\mathrm{S}\,(\mathrm{U}(p,q) \times \mathbb{C}^*)} \longrightarrow \frac{\mathrm{SU}(p+1, q+1)}{\mathrm{S}\,(\mathrm{U}(p,q) \times \mathrm{U}(1,1))}$$

(where $\mathbb{C}^* = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1/\bar{a} \end{smallmatrix}\right) : a \in \mathbb{C}^*\right\} \subset \mathrm{U}(1,1)$).

The coset space $\mathrm{SU}(p+1, q+1) \,/\, \mathrm{S}\,(\mathrm{U}(p,q) \times \mathrm{U}(1,1))$ can be identified with the Grassmannian $\mathrm{Gr}_{1,1}(\mathbb{C}^{p+1,q+1})$ of 2-dimensional complex subspaces $V \subset \mathbb{C}^{n+2}$ which have signature $(1,1)$ with respect to the above pseudo-hermitian metric; an element of G is sent to the subspace V that is spanned by the first and the last column. The twistor fibre over $V \subset \mathbb{C}^{n+2}$ is the set of compatible para-complex structures on V, i.e. the set of decompositions $V = V^+ \oplus V^-$ into complementary complex (totally) degenerate spaces.

The subgroups $P_\pm$ and $N_\pm$ are the groups of lower or upper block diagonal matrices, where the groups $N_\pm$ have unit matrices on the diagonal. The homogenous space $(G/P_-) \times (G/P_+)$ is the space $\bar{Z}$ of *all* pairs (V^+, V^-) of one-dimensional degenerate spaces $V^\pm \subset \mathbb{C}^{p+1,q+1}$. The map $G \times G \to \bar{Z}$ sends a pair (g_+, g_-) to (V^+, V^-), where V^+ is the span of the first column of g_+ and V^- the span of the last column of g_-.

As before, the algebra $\mathfrak{n}_+$ consists of the strictly lower block diagonal matrices. A Lie algebra isomorphism $\mathfrak{h}(p+q, \mathbb{R}) \xrightarrow{\sim} \mathfrak{n}_+$ is given by

$$\left(\begin{array}{ccccc} 0 & & & & \\ u^1 & 0 & & & \\ \vdots & & \ddots & & \\ u^{p+q} & & & 0 & \\ v_0 & v_1 & \cdots & v_{p+q} & 0 \end{array}\right) \longmapsto \left(\begin{array}{c|ccc|c} 0 & & & & \\ \hline \boldsymbol{u} + i\,\boldsymbol{v} & \multicolumn{2}{c}{\mathbf{0}_p} & & \\ \tilde{\boldsymbol{u}} - i\,\tilde{\boldsymbol{v}} & & & \mathbf{0}_q & \\ \hline 2i\,v_0 & -\boldsymbol{u}^t + i\,\boldsymbol{v}^t & & \tilde{\boldsymbol{u}}^t + i\,\tilde{\boldsymbol{v}}^t & 0 \end{array}\right),$$

where

$$\begin{aligned} \boldsymbol{u} &= (u^1, \dots, u^p)^t, \quad \tilde{\boldsymbol{u}} = (u^{p+1}, \dots u^{p+q})^t, \\ \boldsymbol{v} &= (v_1, \dots, v_p)^t, \quad \tilde{\boldsymbol{v}} = (v_{p+1}, \dots v_{p+q})^t. \end{aligned}$$

Exponentiating this, we obtain an isomorphism $\mathrm{H}(p+q, \mathbb{R}) \xrightarrow{\sim} N_+$, and Darboux coordinates on N_+:

$$\left(\begin{array}{c|cc|c} 1 & & & \\ \hline \boldsymbol{z}_+ + i\boldsymbol{w}_+ & \mathbf{1}_p & 0 & \\ \tilde{\boldsymbol{z}}_+ - i\tilde{\boldsymbol{w}}_+ & 0 & \mathbf{1}_q & \\ \hline \frac{1}{2}(\boldsymbol{z}_+^t \boldsymbol{z}_+ - \boldsymbol{w}_+^t \boldsymbol{w}_+ + \tilde{\boldsymbol{z}}_+^t \tilde{\boldsymbol{z}}_+ + \tilde{\boldsymbol{w}}_+^t \tilde{\boldsymbol{w}}_+) + i(2w_0^+ - \boldsymbol{z}_+^t \boldsymbol{w}_+ - \tilde{\boldsymbol{z}}_+^t \tilde{\boldsymbol{w}}_+) & -\boldsymbol{z}_+^t + i\boldsymbol{w}_+^t & \tilde{\boldsymbol{z}}_+^t - i\tilde{\boldsymbol{w}}_+^t & 1 \end{array}\right),$$

where $\boldsymbol{z}_+ = (z_+^1, \dots, z_+^p)^t, \tilde{\boldsymbol{z}}_+ = (z_+^{p+1}, \dots, z_+^{p+q})^t, \boldsymbol{w}_+ = (w_1^+, \dots, w_p^+)^t, \tilde{\boldsymbol{w}}_+ = (w_{p+1}^+, \dots, w_{p+q}^+)^t$.

In the same way for N_-:

$$\left(\begin{array}{c|cc|c} 1 & -\boldsymbol{z}_-^t + i\boldsymbol{w}_-^t & \tilde{\boldsymbol{z}}_-^t - i\tilde{\boldsymbol{w}}_-^t & \frac{1}{2}(-\boldsymbol{z}_-^t \boldsymbol{z}_- - \boldsymbol{w}_-^t \boldsymbol{w}_- + \tilde{\boldsymbol{z}}_-^t \tilde{\boldsymbol{z}}_- + \tilde{\boldsymbol{w}}_-^t \tilde{\boldsymbol{w}}_-) + i(2w_0^- - \boldsymbol{z}_-^t \boldsymbol{w}_- - \tilde{\boldsymbol{z}}_-^t \tilde{\boldsymbol{w}}_-) \\ \hline & \mathbf{1}_p & 0 & \boldsymbol{z}_- + i\boldsymbol{w}_- \\ & 0 & \mathbf{1}_q & \tilde{\boldsymbol{z}}_- - i\tilde{\boldsymbol{w}}_- \\ \hline & & & 1 \end{array}\right).$$

Therefore, the isotropic para-pluriharmonic maps are of the form

$$\begin{aligned} \boldsymbol{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n \supset U_z &\longrightarrow \mathrm{Gr}_{(1,1)}(\mathbb{C}^{p+1,q+1}), \\ (z_\pm^1, \dots, z_\pm^n) &\longmapsto V^+ \oplus V^-, \end{aligned}$$

where V_+ is the span of the first column of N_+ and V_- the span of the last column of N_- (note that these vectors are degenerate):

$$V_+ = \mathbb{C} \cdot \begin{pmatrix} 1 \\ \boldsymbol{z}_+ + i\boldsymbol{w}_+ \\ \tilde{\boldsymbol{z}}_+ - i\tilde{\boldsymbol{w}}_+ \\ \frac{1}{2}(-\boldsymbol{z}_+^t \boldsymbol{z}_+ - \boldsymbol{w}_+^t \boldsymbol{w}_+ + \tilde{\boldsymbol{z}}_+^t \tilde{\boldsymbol{z}}_+ + \tilde{\boldsymbol{w}}_+^t \tilde{\boldsymbol{w}}_+) + i(2w_0^+ - \boldsymbol{z}_+^t \boldsymbol{w}_+ - \tilde{\boldsymbol{z}}_+^t \tilde{\boldsymbol{w}}_+) \end{pmatrix},$$

$$V_- = \mathbb{C} \cdot \begin{pmatrix} \frac{1}{2}(-\boldsymbol{z}_-^t \boldsymbol{z}_- - \boldsymbol{w}_-^t \boldsymbol{w}_- + \tilde{\boldsymbol{z}}_-^t \tilde{\boldsymbol{z}}_- + \tilde{\boldsymbol{w}}_-^t \tilde{\boldsymbol{w}}_-) + i(2w_0^- - \boldsymbol{z}_-^t \boldsymbol{w}_- - \tilde{\boldsymbol{z}}_-^t \tilde{\boldsymbol{w}}_-) \\ \boldsymbol{z}_- + i\boldsymbol{w}_- \\ \tilde{\boldsymbol{z}}_- - i\tilde{\boldsymbol{w}}_- \\ 1 \end{pmatrix},$$

and where $w_0^\pm = h_\pm(z_\pm^1, \dots, z_\pm^n)$, $w_a^\pm = \partial h_\pm / \partial z_\pm^a$ for a para-holomorphic function $h = h_+ \times h_- \colon \mathbb{R}^n \times \mathbb{R}^n \supset U \to \mathbb{R} \times \mathbb{R}$.

3.6.3 $\mathbf{SO(p+2, q+2) / SO(p, q) \times SO(2, 2)}$. Let $G = \mathrm{SO}(p+2, q+2)$ be the group of endomorphism of $\mathbb{R}^{p+q+4}$ which preserve the pseudo-Riemannian metric given by the matrix

$$E = \begin{pmatrix} & & & \mathbf{1}_2 \\ & \mathbf{1}_p & & \\ & & -\mathbf{1}_q & \\ \mathbf{1}_2 & & & \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} \cong \mathfrak{so}(p+2, q+2)$ consists of the E-skew matrices; these are of the form

$$\left(\begin{array}{c|cc|c} a & -C^t & D^t & c \\ \hline A & \multicolumn{2}{c|}{} & C \\ B & \multicolumn{2}{c|}{\mathfrak{so}(p,q)} & D \\ \hline b & -A^t & B^t & -a^t \end{array}\right) \in \mathfrak{gl}(2+n+2), \quad b, c \in \mathfrak{o}(2).$$

The canonical element $\xi := \left(\begin{array}{c|c|c} -\mathbf{1}_2 & & \\ \hline & \mathbf{0}_{p+q} & \\ \hline & & \mathbf{1}_2 \end{array}\right)$ leads to the twistor fibration $G/H \to G/(H \cdot G_0^\sigma)$

$$\frac{\mathrm{SO}(p+2, q+2)}{\mathrm{SO}(p,q) \times \mathrm{GL}(2,\mathbb{R})} \longrightarrow \frac{\mathrm{SO}(p+2, q+2)}{\mathrm{SO}(p,q) \times \mathrm{SO}(2,2)}.$$

The space $M = \mathrm{SO}(p+2, q+2)/\mathrm{SO}(p,q) \times \mathrm{SO}(2,2)$ can be identified with the Grassmannian $\mathrm{Gr}^{\mathrm{or}}_{2,2}(\mathbb{R}^{p+2,q+2})$ of oriented 4-dimensional subspaces of $\mathbb{R}^{p+2,q+2}$ which have signature (2,2) with respect to the above pseudo-Riemannian metric. The map $G \to M$ is defined as follows: Given an element of G, its first two and last two columns form an oriented basis of this (2,2)-subspace.

Note that M has a two-fold quotient, namely the non-oriented Grassmannian $\mathrm{Gr}_{2,2}(\mathbb{R}^{p+2,q+2}) = \mathrm{SO}(p+2, q+2) \,/\, \mathrm{S}(\mathrm{O}(p,q) \times \mathrm{O}(2,2))$, which is *not* para-quaternionic Kähler (the para-quaternionic structure is not $\mathrm{S}(\mathrm{O}(p,q) \times \mathrm{O}(2,2))$-invariant). M also has a two-fold cover, the strongly oriented Grassmannian $\mathrm{Gr}^{\mathrm{stror}}_{2,2}(\mathbb{R}^{p+2,q+2}) = \mathrm{SO}(p+2, q+2) \,/\, \mathrm{SO}(p,q) \times \mathrm{SO}^+(2,2)$, which is para-quaternionic Kähler, but has no twistor fibration in the sense of Proposition 3.3.4 (in other words, the twistor space defined in Section 2 is no adjoint orbit).

The twistor space $Z = G/H$ is the set of pairs (V^+, V^-) of 2-dimensional totally degenerate subspaces $V^\pm$ of $\mathbb{R}^{p+2,q+2}$, such that the pseudo-Euclidean metric is nondegenerate on $V^+ \oplus V^-$ (which is therefore a subspace of signature (2,2)). The map $Z \to M$ maps the pair (V^+, V^-) to the sum $V^+ \oplus V^-$; note that a canonical orientation on $V^+ \oplus V^-$ is obtained through the isomorphism $V^+ \xrightarrow{\sim} (V^-)^*$.

The closed twistor space $\overline{Z} = (G/P_-) \times (G/P_+)$ is the space of all pairs (V^+, V^-) of 2-dimensional totally degenerate subspaces of $\mathbb{R}^{p+2,q+2}$. The map $G \times G \to \overline{Z}$ sends the pair (g_+, g_-) to (V^+, V^-), where V^+ is the span of the first two columns of g_+ and V^- the span of the last two columns of g_-. This map is $P_- \times P_+$-invariant and defines an isomorphism $(G/P_-) \times (G/P_+) \xrightarrow{\sim} \overline{Z}$.

An isomorphism $\mathfrak{h}(n, \mathbb{R}) \to \mathfrak{n}_+$ is given by

$$\left(\begin{array}{ccccc} 0 & & & & \\ u^1 & 0 & & & \\ \vdots & & \ddots & & \\ u^{p+q} & & & 0 & \\ v_0 & v_1 & \cdots & v_{p+q} & 0 \end{array}\right) \longmapsto \left(\begin{array}{cc|cc|c} 0 & & & & \\ & 0 & & & \\ \hline \boldsymbol{u} & -\boldsymbol{v} & \mathbf{0}_p & & \\ \tilde{\boldsymbol{u}} & \tilde{\boldsymbol{v}} & & \mathbf{0}_q & \\ \hline 0 & -v_0 & -\boldsymbol{u}^t & \tilde{\boldsymbol{u}}^t & 0 \\ v_0 & 0 & \boldsymbol{v}^t & \tilde{\boldsymbol{v}}^t & \quad 0 \end{array}\right),$$

where

$$\begin{aligned} \boldsymbol{u} &= (u^1, \dots, u^p)^t, \quad \tilde{\boldsymbol{u}} = (u^{p+1}, \dots u^{p+q})^t, \\ \boldsymbol{v} &= (v_1, \dots, v_p)^t, \quad \tilde{\boldsymbol{v}} = (v_{p+1}, \dots v_{p+q})^t. \end{aligned}$$

Thus we obtain Darboux coordinates on N_+:

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & & & & \\ 0 & 1 & & & & \\ \hline \boldsymbol{z}_+ & -\boldsymbol{w}_+ & \mathbf{1}_p & 0 & & \\ \tilde{\boldsymbol{z}}_+ & \tilde{\boldsymbol{w}}_+ & 0 & \mathbf{1}_q & & \\ \hline \frac{1}{2}(-\boldsymbol{z}_+^t \boldsymbol{z}_+ + \tilde{\boldsymbol{z}}_+^t \tilde{\boldsymbol{z}}_+) & -w_0^+ + \boldsymbol{z}_+^t \boldsymbol{w}_+ + \tilde{\boldsymbol{z}}_+^t \boldsymbol{w}_+ & -\boldsymbol{z}_+^t & \tilde{\boldsymbol{z}}_+^t & 1 & 0 \\ w_0^+ & \frac{1}{2}(-\boldsymbol{w}_+^t \boldsymbol{w}_+ + \tilde{\boldsymbol{w}}_+^t \tilde{\boldsymbol{w}}_+) & \boldsymbol{w}_+^t & \tilde{\boldsymbol{w}}_+^t & 0 & 1 \end{array}\right);$$

and similarly on N_-:

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & -\boldsymbol{z}_-^t & \tilde{\boldsymbol{z}}_-^t & \frac{1}{2}(-\boldsymbol{z}_-^t \boldsymbol{z}_- + \tilde{\boldsymbol{z}}_-^t \tilde{\boldsymbol{z}}_-) & -w_0^- + \boldsymbol{z}_-^t \boldsymbol{w}_- + \tilde{\boldsymbol{z}}_-^t \boldsymbol{w}_- \\ 0 & 1 & \boldsymbol{w}_-^t & \tilde{\boldsymbol{w}}_-^t & w_0^- & \frac{1}{2}(-\boldsymbol{w}_-^t \boldsymbol{w}_- + \tilde{\boldsymbol{w}}_-^t \tilde{\boldsymbol{w}}_-) \\ \hline & & \mathbf{1}_p & 0 & \boldsymbol{z}_- & -\boldsymbol{w}_- \\ & & 0 & \mathbf{1}_q & \tilde{\boldsymbol{z}}_- & \tilde{\boldsymbol{w}}_- \\ \hline & & & & 1 & 0 \\ & & & & 0 & 1 \end{array}\right).$$

Therefore, the isotropic para-pluriharmonic maps are given by

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n \supset U &\longrightarrow \mathrm{Gr}_{2,2}^{\mathrm{or}}(\mathbb{R}^{p+2,q+2}), \\ (z_\pm^1, \dots, z_\pm^n) &\longmapsto V_+(z_+^1, \dots z_+^n) \oplus V_-(z_-^1, \dots z_-^n), \end{aligned}$$

where

$$V_+ = \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ \boldsymbol{z}_+ \\ \tilde{\boldsymbol{z}}_+ \\ \frac{1}{2}(-\boldsymbol{z}_+^t \boldsymbol{z}_+ + \tilde{\boldsymbol{z}}_+^t \tilde{\boldsymbol{z}}_+) \\ w_0^+ \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\boldsymbol{w}_+ \\ \tilde{\boldsymbol{w}}_+ \\ -w_0^+ + \boldsymbol{z}_+^t \boldsymbol{w}_+ + \tilde{\boldsymbol{z}}_+^t \boldsymbol{w}_+ \\ \frac{1}{2}(-\boldsymbol{w}_+^t \boldsymbol{w}_+ + \tilde{\boldsymbol{w}}_+^t \tilde{\boldsymbol{w}}_+) \end{pmatrix} \right\},$$

$$V_- = \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} \frac{1}{2}(-\boldsymbol{z}_-^t \boldsymbol{z}_- + \tilde{\boldsymbol{z}}_-^t \tilde{\boldsymbol{z}}_-) \\ w_0^- \\ \boldsymbol{z}_- \\ \tilde{\boldsymbol{z}}_- \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -w_0^- + \boldsymbol{z}_-^t \boldsymbol{w}_- + \tilde{\boldsymbol{z}}_-^t \boldsymbol{w}_- \\ \frac{1}{2}(-\boldsymbol{w}_-^t \boldsymbol{w}_- + \tilde{\boldsymbol{w}}_-^t \tilde{\boldsymbol{w}}_-) \\ -\boldsymbol{w}_- \\ \tilde{\boldsymbol{w}}_- \\ 0 \\ 1 \end{pmatrix} \right\},$$

and where $w_0^\pm = h_\pm(z_\pm^1, \dots, z_\pm^n)$, $w_a^\pm = \partial h_\pm / \partial z_\pm^a$ for a para-holomorphic function $h = h_+ \times h_- \colon \mathbb{R}^n \times \mathbb{R}^n \supset U \to \mathbb{R} \times \mathbb{R}$.

3.6.4 $\mathbf{SO^*(2n+4)/SO^*(4)\times SO^*(2n)}$. Let $\mathbb{H} = \mathbb{R}\oplus i\mathbb{R}\oplus j\mathbb{R}\oplus k\mathbb{R}$, $i^2 = j^2 = -1$, $k = ij = -ji$, be the algebra of quaternions. We take quaternionic vector spaces to be right $\mathbb{H}$-modules. Accordingly, $\mathbb{H}^n$ is the space of column vectors with scalar multiplication on the right, and left multiplication by quaternionic matrices is $\mathbb{H}$-linear. Regarding $\mathbb{H}^n$ as the complex vector space $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n \cong \mathbb{C}^{2n}$, right multiplication by j becomes a $\mathbb{C}$-antilinear map $r_j : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. A $\mathbb{C}$-linear map $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is $\mathbb{H}$-linear, if and only if it commutes with r_j.

The group $\mathrm{SO}^*(2n)$ – sometimes also denoted $\mathrm{U}^*(n,\mathbb{H})$ – is defined as the group of endomorphisms of $\mathbb{H}^n$ preserving the skew-hermitian form $\omega(h,k) := h^*jk$. Writing $\omega =: \alpha + j\beta$, we obtain maps $\alpha, \beta : \mathbb{C}^{2n}\times\mathbb{C}^{2n} \to \mathbb{C}$, where β is $\mathbb{C}$-bilinear symmetric and α is skew-hermitian (whence $i\alpha$ is hermitian). These two are related by $\alpha(r_j\cdot,\cdot) = \beta$. It follows that an endomorphism of $\mathbb{C}^{2n}$ is an element of $\mathrm{SO}^*(2n)$, if and only if it commutes with r_j and preserves the hermitian form $i\alpha$. Moreover, we have $l_j^*\alpha = \overline{r_j^*\alpha} = -\alpha$. It follows that $i\alpha =: h$ has signature (n,n). Thus we have

$$\mathrm{SO}^*(2n) = \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{U}(n,n).$$

Let $G \cong \mathrm{SO}^*(2n+4)$ be the group of endomorphisms of $\mathbb{C}^{2n}$ which commute with

$$r_j = \bar{\cdot} \circ \left(\begin{array}{cc|cc|cc} & -1 & & & & \\ 1 & & & & & \\ \hline & & & -\mathbf{1}_n & & \\ & & \mathbf{1}_n & & & \\ \hline & & & & & -1 \\ & & & & 1 & \end{array}\right)$$

and preserve the indefinite hermitian scalar product h given by the matrix

$$E := \left(\begin{array}{cc|cc|cc} & & & & 1 & \\ & & & & & -1 \\ \hline & & \mathbf{1}_n & & & \\ & & & -\mathbf{1}_n & & \\ \hline 1 & & & & & \\ & -1 & & & & \end{array}\right).$$

The Lie algebra $\mathfrak{g} \cong \mathfrak{so}^*(2n+4)$ consists of matrices of the form

$$\left(\begin{array}{cc|cc|cc} a & -\bar{b} & -C^* & D^* & d & 0 \\ b & \bar{a} & -D^t & -C^t & 0 & \bar{d} \\ \hline A & -\bar{B} & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathfrak{so}^*(2n)$}} & C & -\bar{D} \\ B & \bar{A} & & & D & \bar{C} \\ \hline c & 0 & -A^* & B^* & -\bar{a} & \bar{b} \\ 0 & \bar{c} & -B^t & -A^t & -b & -a \end{array}\right) \in \mathfrak{gl}(2n+4,\mathbb{C}), \quad c,d \in i\mathbb{R}.$$

The canonical element

$$\xi := \left(\begin{array}{c|c|c} -\mathbf{1}_2 & & \\ \hline & \mathbf{0}_{2n} & \\ \hline & & \mathbf{1}_2 \end{array}\right)$$

leads to the twistor fibration

$$\frac{\mathrm{SO}^*(2n+4)}{\mathrm{GL}(1,\mathbb{H})\times\mathrm{SO}^*(2n)} \longrightarrow \frac{\mathrm{SO}^*(2n+4)}{\mathrm{SO}^*(4)\times\mathrm{SO}^*(2n)}.$$

The coset space $\mathrm{SO}^*(2n+4)/\mathrm{SO}^*(4)\times\mathrm{SO}^*(2n)$ can be identified with the set of complex subspaces which are r_j- invariant and have signature (2,2) with respect to the above pseudo-hermitian product. (In order to see that the $\mathrm{SO}^*(2n+4)$-action on this set is transitive, we have to verify that the orthogonal complement of such a subspace is again r_j-invariant: This follows from $r_j^* h = \bar{h}$.) The projection $G \to G/K$ sends an element of G to the span of its first two and last two columns.

The isomorphism $\mathfrak{h}(2n,\mathbb{R}) \xrightarrow{\sim} \mathfrak{n}_+$ is given by the coordinates

$$\left(\begin{array}{cc|cc|cc} 0 & & & & & \\ & 0 & & & & \\ \hline \boldsymbol{u}+i\boldsymbol{v} & -\tilde{\boldsymbol{u}}-i\tilde{\boldsymbol{v}} & \mathbf{0}_n & & & \\ \tilde{\boldsymbol{u}}-i\tilde{\boldsymbol{v}} & \boldsymbol{u}-i\boldsymbol{v} & & \mathbf{0}_n & & \\ \hline 2iv_0 & 0 & -\boldsymbol{u}^t+i\boldsymbol{v}^t & \tilde{\boldsymbol{u}}^t+i\tilde{\boldsymbol{v}}^t & 0 & \\ 0 & -2iv_0 & -\tilde{\boldsymbol{u}}^t+i\tilde{\boldsymbol{v}}^t & -\boldsymbol{u}^t-i\boldsymbol{v}^t & & 0 \end{array}\right)$$

Thus, the isotropic para-pluriharmonic maps

$$\begin{aligned} \mathbb{R}\times\mathbb{R} \supset U &\longrightarrow \mathrm{Gr}^{\mathbb{H}}_{2,2}(\mathbb{C}^{n+2,n+2}), \\ (z^1_\pm,\ldots,z^{2n}_\pm) &\longmapsto V_+(z^1_+,\ldots,z^{2n}_+)\oplus V_-(z^1_-,\ldots,z^{2n}_-), \end{aligned}$$

are given by

$$V_+ = \mathbb{C}\cdot\begin{pmatrix} 1 \\ 0 \\ \boldsymbol{z}_+ + i\boldsymbol{w}_+ \\ \tilde{\boldsymbol{z}}_+ - i\tilde{\boldsymbol{w}}_+ \\ \frac{1}{2}(-\boldsymbol{z}_+^t\boldsymbol{z}_+ - \boldsymbol{w}_+^t\boldsymbol{w}_+ + \tilde{\boldsymbol{z}}_+^t\tilde{\boldsymbol{z}}_+ + \tilde{\boldsymbol{w}}_+^t\tilde{\boldsymbol{w}}_+) + i(2w_0^+ - \boldsymbol{z}_+^t\boldsymbol{w}_+ - \tilde{\boldsymbol{z}}_+^t\tilde{\boldsymbol{w}}_+) \\ 2(-\boldsymbol{z}^t\tilde{\boldsymbol{z}} - \boldsymbol{w}^t\tilde{\boldsymbol{w}} + i(\boldsymbol{z}^t\tilde{\boldsymbol{w}} - \boldsymbol{w}^t\tilde{\boldsymbol{z}})) \end{pmatrix}$$

$$\oplus\,\mathbb{C}\cdot\begin{pmatrix} 0 \\ 1 \\ -\tilde{\boldsymbol{z}}_+ - i\tilde{\boldsymbol{w}}_+ \\ \boldsymbol{z}_+ - i\boldsymbol{w}_+ \\ 2(\boldsymbol{z}^t\tilde{\boldsymbol{z}} + \boldsymbol{w}^t\tilde{\boldsymbol{w}} + i(\boldsymbol{z}^t\tilde{\boldsymbol{w}} - \boldsymbol{w}^t\tilde{\boldsymbol{z}})) \\ \frac{1}{2}(-\boldsymbol{z}_+^t\boldsymbol{z}_+ - \boldsymbol{w}_+^t\boldsymbol{w}_+ + \tilde{\boldsymbol{z}}_+^t\tilde{\boldsymbol{z}}_+ + \tilde{\boldsymbol{w}}_+^t\tilde{\boldsymbol{w}}_+) - i(2w_0^+ - \boldsymbol{z}_+^t\boldsymbol{w}_+ - \tilde{\boldsymbol{z}}_+^t\tilde{\boldsymbol{w}}_+) \end{pmatrix},$$

$$V_- = \mathbb{C}\cdot\begin{pmatrix} \frac{1}{2}(-z_-^t z_- - w_-^t w_- + \tilde z_-^t \tilde z_- + \tilde w_-^t \tilde w_-) + i(2w_0^- - z_-^t w_- - \tilde z_-^t \tilde w_-) \\ 2(-z_-^t \tilde z_- - w_-^t \tilde w_- + i(z_-^t \tilde w_- - w_-^t \tilde z_-)) \\ z_- + i w_- \\ \tilde z_- - i \tilde w_- \\ 1 \\ 0 \end{pmatrix}$$

$$\oplus\,\mathbb{C}\cdot\begin{pmatrix} 2(z_-^t \tilde z_- + w_-^t \tilde w_- + i(z_-^t \tilde w_- - w_-^t \tilde z_-)) \\ \frac{1}{2}(-z_-^t z_- - w_-^t w_- + \tilde z_-^t \tilde z_- + \tilde w_-^t \tilde w_-) - i(2w_0^- - z_-^t w_- - \tilde z_-^t \tilde w_-) \\ -\tilde z_- - i \tilde w_- \\ z_- - i w_- \\ 0 \\ 1 \end{pmatrix},$$

where $z_\pm = (z_\pm^1, \dots, z_\pm^n)$, $\tilde z_\pm = (z_\pm^{n+1}, \dots, z_\pm^{2n})$, $w_\pm = (w_1^\pm, \dots w_n^\pm)$, $\tilde w_\pm = (w_{n+1}^\pm, \dots w_{2n}^\pm)$ with $w_0^\pm = h_\pm(z_\pm^1, \dots, z_\pm^n)$, $w_a^\pm = \partial h_\pm / \partial z_\pm^a$ for a para-holomorphic function $h = h_+ \times h_-\colon \mathbb{R}^n \times \mathbb{R}^n \supset U \to \mathbb{R} \times \mathbb{R}$.

3.6.5 $\mathbf{Sp(2n+2,\mathbb{R})\,/\,Sp(2,\mathbb{R})\times Sp(2n,\mathbb{R})}$. Recall that $\mathrm{Sp}(2n,\mathbb{R}) \cong \mathrm{Sp}^\pi(n)$, therefore $\mathrm{Sp}(2n+2,\mathbb{R})\,/\,\mathrm{Sp}(2,\mathbb{R})\times\mathrm{Sp}(2n,\mathbb{R})$ can be identified with the para-quaternionic projective space $\boldsymbol{H}\mathrm{P}^n = \{(h_0,\dots,h_n) \in \boldsymbol{H}^{n+1} : \sum_a \bar h_a h_a \neq 0\}/\boldsymbol{H}^*$, cf. Example 2.2.1.

It is useful to regard $\boldsymbol{H}^n$ as a vector space over $\boldsymbol{C}$ and write $\mathrm{Sp}^\pi(n)$ as a subgroup of $\mathrm{GL}(2n,\boldsymbol{C})$: In fact, $\boldsymbol{H}^n = \boldsymbol{C}^n \oplus i\boldsymbol{C}^n$ can be identified with the free $\boldsymbol{C}$ module $\boldsymbol{C}^{2n}$. A $\boldsymbol{C}$-linear map $\boldsymbol{C}^{2n} \to \boldsymbol{C}^{2n}$ is $\boldsymbol{H}$-linear, if and only if it commutes with the right multiplication by i, which is a $\boldsymbol{C}$-antilinear involution. It is easy to see that

$$\mathrm{GL}(n,\boldsymbol{H}) = \left\{\begin{pmatrix} A & -\bar B \\ B & \bar A \end{pmatrix} \in \mathrm{GL}(2n,\boldsymbol{C})\right\}.$$

Let h be a para-quaternionic hermitian metric on $\boldsymbol{H}^n$. The identification $\boldsymbol{H} = \boldsymbol{C} \oplus i\boldsymbol{C}$ yields a decomposition of $h = g \oplus i\omega$ into a para-hermitian scalar product g and a $\boldsymbol{C}$-bilinear symplectic form ω on $\boldsymbol{C}^{2n}$, which are related by $\omega = g(\cdot i, \cdot)$. Therefore an endomorphism of $\boldsymbol{C}^{2n}$ preserves h, if and only if it preserves g and ω, or equivalently, if it preserves ω and is $\boldsymbol{H}$-linear. If h is standard, so is ω. This shows that

$$\mathrm{Sp}^\pi(n) = \mathrm{Sp}(2n,\boldsymbol{C}) \cap \mathrm{GL}(n,\boldsymbol{H}).$$

Let $G = \mathrm{Sp}^\pi(n+1)$. The Lie algebra of G is $\mathfrak{g} = \mathfrak{sp}^\pi(n+1) = \mathfrak{sp}(2n+2,\boldsymbol{C}) \cap \mathfrak{gl}(n+1,\boldsymbol{H})$ and consists of matrices

$$\left(\begin{array}{c|cc|c} a & -A^* & -B^* & -\bar b \\ \hline A & \multicolumn{2}{c|}{\multirow{2}{*}{$\mathfrak{sp}^\pi(n)$}} & -\bar B \\ B & & & \bar A \\ \hline b & B^t & -A^t & \bar a \end{array}\right) \in \mathfrak{gl}(2n+2,\boldsymbol{C}), \quad a \in j\mathbb{R}.$$

The canonical element

$$\xi = \left(\begin{array}{c|c|c} j & & \\ \hline & \mathbf{0}_{2n} & \\ \hline & & -j \end{array}\right)$$

leads to the twistor fibration

$$\frac{\mathrm{Sp}^{\pi}(n+1)}{\mathrm{Sp}^{\pi}(n)\times \mathrm{U}^{\pi}(1)} \longrightarrow \frac{\mathrm{Sp}^{\pi}(n+1)}{\mathrm{Sp}^{\pi}(n)\times \mathrm{Sp}^{\pi}(1)}$$

(cf. Table 1).

This is, of course, the fibration $\boldsymbol{C}\mathrm{P}^{2n+1} \to \boldsymbol{H}\mathrm{P}^n$ of Example 2.2.1. The projection map $\mathrm{Sp}^{\pi}(n+1) \to \boldsymbol{C}\mathrm{P}^{2n+1}$ sends each matrix to the $\boldsymbol{C}$-span of its first column. We have $G^{\boldsymbol{c}} = \mathrm{Sp}(2n+2, \boldsymbol{C})$. The closed twistor space $\bar{Z} = G^{\boldsymbol{c}}/P^{\boldsymbol{c}}_{-}$ is the space of *all* (possibly degenerate) one-dimensional para-complex subspaces in $\boldsymbol{C}\mathrm{P}^{2n+1}$ (by the proof Proposition 2.1.2, any vector that spans such a subspace can be extended to a symplectic basis). The projection $G^{\boldsymbol{c}} \to \bar{Z}$ maps a matrix to the span of its first column.

The isomorphism of the Lie algebras

$$\mathfrak{h}(n, \boldsymbol{C}) \xrightarrow{\sim} \mathfrak{n}^{\boldsymbol{c}}_{+},$$

$$\begin{pmatrix} 0 & & & & \\ u^1 & 0 & & & \\ \vdots & & \ddots & & \\ u^n & & & 0 & \\ v_0 & v_1 & \cdots & v_n & 0 \end{pmatrix} \longmapsto \left(\begin{array}{c|ccc|c} 0 & & & & \\ \hline \boldsymbol{u} & \mathbf{0}_n & & & \\ \boldsymbol{v} & & \mathbf{0}_n & & \\ \hline 2v_0 & \boldsymbol{v}^t & -\boldsymbol{u}^t & & 0 \end{array}\right),$$

(where $\boldsymbol{u} = (u^1, \ldots, u^n)^t$, $\boldsymbol{v} = (v_1, \ldots, v_n)^t$) yields the isomorphism

$$\mathrm{H}(n, \boldsymbol{C}) \xrightarrow{\sim} N^{\boldsymbol{c}}_{+},$$

$$\begin{pmatrix} 1 & & & & \\ z^1 & 1 & & & \\ \vdots & & \ddots & & \\ z^n & & & 1 & \\ w_0 & w_1 & \cdots & w_n & 1 \end{pmatrix} \longmapsto \left(\begin{array}{c|cc|c} 1 & & & \\ \hline \boldsymbol{z} & \mathbf{1}_n & 0 & \\ \boldsymbol{w} & 0 & \mathbf{1}_n & \\ \hline 2w_0 - \boldsymbol{z}^t\boldsymbol{w} & \boldsymbol{w}^t & -\boldsymbol{z}^t & 1 \end{array}\right).$$

The holomorphic horizontal maps $\boldsymbol{C}^n \supset U \to N^{\boldsymbol{c}}_{+}$ are given by the equations $w_0 = h(z^1, \ldots, z^n)$, $w_a = \partial h/\partial z^a$ for a holomorphic function $h : \boldsymbol{C}^n \supset U \to \boldsymbol{C}$.

The projection $N_{+} \subset \boldsymbol{C}\mathrm{P}^{2n+1} \to \boldsymbol{H}\mathrm{P}^n$ maps this matrix to the $\boldsymbol{H}$-span of its first column; in conclusion, the isotropic para-pluriharmonic maps

$$f : \boldsymbol{C}^n \supset U \to \boldsymbol{H}\mathrm{P}^n$$

are given by

$$(z^1,\dots,z^n) \longmapsto \boldsymbol{H}\cdot\begin{pmatrix} 1+i(2h-\sum z^a\partial h/\partial z^a)\\ z^1+i\,\partial h/\partial z^1\\ \vdots\\ z^n+i\,\partial h/\partial z^n\end{pmatrix}.$$

4 tt*-bundles and para-pluriharmonic maps

Let (N,J) be a simply connected para-complex manifold. As shown in [S1] (see also [CS] for the complex case), any para-pluriharmonic map

$$\tilde f\colon N\to \mathrm{GL}(p+q,\mathbb{R})/\mathrm{O}(p,q)=:\mathrm{S}_{p,q}$$

which fulfills $R(d\tilde f(T^{\pm}N), d\tilde f(T^{\pm}N))=0$ defines a metric para-tt*-bundle over N and vice versa. In this paragraph we relate this to the construction of the associated family (cf. Section 3.1.4) and give a criterion for the map $\tilde f$ to be isotropic.

Definition 4.1.1. Let (N,J) be a simply connected para-complex manifold. A *metric para-tt*-bundle* (E,D,S,g) over N is a real vector bundle $E\to N$ endowed with a connection D, a (possibly indefinite) D-parallel fibre metric g and a section S of $T^*N\times\mathrm{End}(E)$ which is g-symmetric (i.e. $g(S_X\cdot,\cdot)=g(\cdot,S_X\cdot)$), such that the *tt*-equation*

$$R^\theta=0\quad\text{for all }\theta\in\mathbb{R}$$

holds, where R^θ is the curvature tensor of the connection $\nabla^\theta_X:=D_X+S_{e^{\theta J}X}$.

We denote by $\mathrm{Sym}_{p,q}(\mathbb{R})$ the space of symmetric $(p+q)\times(p+q)$ matrices with signature (p,q). This space is identified with $\mathrm{S}_{p,q}$ via the map $\mathrm{S}_{p,q}\ni g\cdot\mathrm{O}(p,q)\mapsto (g^{-1})^t\mathbf{1}_{p,q}g^{-1}\in\mathrm{Sym}_{p,q}(\mathbb{R})$, where $\mathbf{1}_{p,q}=\left(\begin{smallmatrix}\mathbf{1}_p & 0\\ 0 & -\mathbf{1}_q\end{smallmatrix}\right)$ is the standard diagonal matrix of signature (p,q). The geodesic reflection at $e\mathrm{O}(p,q)\in\mathrm{S}_{p,q}$ yields the symmetric decomposition $\mathfrak{gl}(p+q,\mathbb{R})=\mathfrak{o}(p,q)\oplus\mathfrak{m}$, where $\mathfrak{m}=\{X\in\mathfrak{gl}(p+q,\mathbb{R}): X^t\mathbf{1}_{p,q}=\mathbf{1}_{p,q}X\}$. The tangent space $T_{gK}\mathrm{S}_{p,q}$ is canonically identified with $\mathrm{Ad}_g\mathfrak{m}=\{X\in\mathfrak{gl}(p+q,\mathbb{R}): X^t(g^{-t}\mathbf{1}_{p,q}g^{-1})=(g^{-t}\mathbf{1}_{p,q}g^{-1})X\}$ via the map $\mathrm{Ad}_g\mathfrak{m}\xrightarrow{(\cdot g)_*}(\mathrm{Ad}_g\mathfrak{m})g=g\mathfrak{m}\xrightarrow{\pi_*}T_{gK}\mathrm{S}_{p,q}$ (which is easily seen to be independent of the choice of $g\in gK$).

Theorem (Theorem 3 and 4 in [S1]). *Let (N,J) be a simply connected para-complex manifold and let (E,D,S,g) be a metric para-tt*-bundle of rank $p+q$ (and signature*

(p,q)) *over* N. *Choose a family* $s_\theta(x) : \mathbb{R}^{p+q} \to E$ *of* ∇^θ*-flat frames and set*

$$\begin{aligned} f_\theta &: N \longrightarrow \mathrm{Sym}_{p,q}(\mathbb{R}), \\ &\quad x \longmapsto s_\theta(x)^* g_x = (g\,(s_\theta(x)_a, s_\theta(x)_b))_{a,b}\,, \\ \tilde{f}_\theta &: N \xrightarrow{f_\theta} \mathrm{Sym}_{p,q}(\mathbb{R}) \xrightarrow{\sim} \mathrm{S}_{p,q}, \\ \tilde{f}_\theta &: N \xrightarrow[\tilde{f}_\theta]{f_\theta} \mathrm{Sym}_{p,q}(\mathbb{R}) \xrightarrow{\sim} \mathrm{S}_{p,q}. \end{aligned}$$

Then $\tilde{f}_\theta$ *is a* 1*-parameter family of para-pluriharmonic maps and*

$$d\tilde{f}_\theta(X) = s_\theta^{-1} \circ S_{e^{\theta J} X} \circ s_\theta. \tag{13}$$

Moreover, for each $x \in N$, $d\tilde{f}_\theta$ *maps* $T_x^+ N$ *and* $T_x^- N$ *into abelian subalgebras of* $T_{\tilde{f}_\theta(x)} \mathrm{S}_{p,q} \subset \mathfrak{gl}(p+q,\mathbb{R})$.

Conversely, any para-pluriharmonic map from a simply-connected para-complex manifold N *into* $\mathrm{S}_{p,q}$ *such that the images of* $T^\pm N$ *are abelian can be obtained in this way.*

Remark. The condition that the images of $d\tilde{f}(T^\pm N)$ are abelian is equivalent to $R(d\tilde{f}(T^\pm N), d\tilde{f}(T^\pm N)) = 0$: Since R is preserved by parallel translation, it suffices to check this assertion at $T_o \mathrm{S}_{p,q} \approx \mathfrak{m}$, where R is given by the Lie triple bracket, so we have to show that for $X, Y \in \mathfrak{m}$,

$$[[X,Y],Z] = 0 \ \text{ for all } Z \in \mathfrak{m} \iff [X,Y] = 0$$

The diagonal matrices form a maximal abelian subalgebra of $\mathfrak{gl}(p+q,\mathbb{R})$, which is contained in $\mathfrak{m}$, so there is no nontrivial element of $\mathfrak{k} = \mathfrak{o}(p,q)$ that commutes with all $Z \in \mathfrak{m}$. Since $[X,Y] \in [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$, the assertion follows. □

This 1-parameter family $\tilde{f}_\theta$ of para-pluriharmonic maps is in fact an associated family in the sense of Theorem 3.1.5:

Proposition 4.1.2. *The map*

$$\begin{aligned} \Phi_\theta &: \tilde{f}_\theta^* T\mathrm{S}_{p,q} \longrightarrow \tilde{f}_0^* T\mathrm{S}_{p,q}, \\ &\quad X \longmapsto (s_\theta^{-1} s_0)^{-1} X (s_\theta^{-1} s_0), \end{aligned}$$

is a parallel bundle homomorphism preserving the curvature of $\mathrm{S}_{p,q}$*, and*

$$\Phi_\theta \circ d\tilde{f}_\theta = d\tilde{f}_0 \circ e^{\theta J}.$$

Proof. The equation $\Phi_\theta \circ d\tilde{f}_\theta = d\tilde{f}_0 \circ e^{\theta J}$ is immediate from (13). $\Phi_\theta(x)$ is the differential of the isometry of the space $\mathrm{S}_{p,q}$, given by left multiplication by $s_\theta^{-1}(x) s_0(x) \in \mathrm{GL}(p+q,\mathbb{R})$, hence it preserves the curvature tensor. In order to verify the parallelity of Φ_θ, let $c : [0,1] \to N$ be a curve and let $\sigma : [0,1] \to \tilde{f}_\theta^* T\mathrm{S}_{p,q}$

be a parallel section of $\tilde{f}_\theta^* T\mathrm{S}_{p,q}$ along c. Any such section is of the form $t \mapsto \gamma(t)\cdot X$, where $X \in \mathfrak{m}$ and γ is the horizontal lift of $\tilde{f}_\theta \circ c$, i.e. $\pi \circ \gamma = \tilde{f}_\theta \circ c$ and $\gamma^{-1}\dot{\gamma} \in \mathfrak{m}$. We have to check that $(\Phi_\theta \circ c)\cdot\sigma$ is parallel, i.e. that $\tilde{\gamma} := ((s_0 \circ c)^{-1}(s_\theta \circ c))\gamma$ is horizontal: By the definition of s_θ we have $D_X s_\theta = -S_{e^{\theta J}X} s_\theta$ and $D_X s_0 = -S_X s_0$, (here D denotes the connection on $E \otimes \mathbb{R}^n$ induced by the given connection D on E and the trivial connection on $\mathbb{R}^n$) so $\frac{d}{dt}(s_0^{-1}s_\theta) = D_{\dot{c}}(s_0^{-1})s_\theta + s_0 D_{\dot{c}} s_\theta = s_0^{-1} S_{\dot{c}-e^{\theta J}\dot{c}} s_\theta$ and $\tilde{\gamma}^{-1}\dot{\tilde{\gamma}} = \gamma^{-1}\dot{\gamma} + \gamma^{-1}\underbrace{s_\theta^{-1} S_{\dot{c}-e^{\theta J}\dot{c}} s_\theta}_{\in \mathrm{Ad}_\gamma \mathfrak{m}}\gamma \in \mathfrak{m}$. □

Proposition 4.1.3. *$\tilde{f}$ is isotropic if and only if there exists a family α_θ of D-parallel, g-orthogonal sections of* $\mathrm{End}(E)$*, such that*

$$S_{e^{\theta J}} = \alpha_\theta \circ S \circ \alpha_\theta^{-1}.$$

Proof. Suppose that $\tilde{f}$ is isotropic, i.e. $\tilde{f}_\theta = \tilde{f}_0$ and therefore $d\tilde{f}_\theta = d\tilde{f}_0$ for all $\theta \in \mathbb{R}$. Define $\alpha_\theta := s_\theta s_0^{-1}$. Then we see that $S_{e^{\theta J}} = \alpha_\theta S \alpha_\theta^{-1}$, $(D\alpha)s_0 = D(\alpha s_0) - \alpha D(s_0) = \nabla^\theta s_\theta - S_{e^{\theta J}} s_\theta - \alpha\nabla^0 s_0 + \alpha S s_0 = 0$ and $g(s_0, s_0) = g(s_\theta, s_\theta) = (\alpha_\theta^* g)(s_0, s_0)$. Conversely, suppose that there exists a family α_θ as above. Choose a ∇^0-flat frame s_0 and set $s_\theta := \alpha_\theta s_0$. Then by a similar calculation it follows that s_θ is ∇^θ-flat and that $f_\theta = f_0$. □

By Corollary 3.3.7, there are no *full* isotropic para-pluriharmonic maps into the symmetric space $\mathrm{GL}(p+q,\mathbb{R})/\mathrm{O}(p,q)$, since the Cartan involution $\sigma(g) = \mathbf{1}_{p,q} g^t \mathbf{1}_{p,q}$ is not an inner automorphism of $\mathrm{GL}(p+q,\mathbb{C})$. Among the totally geodesic submanifolds is the symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R})/\mathrm{O}(n,n)$. Isotropic para-pluriharmonic maps into this space arise from special para-Kähler manifolds:

Example: special Kähler manifolds. Let (N, J, g) be a para-Kähler manifold and let ∇ be a torsion-free flat connection on TN such that $d^\nabla J = 0$ and $\nabla\omega = 0$, where $\omega := g(\cdot, J\cdot)$ denotes the Kähler form of N. The quadruple (N, J, g, ∇) is then called an *affine special para-Kähler manifold.* In [S1] it is shown that $(E := TN,\ D := \nabla - S,\ S := \frac{1}{2}J(\nabla J),\ g)$ is a metric para-tt*-bundle. From the definition of S, one sees after short calculation that $S_{JX}Y = -JS_XY = S_X JY$ and

$$S_{e^{\theta J}} = e^{-(\theta/2)J} \circ S \circ e^{(\theta/2)J}, \tag{14}$$

further $DJ = 0$, thus $e^{-(\theta/2)J}$ is D-parallel and g-orthogonal, and the para-pluriharmonic map $\tilde{f}\colon N \to \mathrm{S}_{n,n}$ is isotropic by the above proposition.

Indeed, $\tilde{f}$ is a *para-holomorphic* map into the para-hermitian symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}^\pi(n) \subset \mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(n,n)$. In [S1] this is proved by relating $\tilde{f}$ to the dual Gauß map of the canonical immersion $N \hookrightarrow \boldsymbol{C}^{2n}$. A direct argument is as follows:

It is a direct consequence of $\nabla\omega = 0$ and $\nabla s = 0$ that the image of $\tilde{f}$ is contained in $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}^\pi(n) = \mathrm{Sp}(2n,\mathbb{R})/(\mathrm{O}(n,n) \cap \mathrm{Sp}(2n,\mathbb{R}))$: The matrix-valued map

$x \mapsto s^*(x)\omega_x$ is constant, and by adjusting the frame s we can assume that its image is the standard skew-symmetric form Ω_{2n}. Now observe that $\gamma \in \mathrm{GL}(2n,\mathbb{R})$ is a representative for $\tilde{f}(x)$ if and only if $s(x) \circ \gamma$ is an orthonormal basis for g_x. In particular, this is the case if $s(x) \circ \gamma$ is an orthonormal basis for the *para-hermitian* metric $g_x + j\omega_x$ (such bases always exist by Proposition 1.3.3). For any such γ we have $\Omega_{2n} = \gamma^*\Omega_{2n}$, i.e. $\gamma \in \mathrm{Sp}_{2n}\mathbb{R}$.

For the holomorphicity it is sufficient to consider one point $x_0 \in M$. Again, by adjusting the frame s, we may assume $\tilde{f}(x_0) = e \cdot \mathrm{U}^\pi(n)$. If g_{x_0} and ω_{x_0} have standard form in the basis $s(x_0)$ then so has J_{x_0}, i.e. $J_{x_0} = s(x_0) \circ J_{2n} \circ s(x_0)^{-1}$. From equation (14) we get $\Phi_\theta = \mathrm{Ad}(s_\theta^{-1} s_0) = \mathrm{Ad}(e^{(\theta/2)J_{2n}})$. Differentiating the equation $\Phi_{-\theta} \circ d\tilde{f} = d\tilde{f} \circ e^{\theta J_N}$ yields

$$\mathrm{ad}(J_{2n}/2) \circ d\tilde{f} = d\tilde{f} \circ J_N,$$

which shows that $\tilde{f}$ is holomorphic, because $\mathrm{ad}(J_{2n}/2)$ is the standard para-complex structure of $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}^\pi(n)$: $\mathrm{ad}(J_{2n}/2)$ is zero on $\mathfrak{k} = \mathfrak{u}_n^\pi = \{X \in \mathfrak{sp}(2n,\mathbb{R}) : XJ - JX = 0\}$ and has eigenvalues ± 1 on $\mathfrak{m} = \{X \in \mathfrak{sp}(2n,\mathbb{R}) : XJ + JX = 0\}$. The twistor bundle is thus trivial (cf. Example 3.3.8).

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Chapter 16

Maximally homogeneous para-CR manifolds of semisimple type

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1 Introduction and notation

Let M be a $2n$-dimensional manifold. An *almost paracomplex structure* on M is a field of endomorphisms $K \in \mathrm{End}(TM)$ of the tangent bundle TM of M such that $K^2 = \mathrm{id}$. It is called an (almost) paracomplex structure in the *strong sense* if its ± 1-eigenspace distributions

$$T^{\pm}M = \{X \pm KX \mid X \in \Gamma(M, TM)\}$$

have the same rank (see e.g. [13], [9]). An almost paracomplex structure K is called a *paracomplex structure*, if it is *integrable*, i.e.,

$$S(X, Y) = [X, Y] + [KX, KY] - K[X, KY] - K[KX, Y] = 0$$

for any vector fields $X,\ Y \in \Gamma(TM)$.

This is equivalent to say that the distributions $T^{\pm}M$ are involutive.

Recall that an *almost* CR-*structure* of codimension k on a $2n+k$-dimensional manifold M is a distribution $HM \subset TM$ of rank $2n$ together with a field of endomorphisms $J \in \mathrm{End}(HM)$ such that $J^2 = -\mathrm{id}$. An almost CR-structure is called CR-*structure*, if the $\pm i$-eigenspace subdistributions $H_{\pm}M$ of the complexified tangent bundle $T^{\mathbb{C}}M$ are involutive.

We define an almost para-CR structure in a similar way.

Definition 1.1. An *almost para-CR structure* of codimension k on a $2n+k$-dimensional manifold M (in the weak sense) is a pair (HM, K), where $HM \subset TM$ is a rank $2n$ distribution and $K \in \mathrm{End}(HM)$ is a field of endomorphisms such that $K^2 = \mathrm{id}$ and $K \neq \pm\mathrm{id}$.

An almost para-CR structure is said to be a para-CR structure, if the eigenspace subdistributions $H_{\pm}M \subset HM$ are integrable or equivalently if the following integrability conditions hold:

$$[KX, KY] + [X, Y] \in \Gamma(HM), \tag{1}$$

$$S(X, Y) := [X, Y] + [KX, KY] - K([X, KY] + [KX, Y]) = 0 \tag{2}$$

for all $X,\ Y \in \Gamma(HM)$.

If the eigenspace distributions

$$H_{\pm}M = \{X \pm KX \mid X \in \Gamma(M, HM)\}$$

of an almost para-CR structure have the same rank, then (HM, K) is called an almost para-CR structure in the *strong sense*.

A straightforward computation shows that the integrability condition is equivalent to the involutivness of the distributions H_+M and H_-M.

A manifold M, endowed with an (almost) para-CR structure, is called an (*almost*) *para*-CR *manifold* .

Note that a direct product of (almost) para-CR manifolds is an (almost) para-CR manifold.

One can associate with a point $x \in M$ of a para-CR manifold (M, HM, K) a fundamental graded Lie algebra $\mathfrak{m}$. A para-CR structure is said to be *regular* if these Lie algebras $\mathfrak{m}_x$ do not depend on x. In this case, a para-CR structure can be considered as a Tanaka structure (see [3] and Section 4). A regular para-CR structure is called a structure of *semisimple type* if the full prolongation

$$\mathfrak{g} = \mathfrak{m}^{\infty} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots$$

of the associated non-positively graded Lie algebra $\mathfrak{g}^{-d} + \cdots + \mathfrak{g}^{-1} + \mathfrak{g}^0$ (which is an analogue of the generalized Levi form of a CR structure) is a semisimple Lie algebra. Such a para-CR structure defines a parabolic geometry and its group of automorphisms $\mathrm{Aut}(M, HM, K)$ is a Lie group of dimension $\leq \dim \mathfrak{g}$.

Recently in [16], P. Nurowski and G. A. J. Sparling consider the natural para-CR structure which arises on the 3-dimensional space M of solutions of a second order ordinary differential equation $y'' = Q(x, y, y')$. Using the Cartan method of prolongation, they construct the full prolongation $\mathcal{G} \to M$ of M with a $\mathfrak{sl}(3, \mathbb{R})$-valued Cartan connection and a quotient line bundle over M with a conformal metric of signature $(2, 2)$. This is a para-analogue of the Feffermann bundle of a CR-structure. They apply these bundles to the initial ODE and get interesting applications.

In [2] we proved that a para-CR structures of semisimple type on a simply connected manifold M with the automorphism group of maximal dimension $\dim \mathfrak{g}$ can be identified with a (real) flag manifold $M = G/P$ where G is the simply connected Lie group with the Lie algebra $\mathfrak{g}$ and P the parabolic subgroup generated by the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$. We gave a classification of maximally homogeneous para-CR structures of semisimple type such that the associated graded semisimple Lie algebra $\mathfrak{g}$ has depth $d = 2$. In the present chapter we classify all maximally homogeneous para-CR structures of semisimple type in terms of graded real semisimple Lie algebras.

2 Graded Lie algebras associated with para-CR structures

2.1 Gradations of a Lie algebra

Recall that a *gradation* (more precisely a $\mathbb{Z}$-gradation) of *depth* k of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i = \mathfrak{g}^{-k} + \mathfrak{g}^{-k+1} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^j + \cdots \tag{3}$$

such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, for any $i, j \in \mathbb{Z}$, and $\mathfrak{g}^{-k} \neq \{0\}$. Note that $\mathfrak{g}^0$ is a subalgebra of $\mathfrak{g}$ and each $\mathfrak{g}^i$ is a $\mathfrak{g}^0$-module.

We say that an element $x \in \mathfrak{g}^j$ has *degree* j and we write $d(x) = j$. The endomorphism δ of $\mathfrak{g}$ defined by

$$\delta_{|\mathfrak{g}_j} = j \cdot \mathrm{id}$$

is a semisimple derivation of $\mathfrak{g}$ (with integer eigenvalues), whose eigenspaces determine the gradation. Conversely, any semisimple derivation δ of $\mathfrak{g}$ with integer eigenvalues defines a gradation where the grading space $\mathfrak{g}^j$ is the eigenspace of δ with eigenvalue j. If $\mathfrak{g}$ is a semisimple Lie algebra, then any derivation δ is inner, i.e., there exists $d \in \mathfrak{g}$ such that $\delta = \mathrm{ad}_d$. The element $d \in \mathfrak{g}$ is called the *grading element*.

Definition 2.1. A gradation $\mathfrak{g} = \sum \mathfrak{g}^i$ of a Lie algebra (or a graded Lie algebra $\mathfrak{g}$) is called

(1) *fundamental*, if the negative part $\mathfrak{m} = \sum_{i<0} \mathfrak{g}^i$ is generated by $\mathfrak{g}^{-1}$;

(2) (*almost*) *effective* or *transitive*, if the non-negative part

$$\mathfrak{g}^{\geq 0} = \mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots$$

contains no non-trivial ideal of $\mathfrak{g}$;

(3) *non-degenerate*, if

$$X \in \mathfrak{g}^{-1},\ [X, \mathfrak{g}^{-1}] = 0 \Longrightarrow X = 0.$$

2.2 Fundamental algebra associated with a distribution

Let $\mathcal{H}$ be a distribution on a manifold M. We recall that to any point $x \in M$ it is possible to associate a Lie algebra $\mathfrak{m}(x)$ in the following way.

First of all, we consider a filtration of the Lie algebra $\mathcal{X}(M)$ of vector fields defined inductively by

$$\begin{aligned}\Gamma(\mathcal{H})_{-1} &= \Gamma(\mathcal{H}),\\ \Gamma(\mathcal{H})_{-i} &= \Gamma(\mathcal{H})_{-i+1} + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})_{-i+1}] \quad \text{for } i > 1.\end{aligned}$$

Then evaluating vector fields at a point $x \in M$, we get a flag

$$T_xM \supset \mathcal{H}_{-d-1}(x) = \mathcal{H}_{-d}(x) \supsetneq \mathcal{H}_{-d+1}(x) \supset \cdots \supset \mathcal{H}_{-2}(x) \supset \mathcal{H}_{-1}(x) = \mathcal{H}_x$$

in T_xM, where

$$\mathcal{H}_{-i}(x) = \{X_{|x} \mid X \in \Gamma(\mathcal{H})_{-i}\}.$$

Let us assume that $\mathcal{H}_{-d}(x) = T_xM$. The commutators of vector fields induce a structure of fundamental negatively graded Lie algebra on the associated graded space

$$\mathfrak{m}(x) = \mathrm{gr}(T_xM) = \mathfrak{m}^{-d}(x) + \mathfrak{m}^{-d+1}(x) + \cdots + \mathfrak{m}^{-1}(x),$$

where $\mathfrak{m}^{-j}(x) = \mathcal{H}_{-j}(x)/\mathcal{H}_{-j+1}(x)$. Note that $\mathfrak{m}^{-1}(x) = \mathcal{H}_x$.

A distribution $\mathcal{H}$ is called a *regular distribution* of *depth* d and *type* $\mathfrak{m}$ if all graded Lie algebras $\mathfrak{m}(x)$ are isomorphic to a given graded fundamental Lie algebra

$$\mathfrak{m} = \mathfrak{m}^{-d} + \mathfrak{m}^{-d+1} + \cdots + \mathfrak{m}^{-1}.$$

In this case $\mathfrak{m}$ is called the *Lie algebra associated* with the distribution $\mathcal{H}$. A regular distribution $\mathcal{H}$ is called *non-degenerate* if the associated Lie algebra is non-degenerate.

2.3 Para-CR algebras and regular para-CR structures

We recall the following

Definition 2.2. A pair $(\mathfrak{m}, K_o)$, where $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ is a negatively graded fundamental Lie algebra and K_o is an involutive endomorphism of $\mathfrak{m}^{-1}$, is called a *para*-CR *algebra* of *depth* d. If, moreover, the ± 1-eigenspaces $\mathfrak{m}^{-1}_{\pm}$ of K_o on $\mathfrak{m}^{-1}$ are commutative subalgebras of $\mathfrak{m}$, then $(\mathfrak{m}, K_o)$ is called an *integrable para-CR algebra*.

Definition 2.3. Let $(\mathfrak{m}, K_o)$ be a para-CR algebra of depth d. An almost para-CR structure (HM, K) on M is called *regular* of *type* $(\mathfrak{m}, K_o)$ and *depth* d if, for any $x \in M$, the pair $(\mathfrak{m}(x), K_x)$ is isomorphic to $(\mathfrak{m}, K_o)$. We say that the regular almost para-CR structure is *non-degenerate* if the graded algebra $\mathfrak{m}$ is non-degenerate.

Note that a regular almost para-CR structure of type $(\mathfrak{m}, K_0)$ is integrable if and only if the Lie algebra $(\mathfrak{m}, K_0)$ is integrable.

3 Prolongations of graded Lie algebras

3.1 Prolongations of negatively graded Lie algebras

The *full prolongation* of a negatively graded fundamental Lie algebra $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ is defined as a maximal graded Lie algebra

$$\mathfrak{g}(\mathfrak{m}) = \mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) + \mathfrak{g}^{0}(\mathfrak{m}) + \mathfrak{g}^{1}(\mathfrak{m}) + \cdots$$

with the negative part

$$\mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) = \mathfrak{m}$$

such that the following transitivity condition holds:

$$\text{if } X \in \mathfrak{g}^{k}(\mathfrak{m}),\ k \geq 0,\ [X, \mathfrak{g}^{-1}(\mathfrak{m})] = \{0\},\ \text{then } X = 0.$$

In [17], N. Tanaka proved that the full prolongation $\mathfrak{g}(\mathfrak{m})$ always exists and that it is unique up to isomorphisms. Moreover, it can be defined inductively by

$$\mathfrak{g}^{i}(\mathfrak{m}) = \begin{cases} \mathfrak{m}^{i}, & i < 0, \\ \{A \in \mathrm{Der}(\mathfrak{m}, \mathfrak{m}) \mid A(\mathfrak{m}^{j}) \subset \mathfrak{m}^{j} \text{ for all } j < 0\}, & i = 0, \\ \{A \in \mathrm{Der}(\mathfrak{m}, \sum_{h<i} \mathfrak{g}^{h}(\mathfrak{m})) \mid A(\mathfrak{m}^{j}) \subset \mathfrak{g}(\mathfrak{m})^{i+j} \text{ for all } j < 0\}, & i > 0, \end{cases}$$

where $\mathrm{Der}(\mathfrak{m}, V)$ denotes the space of derivations of the Lie algebra $\mathfrak{m}$ with values in the $\mathfrak{m}$-module V.

Note that

$$\begin{aligned} \mathfrak{g}^{i}(\mathfrak{m}) = \big\{ & A \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \textstyle\sum_{h<i} \mathfrak{g}^{h}(\mathfrak{m})) \mid A(\mathfrak{g}^{h}(\mathfrak{m})) \subset \mathfrak{g}^{h+i}(\mathfrak{m}) \text{ for all } h < 0, \\ & \text{and } [A(Y), Z] + [Y, A(Z)] = A([Y, Z]) \text{ for all } Y, Z \in \mathfrak{m} \big\}. \end{aligned} \tag{4}$$

3.2 Prolongations of non-positively graded Lie algebras

Consider now a non-positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}^{0} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^{0}$. The *full prolongation* of $\mathfrak{m} + \mathfrak{g}^{0}$ is the subalgebra

$$(\mathfrak{m} + \mathfrak{g}^{0})^{\infty} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1} + \mathfrak{g}^{2} + \cdots$$

of $\mathfrak{g}(\mathfrak{m})$, defined inductively by

$$\mathfrak{g}^i = \{X \in \mathfrak{g}(\mathfrak{m})^i \mid [X, \mathfrak{m}^{-1}] \subset \mathfrak{g}^{i-1}\} \quad \text{for any } i \geq 1.$$

Definition 3.1. A graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$ is called of *finite type* if its full prolongation $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ is a finite dimensional Lie algebra and it is called of *semisimple type* if $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra.

We have the following criterion (see [18], [3]).

Lemma 3.2. *Let* $(\mathfrak{m} = \sum_{i<0} \mathfrak{m}^i, K_o)$ *be an integrable para-CR algebra and* $\mathfrak{g}^0$ *the subalgebras of* $\mathfrak{g}^0(\mathfrak{m})$ *consisting of any* $A \in \mathfrak{g}^0(\mathfrak{m})$ *such that* $A|_{\mathfrak{m}^{-1}}$ *commutes with* K_o*. Then the graded Lie algebra* $(\mathfrak{m} + \mathfrak{g}^0)$ *is of finite type if and only if* $\mathfrak{m}$ *is non-degenerate.*

The following result will be used in the last section (see e.g. [14], Theorem 3.21)

Lemma 3.3. *Let* $\mathfrak{g} = \sum_i \mathfrak{g}_i$ *be a fundamental effective semisimple graded Lie algebra such that* $\mathfrak{m} + \mathfrak{g}^0$ *is of finite type. Then* $\mathfrak{g}$ *coincides with the full prolongation* $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ *of* $\sum_{i \leq 0} \mathfrak{g}_i = \mathfrak{m} + \mathfrak{g}_0$.

4 Standard almost para-CR manifolds

4.1 Maximally homogeneous Tanaka structures

A regular para-CR structure of type $(\mathfrak{m}, K_0)$ is of *finite type* or, respectively, of *semisimple type*, if the Lie algebra $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ is finite-dimensional or, respectively, semisimple. Recall that $\mathfrak{g}^0 = \mathrm{Der}(\mathfrak{m}, K_0)$ is the Lie algebra of Lie group $\mathrm{Aut}(\mathfrak{m}, K_0)$.

We recall the following (see [3])

Definition 4.1. Let $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ be a negatively graded Lie algebra generated by $\mathfrak{m}^{-1}$ and G^0 a closed Lie subgroup of (grading preserving) automorphisms of $\mathfrak{m}$. A *Tanaka structure* of *type* $(\mathfrak{m}, G^0)$ on a manifold M is a regular distribution $\mathcal{H} \subset TM$ of type $\mathfrak{m}$ together with a principal G^0-bundle $\pi\colon Q \to M$ of adapted coframes of $\mathcal{H}$. A coframe $\varphi\colon \mathcal{H}_x \to \mathfrak{m}^{-1}$ is called *adapted* if it can be extended to an isomorphism $\varphi\colon \mathfrak{m}_x \to \mathfrak{m}$ of graded Lie algebras.

We say that the Tanaka structure of type $(\mathfrak{m}, G^0)$ is of *finite type* (respectively *semisimple type* $(\mathfrak{m}, G^0)$), if the graded Lie algebra $\mathfrak{m}+\mathfrak{g}^0$ is of finite type (respectively semisimple type). Let P be a Lie subgroup of a connected Lie group G and $\mathfrak{p}$ (respectively $\mathfrak{g}$) the Lie algebra of P (respectively G).

Theorem 4.2. *Let* $(\pi\colon Q \to M, \mathcal{H})$ *be a Tanaka structure on* M *of semisimple type* $(\mathfrak{m}, G^0)$*. Then the Tanaka prolongation of* $(\pi, \mathcal{H})$ *is a* P*-principal bundle* $\mathcal{G} \to M$*,*

with the parabolic structure group P, equipped with a Cartan connection $\kappa\colon T\mathcal{G} \to \mathfrak{g}$, where $\mathfrak{g}$ is the full prolongation of $\mathfrak{m} + \mathfrak{g}^0$ and $\mathrm{Lie}\, P = \mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}_i$. *Moreover,* $\mathrm{Aut}(\mathcal{H}, \pi)$ *is a Lie group and*

$$\dim \mathrm{Aut}(\mathcal{H}, \pi) \leq \dim \mathfrak{g}.$$

Let $(\mathcal{H}, \pi\colon Q \to M)$ be a Tanaka structure of semisimple type $(\mathfrak{m}, G^0)$ and $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty = \mathfrak{m} + \mathfrak{p}$ be the full prolongation of the non-positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$.

Definition 4.3. A semisimple Tanaka structure $(\mathcal{H}, \pi\colon Q \to M)$ is called *maximally homogeneous* if the dimension of its automorphism group $\mathrm{Aut}(\mathcal{H}, \pi)$ is equal to $\dim \mathfrak{g}$.

4.2 Tanaka structures of semisimple type

We construct maximally homogeneous Tanaka structures of semisimple type $(\mathfrak{m}, G^0)$ as follows. Let $G = \mathrm{Aut}(\mathfrak{g})$ be the Lie group of automorphisms of the graded Lie algebra $\mathfrak{g}$. Recall that G^0 is a closed subgroup of the automorphism group of the graded Lie algebra $\mathfrak{g}_- = \sum_{i<0} \mathfrak{g}_i = \mathfrak{m}$ Since the Lie algebra $\mathfrak{g}$ is canonically associated with $\mathfrak{m}$, we can canonically extend the action of G^0 on $\mathfrak{m}$ to the action of G^0 on $\mathfrak{g}$ by automorphisms. In other words, we have an embedding $G^0 \hookrightarrow \mathrm{Aut}(\mathfrak{g}) = G$ as a closed subgroup. We denote by G^+ the connected (closed) subgroup of G with Lie algebra $\mathfrak{g}_+ = \sum_{p>0} \mathfrak{g}^p$. Then $P = G^0 \cdot G^+ \subset G$ is a (closed) parabolic subgroup of G. Let G/P be the corresponding flag manifold. We have a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{p}$ and we identify $\mathfrak{m}$ with the tangent space $T_o(G/P)$. Then the natural action of G^0 on $\mathfrak{m}$ is the isotropy representation of G^0. We have a natural Tanaka structure $(\mathcal{H}, \pi\colon Q \to G/P)$ of type $(\mathfrak{m}, G^0)$, where $\mathcal{H}$ is the G-invariant distribution defined by $\mathfrak{m}^{-1}$ and Q is the G^0-bundle of coframes on $\mathcal{H}$.

Hence, the flag manifold G/P carries a natural maximally homogeneous Tanaka structure $(\mathcal{H}, \pi\colon Q \to G/P)$.

The universal covering F of the manifold G/P also has the induced Tanaka structure $(H_F, \pi_F\colon Q_F \to F)$ of type $(\mathfrak{m}, G^0)$ and the simply connected (connected) Lie group $\widetilde{G}$ with the Lie algebra $\mathfrak{g}$ acts transitively and almost effectively on F as a group of automorphisms of this Tanaka structure. Moreover, the stabilizer in $\widetilde{G}$ of an appropriate point $o \in F$ is the (connected) parabolic subgroup $\widetilde{P}$ generated by the subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$. The Tanaka structure $(\mathcal{H}, \pi\colon Q \to F = \widetilde{G}/\widetilde{P})$ is obviously maximally homogeneous and it is called the *standard* (*simply connected maximally homogeneous*) *Tanaka structure* of type $(\mathfrak{m}, G^0)$. We can state the following (see e.g. Theorem 4.8 in [2])

Theorem 4.4. *Any maximally homogeneous Tanaka structure of semisimple type $(\mathfrak{m}, G_0)$ is locally isomorphic to the standard Tanaka structure on the simply connected flag manifold $F = \widetilde{G}/\widetilde{P}$ where $\widetilde{G}$ is the simply connected semisimple Lie*

group with the Lie algebra $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ *and* $\widetilde{P}$ *is the parabolic subgroup generated by the subalgebra* $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$.

Let (HM, K) be a regular almost para-CR structure of type $(\mathfrak{m}, K_0)$. Assume that it has finite type, i.e., $m = \dim(\mathfrak{m} + \mathfrak{g}^0)^\infty < \infty$. According to the above definition, (HM, K) is *maximally homogeneous*, if it admits a (transitive) Lie group of automorphisms of dimension m.

By Theorem 4.4, a maximally homogeneous almost para-CR structure of semisimple type is locally equivalent to the standard structure associated with a gradation of a semisimple Lie algebra. In the following subsection we describe this correspondence in more details.

4.3 Models of almost para-CR manifolds

Let $\mathfrak{g} = \sum_{-d}^{d} \mathfrak{g}^i = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+$ be an effective fundamental gradation of a semisimple Lie algebra $\mathfrak{g}$ with negative part $\mathfrak{m} = \mathfrak{g}^- = \sum_{i<0} \mathfrak{g}^i$ and positive part $\mathfrak{g}^+ = \sum_{i>0} \mathfrak{g}^i$.

Denote by $F = \widetilde{G}/\widetilde{P}$ the simply connected real flag manifold associated with the graded Lie algebra $\mathfrak{g}$ where $\widetilde{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\widetilde{P} = G^0 G^+$ is the connected subgroup generated by the Lie subalgebra $\mathfrak{g}^0 + \mathfrak{g}^+$.

We will identify the tangent space $T_o F$ at the point $o = eP$ with the subspace

$$\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{m}.$$

Since the subspace $(\mathfrak{g}^{-1} + \mathfrak{p})/\mathfrak{p} \subset T_o F$ is invariant under the isotropy representation of P, it defines an invariant distribution $\mathcal{H}$ on F. Since the gradation is fundamental, one can easily check that, for any $x \in F$, the negatively graded Lie algebra $\mathfrak{m}(x)$ associated with $\mathcal{H}$ is isomorphic to the Lie algebra $\mathfrak{m}$. Moreover, let

$$\mathfrak{g}^{-1} = \mathfrak{g}_+^{-1} + \mathfrak{g}_-^{-1} \tag{5}$$

be a decomposition of the G^0-module $\mathfrak{g}^{-1}$ into a sum of two submodules and K_0 the associated $\mathrm{ad}_{\mathfrak{g}_0}$-invariant endomorphism such that $\mathfrak{g}_\pm^{-1}$ are ± 1-eigenspaces of K_0.

The decomposition (5) defines two invariant complementary subdistributions $\mathcal{H}_\pm$ of the distribution $\mathcal{H} \subset TF$ associated with $\mathfrak{g}^{-1}$ and K_0 defines $\widetilde{G}$-invariant para-CR structure (HF, K) on F. It is the standard para-CR structure associated with the graded Lie algebra $\mathfrak{g}$ and the decomposition (5). We get the following theorem (see also Theorem 5.1 in [2]).

Theorem 4.5. *Let* $F = \widetilde{G}/\widetilde{P}$ *be the simply connected flag manifold associated with a (real) semisimple effective fundamental graded Lie algebra* $\mathfrak{g}$. *A decomposition* $\mathfrak{g}^{-1} = \mathfrak{g}_+^{-1} + \mathfrak{g}_-^{-1}$ *of* $\mathfrak{g}^{-1}$ *into complementary* G^0*-submodules* $\mathfrak{g}_\pm^{-1}$ *determines an invariant almost para-CR structure* (HM, K) *such that* ± 1*-eigenspaces* $H_\pm M$ *of* K *are subdistributions of* HM *associated with* $\mathfrak{g}_\pm^{-1}$. *Conversely, any standard almost para-CR structure* (HM, K) *on* F *can be obtained in such a way.*

Moreover, (HM, K) *is*

(1) *an almost para-CR structure in the strong sense if* $\mathfrak{g}_+^{-1}$ *and* $\mathfrak{g}_-^{-1}$ *have the same dimensions,*

(2) *a para-CR structure if and only if* $\mathfrak{g}_+^{-1}$ *and* $\mathfrak{g}_-^{-1}$ *are commutative subalgebras of* $\mathfrak{g}$*,*

(3) *non-degenerate if and only if* $\mathfrak{g}$ *has no graded ideals of depth one.*

By Theorem 4.5, the classification of maximally homogeneous para-CR structures of semisimple type, up to local isomorphisms (i.e., up to coverings), reduces to the description of all gradation of semisimple Lie algebras $\mathfrak{g}$ and to decomposition of the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules. We will give such a description for complex and real semisimple Lie algebras in the next two sections.

5 Fundamental gradations of a complex semisimple Lie algebra

We recall here the construction of a gradation of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote by

$$\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset R$$

a system of simple roots of the root system R and associate to each simple root α_i (or corresponding vertex of the Dynkin diagram) a non-negative integer d_i. Using the *label vector* $\vec{d} = (d_1, \dots, d_\ell)$, we define the *degree* of a root $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ by

$$d(\alpha) = \sum_{i=1}^{\ell} k_i d_i.$$

This defines a gradation of $\mathfrak{g}$ by the conditions

$$d(\mathfrak{h}) = 0, \quad d(\mathfrak{g}_\alpha) = d(\alpha) \ \text{ for all } \alpha \in R,$$

which is called the *gradation associated with the label vector* $\vec{d}$.

We denote by $d \in \mathfrak{h}$ the corresponding grading element. Then $d(\alpha) = \alpha(d)$. Any gradation of a complex semisimple Lie algebra $\mathfrak{g}$ is conjugated to a gradation of such a type (see [11]). In particular, it has the form

$$\mathfrak{g} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^k,$$

where $\mathfrak{g}^0$ is a reductive subalgebra of $\mathfrak{g}$ and the grading spaces $\mathfrak{g}^{-i}$ and $\mathfrak{g}^i$ are dual with respect to the Killing form. It is clear now that any graded semisimple Lie algebra is a direct sum of graded simple Lie algebras. Hence, it is sufficient to describe gradations of simple Lie algebras.

We need the following (see [19])

Lemma 5.1. *The gradation of a complex semisimple Lie algebra* $\mathfrak{g}$ *associated with a label vector* $\vec{d} = (d_1, \dots, d_\ell)$ *is fundamental if and only if all labels* $d_i \in \{0, 1\}$.

Let $\Pi^1 \subset \Pi$ be a set of simple roots. We denote by $\vec{d}_{\Pi^1}$ the label vector which associates label one to the roots in Π^1 and label zero to the other simple roots.

Now we describe the depth of a fundamental gradation.

Let μ be the maximal root with respect to the fundamental system Π. It can be written as a linear combination

$$\mu = m_1\alpha_1 + \cdots + m_\ell\alpha_\ell \tag{6}$$

of fundamental roots, where the coefficient m_i is a positive integer called the *Dynkin mark associated* with α_i.

Lemma 5.2. *Let* $\Pi^1 = \{\alpha_{i_1}, \dots, \alpha_{i_s}\} \subset \Pi$ *be a set of simple roots. Then the depth* k *of the fundamental gradation defined by the label vector* $\vec{d}_{\Pi^1}$ *is given by*

$$k = m_{i_1} + m_{i_2} + \cdots + m_{i_s}.$$

Proof. The depth k of the gradation is equal to the maximal degree $d(\alpha)$, α being a root. If $\alpha = k_1\alpha_1 + \cdots + k_\ell\alpha_\ell$ is the decomposition of a root α with respect to simple roots, then

$$d(\alpha) = k_{i_1} + \cdots + k_{i_s} \le d(\mu) = m_{i_1} + \cdots + m_{i_s} = k. \qquad \square$$

Irreducible submodules of the $\mathfrak{g}^0$-module $\mathfrak{g}^1$. Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a fundamental gradation of a complex semisimple Lie algebra $\mathfrak{g}$, defined by a label vector $\vec{d}$. Following [11], we describe the decomposition of a $\mathfrak{g}^0$-module into irreducible submodules. Set

$$R^i = \{\alpha \in R \mid d(\alpha) = i\} = \{\alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{g}^i\}$$

and

$$\Pi^i = \Pi \cap R^i = \{\alpha \in \Pi \mid d(\alpha) = i\}.$$

For any simple root $\gamma \in \Pi$, we put

$$R(\gamma) = \{\gamma + (R^0 \cup \{0\})\} \cap R = \{\alpha = \gamma + \phi^0 \in R,\ \ \phi^0 \in R^0 \cup \{0\}\}.$$

We associate to any set of roots $Q \subset R$ a subspace

$$\mathfrak{g}(Q) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \subset \mathfrak{g}.$$

Proposition 5.3 ([11]). *The decomposition of a* $\mathfrak{g}^0$*-module* $\mathfrak{g}^1$ *into irreducible submodules is given by*

$$\mathfrak{g}^1 = \sum_{\gamma\in\Pi^1} \mathfrak{g}(R(\gamma)).$$

Moreover, γ *is a lowest weight of the irreducible submodule* $\mathfrak{g}(R(\gamma))$*. In particular, the number of the irreducible components is equal to the number* $\#\Pi^1$ *of the simple roots of degree* 1.

Since the $\mathfrak{g}^0$-modules $\mathfrak{g}^i$ and $\mathfrak{g}^{-i}$ are dual, Proposition 5.3 gives also the decomposition of the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules.

6 Fundamental gradations of a real semisimple Lie algebra

6.1 Real forms of a complex semisimple Lie algebra

Now we recall the description of a real form of a complex semisimple Lie algebra in terms of Satake diagrams. It is sufficient to do this for complex simple Lie algebras.

Any real form of a complex semisimple Lie algebra $\mathfrak{g}$ is the fixed points set $\mathfrak{g}^\sigma$ of an antilinear involution σ, that is, an antilinear map $\sigma\colon \mathfrak{g}\to\mathfrak{g}$, which is an automorphism of $\mathfrak{g}$ as a real algebra, such that $\sigma^2 = \mathrm{id}$. We fix a Cartan decomposition

$$\mathfrak{g}^\sigma = \mathfrak{k} + \mathfrak{m}$$

of the real form $\mathfrak{g}^\sigma$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}^\sigma$ and $\mathfrak{m}$ is its orthogonal complement with respect to the Killing form B. Let

$$\mathfrak{h}^\sigma = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{m}$$

be a Cartan subalgebra of $\mathfrak{g}^\sigma$ which is consistent with this decomposition and such that $\mathfrak{h}_\mathfrak{m} = \mathfrak{h}\cap\mathfrak{m}$ has maximal dimension. Then the root decomposition of $\mathfrak{g}^\sigma$, with respect to the subalgebra $\mathfrak{h}^\sigma$, can be written as

$$\mathfrak{g}^\sigma = \mathfrak{h}^\sigma + \sum_{\lambda\in\Sigma} \mathfrak{g}^\sigma_\lambda,$$

where $\Sigma\subset(\mathfrak{h}^\sigma)^*$ is a (non-reduced) root system. The number $m_\lambda = \dim\mathfrak{g}_\lambda$ is the *multiplicity* of a root $\lambda\in\Sigma$.

Denote by $\mathfrak{h} = (\mathfrak{h}^\sigma)^\mathbb{C}$ the complexification of $\mathfrak{h}^\sigma$ which is a σ-invariant Cartan subalgebra. We denote by σ^* the induced antilinear action of σ on $\mathfrak{h}^*$ given by

$$\sigma^*\alpha = \overline{\alpha\circ\sigma},\quad \alpha\in\mathfrak{h}^*.$$

Consider the root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in R}\mathfrak{g}_\alpha$$

of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Note that σ^* preserves the root system R, i.e., $\sigma^* R = R$. Now we relate the root space decomposition of $\mathfrak{g}^\sigma$ and $\mathfrak{g}$. We define the subsystem of compact roots $R_\bullet$ by

$$R_\bullet = \{\alpha \in R \mid \sigma^*\alpha = -\alpha\} = \{\alpha \mid \alpha(\mathfrak{h}_\mathfrak{m}) = 0\}$$

and denote by $R' = R \setminus R_\bullet$ the complementary set of non-compact roots. We can choose a system Π of simple roots of R such that the corresponding system of positive roots R_+ satisfies the condition: $R'_+ = R' \cap R_+$ is σ-invariant. In this case, Π is called a *σ-fundamental system* of roots. We denote by $\Pi_\bullet = \Pi \cap R_\bullet$ the set of compact simple roots (which are also called black) and by $\Pi' = \Pi \setminus \Pi_\bullet$ the non-compact simple roots (called white). The action of σ^* on white roots satisfies the following property:

For any $\alpha \in \Pi'$ there exists a unique $\alpha' \in \Pi'$ such that $\sigma^*\alpha - \alpha'$ is a linear combination of black roots, i.e.,

$$\sigma^*\alpha = \alpha' + \sum_{\beta \in \Pi_\bullet} k_\beta \beta, \quad k_\beta \in \mathbb{N}.$$

In this case, we say that the roots α, α' are *σ-equivalent* and we will write $\alpha \sim \alpha'$. The information about fundamental system ($\Pi = \Pi_\bullet \cup \Pi'$) together with the σ-equivalence can be visualized in terms of the *Satake diagram*, which is defined as follows.

On the Dynkin diagram of the system of simple roots Π, we paint the vertices which correspond to black roots into black and we join the vertices which correspond to σ-equivalent roots α, α' by a curved arrow.

By a slight abuse of notation, we will refer to the σ-fundamental system $\Pi = \Pi_\bullet \cup \Pi'$, together with the σ-equivalence $\sim$, as the *Satake diagram*. This diagram is determined by the real form $\mathfrak{g}^\sigma$ of a complex simple Lie algebra $\mathfrak{g}$ and does not depend on the choice of a Cartan subalgebra and a σ-fundamental system. The list of Satake diagram of real simple Lie algebras is known (see e.g. [11]).

Conversely, Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$) allows to reconstruct the action of σ^* on Π, hence on $\mathfrak{h}^*$. This action can be canonically extended to the antilinear involution σ of the complex Lie algebra $\mathfrak{g}$. Hence, *there is a natural* 1-1 *correspondence between Satake diagrams subordinated to the Dynkin diagram of a complex semisimple Lie algebra* $\mathfrak{g}$*, up to isomorphisms, and real forms* $\mathfrak{g}^\sigma$ *of* $\mathfrak{g}$*, up to conjugations.*

6.2 Gradations of a real semisimple Lie algebra

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{g}^\sigma$ be a real form of $\mathfrak{g}$ with a Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$).

We identify $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ with a σ-fundamental system, which is a system of simple roots of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ and $\Pi_\bullet$ and Π' with the set of black and white roots respectively.

Let $\vec{d} = (d_1, \dots, d_\ell)$ be a label vector of the simple roots system Π and $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}^i$ be the corresponding gradation of $\mathfrak{g}$, with the grading element $d \in \mathfrak{h} \subset \mathfrak{g}$.

The following theorem gives necessary and sufficient conditions in order that this gradation induces a gradation

$$\mathfrak{g}^\sigma = \sum_{i\in\mathbb{Z}} \mathfrak{g}^\sigma \cap \mathfrak{g}^i$$

of the real form $\mathfrak{g}^\sigma$. This means that the grading element d belongs to $\mathfrak{g}^\sigma$. We denote by $\Pi^0 \subset \Pi$ the set of simple roots with label zero.

Theorem 6.1 ([10]). *A gradation of a complex semisimple Lie algebra $\mathfrak{g}$, associated with a label vector $\vec{d} = (d_1, \dots, d_\ell)$, induces a gradation of the real form $\mathfrak{g}^\sigma$, which corresponds to a Satake diagram $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ if and only if the following two conditions hold:*

i) *$\Pi_\bullet \subset \Pi^0$, i.e., any black vertex of the Satake diagram has label zero;*

ii) *if $\alpha \sim \alpha'$ for $\alpha, \alpha' \in \Pi \setminus \Pi_\bullet$, then $d(\alpha) = d(\alpha')$, i.e., white vertices of the Satake diagram which are joint by a curved arrow have the same label.*

A label vector $\vec{d} = (d_1, \dots, d_\ell)$ of a Satake diagram $(\Pi = \{\alpha_1, \dots, \alpha_\ell\} = \Pi_\bullet \cup \Pi', \sim)$ and the corresponding gradation of $\mathfrak{g}$ are called of *real type* if they satisfy conditions i) and ii) of the theorem above, that is, black vertices have label zero and vertices related by a curved arrow have the same label. Hence, we can state Theorem 6.1 as follows.

Corollary 6.2. *There exists a natural* 1-1 *correspondence between label vectors $\vec{d}$ of real type of a Satake diagram of a real semisimple Lie algebra $\mathfrak{g}^\sigma$ and gradations of $\mathfrak{g}^\sigma$. The gradation of $\mathfrak{g}^\sigma$ is fundamental if and only if the corresponding gradation of $\mathfrak{g}$ is fundamental, i.e., $\vec{d} = \vec{d}_{\Pi^1}$.*

Irreducible submodules of the $\mathfrak{g}^0$-module $\mathfrak{g}^1$. Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a gradation of a complex semisimple Lie algebra $\mathfrak{g}$ with grading element d and $\mathfrak{g}^\sigma = \sum (\mathfrak{g}^\sigma)^i = \sum \mathfrak{g}^i \cap \mathfrak{g}^\sigma$ be a real form of $\mathfrak{g}$, consistent with this gradation. We denote by $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ the Satake diagram of $\mathfrak{g}^\sigma$.

By Proposition 5.3, the decomposition of $\mathfrak{g}^1$ into irreducible $\mathfrak{g}^0$-submodules is given by $\mathfrak{g}^1 = \sum_{\gamma\in\Pi^1} \mathfrak{g}(R(\gamma))$, where Π^1 is the set of simple roots of label one. The following obvious proposition describes the decomposition of $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^1$ into irreducible submodules.

Proposition 6.3. *For any simple root $\gamma \in \Pi^1$ of label one, there are two possibilities:*

i) *$\sigma^*\gamma = \gamma + \sum_{\beta\in\Pi_\bullet} k_\beta \beta$. Then $\sigma^*\gamma \in R(\gamma)$ and the $\mathfrak{g}^0$-module $\mathfrak{g}(R(\gamma))$ is σ-invariant;*

ii) $\sigma^*\gamma = \gamma' + \sum_{\beta\in\Pi_\bullet} k_\beta\beta$, *where* $\gamma \neq \gamma' \in \Pi^1$. *Then,* $\sigma^* R(\gamma) = R(\gamma')$ *and the two irreducible* $\mathfrak{g}^0$*-modules* $\mathfrak{g}(R(\gamma))$ *and* $\mathfrak{g}(R(\gamma'))$ *determine one irreducible submodule* $\mathfrak{g}^\sigma \cap (\mathfrak{g}(R(\gamma)) + \mathfrak{g}(R(\gamma')))$ *of* $\mathfrak{g}^\sigma$.

Corollary 6.4. *Let* $\mathfrak{g}^\sigma = \sum(\mathfrak{g}^\sigma)^i$ *be the gradation of a real semisimple Lie algebra* $\mathfrak{g}^\sigma$*, associated with a label vector* $\vec{d}$ *of real type. Then irreducible submodules of the* $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^{-1}$ *correspond to vertices* γ *with label one without curved arrow and to pairs* (γ, γ') *of vertices with label one related by a curved arrow. In particular, a decomposition of the* $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^{-1}$ *is determined by a decomposition of the set* Π^1 *of vertices with label one into a disjoint union* $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ *such that equivalent vertices belong to the same component. The corresponding submodules* $(\mathfrak{g}^\sigma)^{-1}_+$ *and* $(\mathfrak{g}^\sigma)^1_-$ *are given by*

$$(\mathfrak{g}^\sigma)^{-1}_\pm = \mathfrak{g}^\sigma \cap \sum_{\gamma\in\Pi^1_\pm} \mathfrak{g}(R(-\gamma)). \tag{7}$$

We will always assume that a decomposition of Π^1 satisfies the above property.

7 Classification of maximally homogeneous para-CR manifolds

Let $\mathfrak{g}^\sigma$ be a real semisimple Lie algebra associated with a Satake diagram $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ with the fundamental gradation defined by a subset $\Pi^1 \subset \Pi'$ and $F = \widetilde{G}/\widetilde{P}$ be the associated flag manifold.

By Theorem 4.5, an almost para-CR structure on $F = \widetilde{G}/\widetilde{P}$ associated with a decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ is integrable (i.e., a para-CR structure) if and only if the $(\mathfrak{g}^\sigma)^0$-submodules $(\mathfrak{g}^\sigma)^{-1}_+$ and $(\mathfrak{g}^\sigma)^{-1}_-$ given by (7) are Abelian subalgebras of $\mathfrak{g}^\sigma$. In order to give an integrability criterion, we introduce the following definitions.

Definition 7.1. Let R be a system of roots and Π be a system of simple roots. A subset $\Pi^1 \subset \Pi$ is said to be *admissible* if Π^1 contains at least two roots and there are no roots of R of the form

$$2\alpha + \sum k_i\phi_i, \quad \text{with } \alpha \in \Pi^1,\ \phi_i \in \Pi_0 = \Pi \setminus \Pi^1. \tag{8}$$

Definition 7.2. Let $\mathfrak{g}^\sigma$ be a real semisimple Lie algebra with a fundamental gradation defined by a subset $\Pi^1 \subset \Pi'$. We say that a decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ is *alternate* if the following conditions hold:

i) If $\alpha \in \Pi^1_\pm$ and $\alpha' \sim \alpha$, then $\alpha' \in \Pi^1_\pm$.

ii) The vertices in Π^1_+ and Π^1_- appear in the Satake diagram in alternate order. This means that each connected component of the graph obtained by deleting vertices of Π^1_+ (respectively, of Π^1_-) has no two connected vertices from Π^1_- (respectively, from Π^1_+).

We are ready to state the following

Proposition 7.3. *Let $\mathfrak{g}^\sigma$ be a semisimple real Lie algebra with the fundamental gradation associated with a subset $\Pi^1 \subset \Pi$ and $F = \widetilde{G}/\widetilde{P}$ the associated flag manifold. A decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ defines a para-CR structure on the flag manifold F if and only if the subset Π^1 is admissible and the decomposition of Π^1 is alternate.*

For the proof we need the following lemma.

Lemma 7.4. *The subspace $\mathfrak{g}^1_+ = \sum_{\gamma\in\Pi^1_+} \mathfrak{g}(R(\gamma))$ (hence also the subspace $(\mathfrak{g}^\sigma)^1_+ = \mathfrak{g}^\sigma \cap \mathfrak{g}^1_+$) which corresponds to a subset $\Pi^1_+ \subset \Pi^1$ is an Abelian subalgebra if and only if there is no root β of the form*

$$\beta = \alpha + \alpha' + \sum k_i \phi_i \tag{9}$$

where $\alpha, \alpha' \in \Pi^1_+$ and $\phi_i \in \Pi^0$. The case $\alpha = \alpha'$ is allowed.

Proof. If such a root β exists, then $[\mathfrak{g}(R(\alpha), \mathfrak{g}(R(\alpha'))] \neq 0$ and $\mathfrak{g}^1_+$ is not an Abelian subalgebra. The converse is also clear. □

Proof of Proposition 7.3. Let $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ be a decomposition of Π^1. The condition (9) for $\alpha = \alpha'$ is fulfilled if and only if Π^1 is admissible. Assume now that two different vertices α, α' in Π^1_+ are connected in the Satake diagram by vertices in $\Pi^0 = \Pi \setminus \Pi^1$. Then there is a root of the form (9) and $\mathfrak{g}^1_+$ is not a commutative subalgebra. This shows that the decomposition which defines a para-CR structure on F must be alternate.

Conversely, assume that the decomposition is alternate. Then any two vertices $\alpha, \alpha' \in \Pi^1_+$ belong to different connected components of the Satake graph with deleting Π^1_-. This implies that there is no root of the form (9) for $\alpha \neq \alpha'$. Then Lemma 7.4 shows that $(\mathfrak{g}^\sigma)^1_+$ is a commutative subalgebra. The same argument is applied also for $(\mathfrak{g}^\sigma)^1_-$. □

The following proposition describes admissible subsystems Π^1 of a system Π of simple roots for any indecomposable root system R.

We enumerate simple roots of complex simple Lie $\mathfrak{g}$ algebras as in [4]. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the simple roots of $\mathfrak{g}$, which are identified with vertices of the corresponding Dynkin diagram. We denote the elements of a subset $\Pi^1 \subset \Pi$ (respectively $\Pi^1 \subset \Pi'$) which defines a fundamental gradation of $\mathfrak{g}$ (respectively $\mathfrak{g}^\sigma$) by

$$\alpha_{i_1}, \ldots, \alpha_{i_k}, \quad i_1 < i_2 < \cdots < i_k.$$

Proposition 7.5. *Let Π be a system of simple roots of a root system R of a complex simple Lie algebra $\mathfrak{g}$. Then a subset $\Pi^1 \subset \Pi$ of at least two elements is admissible (see Definition 7.1) in the following cases:*

- *for* $\mathfrak{g} = A_\ell$, *in all cases;*
- *for* $\mathfrak{g} = B_\ell$, *under the condition:* $i_k = i_{k-1} + 1$;
- *for* $\mathfrak{g} = C_\ell$, *under the condition:* $i_k = \ell$;
- *for* $\mathfrak{g} = D_\ell$, *under the condition: if* $i_k < \ell - 1$, *then* $i_k = i_{k-1} + 1$;
- *for* $\mathfrak{g} = E_6$, *in all cases except the following ones:*

$$\{\alpha_1, \alpha_4\},\ \{\alpha_1, \alpha_5\},\ \{\alpha_3, \alpha_6\},\ \{\alpha_4, \alpha_6\},\ \{\alpha_1, \alpha_4, \alpha_6\};$$

- *for* $\mathfrak{g} = E_7$, *in all cases except the following ones:*

$$\begin{gathered}\{\alpha_1, \alpha_4\},\ \{\alpha_1, \alpha_5\},\ \{\alpha_3, \alpha_6\},\ \{\alpha_4, \alpha_6\},\ \{\alpha_1, \alpha_6\},\\ \{\alpha_2, \alpha_7\},\ \{\alpha_3, \alpha_7\},\ \{\alpha_4, \alpha_7\},\ \{\alpha_5, \alpha_7\},\\ \{\alpha_1, \alpha_4, \alpha_6\},\ \{\alpha_1, \alpha_4, \alpha_7\},\ \{\alpha_1, \alpha_5, \alpha_7\},\ \{\alpha_3, \alpha_6, \alpha_7\},\ \{\alpha_4, \alpha_6, \alpha_7\},\\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\};\end{gathered}$$

- *for* $\mathfrak{g} = E_8$, *in all cases except the following ones:*

$$\begin{gathered}\{\alpha_1, \alpha_4\},\ \{\alpha_1, \alpha_5\},\ \{\alpha_3, \alpha_6\},\ \{\alpha_4, \alpha_6\},\ \{\alpha_1, \alpha_6\},\\ \{\alpha_2, \alpha_7\},\ \{\alpha_3, \alpha_7\},\ \{\alpha_4, \alpha_7\},\ \{\alpha_5, \alpha_7\},\\ \{\alpha_1, \alpha_4, \alpha_6\},\ \{\alpha_1, \alpha_4, \alpha_7\},\ \{\alpha_1, \alpha_5, \alpha_7\},\ \{\alpha_3, \alpha_6, \alpha_7\},\ \{\alpha_4, \alpha_6, \alpha_7\},\\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\},\\ \{\alpha_1, \alpha_7\},\ \{\alpha_1, \alpha_8\},\ \{\alpha_2, \alpha_8\},\ \{\alpha_3, \alpha_8\},\ \{\alpha_4, \alpha_8\},\ \{\alpha_5, \alpha_8\},\ \{\alpha_6, \alpha_8\},\\ \{\alpha_1, \alpha_4, \alpha_8\},\ \{\alpha_1, \alpha_5, \alpha_8\},\ \{\alpha_3, \alpha_6, \alpha_8\},\ \{\alpha_4, \alpha_6, \alpha_8\},\ \{\alpha_1, \alpha_6, \alpha_8\},\\ \{\alpha_2, \alpha_7, \alpha_8\},\ \{\alpha_3, \alpha_7, \alpha_8\},\ \{\alpha_4, \alpha_7, \alpha_8\},\ \{\alpha_5, \alpha_7, \alpha_8\},\\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_8\},\ \{\alpha_1, \alpha_4, \alpha_7, \alpha_8\},\ \{\alpha_1, \alpha_5, \alpha_7, \alpha_8\},\ \{\alpha_3, \alpha_6, \alpha_7, \alpha_8\},\\ \{\alpha_4, \alpha_6, \alpha_7, \alpha_8\},\\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_8\};\end{gathered}$$

- *for* $\mathfrak{g} = F_4$, *in all cases except the following ones:*

$$\{\alpha_1, \alpha_3\},\ \{\alpha_1, \alpha_4\},\ \{\alpha_2, \alpha_4\},\ \{\alpha_3, \alpha_4\},\ \{\alpha_1, \alpha_3, \alpha_4\};$$

- *for* $\mathfrak{g} = G_2$, *in the case* $\{\alpha_1, \alpha_2\}$.

In cases different from D_ℓ, E_6, E_7 *and* E_8, *for any* Π^1 *given as above it is possible to give an alternate decomposition* $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$.

For D_ℓ, *an alternate decomposition of* Π^1 *can be given in the following cases:*

- $\alpha_{\ell-2} \in \Pi^1$,
- Π^1 *is contained in at most two of the branches issuing from* $\alpha_{\ell-2}$.

For E_6, E_7 *and* E_8, *an alternate decomposition of* Π^1 *can be given in the following cases:*

- $\alpha_4 \in \Pi^1$,
- Π^1 *is contained in at most two of the branches issuing from* α_4.

Proof. We have to describe all subsets Π^1 of Π which satisfy (8). This condition can be reformulated as follows. For any $\alpha \in \Pi^1$, denote by Π_α the connected component of the subdiagram of the Dynkin diagram Π obtained by deleting vertices in $\Pi^1 \setminus \{\alpha\}$ and containing α. Then the root system associated with Π_α has no roots of the form

$$\beta = 2\alpha + \sum_{\phi \in \Pi_\alpha \setminus \{\alpha\}} k_\phi \phi.$$

Using this condition and the decomposition of any root into a linear combination of simple roots, one can prove the proposition.

In the case of A_ℓ, any root has coefficient $0, 1$ in the decomposition into simple roots. Hence, any decomposition satisfies the property (8).

In the case of B_ℓ, any root which has coefficient 2 has the form

$$\sum_{i \le h < j} \alpha_h + 2 \sum_{j \le h \le \ell} \alpha_h \quad (1 \le i < j \le \ell).$$

Hence the condition (8) holds if and only if the last two roots in Π^1 are consecutive, i.e., $i_{k-1} + 1 = i_k$.

In the case of C_ℓ, the roots with a coefficient 2 are given by

$$\sum_{i \le h < j} \alpha_h + 2 \sum_{j \le h < \ell} \alpha_h + \alpha_\ell \quad (1 \le i < j \le \ell),$$
$$2 \sum_{i \le h < \ell} \alpha_h + \alpha_\ell \quad (1 \le i < \ell).$$

The second formula implies that there are no roots of the form given in (8) if and only if $i_k = \ell$.

In the case of D_ℓ, the roots with a coefficient 2 are

$$\sum_{i \le h < j} \alpha_h + 2 \sum_{j \le h < \ell - 1} \alpha_h + \alpha_{\ell-1} + \alpha_\ell \quad (1 \le i < j < \ell - 1).$$

The condition (8) fails if and only if the last two roots $\alpha_{i_{k-1}}, \alpha_{i_k}$ satisfy $i_{k-1} < i_k - 1$ and $i_k < \ell - 1$.

The case of exceptional Lie algebras can be treated in a similar way, by using tables in [4]. □

Let $\Pi^1 \subset \Pi'$ be an admissible subset which defines a fundamental gradation of $\mathfrak{g}^\sigma$. An alternate decomposition of $\Pi^1 = \Pi^1_+ \cup \Pi^1_+$ can be given if the conditions of Proposition 7.5 are satisfied and, in addition, the following ones hold:

- for $\mathfrak{su}(p, q)$, it has to be $q = p$ and $\alpha_p \in \Pi^1$;
- for $\mathfrak{so}(\ell - 1, \ell + 1)$, it has to be $\Pi^1 \cap \{\alpha_{\ell-1}, \alpha_\ell\} = \emptyset$ or $\{\alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell\} \subset \Pi^1$;

- for E_6II, it has to be $\alpha_4 \in \Pi^1$ and if $\alpha_2 \notin \Pi^1$, then $\{\alpha_3, \alpha_5\} \subset \Pi^1$;

while for $\mathfrak{so}^*(2\ell)$ and E_6III there is no alternate decomposition of Π^1.

Proposition 7.3 implies the following final theorem.

Theorem 7.6. *Let $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ be a Satake diagram of a simple real Lie algebra $\mathfrak{g}^\sigma$ and $\Pi^1 \subset \Pi'$ be an admissible subset as described above. Let $\widetilde{G}$ be the simply connected Lie group with the Lie algebra $\mathfrak{g}^\sigma$ and $\widetilde{P}$ be the parabolic subgroup of $\widetilde{G}$ generated by the non-negatively graded subalgebra*

$$\mathfrak{p} = \sum_{i \geq 0} (\mathfrak{g}^\sigma)^i$$

associated with the grading element $\vec{d}_{\Pi^1}$. Then the alternate decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ defines a decomposition

$$(\mathfrak{g}^\sigma)^1 = (\mathfrak{g}^\sigma)^1_+ + (\mathfrak{g}^\sigma)^1_-$$

of the $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^1$ into a sum of two commutative subalgebras. This decomposition determines an invariant para-CR structure on the simply connected flag manifold $F = \widetilde{G}/\widetilde{P}$. Moreover, any simply connected maximally homogeneous para-CR manifolds of semisimple type is a direct product of such manifolds.

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Part E

Holonomy theory

Chapter 17

Recent developments in pseudo-Riemannian holonomy theory

Anton Galaev and Thomas Leistner

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1 Introduction

In the study of geometric structures of manifolds equipped with a non-degenerate metric the notion of a holonomy group turned out to be very useful. It links geometric and algebraic properties and allows to apply the tools of algebra to geometric questions. In particular, it enables us describe parallel sections in geometric vector bundles associated to the manifold, such as the tangent bundle, tensor bundles, or the spin bundle, as holonomy-invariant objects and by algebraic means. Hence, a clas-

sification of holonomy groups gives a framework in which geometric structures on semi-Riemannian manifolds can be studied.

Many interesting developments in differential geometry were initiated or driven by the study and the knowledge of holonomy groups, such as the study of so-called *special geometries* in Riemannian geometry. These developments were based on the classification result of Riemannian holonomy groups, which was achieved by the de Rham decomposition theorem [36] and the Berger list of irreducible pseudo-Riemannian holonomy groups [14] (see Section 2.2 of the present work).

For manifolds with indefinite metrics this question was for a long time widely open and untackled, apart from a classification regarding 4-dimensional Lorentzian metrics by J. F. Schell [75] and R. Shaw [79] (presented in Section 3.7). The main difficulty in the case of pseudo-Riemannian manifolds is the situation that the holonomy group preserves a degenerate subspace of the tangent space. In this situation the de Rham theorem does not apply, and one cannot reduce the algebraic aspect of the classification problem to irreducible representations. However, the Wu theorem [86] and the results of Berger [14] reduce the task of classifying holonomy groups of pseudo-Riemannian manifolds to the case of indecomposably, non-irreducibly acting groups (any such group does not preserve any proper non-degenerate subspace of the tangent space, but preserves an isotropic subspace of the tangent space, see Section 2.2). In this chapter we want to present how this problem can be dealt with and which classifications results have been obtained recently by applying this method.

After explaining the basic properties of affine and semi-Riemannian holonomy groups in Section 2, we discuss the classification of the holonomy algebras (equivalently, connected holonomy groups) of Lorentzian manifolds in Section 3. The first step in this classification was done in 1993 by L. Bérard-Bergery and A. Ikemakhen who divided indecomposable, non-irreducible subalgebras of $\mathfrak{so}(1,n+1)$ into 4 types, see [11]. In [47] a more geometric proof of this result is given, which is presented in Section 3.2. To each indecomposable, non-irreducible subalgebra of $\mathfrak{h} \subset \mathfrak{so}(1,n+1)$ one can associate a subalgebra of $\mathfrak{so}(n)$, which is called the orthogonal part of $\mathfrak{h}$. In [66], [68], [69] (see also [70], [73]) it is proved that the orthogonal part of an indecomposable, non-irreducible holonomy algebra of a Lorentzian manifold is the holonomy algebra of a Riemannian manifold (see Section 3.3). In [48] metrics for all possible holonomy algebras of Lorentzian manifold were constructed (Section 3.4). This completes the classification of holonomy algebras for Lorentzian manifolds. There are many applications of holonomy theory for Lorentzian manifolds such as the study of equations motivated by physics in relation to the possible holonomy groups. On the one hand these are the Einstein equations, on the other hand certain spinor field equations in supergravity theories (see e.g., [42]). In Section 3.5 we present the classification of holonomy groups of indecomposable Lorentzian manifolds which admit a parallel spinor. In Section 3.6 we describe holonomy related geometric structures on Lorentzian manifolds.

Widely open is the classification problem of holonomy groups in signatures other than Riemannian and Lorentzian apart from some results in certain signatures.

In Section 4 we discuss the holonomy of pseudo-Riemannian manifolds of index 2. For indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2, n+2)$ which satisfy a certain condition Ikemakhen gave in [56] a distinction into different types similar to the Lorentzian case. In [44] the analog of the orthogonal part of an indecomposable, non-irreducible subalgebra in $\mathfrak{so}(2, n+2)$ is studied. The surprising result is that, unlike to the Lorentzian case, there is no additional condition on the subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$ induced by the Bianchi-identity and replacing the property which turned out to be essential in Lorentzian signature. Instead, any subalgebra of $\mathfrak{so}(n)$ can be realised as this part of a holonomy algebra, see Section 4.1. Furthermore, in [45] indecomposable, non-irreducible holonomy algebras of pseudo-Kählerian manifolds of index 2, i.e., holonomy algebras contained in $\mathfrak{u}(1, n+1) \subset \mathfrak{so}(2, 2n+2)$, were classified. We present this classification with the idea of the proof in Section 4.2. In Section 4.3 examples of 4-dimensional Lie groups with left-invariant pseudo-Kählerian metrics are given.

In Section 5 we review the results known about holonomy for metrics of neutral signature (n, n), again obtained by Bérard-Bergery and Ikemakhen in [12]. They also gave a list of possible holonomy groups in signature $(2, 2)$, and realised those in the list which leave invariant two complementary totally isotropic planes as holonomy algebras. Our results in [45] enables us to realise all subalgebras of $\mathfrak{u}(1, 1)$ from this list. As a new result we give metrics realising two algebras which were still in question to be realised. We should remark, that some of these metrics disprove claims made in [49] that the corresponding algebras cannot be realised as holonomy algebras. This result completes the classification of holonomy groups in signature (2,2), apart from one exception, for which we could not find a metric. Upon completion of this exposition we were informed that L. Bérard-Bergery and T. Krantz have developed a construction on the cotangent bundle of a surface which ensures that even this last group can be realised as a holonomy group [13].

2 Holonomy groups

2.1 Holonomy groups of linear connections

If a smooth manifold M of dimension m is equipped with a linear connection ∇ on the tangent bundle TM, we can parallel translate a tangent vector $X \in T_pM$ at a point $p \in M$ along any given curve $\gamma\colon [0,1] \to M$ starting at p, i.e., $\gamma(0) = p$. The parallel displacement, denoted by $X(t)$, is a vector field along γ satisfying the equation $\nabla_{\dot\gamma(t)} X(t) = 0$ for all t in the domain of the curve. This is a linear ordinary differential equation, and thus, for any curve γ the map

$$\begin{aligned} \mathcal{P}_{\gamma(t)}\colon T_{\gamma(0)}M &\longrightarrow T_{\gamma(t)}M, \\ X &\longmapsto \mathcal{P}_{\gamma(t)}(X) := X(t) \end{aligned}$$

is a vector space isomorphism which is called *parallel displacement*. Hence, ∇ enables us to link the tangent spaces in different points, which is the reason why it bears the name *connection*. Then the *holonomy group* of ∇ at p is the group defined by parallel displacements along loops about this point,

$$\mathrm{Hol}_p(M,\nabla) := \{\mathcal{P}_{\gamma(1)} \mid \gamma(0) = \gamma(1) = p\}.$$

This group is a Lie group which is connected if the manifold is simply connected. Its connected component is called *connected holonomy group*, denoted by $\mathrm{Hol}_p^0(M,\nabla)$. This is the group generated by parallel displacements along homotopically trivial loops. Its Lie algebra $\mathfrak{hol}_p(M,\nabla)$ its called *holonomy algebra*. Obviously, both are given together with their representation on the tangent space T_pM which is usually identified with $\mathbb{R}^m$. In this sense we have that $\mathrm{Hol}_p(M,\nabla) \subset \mathrm{GL}(m,\mathbb{R})$, but only defined up to conjugation. Holonomy groups at different points in a connected component of the manifold are conjugated by an element in $\mathrm{GL}(m,\mathbb{R})$, which is obtained by the parallel displacement along a curve joining these different points. It is worthwhile to note that the holonomy group is closed if it acts irreducibly (for a proof of this fact see [82] or [38]). This is not true in general, there are examples of non-closed holonomy groups.

The calculation of holonomy groups uses the *Ambrose–Singer holonomy theorem*, which states that for a manifold M with linear connection ∇ the holonomy algebra $\mathfrak{hol}_p(M,\nabla)$ is equal to

$$\mathrm{span}\big\{\mathcal{P}_{\gamma(t)}^{-1} \circ \mathcal{R}\big(\mathcal{P}_{\gamma(t)}X, \mathcal{P}_{\gamma(t)}Y\big) \circ \mathcal{P}_\gamma(t) \mid \gamma(0) = p,\ X, Y \in T_pM\big\},$$

where $\mathcal{R}$ is the curvature of ∇, $\mathcal{R}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. For connections that have no torsion, i.e., $\nabla_X Y - \nabla_Y X = [X,Y]$, the curvature satisfies the first Bianchi-identity

$$\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0,$$

which imposes very strong algebraic conditions on the holonomy algebra which can be described in terms of *curvature endomorphisms*. Let $\mathbb{K}$ be the real or complex numbers. The curvature endomorphisms of a subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{K})$ are defined as

$$\mathcal{K}(\mathfrak{g}) := \{R \in \Lambda^2(\mathbb{K}^n)^* \otimes \mathfrak{g} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0\}. \quad (1)$$

$\mathcal{K}(\mathfrak{g})$ is a $\mathfrak{g}$-module, and the space $\mathfrak{g}^{\mathcal{K}} := \mathrm{span}\{R(x,y) \mid x, y \in \mathbb{K}^n, R \in \mathcal{K}(\mathfrak{g})\}$ is an ideal in $\mathfrak{g}$. One defines:

Definition 2.1. $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{K})$ is called *Berger algebra* if $\mathfrak{g}^{\mathcal{K}} = \mathfrak{g}$.

Then, the Ambrose–Singer holonomy theorem implies the following result.

Theorem 2.2. *The Lie algebra of a holonomy group of a torsion-free connection on a smooth manifold is a Berger algebra.*

Hence, there are two steps involved in the classification of holonomy groups of torsion-free connections. The first is to classify Berger algebras, and the second to

find torsion-free connections with the algebras obtained as holonomy algebras. The first problem can be solved in full generality when the Berger algebra acts *irreducibly*. This was done recently by S. Merkulov and L. Schwachhöfer in [74], [77], [78], also providing examples of torsion-free connections realising all the algebras obtained. This classification extends the well-known *Berger list* of irreducible holonomy groups of pseudo-Riemannian manifolds (c.f. next section). However, the assumption of irreducibility is essential for these classification results because their proof uses the theory of irreducible representations of Lie algebras. An overview about results on irreducible holonomy groups is also given in [25] and [24]. As we will see later, the approach we have to take for connections of indefinite metrics has to deal with non-irreducible representations.

One should remark that a classification problem for holonomy groups only arises if one poses further conditions on the linear connection, such as conditions on the torsion because of the following result of J. Hano and H. Ozeki [54]: Any closed subgroup of $\mathrm{GL}(m,\mathbb{R})$ can be obtained as a holonomy group of a linear connection, but possibly a connection with torsion. They also gave examples of holonomy groups which were not closed. We will return to this question later.

Concluding this introductory section we want to point out a general principle in holonomy theory which says that any subspace which is invariant under the holonomy group corresponds to a distribution (i.e., a subbundle of the tangent bundle) which is invariant under parallel transport. Obviously, this distribution is obtained by parallel transporting the invariant subspace, and this procedure is independent of the chosen path because of the holonomy invariance of the subspace. This distribution is called *parallel*, which means that its sections are mapped onto its sections under ∇_X for any $X \in TM$.

2.2 Holonomy groups of semi-Riemannian manifolds

(M,g) is a semi-Riemannian manifold of dimension $m = r + s$ and signature (r,s) if g is a metric of signature (r,s). If the metric is positive definite ($r = 0$ in our convention) it is called *Riemannian*, otherwise *pseudo-Riemannian*. For a semi-Riemannian manifold, there exists a uniquely defined linear torsion-free connection $\nabla = \nabla^g$ which parallelises the metric, called *Levi-Civita connection*. The holonomy group of a semi-Riemannian manifold then is the holonomy group of this connection, $\mathrm{Hol}_p(M,g) := \mathrm{Hol}_p(M,\nabla)$. As the Levi-Civita connection is metric, the parallel displacement preserves the metric. This implies on the one hand that the holonomy group is a subgroup of $\mathrm{O}(T_pM,g)$, and can be understood as a subgroup of $\mathrm{O}(r,s)$ which is only defined up to conjugation in $\mathrm{O}(r,s)$. On the other hand it ensures that for a subspace $\mathcal{V} \subset T_pM$ which is invariant under the holonomy group the orthogonal complement $\mathcal{V}^\perp$ is invariant as well.

For a Riemannian metric the holonomy group acts completely reducibly, i.e., the tangent space decomposes into subspaces on which it acts trivially or irreducibly, but for indefinite metrics the situation is more subtle. We say that the holonomy group

acts *indecomposably* if the metric is degenerate on any invariant proper subspace. In this case we also say that the manifold is *indecomposable*. Of course, for Riemannian manifolds, this is the same as irreducibility.

A remarkable property is that the holonomy group of a product of Riemannian manifolds (i.e., equipped with the product metric) is the product of the holonomy groups of these manifolds (with the corresponding representation on the direct sum). Even more remarkable is the fact that a converse of this statement is true in the following sense: Any pseudo-Riemannian manifold whose tangent space at a point admits a decomposition into *non-degenerate*, holonomy-invariant subspaces is locally isometric to a product of pseudo-Riemannian manifolds corresponding to the invariant subspaces, and moreover, the holonomy group is a product of the groups acting on the corresponding invariant subspaces. These groups are the holonomy groups of the manifolds in the local product decomposition if the original manifold is complete (see Theorem 10.38 and Remark 10.42 in [16]). This was proven by A. Borel and A. Lichnerowicz [18], and the property that a decomposition of the representation space entails a decomposition of the acting group is sometimes called *Borel–Lichnerowicz property*. A global version of this statement was proven under the assumption that the manifold is simply-connected and complete by G. de Rham [36] (for Riemannian manifolds) and H. Wu [86] (in arbitrary signature). Summarizing we have the following result:

Theorem 2.3 (G. de Rham [36] and H. Wu [86]). *Any simply-connected, complete pseudo-Riemannian manifold (M, g) is isometric to a product of simply connected, complete pseudo-Riemannian manifolds one of which can be flat and the others have an indecomposably acting holonomy group and the holonomy group of (M, g) is the product of these indecomposably acting holonomy groups.*

The other groundbreaking result in the holonomy theory of semi-Riemannian manifolds is the list of irreducible holonomy groups of non locally-symmetric pseudo-Riemannian manifolds, which was obtained by M. Berger [14]. This list as it appears here is a result of the efforts of several other authors, simplifying the proof in Riemannian signature [80], eliminating groups of locally symmetric metrics [1], [22], realising the exceptional groups as holonomy groups [23], eliminating groups which were not Berger algebras and finding missing entries (for an overview see [24]).

Theorem 2.4 (M. Berger [14]). *Let (M, g) be a simply connected pseudo-Riemannian manifold of dimension $m = r + s$ and signature (r, s), which is not locally-symmetric. If the holonomy group of (M, g) acts irreducibly, then it is either $\mathrm{SO}_0(r, s)$ or one of the following (modulo conjugation in $\mathrm{O}(r, s)$):*

$$\mathrm{U}(p, q) \text{ or } \mathrm{SU}(p, q) \subset \mathrm{SO}(2p, 2q), \quad m \geq 4,$$

$$\begin{aligned}
\mathrm{Sp}(p,q) \text{ or } \mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1) &\subset \mathrm{SO}(4p,4q), && m \geq 8,\\
\mathrm{SO}(r,\mathbb{C}) &\subset \mathrm{SO}(r,r), && m \geq 4,\\
\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R}) &\subset \mathrm{SO}(2p,2p), && m \geq 8,\\
\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C}) &\subset \mathrm{SO}(4p,4p), && m \geq 16,\\
\mathrm{G}_2 &\subset \mathrm{SO}(7),\\
\mathrm{G}^*_{2(2)} &\subset \mathrm{SO}(4,3),\\
\mathrm{G}_2^{\mathbb{C}} &\subset \mathrm{SO}(7,7),\\
\mathrm{Spin}(7) &\subset \mathrm{SO}(8),\\
\mathrm{Spin}(4,3) &\subset \mathrm{SO}(4,4),\\
\mathrm{Spin}(7)^{\mathbb{C}} &\subset \mathrm{SO}(8,8).
\end{aligned}$$

We should remark that a lot of progress has been made in constructing Riemannian manifolds with given holonomy group and certain *topological properties*. We only mention a few: compact manifolds with holonomy $\mathrm{Sp}(q)$ have been constructed in [8], see also [16]. In [64] complete manifolds with holonomy $\mathrm{Sp}(1)\cdot\mathrm{Sp}(q)$ were constructed, for compact examples with this holonomy group see [65]. Complete examples with exceptional holonomy G_2 and $\mathrm{Spin}(7)$ were constructed in [27] and compact ones in [59], [58]. For an overview of related results see [24]. For indefinite metrics these global questions are widely open, apart from attempts in [7], where globally hyperbolic Lorentzian metrics with indecomposable, non-irreducible holonomy group (see next section) were constructed.

Returning to the classification problem, for Riemannian manifolds one combines these classification results with the de Rham decomposition in order to obtain a comprehensive holonomy classification.

Theorem 2.5. *Any simply-connected, complete Riemannian manifold (M,g) is isometric to a product of simply-connected, complete Riemannian manifolds one of which may be flat and the others are either locally symmetric or have one of the following groups as holonomy,* $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, G_2, *or* $\mathrm{Spin}(7)$. *The holonomy group of (M,g) is a product of these groups.*

In case of symmetric spaces, the holonomy group is equal to the isotropy group of the symmetric space and in many cases determines the space up to duality. Simply connected pseudo-Riemannian symmetric spaces with irreducible holonomy groups were classified by E. Cartan [33] (for Riemannian signature) and M. Berger [15] (for arbitrary signature). In the case of indecomposable, non-irreducible holonomy group the classification exists in the following cases: Lorentzian signature [30], signature $(2,q)$ [28], [29], [60], hyper-Kählerian manifolds of signature $(4,4q)$ [4], [61]. For more details see the overview [62].

For indefinite metrics there is the possibility that one of the factors in Theorem 2.3 is indecomposable, but non-irreducible. This means that the holonomy representa-

tion admits an invariant subspace on which the metric is degenerate, but no proper non-degenerate invariant subspace. An attempt to classify holonomy groups for indefinite metric has to provide a classification of these indecomposable, non-irreducible holonomy groups.

If a holonomy group $\mathrm{Hol}_p(M,h) =: H \subset \mathrm{SO}_0(p,q)$ acts indecomposably, but non-irreducibly, with an degenerate invariant subspace $\mathcal{V} \subset T_pM$, it admits a totally isotropic invariant subspace

$$\mathcal{I} := \mathcal{V} \cap \mathcal{V}^{\perp}.$$

This implies that H is contained in the stabiliser of this totally isotropic subspace,

$$H \subset \mathrm{SO}_0(p,q)_{\mathcal{I}} := \{A \in \mathrm{SO}_0(p,q) \mid A\mathcal{I} \subset \mathcal{I}\},$$

or in terms of the corresponding Lie algebras

$$\mathfrak{h} \subset \mathfrak{so}(p,q)_{\mathcal{I}} := \{X \in \mathfrak{so}(p,q) \mid X\mathcal{I} \subset \mathcal{I}\}.$$

In the following sections we will present results about the classification of holonomy groups contained in $\mathfrak{so}(p,q)_{\mathcal{I}}$ for $\mathcal{I}$ a totally isotropic subspace.

Finally we should mention that by the general principle, $\mathcal{I}$ defines a totally isotropic distribution on M, i.e., a subbundle of TM which is invariant under parallel transport. This parallel distribution ensures the existence of so-called *Walker coordinates* (first in [83], [84], [85], see also [37]).

Theorem 2.6 (Walker [83], [84], [85]). *Let (M,h) be a pseudo-Riemannian manifold of dimension n with a parallel r-dimensional totally isotropic distribution $\mathcal{I}$. The exist coordinates $(x_1,\dots,x_n)$ such that*

$$\left(h\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\right)_{i,j=1}^{n} = \begin{pmatrix} 0 & 0 & \mathbb{I}_r \\ 0 & G & F \\ \mathbb{I}_r & F^t & H \end{pmatrix},$$

where F, G, H are matrices of smooth functions, G is a symmetric $(n-2r)\times(n-2r)$ matrix, H is a symmetric $r\times r$ matrix, and F is an $r\times(n-2r)$ matrix, such that G and F are independent of the coordinates $(x_1,\dots,x_r)$.

These coordinates will be useful in order to obtain metrics which realise the possible indecomposable, non-irreducible holonomy groups.

3 Lorentzian holonomy groups

3.1 The classification result

In this section we want to describe the classification of reduced holonomy groups of Lorentzian manifolds. First of all, the Berger list in Theorem 2.4 implies that the

only *irreducible* holonomy group of Lorentzian manifolds is the full $\mathrm{SO}_0(1,n)$. This is due to the algebraic fact that the only connected irreducible subgroup of $\mathrm{O}(1,n)$ is $\mathrm{SO}_0(1,n)$ which was proven by A. J. Di Scala and C. Olmos [39] (see also [20], [10], [38] for other proofs). Hence, if one is interested in Lorentzian manifolds with *special holonomy*, i.e., with proper subgroups of $\mathrm{SO}_0(1,n)$ as holonomy but not being a product, one has to look at manifolds admitting a holonomy-invariant subspace. Using this fact, the decomposition in Theorem 2.3 gives the following result for Lorentzian manifolds.

Corollary 3.1. *Any simply-connected, complete Lorentzian manifold (M,h) is isometric to the following product of simply-connected complete pseudo-Riemannian manifolds,*

$$(N,h)\times(M_1,g_1)\times\cdots\times(M_k,g_k),$$

where the (M_i,g_i) are either flat or irreducible Riemannian manifolds and (N,h) is either $(\mathbb{R},-dt^2)$ or an indecomposable Lorentzian manifold, the holonomy of which is either $\mathrm{SO}_0(1,n)$ or contained in the stabiliser $\mathrm{SO}_0(1,n)_{\mathcal{I}}$ of a light-like line $\mathcal{I}$. The holonomy group of (M,h) is the product of the holonomy groups of (N,h) and the (M_i,g_i)'s.

Hence, we have to focus on the classification of Lorentzian holonomy groups which act indecomposably, but non-irreducibly, i.e., which are contained in the stabiliser in $\mathrm{SO}_0(T_pM)$ of a light-like line $\mathcal{I}$ in T_pM. This stabiliser is the parabolic group $\mathrm{SO}_0(1,n+1)_{\mathcal{I}}$ in the conformal group $\mathrm{SO}_0(1,n+1)$ if $(n+2)$ is the dimension of M. To describe this stabiliser further we identify T_pM with the Minkowski space $\mathbb{R}^{1,n+1}$ of dimension $(n+2)$ and fix a basis $(X,E_1,\dots,E_n,Z)$ in which the scalar product has the form

$$\begin{pmatrix}0&0^t&1\\0&\mathbb{I}_n&0\\1&0^t&0\end{pmatrix}, \tag{1}$$

where $\mathbb{I}_n$ is the n-dimensional identity matrix. The Lie algebra of the connected stabiliser of $\mathcal{I}=\mathbb{R}\cdot X$ inside the conformal group $\mathrm{SO}_0(1,n+1)$ can be written as follows

$$\mathfrak{so}(1,n+1)_{\mathcal{I}}=\left\{\begin{pmatrix}a&v^t&0\\0&A&-v\\0&0^t&-a\end{pmatrix}\;\middle|\;a\in\mathbb{R},\ v\in\mathbb{R}^n,\ A\in\mathfrak{so}(n)\right\}. \tag{2}$$

This Lie algebra is a semi-direct sum in an obvious way, $\mathfrak{so}(1,n+1)_{\mathcal{I}}=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n$, the commutator relations are given as follows:

$$[(a,A,v),(b,B,w)]=\big(0,[A,B]_{\mathfrak{so}(n)},(A+a\,\mathrm{Id})\,w-(B+b\,\mathrm{Id})\,v\big). \tag{3}$$

In this sense we will refer to $\mathbb{R}$, $\mathbb{R}^n$ and $\mathfrak{so}(n)$ as subalgebras of $\mathfrak{so}(1,n+1)_{\mathcal{I}}$. $\mathbb{R}$ is an Abelian subalgebra of $\mathfrak{so}(1,n+1)_{\mathcal{I}}$, commuting with $\mathfrak{so}(n)$, and $\mathbb{R}^n$ an Abelian ideal in $\mathfrak{so}(1,n+1)_{\mathcal{I}}$. $\mathfrak{so}(n)$ is the semisimple part and $\mathfrak{co}(n)=\mathbb{R}\oplus\mathfrak{so}(n)$

the reductive part of $\mathfrak{so}(1,n+1)_I$. The corresponding connected Lie groups in $\mathrm{SO}_0(1,n+1)_I$ are $\mathbb{R}_+$, $\mathrm{SO}(n)$, and $\mathbb{R}^n$, and $\mathrm{SO}_0(1,n+1)_I$ is equal to the semidirect product $(\mathbb{R}_+ \times \mathrm{SO}(n)) \ltimes \mathbb{R}^n$.

Now one can assign to a subalgebra $\mathfrak{h} \subset \mathfrak{so}(1,n+1)_I$ the projections $\mathrm{pr}_{\mathbb{R}}(\mathfrak{h})$, $\mathrm{pr}_{\mathbb{R}^n}(\mathfrak{h})$ and $\mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$. The subalgebra $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$ associated to a $\mathfrak{h}$ is called *the orthogonal part of* $\mathfrak{h}$. Note that if $\mathfrak{h} \subset \mathfrak{so}(1,n+1)_I$ acts indecomposably , then $\mathrm{pr}_{\mathbb{R}^n}(\mathfrak{h}) = \mathbb{R}^n$, and $\mathfrak{h}$ is Abelian if and only if $\mathfrak{h} = \mathbb{R}^n$. Moreover, $\mathfrak{h}$ has a trivial subrepresentation if and only if $\mathrm{pr}_{\mathbb{R}}\mathfrak{h} = 0$. In this case (M,h) admits a parallel light-like vector field. Finally $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{h}) \subset \mathfrak{so}(n)$ is compact, i.e., there exists a positive definite invariant symmetric bilinear form on it. This implies that $\mathfrak{g}$ is *reductive*, i.e., its Levi decomposition is $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}' := [\mathfrak{g},\mathfrak{g}] = \mathfrak{z}^{\perp}$ is the derived Lie algebra, which is semisimple.

Now, the classification of indecomposable, non-irreducible Lorentzian holonomy algebras consists of two main results. The first is the distinction of indecomposable subalgebras of $\mathfrak{so}(1,n+1)_I$ into four types due to the relation between their projections obtained by L. Bérard-Bergery and A. Ikemakhen [11].

Theorem 3.2 ([11]). *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{so}(1,n+1)_I = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$, acting indecomposably on $\mathbb{R}^{n+2}$, and let $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{h}) = \mathfrak{z} \oplus \mathfrak{g}'$ be the Levi-decomposition of its orthogonal part. Then $\mathfrak{h}$ belongs to one of the following types.*

(1) *If $\mathfrak{h}$ contains $\mathbb{R}^n$, then there are three types:*

Type 1. *$\mathfrak{h}$ contains $\mathbb{R}$. Then $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$.*

Type 2. *$\mathrm{pr}_{\mathbb{R}}(\mathfrak{h}) = 0$, i.e., $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$.*

Type 3. *Neither type* 1 *nor type* 2. *In this case there exists an epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, such that*

$$\mathfrak{h} = \left(\mathfrak{l} \oplus \mathfrak{g}'\right) \ltimes \mathbb{R}^n,$$

where $\mathfrak{l} := \mathrm{graph}\ \varphi = \{(\varphi(T), T) \mid T \in \mathfrak{z}\} \subset \mathbb{R} \oplus \mathfrak{z}$. Or, written in matrix form:

$$\mathfrak{h} = \left\{ \begin{pmatrix} \varphi(A) & v^t & 0 \\ 0 & A+B & -v \\ 0 & 0 & -\varphi(A) \end{pmatrix} \;\middle|\; A \in \mathfrak{z},\ B \in \mathfrak{g}',\ v \in \mathbb{R}^n \right\}.$$

(2) *In the case where $\mathfrak{h}$ does not contain $\mathbb{R}^n$ we have*

Type 4. *There exists a non-trivial decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$, $0 < k,l < n$ and a epimorphism $\psi : \mathfrak{z} \to \mathbb{R}^l$, such that $\mathfrak{g} \subset \mathfrak{so}(k)$ and $\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{g}') \ltimes \mathbb{R}^k \subset \mathfrak{so}(1,n+1)_I$ where $\mathfrak{l} := \{(\varphi(T), T) \mid T \in \mathfrak{z}\} = \mathrm{graph}\ \psi \subset \mathbb{R}^l \oplus \mathfrak{z}$. Or, written in matrix form:*

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(A)^t & v^t & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A+B & -v \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{z},\ B \in \mathfrak{g}',\ v \in \mathbb{R}^k \right\}.$$

The distinction of Theorem 3.2 obviously gives four types for the corresponding connected, indecomposable groups in the parabolic $\mathrm{SO}_0(1,n+1)_I$. We should remark that these types are independent of conjugation within $\mathrm{O}(1,n+1)$.

The second result gives a classification of the orthogonal part and was proved in [66], [68], [69, 48].

Theorem 3.3. *Let H be a connected subgroup of $\mathrm{SO}_0(1,n+1)$ which acts indecomposably and non-irreducibly. Then H is a Lorentzian holonomy group if and only if its orthogonal part is a Riemannian holonomy group.*

Naturally, the proof of this theorem consists of two main steps. The first is to show that the orthogonal part of H has to be a Riemannian holonomy group. This involves the notion of weak-Berger algebras and their classification, which is explained in Section 3.3. This step uses similar methods as the classification of irreducible holonomy groups of torsion-free connections and was done in [66], [68], [69]. The second step consists of showing that each of the arising groups can actually be realised as holonomy group. This is easy for the types 1 and 2 in Theorem 3.2 (see for example [67], [70]) but more involved for the coupled types 3 and 4 and was achieved recently in [48]. This method is explained in Section 3.4.

3.2 Indecomposable subalgebras of $\mathfrak{so}(1,n+1)_I$

The proof of Theorem 3.2 given by L. Bérard-Bergery and A. Ikemakhen was purely algebraic. We describe now a more geometric proof of this theorem given in [47], which works directly for the groups and provides a geometric interpretation for the different types. Therefore we equip $\mathbb{R}^n$ with the Euclidean scalar product. Denote by $\mathrm{Sim}(n)$ the connected component of the Lie group of similarity transformations of $\mathbb{R}^n$. Then $\mathbb{R}_+$, $\mathrm{SO}(n)$, and $\mathbb{R}^n$ in $\mathrm{Sim}(n)$ are the connected identity components of the Lie groups of homothetic transformations, rotations and translations, respectively. We obtain for $\mathrm{Sim}(n)$ the same decomposition as for $P := \mathrm{SO}_0(1,n+1)_I$, i.e., we have a Lie group isomorphism $\Gamma\colon P \to \mathrm{Sim}(n)$. The isomorphism Γ can be defined geometrically. For this consider the vector model of the real hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}^{1,n+1}$ and its boundary $\partial\mathbb{H}^{n+1} \subset \mathbb{P}\mathbb{R}^{1,n+1}$ that consists of isotropic lines of $\mathbb{R}^{1,n+1}$ and is isomorphic to the n-dimensional sphere. Any element $f \in P$ induces a transformation $\Gamma(f)$ of the Euclidean space $\partial\mathbb{H}^{n+1}\backslash\{\mathbb{R}\cdot X\} \simeq \mathbb{R}^n$. In fact, $\Gamma(f)$ is a similarity transformation of $\mathbb{R}^n$. This defines the isomorphism Γ. Now, in [47] we have proven that *a connected Lie subgroup $H \subset P$ is indecomposable if and only if the subgroup $\Gamma(H) \subset \mathrm{Sim}(n)$ acts transitively on $\mathbb{R}^n$*. Then, using a description for connected transitive subgroups of $\mathrm{Sim}(n)$ given in [2] and [3], one can show the following theorem.

Theorem 3.4. *A connected Lie subgroup $H \subset \mathrm{Sim}(n)$ is transitive if and only if H belongs to one of the following types for which $G \subset \mathrm{SO}(n)$ is a connected Lie subgroup:*

Type 1. $H = (\mathbb{R}_+ \times G) \ltimes \mathbb{R}^n$;

Type 2. $H = G \ltimes \mathbb{R}^n$;

Type 3. $H = (\mathbb{R}_+^\Phi \times G) \ltimes \mathbb{R}^n$, *where* $\Phi\colon \mathbb{R}_+ \to \mathrm{SO}(n)$ *is a non-trivial homomorphism and*

$$\mathbb{R}_+^\Phi = \{a \cdot \Phi(a) \mid a \in \mathbb{R}_+\} \subset \mathbb{R}_+ \times \mathrm{SO}(n)$$

is a group of screw dilations of $\mathbb{R}^n$ *that commutes with* G;

Type 4. $H = (G \times (\mathbb{R}^{n-m})^\Psi) \ltimes \mathbb{R}^m$, *where* $0 < m < n$, $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ *is an orthogonal decomposition,* $\Psi\colon \mathbb{R}^{n-m} \to \mathrm{SO}(m)$ *is a homomorphism with* $\ker \mathrm{d}\Psi = \{0\}$, *and*

$$(\mathbb{R}^{n-m})^\Psi = \{\Psi(u) \cdot u \mid u \in \mathbb{R}^{n-m}\} \subset \mathrm{SO}(m) \times \mathbb{R}^{n-m}$$

is a group of screw isometries of $\mathbb{R}^n$ *that commutes with* G.

The indecomposable Lie algebras of the corresponding Lie subgroups of the group $\mathrm{SO}_0(1, n+1)_{\mathcal{I}}$ *have the same type as in Theorem* 3.2.

Now we want to describe the curvature endomorphisms $\mathcal{K}(\mathfrak{h})$ for a subalgebra $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(1, n+1)$ with respect to these four types. In addition to the space $\mathcal{K}(\mathfrak{h})$ defined in (1) we define another kind of curvature endomorphisms. Let $\mathbb{K}$ be the real or complex numbers. For a subalgebra $\mathfrak{g} \subset \mathfrak{so}(n, \mathbb{C})$ or $\mathfrak{g} \subset \mathfrak{so}(r, s)$ we set

$$\mathcal{B}(\mathfrak{g}) := \{Q \in (\mathbb{K}^n)^* \otimes \mathfrak{g} \mid \langle Q(x)y, z\rangle + \langle Q(y)z, x\rangle + \langle Q(z)x, y\rangle = 0\}, \quad (4)$$

where $\langle \cdot, \cdot \rangle$ is the corresponding scalar product. $\mathcal{B}(\mathfrak{g})$ is a $\mathfrak{g}$-modules of *curvature endomorphisms*. In order to distinguish it from $\mathcal{K}(\mathfrak{g})$, one may call $\mathcal{B}(\mathfrak{g})$ the space of *weak curvature endomorphisms*. In [46] the following theorem was proved, for which we fix a basis $(X, E_1, \dots, E_n, Z)$ of $\mathbb{R}^{1,n+1}$ as in the previous section.

Theorem 3.5. *Suppose that* $\mathfrak{h}$ *is a subalgebra of the parabolic algebra* $\mathfrak{so}(1, n+1)_{\mathcal{I}}$ *in* $\mathfrak{so}(1, n+1)$ *and let* $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ *be its orthogonal part. Then it holds:*

(1) *Any* $R \in \mathcal{K}(\mathfrak{h})$ *is uniquely given by*

$$\lambda \in \mathbb{R},\ L \in (\mathbb{R}^n)^*,\ Q \in \mathcal{B}(\mathfrak{g}),\ R_0 \in \mathcal{K}(\mathfrak{g}),\ \text{and } T \in \mathrm{End}(\mathbb{R}^n) \text{ with } T^* = T$$

in the following way,

$$\begin{aligned} R(X, Z) &= (\lambda, 0, L^*(1)),\\ R(U, V) &= (0, R_0(U, V), -\tfrac{1}{2}Q^*(U \wedge V)),\\ R(U, Z) &= (L(U), Q(U), T(U)),\\ R(X, U) &= 0, \end{aligned}$$

where $U, V \in \mathrm{span}(E_1, \dots, E_n)$.

(2) *If* $\mathfrak{h}$ *is indecomposable of type* 2, *any* $R \in \mathcal{K}(\mathfrak{h})$ *is given as in* (1) *with* $\lambda = 0$ *and* $L = 0$.

(3) *If* $\mathfrak{h}$ *is indecomposable of type* 3 *defined by the epimorphism* $\varphi\colon \mathfrak{z} \to \mathbb{R}$, *any* $R \in \mathcal{K}(\mathfrak{h})$ *is given as in* (1) *with* $\lambda = 0$, $L = \tilde{\varphi} \circ Q$ *and* $R_0 \in \mathcal{K}(\mathfrak{g}' \oplus \ker\varphi)$, *where* $\tilde{\varphi}$ *is the extension of* φ *to* $\mathfrak{g}$ *set to zero on* $\mathfrak{g}'$.

(4) *If* $\mathfrak{h}$ *is of type* 4 *defined by the epimorphism* $\psi\colon \mathfrak{z} \to \mathbb{R}^{n-k}$, *any* $R \in \mathcal{K}(\mathfrak{h})$ *is given as in* (1) *with* $\lambda = 0$, $L = 0$, $\mathrm{pr}_{\mathbb{R}^{n-k}} \circ T = \tilde{\psi} \circ Q$ *and* $R_0 \in \mathcal{K}(\mathfrak{g}' \oplus \ker\psi)$, *where* $\tilde{\psi}$ *is the extension of* ψ *to* $\mathfrak{g}$ *set to zero on* $\mathfrak{g}'$.

Here $U \wedge V$ denotes the identification of $\Lambda^2\mathbb{R}^n$ with $\mathfrak{so}(n)$ and the * denotes the adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$ and the Killing form in $\mathfrak{so}(n)$. In particular T is a symmetric matrix and $Q^*\colon \mathfrak{g} \to \mathbb{R}^n$ is given by $Q^*(U \wedge V) = 2\langle Q(E_i)U, V\rangle E_i$.

Now we can apply the above results to the (connected) holonomy group $H := \mathrm{Hol}_p(M,h)$ of an indecomposable, non-irreducible Lorentzian manifold (M,h). H belongs to one of the four types corresponding to the characterization of the Lie algebra $\mathfrak{h}$ in Theorem 3.2. The Lie group corresponding to the orthogonal part $\mathfrak{g}$ is denoted by $G \subset \mathrm{SO}(n)$. If $\mathfrak{h}$ is of uncoupled type 1 or 2, then we have either

$$H = (\mathbb{R}_+ \times G) \ltimes \mathbb{R}^n, \quad \text{or} \quad H = G \ltimes \mathbb{R}^n, \tag{5}$$

respectively. If $\mathfrak{h}$ is of one of the coupled types 3 or 4 it is defined by a epimorphism $\varphi\colon \mathfrak{z} \to \mathbb{R}$ or $\psi\colon \mathfrak{z} \to \mathbb{R}^l$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ due to Theorem 3.2. For type 3 or 4 we have that

$$H = L \cdot G' \ltimes \mathbb{R}^n, \quad \text{or} \quad H = L \cdot G' \ltimes \mathbb{R}^{n-l}, \tag{6}$$

where G' is the Lie group corresponding to the derived Lie algebra $\mathfrak{g}'$ of $\mathfrak{g}$ and L is the Lie group corresponding to the graph of φ or ψ, respectively. We get some immediate consequences.

Proposition 3.6. *A Lorentzian manifold with indecomposable, non-irreducible holonomy group* H *admits a parallel light-like vector field if and only if* $\mathrm{pr}_{\mathbb{R}_+}(H) = 0$, *i.e., if and only if its Lie algebra is of type* 2 *or* 4.

Regarding Lorentzian Einstein manifolds, we get another consequence by (1) of Theorem 3.5 which implies that the Ricci-trace $\mathrm{Ric} = \mathrm{tr}_{(1,4)}\, R$ is given by

$$\begin{aligned} &\mathrm{Ric}(X,Z) = -\lambda,\\ &\mathrm{Ric}(U,V) = \mathrm{Ric}_0(U,V), \quad \text{where } \mathrm{Ric}_0 = \mathrm{tr}_{(1,4)}\, R_0,\\ &\mathrm{Ric}(U,Z) = -L(U) - \sum_{i=1}^{n} \langle Q(E_i)U, E_i\rangle\\ &\mathrm{Ric}(Z,Z) = \mathrm{tr}(T) \end{aligned} \tag{7}$$

for $U, V \in \mathrm{span}(E_1,\dots,E_n)$. Evaluating these formulas we get in [43] the following consequence.

Theorem 3.7. *Let (M,h) be an indecomposable non-irreducible Lorentzian Einstein manifold. Then the holonomy of (M,h) is of uncoupled type 1 or 2. If the Einstein constant of (M,h) is non-zero, then the holonomy of (M,h) is of type 1.*

3.3 Lorentzian holonomy, weak-Berger algebras, and their classification

In this section we want to present the classification of possible Lorentzian holonomy groups. In particular, we want to describe how to prove the following Theorem.

Theorem 3.8. *Let H be the connected holonomy group of an indecomposable, non-irreducible Lorentzian manifold. Then its orthogonal part $G := \mathrm{pr}_{\mathrm{SO}(n)}(H)$ is a Riemannian holonomy group.*

To this end we will explain the notion of weak-Berger algebras, which was introduced and studied in [66]. As for the definition of $\mathcal{K}(\mathfrak{g})$ in Section 2.1, let $\mathbb{K}$ be the real or complex numbers. For $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{g} \subset \mathfrak{so}(r,s)$ and $\mathcal{B}(\mathfrak{g})$ the space of weak curvature endomorphism we define:

$$\mathfrak{g}^{\mathcal{B}} := \mathrm{span}\{Q(x) \mid x \in \mathbb{K}^n, Q \in \mathcal{B}(\mathfrak{g})\}.$$

Just as $\mathfrak{g}^{\mathcal{K}}$ in Definition 2.1, $\mathfrak{g}^{\mathcal{B}}$ is an ideal in $\mathfrak{g}$.

Definition 3.9. $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{g} \in \mathfrak{so}(r,s)$ is a *weak-Berger algebra* if $\mathfrak{g}^{\mathcal{B}} = \mathfrak{g}$.

Equivalent to the (weak-)Berger property is the fact that there is no proper ideal $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathcal{K}(\mathfrak{h}) = \mathcal{K}(\mathfrak{g})$ (resp. $\mathcal{B}(\mathfrak{h}) = \mathcal{B}(\mathfrak{g})$). One easily verifies that the vector space $\mathcal{R}(\mathfrak{g})$ spanned by $\{R(x,\cdot) \in \mathcal{B}(\mathfrak{g}) \mid R \in \mathcal{K}(\mathfrak{g}),\ x \in \mathbb{K}^n\}$ is a $\mathfrak{g}$-submodule of $\mathcal{B}(\mathfrak{g})$. This implies $\mathfrak{g}^{\mathcal{K}} \subset \mathfrak{g}^{\mathcal{B}}$, and thus:

Proposition 3.10. *Every orthogonal Berger algebra is a weak-Berger algebra.*

For a weak-Berger algebra the Bianchi-identity which defines $\mathcal{B}(\mathfrak{g})$ yields a decomposition property similar to the Borel–Lichnerowicz property mentioned in Section 2.2.

Theorem 3.11. *Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a weak-Berger algebra. To the decomposition of $\mathbb{R}^n$ into invariant subspaces $\mathbb{R}^n = E_0 \oplus E_1 \oplus \cdots \oplus E_k$, where E_0 is a trivial submodule and the E_i are irreducible for $i = 1,\dots,k$, corresponds a decomposition of $\mathfrak{g}$ into ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ such that $\mathfrak{g}_i$ acts irreducibly on E_i and trivially on E_j for an $i \neq j$. Each of the $\mathfrak{g}_i \subset \mathfrak{so}(\dim E_i)$ is a weak Berger algebra and it holds that $\mathcal{B}(\mathfrak{g}) = \mathcal{B}(\mathfrak{g}_1) \oplus \cdots \oplus \mathcal{B}(\mathfrak{g}_k)$.*

We should point out that the same statement holds for orthogonal Berger algebras for a decomposition of $\mathbb{R}^n$ into $\mathfrak{g}$-invariant orthogonal subspaces. This corresponds to

the algebraic aspect of Theorem 2.3. Using the description of the curvature endomorphisms in Theorem 3.5 and by Proposition 3.10 one obtains the following consequence, the 'only if'-direction of which was proved in [66] independently of the description of the space of curvature endomorphisms by restricting the Bianchi-identity to the space $\operatorname{span}(E_1, \dots, E_n)$.

Corollary 3.12. *An indecomposable subalgebra $\mathfrak{h} \subset \mathfrak{p}$ is a Berger algebra if and only if its orthogonal part $\mathfrak{g} \subset \mathfrak{so}(n)$ is a weak-Berger algebra.*

Holonomy algebras of torsion-free connections are Berger algebras but the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian manifold *a priori* is no holonomy algebra, and therefore not necessarily a Berger algebra. But from Corollary 3.12 and the Ambrose–Singer holonomy theorem it follows that it is a weak-Berger algebra.

Theorem 3.13. *The orthogonal part $\mathfrak{g}$ of an indecomposable, non-irreducible Lorentzian holonomy algebra is a weak-Berger algebra. In particular, $\mathfrak{g}$ decomposes into irreducibly acting weak-Berger algebras as in Theorem 3.11.*

This theorem has several important consequences. It not only gives an algebraic criterion for the orthogonal part from which a classification attempt can start, it also provides a proof of the Borel–Lichnerowicz decomposition property for the orthogonal part proved by L. Bérard-Bergery and A. Ikemakhen [11], Theorem II, which is given in our Theorem 3.11. This ensures that we are at a similar point as in the Riemannian situation, that means left with the task of classifying irreducible weak-Berger algebras instead of Berger algebras.

But Theorem 3.13 also has implication for algebras of coupled type 4. They were defined by an epimorphism $\psi : \mathfrak{z} \to \mathbb{R}^p$ for $0 < p < n$ where $\mathfrak{z}$ is the center of the orthogonal part $\mathfrak{g}$. If $\mathbb{R}^n$ decomposes as

$$\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_s},$$

where $\mathfrak{g}$ acts irreducibly on the $\mathbb{R}^{n_i}$ and trivial on $\mathbb{R}^{n_0}$, inducing the decomposition of $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ as in Theorem 3.11, then, first of all, we have that $0 < p \leq n_0 < n - 1$. Moreover, as the $\mathfrak{g}_i$'s act irreducibly, their center has to be at most one-dimensional. Since ψ is surjective this implies that $0 < p \leq s$. In particular, type 4 only occurs for $n \geq 3$, i.e., $\dim M \geq 5$.

Before we explain the classification of weak-Berger algebras we want to present implications of Theorem 3.13 about the conditions under which indecomposable, non-irreducible holonomy groups are *closed*. Let H be an indecomposable, non-irreducible Lorentzian holonomy group and, as above, let G be its orthogonal part. For the uncoupled types 1 and 2 it depends only on G if H is closed. But due to Theorem 3.13 G is a product of irreducibly and orthogonally acting Lie groups, which are closed. Therefore G, and thus H is closed in this case. If H is of one of the coupled types 3 or 4 it is closed if and only if L as in formulae (6) is closed. But L

is closed if and only if its intersection with the Torus Z, which is the center of G, is closed. This can be summarised in the following result obtained in [11].

Corollary 3.14. *If the Lie algebra of an indecomposable, non-irreducible Lorentzian holonomy group H is of type* 1 *and* 2*, then it is closed. If it is of type* 3 *or* 4*, defined by an epimorphism φ , then it is closed if and only if the Lie group generated by the subalgebra* $\ker(\varphi)$ *is a compact subgroup of the torus.*

This corollary implies that holonomy groups of Lorentzian manifolds of dimension less or equal to 5 are closed.

Now we turn to the classification of irreducible weak-Berger algebras obtained in [66], [68], [69]. As we use representation theory of complex semisimple Lie algebras, we have to describe the transition of a real weak-Berger algebra to its complexification. The spaces $\mathcal{K}(\mathfrak{g})$ and $\mathcal{B}_h(\mathfrak{g})$ for $\mathfrak{g} \subset \mathfrak{so}(r+s)$ are defined by the following exact sequences:

$$0 \longrightarrow \mathcal{K}(\mathfrak{g}) \hookrightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{g} \xrightarrow{\lambda} \Lambda^3(\mathbb{R}^n)^* \otimes \mathbb{R}^n,$$
$$0 \longrightarrow \mathcal{B}(\mathfrak{g}) \hookrightarrow (\mathbb{R}^n)^* \otimes \mathfrak{g} \xrightarrow{\lambda^*} \Lambda^3(\mathbb{R}^n)^*,$$

where the map λ is the skew-symmetrization and λ^* the dualization by the scalar product and the skew-symmetrization. If we consider a real Lie algebra $\mathfrak{g}$ acting orthogonally on $\mathbb{R}^n$, then the scalar product extends by complexification to a complex-linear scalar product which is invariant under $\mathfrak{g}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(r+s,\mathbb{C})$. The complexification of the above exact sequences gives

$$\mathcal{K}(\mathfrak{g})^{\mathbb{C}} = \mathcal{K}(\mathfrak{g}^{\mathbb{C}}) \quad \text{and} \quad \mathcal{B}(\mathfrak{g})^{\mathbb{C}} = \mathcal{B}(\mathfrak{g}^{\mathbb{C}}) \tag{8}$$

and leads to the following statement.

Proposition 3.15. *$\mathfrak{g} \subset \mathfrak{so}(r,s)$ is a (weak-) Berger algebra if and only if $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(r+s,\mathbb{C})$ is a (weak-) Berger algebra.*

Thus complexification preserves the weak-Berger as well as the Berger property. But irreducibility is *not* preserved under complexification. In order to deal with this problem we have to recall the following distinctions (for details of the following see [66] or [73]). Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a real orthogonal Lie algebra which acts irreducibly on $\mathbb{R}^n$. Then one can consider the complexification of this representation, i.e., the representation of the real Lie algebra $\mathfrak{g}$ on $\mathbb{C}^n$ given by $\mathfrak{g} \subset \mathfrak{so}(n) \subset \mathfrak{so}(n,\mathbb{C})$. This representations can still be irreducible, in which case we say that $\mathfrak{g}$ is of *real type*, or it can be reducible, and we say it is of *unitary type*. In the second case $n = 2k$ has to be even and $\mathbb{C}^n$ decomposes into two $\mathfrak{g}$-invariant subspaces, $\mathbb{C}^n = \mathbb{C}^k \oplus \overline{\mathbb{C}}^k$ for which we obtain that $\mathfrak{g} \subset \mathfrak{u}(k)$ is unitary and irreducible. Note that this implies that $\mathfrak{g} \not\subset \mathfrak{so}(k,\mathbb{C})$. This distinction was made by E. Cartan in [32] (see also [57] and [51])

for arbitrary irreducible real representations, where these are called "representations of first type" and of "second type".

Now we complexify also the Lie algebra $\mathfrak{g}$ because we want to use the tools of the theory of irreducible representations of complex Lie algebras. Of course, it holds that $\mathfrak{g} \subset \mathfrak{so}(n)$ is irreducible if and only if $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(n,\mathbb{C})$ is irreducible. Hence, for an irreducible $\mathfrak{g} \subset \mathfrak{so}(n)$ we end up with two cases: If $\mathfrak{g}$ is of real type, then $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(n,\mathbb{C})$ is irreducible, or if $\mathfrak{g}$ is of unitary type, i.e., $n = 2k$, then $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(k,\mathbb{C})$ with $\mathfrak{g}^{\mathbb{C}} \not\subset \mathfrak{so}(k,\mathbb{C})$.

For a *unitary weak-Berger algebra* $\mathfrak{g}_0 \subset \mathfrak{u}(k) \subset \mathfrak{so}(2k)$ in [66] it is shown that there is an isomorphism between the complexified weak curvature endomorphisms $\mathcal{B}(\mathfrak{g}_0^{\mathbb{C}})$ and the *first prolongation*

$$\mathfrak{g}^{(1)} \quad = \quad \{Q \in \mathrm{Hom}(\mathbb{C}^k, \mathfrak{g}) \mid Q(u)v = Q(v)u\},$$

where $\mathfrak{g} \simeq \mathfrak{g}_0^{\mathbb{C}}$ is the complexification of $\mathfrak{g}_0$ restricted to the irreducible module $\mathbb{C}^k$, as explained above, i.e., $\mathfrak{g} \subset \mathfrak{gl}(k,\mathbb{C})$ irreducibly. We should point out that an analogous result can be obtained for Berger algebras leading to a classification of irreducible Berger algebras of unitary type. In this situation one can use the classification of irreducible complex linear Lie algebras with non-vanishing first prolongation, which is due to E. Cartan [31], and S. Kobayashi and T. Nagano [63] (see also the list in [78]). Checking the entries in this list one finds that all but one are complexifications of Riemannian holonomy algebras, either of non-symmetric Kählerian ones or of hermitian symmetric spaces. The only exception is $\mathbb{C} \oplus \mathfrak{sp}(k/2,\mathbb{C})$, but it can be shown that this is not a weak-Berger algebra. Hence, in this case we end up with:

Proposition 3.16. *If $\mathfrak{g} \subset \mathfrak{u}(k) \subset \mathfrak{so}(2k)$ is an irreducible weak-Berger algebra of unitary type, then it is a Riemannian holonomy algebra, in particular a Berger algebra.*

Now we turn to the case of a *weak-Berger algebra of real type*, i.e., an irreducible real Lie algebra $\mathfrak{g}_0 \subset \mathfrak{so}(n)$ such that $\mathfrak{g} := \mathfrak{g}_0^{\mathbb{C}} \subset \mathfrak{so}(n,\mathbb{C})$ is irreducible as well. The first thing to notice is that by the Schur lemma, $\mathfrak{g}$ has no center, and thus $\mathfrak{g}$ is not only reductive but semisimple. Considering the four different types of indecomposable, non-irreducible holonomy algebras from Theorem 3.2, this fact already yields the observation that the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian holonomy algebra can only be of coupled type 3 or 4 if at least one of the irreducibly acting ideals of $\mathfrak{g} \subset \mathfrak{so}(n)$ is of non-real type.

As $\mathfrak{g}$ is semisimple, the weak-Berger property can be transformed into conditions on roots and weights of the corresponding representation as follows (for the proofs see [68], [69], [73]). Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{t}^*$ be the roots of $\mathfrak{g}$, and set $\Delta_0 := \Delta \cup \{0\}$. $\mathfrak{g}$ decomposes into its root spaces $\mathfrak{g}_\alpha := \{A \in \mathfrak{g} \mid [T,A] = \alpha(T)\cdot A \text{ for all } T \in \mathfrak{t}\} \neq \{0\}$. If $\Omega \subset \mathfrak{t}^*$ are the weights of $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$, then $\mathbb{C}^n$ decomposes into weight spaces $V_\mu := \{v \in V \mid T(v) = \mu(T)\cdot v \text{ for all } T \in \mathfrak{t}\} \neq \{0\}$, which satisfy that $V_\mu \perp V_\lambda$ if and only if $\lambda \neq -\mu$. In particular, if μ is a weight, then $-\mu$ too. If we denote by Π the weights of the $\mathfrak{g}$–module $\mathcal{B}(\mathfrak{g})$ we define a subset

of $\mathfrak{t}^*$ by

$$\Gamma := \left\{ \mu + \phi \;\middle|\; \begin{array}{l} \mu \in \Omega,\ \phi \in \Pi \ \text{ and there is an } u \in V_\mu \\ \text{and a } Q \in \mathcal{B}_\phi \text{ such that } Q(u) \neq 0 \end{array} \right\} \subset \mathfrak{t}^*.$$

It is not difficult to see that $\Gamma \subset \Delta_0$. But for a weak-Berger algebra, the fact that $Q_\phi(u_\mu) \in \mathfrak{g}_{\phi+\mu}$, implies that

$$\mathfrak{g}^{\mathcal{B}} = \operatorname{span}\{Q_\phi(u_\mu) \mid \phi + \mu \in \Gamma\} \subset \bigoplus_{\beta \in \Gamma} \mathfrak{g}_\beta.$$

Hence for a weak-Berger algebra of real type we have even that

$$\Gamma = \Delta_0.$$

This property then can be tested for the representations of all the simple [68], and based on this, the semisimple Lie algebras [69], with the following result.

Proposition 3.17. *Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be an irreducible weak-Berger algebra of real type. Then $\mathfrak{g}$ is a Riemannian holonomy algebra, and in particular a Berger algebra.*

Propositions 3.16 and 3.17 yield Theorem 3.8 at the beginning of this section.

3.4 Metrics realizing all possible Lorentzian holonomy groups

In this section we shall present Lorentzian metrics that realise all possible groups obtained in the previous section. But first we want to specify the Walker coordinates of Theorem 2.6 to the Lorentzian situation.

For an indecomposable, non-irreducible Lorentzian manifold, the holonomy-invariant light-like line $\mathcal{I} \subset T_pM$ corresponds to a distribution Ξ of light-like lines which are invariant under parallel transport. Locally, this distribution is spanned by a *recurrent light-like vector field*. A vector field X is called *recurrent* if there is a one-form ξ such that $\nabla X = \xi \otimes X$. If $d\xi = 0$, e.g., if the length of X is not zero, X can be re-scaled to a parallel vector field. Now Theorem 2.6 reads as follows.

Proposition 3.18. *Let (M, h) be a Lorentzian manifold of dimension $(n+2)$.*

(1) *(M, h) admits a parallel, light-like line if and only if there exists coordinates $(x, (y_i)_{i=1}^n, z)$ such that*

$$h = 2\,dxdz + \sum_{i=1}^{n} u_i dy_i dz + f dz^2 + \sum_{i,j=1}^{n} g_{ij} dy_i\, dy_j \tag{9}$$

*with $\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0$, $f \in C^\infty(M)$. There is a parallel vector field if and only if $\frac{\partial f}{\partial x} = 0$. In this case the coordinates are called **Brinkmann coordinates** [21], [41].*

(2) *If (M,h) is a Lorentzian manifold with parallel light-like vector field, then there exists coordinates $(x,(y_i)_{i=1}^n,z)$ such that*

$$h = 2\,dxdz + \sum_{i,j=1}^{n} g_{ij}dy_i\,dy_j\ ,\ \textit{with}\ \frac{\partial g_{ij}}{\partial x} = 0. \tag{10}$$

These coordinates are due to R. Schimming [76].

In these coordinates the vector field $\frac{\partial}{\partial x}$ corresponds to the recurrent/parallel light-like vector field. We should remark that if f is sufficiently general (e.g., $\det(\mathrm{Hess}(f)) \neq 0$ at one point), then (M,h) is indecomposable.

Walker coordinates define n-dimensional submanifolds $W_{(x,z)}$ through a point p with coordinates $(x,y_1,\dots,y_n,z)$ by varying only the y_i and keeping x and z constant. One can understand the g_{ij} as coefficients of a family of Riemannian metrics g_z and the u_i as coefficients of a family of 1-forms ϕ_z on $W_{(x,z)}$ depending on a parameter z. A direct calculation shown in [55] gives a relation between the holonomy of these Riemannian metrics and the orthogonal component of the Lorentzian holonomy, namely

$$\mathrm{Hol}_{(x,y,z)}(W_{(x,z)},g_z) \quad\subset\quad \mathrm{pr}_{\mathfrak{so}(n)}\left(\mathrm{Hol}_{(x,y,z)}(M,h)\right).$$

Led by this description, for a given Riemannian holonomy group G, it is not difficult to construct an indecomposable, non-irreducible Lorentzian manifold having G as orthogonal component of its holonomy. In fact, the following is true [67], [70].

Proposition 3.19. *Let (N,g) be a n-dimensional Riemannian manifold with holonomy group G and let $f \in C^\infty(\mathbb{R}\times N)$ a smooth function on M also depending on the parameter x, and φ a smooth real function of the parameter z. Then the Lorentzian manifold $\left(M := \mathbb{R}\times N\times\mathbb{R}, h = 2dxdz + fdz^2 + \mathrm{e}^{2\varphi}g\right)$ has holonomy $(\mathbb{R}\times G)\ltimes\mathbb{R}^n$ if f is sufficiently generic, and $G\ltimes\mathbb{R}^n$ if f does not depend on x.*

This obviously gives a construction method for any Lorentzian holonomy group of uncoupled type 1 or 2. This procedure was used in physics literature to construct examples of Lorentzian manifolds in special cases [42].

Although in [11] some examples of metrics with holonomy of coupled types 3 and 4 were constructed in order to verify that there are metrics of this type, after the classification of possible Lorentzian holonomies, the following question arose: *Given a Riemannian holonomy group G with Lie algebra $\mathfrak{g}$ having a non-trivial center $\mathfrak{z}$, and given an epimorphism $\varphi\colon\mathfrak{z}\to\mathbb{R}^l$, for $0<l<n$, does there exist a Lorentzian manifold with holonomy algebra of type 3 or 4 defined by φ?* In [48] this question was set in the affirmative by providing a unified construction of local polynomial metrics realizing all possible indecomposable, non-irreducible holonomy algebras of Lorentzian manifolds. We will now sketch this method.

Let $\mathfrak{g}\subset\mathfrak{so}(n)$ be the holonomy algebra of a Riemannian manifold. As seen above we have an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^{n_0}\oplus\mathbb{R}^{n-n_0}$, where $\mathfrak{g}$ acts triv-

ially on $\mathbb{R}^{n_0}$ and $\mathbb{R}^{n-n_0}$ decomposes further into irreducible modules. We choose an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ compatible with this decomposition. Obviously, $\mathfrak{g} \subset \mathfrak{so}(n-n_0)$ does not annihilate any proper subspace of $\mathbb{R}^{n-n_0}$. If $\mathfrak{h}$ is an indecomposable subalgebra of $\mathfrak{so}(1,n+1)_I$ with orthogonal part $\mathfrak{g}$ having center $\mathfrak{z}$ and of coupled type 3 defined by an epimorphism $\varphi\colon \mathfrak{z} \to \mathbb{R}$, then we denote $\mathfrak{h}$ by $\mathfrak{h}(\mathfrak{g},\varphi)$. If $\mathfrak{h}$ is of coupled type 4 defined by an epimorphisms $\psi\colon \mathfrak{z} \to \mathbb{R}^p$ for $0 < p < n$ we denote $\mathfrak{h}$ by $\mathfrak{h}(\mathfrak{g},\psi,p)$. Note that in the latter case we have $0 < p \le n_0 < n$.

First, for a weak-Berger algebra $\mathfrak{g} \subset \mathfrak{so}(n)$ one fixes weak curvature endomorphisms $Q_A \in \mathcal{B}(\mathfrak{g})$ for $A = 1, \dots, N$ such that $\{Q_A\}_{A=1\dots N}$ span $\mathcal{B}(\mathfrak{g})$. Now one defines the following polynomials on $\mathbb{R}^{n+1}$:

$$u_i(y_1,\dots,y_n,z) := \sum_{A=1}^{N}\sum_{k,l=1}^{n} \frac{1}{3(A-1)!} \underbrace{\big\langle Q_A(e_k)e_l + Q_A(e_l)e_k, e_i\big\rangle}_{=:Q^i_{Akl}} y_k y_l z^A. \tag{11}$$

Then we define the following Lorentzian metric on $\mathbb{R}^{n+2}$:

$$h = 2dxdz + fdz^2 + 2\sum_{i=1}^{n} u_i dy_i dz + \sum_{k=1}^{n} dy_k^2, \tag{12}$$

where f is a function on $\mathbb{R}^{n+2}$ to be specified. If $\mathfrak{h}$ is of type 3 defined by an epimorphism $\varphi\colon \mathfrak{z} \to \mathbb{R}$, i.e., $\mathfrak{h} = \mathfrak{h}(\mathfrak{g},\varphi)$, first we extend φ to the whole of $\mathfrak{g}$ by setting it to zero on $\mathfrak{g}'$, i.e., we set $\tilde{\varphi}(Z+U) = \varphi(Z)$ for $Z \in \mathfrak{z}$ and $U \in \mathfrak{g}'$. Then, for $A = 1, \dots, N$ and $i = n_0+1, \dots, n$ we define the numbers

$$\varphi_{Ai} = \frac{1}{(A-1)!}\tilde{\varphi}(Q_A(e_i)).$$

If $\mathfrak{h}$ is of type 4 defined by an epimorphism $\psi\colon \mathfrak{z} \to \mathbb{R}^p$, i.e., $\mathfrak{h} = \mathfrak{h}(\mathfrak{g},\psi,p)$, again we extend ψ to an epimorphism $\tilde{\psi}$ to the whole of $\mathfrak{g}$ as above, and define the following numbers,

$$\psi_{Aib} : \frac{1}{(A-1)!}\big\langle \tilde{\psi}(Q_A(e_i)), e_b\big\rangle,$$

for $A = 1, \dots, N$, $i = n_0+1, \dots, n$, and $b = 1, \dots, p$. Then in [48] the following is proved.

Theorem 3.20. *Let $\mathfrak{h} \subset \mathfrak{so}(1,n+1)$ be indecomposable and non-irreducible with a Riemannian holonomy algebra $\mathfrak{g}$ as orthogonal part. If $\mathfrak{h}$ is given by the left-hand-side of the following table, then the holonomy algebra at the origin $0 \in \mathbb{R}^{n+2}$ of the Lorentzian metric h given in (11) is equal to $\mathfrak{h}$ if the function f is defined as in the right-hand side of the table:*

$\mathfrak{h}$	f
Type 1: $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$	$x^2 + \sum_{i=1}^{n_0} y_i^2$
Type 2: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$	$\sum_{i=1}^{n_0} y_i^2$
Type 3: $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \varphi)$	$2x \sum_{A=1}^{N} \sum_{i=n_0+1}^{n} \varphi_{Ai} y_i z^{A-1} + \sum_{k=1}^{n_0} y_k^2$
Type 4: $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \psi, p)$	$2 \sum_{A=1}^{N} \sum_{i=n_0+1}^{n} \sum_{b=1}^{p} \psi_{Aib} y_i y_b z^{A-1} + \sum_{k=p+1}^{n_0} y_k^2$

Obviously, this theorem implies the 'if'-direction of the main classification result of Theorem 3.3. The idea of its proof is the following: The metric h given in (11) with a function f given as in the theorem is analytic. Hence, its holonomy at $0 \in \mathbb{R}^{n+2}$ is generated by the derivations of the curvature tensor at 0. But the metric is constructed in a way such that the only non-vanishing $\mathfrak{so}(n)$-parts of the curvature and its derivatives satisfy at $0 \in \mathbb{R}^{n+2}$:

$$\mathrm{pr}_{\mathfrak{so}(n)}\big[\big(\underbrace{\nabla_{\partial_z} \dots \nabla_{\partial_z}}_{(A-1)-\text{times}} \mathcal{R}\big)(\partial_i, \partial_z)_0\big] = Q_A(e_i), \tag{13}$$

for $A = 1, \dots, N$, $i = n_0 + 1, \dots, n$, and writing ∂_z for $\frac{\partial}{\partial z}$ and ∂_i for $\frac{\partial}{\partial y_i}$. But $Q_1, \dots, Q_N$ span $\mathcal{B}(\mathfrak{g})$, hence, the derivatives of the curvature will span $\mathfrak{g}$, since this is a weak-Berger algebra. Therefore the orthogonal part $\mathfrak{g}$ of $\mathfrak{h}$ we started with is the orthogonal part of $\mathfrak{hol}_0(\mathbb{R}^{n+2}, h)$. A more detailed analysis also shows that (13) implies that $\mathbb{R}^{n-n_0}$ is contained in $\mathfrak{hol}_0(\mathbb{R}^{n+2}, h)$. But, more importantly, one can show that for the different choices of the function f the derivatives of the curvature generate holonomy algebras of the corresponding types.

We want to conclude this section with some remarks and examples constructed by the method of Theorem 3.20.

First of all one notices that the resulting Lorentzian manifolds are of a special type, introduced in [72] as *manifolds with light-like hypersurface curvature*. They are defined by the condition that their curvature tensor $\mathcal{R}$ vanishes on $\mathcal{I}^\perp \times \mathcal{I}^\perp \times \mathcal{I}^\perp \times \mathcal{I}^\perp$. It is remarkable that this rather strong condition on the curvature does not prevent these manifolds from having any possible indecomposable, non-irreducible Lorentzian holonomy. All the following examples, including the ones with non-closed holonomy, will be of this type.

The method of Theorem 3.20 works for any Riemannian holonomy algebra, as soon as one is able to calculate $\mathcal{B}(\mathfrak{g})$. Sometimes it is not necessary to calculate the whole of $\mathcal{B}(\mathfrak{g})$ but a submodule which is sufficient to generate the Lie algebra $\mathfrak{g}$. This could be the sub-module $\mathcal{R}(\mathfrak{g})$. For instance, in [72] a Riemannian symmetric

space G/K with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is considered. The curvature endomorphisms of $\mathfrak{k}$ satisfy $\mathcal{K}(\mathfrak{k}) = \mathbb{R} \cdot [\cdot,\cdot]$, where $[\cdot,\cdot]$ is the commutator of $\mathfrak{g}$. Since $\mathfrak{k}$ is the holonomy algebra of this space we get $\mathfrak{k} = \operatorname{span}\{[X,Y] \mid X, Y \in \mathfrak{m}\}$. Hence for a basis $X_1, \dots, X_n$ of $\mathfrak{m}$, the $Q_j := \operatorname{ad}(X_j)$ are spanning the submodule $\mathcal{R}(\mathfrak{k})$ in $\mathcal{B}(\mathfrak{k})$ and generate the whole Lie algebra $\mathfrak{k}$. In this situation, the polynomials u_i defined in (11) can be written in terms of the basis X_i and the Killing form of $\mathfrak{g}$,

$$u_i^{(G,K)}(y_1, \dots, y_n, z) := \sum_{j,k,l=1}^{n} \frac{1}{3(j-1)!} \big(B\big([X_j, X_k], [X_l, X_i]\big) + B\big([X_j, X_l], [X_k, X_i]\big)\big) y_k y_l z^j,$$

where $[\cdot,\cdot]$ is the commutator in $\mathfrak{g}$ and B the Killing form. In this way one obtains a Lorentzian manifold with the isotropy group K of a symmetric space G/K as orthogonal part of the holonomy. Examples where the orthogonal part is given by the Riemannian symmetric pair $\mathfrak{so}(3) \subset \mathfrak{so}(5)$ have been constructed in [55], [70], and [48].

For non-symmetric Riemannian holonomy algebras, $\mathcal{K}(\mathfrak{g})$ can be very big and thus the calculations complicated. As sketched in [72], another way is to use other, easier submodules of $\mathcal{B}(\mathfrak{g})$. This methods works if $\mathfrak{g}$ is simple, since any sub-module of $\mathcal{B}(\mathfrak{g})$ generates a non-trivial ideal in $\mathfrak{g}$ which has to be equal to $\mathfrak{g}$ if $\mathfrak{g}$ is $\mathfrak{g}$ simple. For example, in the case of the exceptional Lie algebra $\mathfrak{g}_2 \subset \mathfrak{so}(V)$, with $V = \mathbb{R}^7$, the $\mathfrak{g}_2$-module $\operatorname{Hom}(V, \mathfrak{g}_2)$ which contains $\mathcal{B}(\mathfrak{g}_2)$ splits into the direct sum of $V_{[1,1]}$, $\odot_0^2 V^*$ and V, where $V_{[1,1]}$ is the 64-dimensional $\mathfrak{g}_2$-module of highest weight $(1,1)$, and $\odot_0^2 V^*$ is the 27-dimensional module of highest weight $(2,0)$. Since $\mathcal{B}(\mathfrak{g}_2)$ is the kernel of the skew-symmetrization

$$\begin{array}{ccc} \lambda\colon \operatorname{Hom}(V, \mathfrak{g}_2) & \longrightarrow & \Lambda^3 V^* \\ \| & & \| \\ V_{[1,1]} \oplus \odot_0^2 V^* \oplus V & \longrightarrow & \odot_0^2 V^* \oplus V \oplus \mathbb{R}, \end{array}$$

a dimension analysis shows that $\mathcal{B}(\mathfrak{g})$ must contain $V_{[1,1]}$. Thus, by choosing a basis of $V_{[1,1]}$ a metric of the form (12) with coefficients as in (11) can be defined and one obtains a Lorentzian manifold with orthogonal holonomy part G_2.

Finally, we want to return to the question of closedness of holonomy groups. In [11] Lorentzian manifolds with indecomposable, non-irreducible holonomy of coupled type 3 and 4 are constructed which have a *non-closed* holonomy group. These examples use a dense immersion of the real line into the 2-torus. They are constructed similar to our construction method. Consider the metric

$$h = 2dxdz - \sum_{i=1}^{4} dy_i^2 + 2x\,(y_1 y_2 + \alpha\, y_3 y_4)\, dz^2 + 2\left(y_2^2 y_1 dy_1 - y_1^2 y_2 dy_2 + y_4^2 y_3 dy_3 - y_3^2 y_4 dy_4\right) dz$$

on $\mathbb{R}^6$ depending on the parameter α. For this metric one can show that it is of coupled type 3 defined by an epimorphism $\varphi\colon \mathfrak{z} \to \mathbb{R}$, its orthogonal part is the torus T^2, and that the kernel of φ defines a closed subgroup in T^2 if and only if α is rational. Hence, for α irrational, the holonomy group of h is not closed in $\mathrm{SO}_0(1,5)$. Similarly, the metric

$$h = 2dxdz - \sum_{i=1}^{5} dy_i^2 + dz^2 + 2\big(y_2^2 y_1 dy_1 - y_1^2 y_2 dy_2 + y_4^2 y_3 dy_3 \\ - y_3^2 y_4 dy_4 + z\,(y_1 y_2 + \alpha\, y_3 y_4)\, dy_5\big)dz$$

on $\mathbb{R}^7$ has a holonomy group of coupled type 4, with T^2 as orthogonal part, and which is non-closed if α is irrational.

3.5 Applications to parallel spinors

The existence of a parallel spinor field on a Lorentzian spin manifold (M,h) implies the existence of a parallel vector field in the following way: To a spinor field φ, one may associate a vector field X_φ, defined by the equation $h(V_\varphi, U) = \langle U \cdot \varphi, \varphi\rangle$ for any $U \in TM$, where $\langle\cdot,\cdot\rangle$ is the inner product on the spin bundle and $\cdot$ is the Clifford multiplication. X_φ sometimes is referred to as *Dirac current*. Now, the vector field associated to a spinor in this way is light-like or time-like. If the spinor field is parallel, so is the vector field. In the case where it is time-like, the manifold splits by the de-Rham decomposition theorem into a factor $(\mathbb{R}, -dt^2)$ and Riemannian factors which are flat or irreducible with a parallel spinor, i.e., with holonomy $\{1\}$, G_2, $\mathrm{Spin}(7)$, $\mathrm{Sp}(k)$ or $\mathrm{SU}(k)$.

In the case where the parallel vector field is light-like we have a Lorentzian factor which is indecomposable, but with parallel light-like vector field (and parallel spinor) and flat or irreducible Riemannian manifolds with parallel spinors. Hence, in this case one has to know which indecomposable Lorentzian manifolds admit a parallel spinor. The existence of the light-like parallel vector field forces the holonomy of such a manifold with parallel spinor to be contained in $\mathrm{SO}(n) \ltimes \mathbb{R}^n$ i.e., to be of type 2 or 4.

Furthermore, the spin representation of the orthogonal part $\mathfrak{g} \subset \mathfrak{so}(n)$ of $\mathfrak{h}$ must admit a trivial subrepresentation. In fact, the dimension of the space of parallel spinor fields is equal to the dimension of the space of spinors which are annihilated by $\mathfrak{g}$ [67]. But for the coupled type 4, the orthogonal part $\mathfrak{g}$ has to have a non-trivial center. Due to the decomposition of $\mathfrak{g}$ into irreducible acting ideals at least one irreducible acting ideal is equal to $\mathfrak{u}(p)$. But a direct calculation shows that $\mathfrak{u}(p)$ cannot annihilate a spinor. Hence we obtain the following consequence.

Corollary 3.21. *Let (M,h) be an indecomposable Lorentzian spin manifold of dimension $n+2>2$ with holonomy group H admitting a parallel spinor field. Then it is $H = G \ltimes \mathbb{R}^n$ where G is the holonomy group of an n-dimensional Riemannian manifold with parallel spinor, i.e., G is a product of* $\mathrm{SU}(p)$, $\mathrm{Sp}(q)$, G_2 *or* $\mathrm{Spin}(7)$.

This generalises a result of R. L. Bryant in [26] (see also [42]) where it is shown up to $n \leq 9$ that the maximal subalgebras of the parabolic algebra admitting a trivial subrepresentation of the spin representation are of type (Riemannian holonomy)$\ltimes\mathbb{R}^n$. Combining Corollary 3.21 with the de Rham–Wu decomposition theorem we obtain the following conclusion.

Theorem 3.22. *Let (M,h) be a simply connected, complete Lorentzian spin manifold which admits a parallel spinor. Then (M,h) is isometric to a product $(M',h')\times(N_1,g_1)\times\cdots\times(N_k,g_g)$, where the (N_i,g_i) are flat or irreducible Riemannian manifolds with a parallel spinor and (M',h') is either $(\mathbb{R},-dt)$ or it is an indecomposable, non-irreducible Lorentzian manifold of dimension $n+2>2$ with holonomy $G\ltimes\mathbb{R}^n$ where G is the holonomy group of a Riemannian manifold with parallel spinor. In particular, the holonomy group of (M,h) is the product*

$$(G\ltimes\mathbb{R}^n)\times\widehat{G},$$

for some $n\geq 0$, and G and $\widehat{G}$ being holonomy groups of Riemannian manifolds admitting a parallel spinor, i.e., both being a product of the possible factors $\{1\}$, $\mathrm{SU}(p)$, $\mathrm{Sp}(q)$, G_2, or $\mathrm{Spin}(7)$ (with G trivial if $n<2$).

3.6 Holonomy related geometric structures

In this section we want to present some remarks about the geometric structures corresponding to the possible holonomy groups. The parallel distribution Ξ of light-like lines equips a Lorentzian manifold (M,g) with further, holonomy related structure. As Ξ is light-like, it is contained in its orthogonal complement $\Xi^\perp$. Hence, the tangent bundle admits a filtration

$$\Xi\ \subset\ \Xi^\perp\ \subset\ TM, \tag{14}$$

which enables us to define a vector bundle $\mathcal{S}$ whose fibers are the quotients $\Xi_p^\perp/\Xi_p$, and equip it with a metric $g^{\mathcal{S}}$ induced by the Lorentzian metric g. Since both distributions are parallel, the Levi-Civita connection of g equips $\mathcal{S}$ also with a metric connection $\nabla^{\mathcal{S}}$.

Definition 3.23. If (M,g,Ξ) is a Lorentzian manifold with a parallel distribution Ξ of light-like lines, the vector bundle $(\mathcal{S},g^{\mathcal{S}},\nabla^{\mathcal{S}})$ is called *screen bundle* of (M,g,Ξ). The holonomy group of the vector bundle connection $\nabla^{\mathcal{S}}$ is called *screen holonomy group*.

The screen holonomy was introduced in [70] and studied further in [72], where it is shown that the orthogonal part of an indecomposable, non-irreducible Lorentzian holonomy group is equal to the screen holonomy. By the above results we know that the screen holonomy has to be a Riemannian holonomy group. More importantly, algebraic properties of the orthogonal part of the holonomy now can be described by invariant structures of the screen bundle. E.g., if the orthogonal part is contained in

the unitary group, then there is a parallel complex structure on the screen bundle, if the orthogonal part is contained in G_2 the screen bundle admits a parallel 3-form, etc. Moreover, we have seen that its fiber $\mathcal{S}_p = \Xi_p^\perp / \Xi_p$ decomposes into subspaces which are invariant under the orthogonal component $\mathfrak{g}$ of the holonomy algebra $\mathfrak{h}$, $\mathcal{S}_p = E_0 \oplus E_1 \oplus \cdots \oplus E_s$, such that $\mathfrak{g}$ acts trivial on E_0 and irreducibly on E_i for $1 \leq i \leq s$. Now, let the spaces Υ_p^i be the pre-image under the canonical projection $\Xi_p^\perp \to \mathcal{S}_p$ of those E_i. They have common intersection Ξ_p. and are holonomy invariant. Therefore they are the fibers of parallel distributions $\Upsilon^0, \ldots, \Upsilon^k$ on M with

$$\Xi = \Upsilon^0 \cap \cdots \cap \Upsilon^s. \tag{15}$$

All the foliations $\Xi \subset \Upsilon^i \subset \Xi^\perp$ are parallel, hence, they are involutive and therefore integrable. I.e. for every point $p \in M$, there are integral manifolds $\mathcal{X}_p$, $\mathcal{Y}_p^i$ and $\mathcal{X}_p^\perp$ of Ξ and $\Xi^\perp$ passing through it. Each leaf of $\mathcal{Y}^i$ and $\mathcal{X}^\perp$ again is foliated in leaves of $\mathcal{X}$, the latter being light-like geodesic lines. One can prove the existence of coordinates which respect this foliation. This is done by C. Boubel in [19], where also an additional condition was found under which these coordinates are unique. We should also point out that the definition of a screen bundle does not require a choice as it does the notion of a screen distribution introduced in [9]. The relation of the screen bundle to the preferred choices in [6] and [40] is not yet studied. Finally, in [17] the relation between the light-like hypersurfaces and the four types of holonomy groups is studied.

In the reminder of this section we want to characterise Lorentzian manifolds for which the screen holonomy is trivial and some of their generalizations. These results were obtained in [71] and [72]. A Lorentzian manifold with parallel light-like vector field is called *Brinkmann wave*. A Brinkmann wave admits coordinates as in Proposition 3.18. A Brinkmann-wave is called *pp-wave* if its curvature tensor $\mathcal{R}$ satisfies the trace condition $\mathrm{tr}_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0$. R. Schimming [76] proved that an $(n+2)$-dimensional pp-waves admits coordinates $(x, (y_i)_{i=1}^n, z)$ such that

$$h = 2\, dxdz + f dz^2 + \sum_{i=1}^{n} dy_i^2 \text{ , with } \tfrac{\partial f}{\partial x} = 0. \tag{16}$$

In [71] we gave another equivalence for the definition which seems to be simpler than any of the trace conditions and which allows for generalizations.

Proposition 3.24. *A Brinkmann-wave (M, h) with parallel light-like vector field X and induced parallel distributions Ξ and $\Xi^\perp$ is a pp-wave if and only if its curvature tensor satisfies*

$$\mathcal{R}(U, V) : \Xi^\perp \longrightarrow \Xi \quad \textit{for all } U, V \in TM, \tag{17}$$

or equivalently $\mathcal{R}(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in \Xi^\perp$.

From this description one obtains easily that a pp-wave is *Ricci-isotropic*, which means that the image of the Ricci-endomorphism is totally light-like, and has vanishing

scalar curvature. But it also enables us to introduce a generalization of pp-waves by supposing (17) but only the existence of a recurrent light-like vector field. Assuming that the abbreviation 'pp' stands for 'plane fronted with parallel rays' we call them *pr-waves*, 'plane fronted with recurrent rays'.

Definition 3.25. A Lorentzian manifold with recurrent light-like vector field X it called *pr-wave* if $\mathcal{R}(U,V)\colon \Xi^{\perp} \to \Xi$ for all $U, V \in TM$, or, equivalently, $\mathcal{R}(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in X^{\perp}$.

Since X is not parallel the trace condition which was true for a pp-wave, fails to hold for a pr-wave. But similar to a pp-wave, a Lorentzian manifold (M,h) is a pr-wave if and only if there are coordinates $(x,(y_i)_{i=1}^n,z)$ such that

$$h = 2\,dxdz + fdz^2 + \sum_{i=1}^{n} dy_i^2 \quad \text{with } f \in C^{\infty}(M). \tag{18}$$

Regarding the vanishing of the screen holonomy the following result can be obtained by the description of Proposition 3.24 and the definition of a pr-wave.

Proposition 3.26. *A Lorentzian manifold (M,h) with recurrent light-like vector field is a pr-wave if and only if the following equivalent conditions are satisfied:*

(1) *The screen holonomy of (M,h) is trivial.*

(2) *(M,h) has solvable holonomy contained in $\mathbb{R} \ltimes \mathbb{R}^n$.*

In addition, (M,h) is a pp-wave if and only if its holonomy is Abelian, i.e., contained in $\mathbb{R}^n$.

Finally, in [71] it is proved that a pr-wave is a pp-wave if and only if it is Ricci-isotropic. There are very important subclasses of pp-waves. The first are the *plane waves* which are pp-waves with quasi-recurrent curvature, i.e., $\nabla\mathcal{R} = \xi \otimes \widetilde{\mathcal{R}}$ where $\xi = h(X,\cdot)$ and $\widetilde{\mathcal{R}}$ a $(4,0)$-tensor. For plane waves the function f in the local form of the metric is of the form $f = \sum_{i,j=1}^{n} a_{ij}\,y_i\,y_j$ where the a_{ij} are functions of z. A subclass of plane waves are the Lorentzian symmetric spaces with solvable transvection group, the so-called *Cahen–Wallach spaces* (see [30], also [11]). For these the function f satisfies $f = \sum_{i,j=1}^{n} a_{ij}\,y_i\,y_j$ where the a_{ij} are constants. Manifolds with light-like hypersurfaces curvature mentioned above are further generalizations of pp-waves [72].

3.7 Holonomy of space-times

To conclude this section about Lorentzian holonomy we want to recall results about the holonomy of space-times, i.e., 4-dimensional Lorentzian manifolds of signature $(-+++)$. J. F. Schell [75] and R. Shaw [79] (see also [52] and [53]) found that

there are 14 types of possible space-times holonomy groups. These 14 types can be derived by the following case study, in which we will also give examples of metrics realizing these groups. Let H be the connected holonomy group of a 4-dimensional Lorentzian manifold.

(1) H acts irreducibly, i.e., $H = \mathrm{SO}_0(1,3)$, which can be realised by the 4-dimensional de Sitter space $S^{1,3}$.

(2) H acts indecomposably, but non-irreducibly. Then H is either:

  (a) $H = (\mathbb{R}_+ \times \mathrm{SO}(2)) \ltimes \mathbb{R}^2$,

  (b) $H = \mathrm{SO}(2) \ltimes \mathbb{R}^2$.

  (c) H is of type 3, i.e., $H = L \ltimes \mathbb{R}^2$ with L given by the graph of an epimorphism $\varphi\colon \mathfrak{so}(2) \to \mathbb{R}$.

  (d) $H = \mathbb{R}^2$, i.e., the holonomy of a 4-dimensional pp-wave.

  (e) $H = \mathbb{R} \ltimes \mathbb{R}^2$, i.e., the holonomy of an 4-dimensional pr-wave.

  In all these cases the previous section gives examples of metrics realizing H.

(3) H acts decomposably. Then H is either:

  (a) $H = \mathrm{SO}(2)$, i.e., the holonomy of the product of the 2-sphere S^2 with the 2-dimensional Minkowski space $\mathbb{R}^{1,1}$.

  (b) $H = \mathrm{SO}(1,1)$, i.e., the holonomy of the product of the 2-dimensional de Sitter space $S^{1,1}$ with the flat $\mathbb{R}^2$.

  (c) $H = \mathrm{SO}(3)$, i.e., the holonomy of the product of $(\mathbb{R}, -dt^2)$ with the 3-sphere S^3.

  (d) $H = \mathrm{SO}(1,2)$, i.e., the holonomy of the product of the line $\mathbb{R}$ with the 3-dimensional de Sitter space $S^{1,2}$.

  (e) $H = \mathrm{SO}(1,1) \times \mathrm{SO}(2)$, i.e., the holonomy of the product of the 2-dimensional de Sitter space $S^{1,1}$ with the 2-sphere S^2 .

  (f) $H = \mathbb{R} \ltimes \mathbb{R}$. This is the holonomy of the product of $\mathbb{R}$ with a 3-dimensional Lorentzian manifold with a recurrent but not parallel light-like vector field, i.e., with a 3-dimensional pr-wave metric. The latter is of the form $h = 2dxdz + g(y)dy^2 + f(x,y,z)dz^2$.

  (g) $H = \mathbb{R}$. This is the holonomy of the product of $\mathbb{R}$ with a 3-dimensional Lorentzian manifold with a parallel light-like vector field, i.e., with a 3-dimensional pp-wave metric. The latter is of the form $h = 2dxdz + g(y)dy^2 + f(y,z)dz^2$.

  (h) H is trivial, i.e., the holonomy of the flat Minkowski space $\mathbb{R}^{1,3}$.

We should point out that there is another type of subgroup in $\mathrm{SO}(1,3)$, which is a one-parameter subgroup of $\mathrm{SO}(1,1) \times \mathrm{SO}(2)$, not equal to either of the factors. But this cannot be a holonomy of a Lorentzian manifold because it does not satisfy the de Rham–Wu decomposition of Theorem 2.3. This is explained in [16], Section 10.J,

where also the question is asked whether there is a space-time with holonomy of coupled type 3, in [16] denoted by B_θ^3. This question is answered affirmatively by A. Ikemakhen in [56], Section 4.2.3, by the metric

$$h = 2dxdz - dy_1^2 + dy_2^2 + 4\alpha y_1 y_2 dy_1 dz + 2xy_1 dz^2,$$

and by the general method given in [48] described in Theorem 3.20 in Section 3.4. Further results can be found in [50].

4 Holonomy in signature $(2, n+2)$

In this section we discuss the holonomy algebras of pseudo-Riemannian manifolds of signature $(2, n+2)$. From the Berger list one reads off that in this signature the only non-symmetric irreducible holonomy groups are $\mathrm{U}(1, \frac{n}{2}+1)$ and $\mathrm{SU}(1, \frac{n}{2}+1)$.

Again we consider indecomposable, non-irreducible holonomy algebras of manifolds of signature $(2, n+2)$. As explained in Section 2.2, if $\mathfrak{h} \subset \mathfrak{so}(2, n+2)$ is an indecomposable, non-irreducible subalgebra, then it preserves a proper degenerate subspace $\mathcal{V} \subset \mathbb{R}^{2,n+2}$ and the non-trivial isotropic subspace $\mathcal{I} := \mathcal{V} \cap \mathcal{V}^\perp \subset \mathbb{R}^{2,n+2}$. Obviously, $\dim(\mathcal{I}) = 1$ or 2. Thus for an indecomposable, non-irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(2, n+2)$ we have two possibilities:

(1) $\mathfrak{h}$ preserves an isotropic plane;

(2) $\mathfrak{h}$ preserves an isotropic line and does not preserve any isotropic plane;

Until now only the first case has been considered. To explain the results in this case, let $\mathbb{R}^{2,n+2}$ be an $n+4$-dimensional real vector space endowed with a symmetric bilinear form $\langle\,,\,\rangle$ of signature $(2, n+2)$. We fix a basis $X_1, X_2, E_1, \dots, E_n, Z_1, Z_2$ of $\mathbb{R}^{2,n+2}$ such that the Gram matrix of $\langle\,,\,\rangle$ has the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbb{I}_n & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{so}(2, n+2)_{\mathcal{I}} \subset \mathfrak{so}(2, n+2)$ be the subalgebra that preserves the isotropic plane $\mathcal{I} = \mathrm{span}\{X_1, X_2\}$. $\mathfrak{so}(2, n+2)_{\mathcal{I}}$ can be identified with the following matrix algebra:

$$\mathfrak{so}(2, n+2)_{\mathcal{I}} = \left\{ \begin{pmatrix} B & & X^t & 0 & -c \\ & & Y^t & c & 0 \\ 0 & 0 & A & -X & -Y \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & -B^t \end{pmatrix} \;\middle|\; \begin{array}{c} B \in \mathfrak{gl}(2, \mathbb{R}), \\ A \in \mathfrak{so}(n), \\ X, Y \in \mathbb{R}^n, \\ c \in \mathbb{R} \end{array} \right\}.$$

In [56] A. Ikemakhen classified indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2,n+2)_{\mathcal{I}}$ that contain the ideal

$$\mathcal{A} := \mathbb{R}\begin{pmatrix} 0 & & 0 & 0 & -1 \\ & & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \end{pmatrix} \subset \mathfrak{so}(2,n+2)_{\mathcal{I}}.$$

As in the Lorentzian case, to each subalgebra $\mathfrak{h} \subset \mathfrak{so}(2,n+2)_{\mathcal{I}}$ one can associate its projection onto $\mathfrak{so}(n) \subset \mathfrak{so}(2,n+2)_{\mathcal{I}}$. In [56] it was noted that such projection $\mathfrak{g}$ of the holonomy algebra may not be a holonomy algebra of a Riemannian manifold. Moreover, in the next section we will see that there is no additional condition on the subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$ induced by the Bianchi-identity and replacing the weak-Berger property.

4.1 The orthogonal part of indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2,n+2)$

Let us now consider one type of indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2,n+2)$. For any subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$ define the Lie algebra

$$\mathfrak{h}^{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & X^t & 0 & -c \\ 0 & 0 & Y^t & c & 0 \\ 0 & 0 & A & -X & -Y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{g},\ X, Y \in \mathbb{R}^n,\ c \in \mathbb{R} \right\} \subset \mathfrak{so}(2,n+2)_{\mathcal{I}}.$$

We identify an element of the Lie algebra $\mathfrak{h}^{\mathfrak{g}}$ with the 4-tuple (A, X, Y, Z). It is easy to see that the subalgebra $\mathfrak{h}^{\mathfrak{g}} \subset \mathfrak{so}(2,n+2)_{\mathcal{I}}$ is indecomposable. The following theorem was proved in [44].

Theorem 4.1. *For any subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$, the Lie algebra $\mathfrak{h}^{\mathfrak{g}}$ can be realised as the holonomy algebra of a pseudo-Riemannian manifold of signature $(2,n+2)$.*

Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a subalgebra. To prove Theorem 4.1, in [44] a polynomial metric on $\mathbb{R}^{n+4}$ is constructed, the holonomy algebra of which at the point 0 is exactly $\mathfrak{h}^{\mathfrak{g}}$. This was done in the following way. First we will explain why for any subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$, the Lie algebra $\mathfrak{h}^{\mathfrak{g}}$ is a Berger algebra. For $u, v \in E := \operatorname{span}\{E_1, \dots, E_n\}$, any symmetric linear maps $T_1, T_2 \colon E \to E$ and any linear map $S \colon E \to E$ such that $S - S^* \in \mathfrak{g}$ let

$$\begin{aligned} R^{T_1}(Z_1,u) &= (0, T_1(u), 0, 0), & R^{T_2}(Z_2,u) &= (0, 0, T_2(u), 0), \\ R^{S}(Z_1,Z_2) &= (S - S^*, 0, 0, 0), & R^{S}(Z_1,u) &= (0, 0, S(u), 0), \\ R^{S}(Z_2,u) &= (0, S^*(u), 0, 0), & R^{S}(u,v) &= (0, 0, 0, \langle S(u), v\rangle - \langle u, S(v)\rangle) \end{aligned}$$

and extend these linear maps to $\mathbb{R}^{2,n+2} \otimes \mathbb{R}^{2,n+2}$ in the trivial way. It easy to check that $R^{T_1}, R^{T_2}, R^S \in \mathcal{K}(\mathfrak{h}^{\mathfrak{g}})$. Obviously, curvature endomorphisms of these forms generate $\mathfrak{h}^{\mathfrak{g}}$.

Now let $\dim \mathfrak{g} = N$ and let $A_1, \dots, A_N$ be a basis of the vector space $\mathfrak{g}$. Denote by $A^i_{j\alpha}$ the elements of the matrices A_α. Let $(x_1, x_2, y_1 \dots, y_n, z_1, z_2)$ be the canonical coordinates on $\mathbb{R}^{n+4}$. Consider the following metric on $\mathbb{R}^{n+4}$:

$$g = 2dx_1dz_1 + 2dx_2dz_2 + \sum_{i=1}^{n}(dy_i)^2 + 2\sum_{i=1}^{n} u_i dy_i dz_2 + f(dz_1)^2,$$

where

$$u_i = \sum_{\alpha=1}^{N}\sum_{j=1}^{n} \frac{1}{\alpha!} A^i_{j\alpha} y_j z_1^\alpha \quad \text{and} \quad f = \sum_{i=1}^{n}(y_i)^2.$$

This metric is constructed in such a way that

$$\mathrm{pr}_{\mathfrak{so}(n)}\big[\big(\underbrace{\nabla_{\partial_{z_1}} \dots \nabla_{\partial_{z_1}}}_{(r-1)-\text{times}} \mathcal{R}\big)\big(\partial_{z_1}, \partial_{z_2}\big)_0\big] = S_r - S_r^* = A_r, \tag{1}$$

for $r = 1, \dots, N$, where $S_r = \frac{1}{2}A_r$, and the images of other covariant derivatives of $\mathcal{R}$ are contained in $\mathfrak{h}^{\mathfrak{g}}$.

4.2 Holonomy groups of pseudo-Kählerian manifolds of index 2

A pseudo-Riemannian manifold (M, g) is called *pseudo-Kählerian* if there exists a parallel smooth field of endomorphisms J of the tangent bundle of M that satisfies $J^2 = -\mathrm{id}$ and $g(JX, JY) = g(X, Y)$ for all vector fields X and Y on M. The holonomy algebra of a pseudo-Kählerian manifold of signature $(2, 2n+2)$ is contained in $\mathfrak{u}(1, n+1)$.

From Theorem 2.3 it follows that the holonomy algebra of a pseudo-Kählerian manifold of $(2, 2n+2)$ is a direct sum of irreducible holonomy algebras of Kählerian manifolds and of the indecomposable holonomy algebra of a pseudo-Kählerian manifold of signature $(2, 2k+2)$. If the last algebra is irreducible, The Berger list in Theorem 2.4 implies that it is either $\mathfrak{u}(1, k+1)$ or $\mathfrak{su}(1, k+1)$, or that it is the holonomy algebra of an irreducible hermitian symmetric space and listed in [15]. If the last algebra is not irreducible, again we are left with the problem of classifying indecomposable, non-irreducible holonomy algebras in $\mathfrak{u}(1, k+1)$.

In order to describe this classification we denote by $\mathbb{R}^{2,2n+2}$ the vector space $\mathbb{R}^{2n+4}$ endowed with a complex structure J and with a J-invariant metric $\langle\,,\,\rangle$ of signature $(2, 2n+2)$, i.e., $\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^{2,2n+2}$. We fix a basis $X_1, X_2, E_1, \dots, E_n, F_1, \dots, F_n, Z_1, Z_2$ of $\mathbb{R}^{2,2n+2}$ such that the Gram matrix of

the metric $\langle\,,\,\rangle$ and the complex structure J have the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbb{I}_{2n} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{I}_n & 0 & 0 \\ 0 & 0 & \mathbb{I}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{respectively.}$$

We denote by $\mathfrak{u}(1,n+1)_{\mathcal{I}}$ the subalgebra of $\mathfrak{u}(1,n+1)$ that preserves the J-invariant 2-dimensional isotropic subspace $\mathcal{I} = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathbb{R}^{2,2n+2}$. The Lie algebra $\mathfrak{u}(1,n+1)_{\mathcal{I}}$ can be identified with the following matrix algebra

$$\mathfrak{u}(1,n+1)_{\mathcal{I}} = \left\{ \begin{pmatrix} a_1 & -a_2 & -z_1^t & -z_2^t & 0 & -c \\ a_2 & a_1 & z_2^t & -z_1^t & c & 0 \\ 0 & 0 & B & -C & z_1 & -z_2 \\ 0 & 0 & C & B & z_2 & z_1 \\ 0 & 0 & 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & 0 & 0 & a_2 & -a_1 \end{pmatrix} \;\middle|\; \begin{array}{l} a_1, a_2, c \in \mathbb{R}, \\ z_1, z_2 \in \mathbb{R}^n, \\ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}(n) \end{array} \right\}.$$

Recall that

$$\mathfrak{u}(n) = \left\{ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \;\middle|\; B \in \mathfrak{so}(n), C \in \mathfrak{gl}(n,\mathbb{R}), C^t = C \right\}$$

and

$$\mathfrak{su}(n) = \left\{ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}(n) \;\middle|\; \operatorname{tr} C = 0 \right\}.$$

We identify an element of $\mathfrak{u}(1,n+1)_{\mathcal{I}}$ with the 4-tuple

$$(a_1 + ia_2, B + iC, z_1 + iz_2, c).$$

The non-vanishing components of the Lie brackets in $\mathfrak{u}(1,n+1)_{\mathcal{I}}$ are the following:

$$\begin{aligned} [(0, B+iC, 0, 0), (0, B_1+iC_1, 0, 0)] &= (0, [B+iC, B_1+iC_1]_{\mathfrak{u}(n)}, 0, 0), \\ [(a_1, 0, 0, 0), (0, 0, z_1+iz_2, c)] &= (0, 0, a_1(z_1+iz_2), 2a_1c), \\ [(ia_2, 0, 0, 0), (0, 0, z_1+iz_2, 0)] &= (0, 0, a_2(z_2-iz_1), 0), \\ [(0, B+iC, 0, 0), (0, 0, z_1+iz_2, 0)] &= (0, 0, Bz_1 - Cz_2 + i(Cz_1 + Bz_2), 0), \\ [(0, 0, z_1+iz_2, 0), (0, 0, w_1+iw_2, 0)] &= (0, 0, 0, 2(-z_1w_2^t + z_2w_1^t)). \end{aligned}$$

Hence we obtain the decomposition

$$\mathfrak{u}(1,n+1)_{\mathcal{I}} = (\mathbb{C} \oplus \mathfrak{u}(n)) \ltimes (\mathbb{C}^n \ltimes \mathbb{R}).$$

Denote by $\mathfrak{su}(1,n+1)_{\mathcal{I}}$ the subalgebra of $\mathfrak{su}(1,n+1)$ that preserves the subspace $\mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathbb{R}^{2,2n+2}$. Then

$$\mathfrak{su}(1,n+1)_{\mathcal{I}} = \{(a_1+ia_2, B+iC, z_1+iz_2, c) \in \mathfrak{u}(1,n+1)_{\mathcal{I}} \mid 2a_2 + \operatorname{tr}_{\mathbb{R}} C = 0\}$$

and

$$\mathfrak{u}(1,n+1)_{\mathcal{I}} = \mathfrak{su}(1,n+1)_{\mathcal{I}} \oplus \mathbb{R}J.$$

Note that

$$\mathfrak{u}(1,1)_{\mathcal{I}} = \left\{ \begin{pmatrix} a_1 & -a_2 & 0 & -c \\ a_2 & a_1 & c & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & a_2 & -a_1 \end{pmatrix} \;\middle|\; a_1, a_2, c \in \mathbb{R} \right\}$$

If an indecomposable subalgebra $\mathfrak{h} \subset \mathfrak{u}(1, n+1)$ preserves a degenerate proper subspace $W \subset \mathbb{R}^{2,2n+2}$, then $\mathfrak{h}$ preserves the J-invariant 2-dimensional isotropic subspace $W_1 \subset \mathbb{R}^{2,2n+2}$, where $W_1 = (W \cap JW) \cap (W \cap JW)^{\perp}$ if $W \cap JW \neq \{0\}$ and $W_1 = (W \oplus JW) \cap (W \oplus JW)^{\perp}$ if $W \cap JW = \{0\}$. Therefore $\mathfrak{h}$ is conjugated to an indecomposable subalgebra of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$.

The classification of indecomposable, non-irreducible holonomy algebras in $\mathfrak{u}(2, n+2)$ is given by the following theorem, for the second part of which we fix some notation: $0 \leq m \leq n$ is an integer, $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra (as above, $\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{u}'$), we fix the decomposition $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}$, $\mathbb{R}^{n-m} \subset \mathbb{C}^{n-m}$ is a real form and $J_{n-m} \subset \mathfrak{u}(n-m) \subset \mathfrak{u}(n) \subset \mathfrak{u}(1, n+1)_{\mathcal{I}}$ is the complex structure on $\mathbb{C}^{n-m}$. Let $\varphi, \phi \colon \mathfrak{u} \to \mathbb{R}$ be linear maps with $\varphi|_{\mathfrak{u}'} = \phi|_{\mathfrak{u}'} = 0$.

Theorem 4.2. 1) *A subalgebra $\mathfrak{h} \subset \mathfrak{u}(1,1)$ is the indecomposable, non-irreducible holonomy algebra of a pseudo-Kählerian manifold of signature $(2,2)$ if and only if $\mathfrak{h}$ is conjugated to $\mathfrak{u}(1,1)_{\mathcal{I}}$ or one of the following subalgebras of it:*

$$\mathfrak{h}^2_{n=0} = \left\{ \begin{pmatrix} a_1 & -a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & a_2 & -a_1 \end{pmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R} \right\};$$

$$\mathfrak{h}^{\gamma_1,\gamma_2}_{n=0} = \left\{ \begin{pmatrix} a\gamma_1 & -a\gamma_2 & 0 & -c \\ a\gamma_2 & a\gamma_1 & c & 0 \\ 0 & 0 & -a\gamma_1 & -a\gamma_2 \\ 0 & 0 & a\gamma_2 & -a\gamma_1 \end{pmatrix} \;\middle|\; a, c \in \mathbb{R} \right\}, \quad \text{where } \gamma_1, \gamma_2 \in \mathbb{R};$$

2) *Let $n \geq 1$. Then a subalgebra $\mathfrak{h} \subset \mathfrak{u}(1, n+1)$ is the indecomposable, non-irreducible holonomy algebra of a pseudo-Kählerian manifold of signature $(2, 2n+2)$ if and only if $\mathfrak{h}$ is conjugated to one of the following subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$:*

$$\begin{aligned}
\mathfrak{h}^{m,\mathfrak{u}} &= (\mathbb{R} \oplus \mathbb{R}(i + J_{n-m}) \oplus \mathfrak{u}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}); \\
\mathfrak{h}^{m,\mathfrak{u},\phi} &= (\mathbb{R} \oplus \{\phi(A)(i + J_{n-m}) + A \mid A \in \mathfrak{u}\}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}); \\
\mathfrak{h}^{m,\mathfrak{u},\varphi,\phi} &= \{\varphi(A) + \phi(A)(i + J_{n-m}) + A \mid A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}); \\
\mathfrak{h}^{m,\mathfrak{u},\varphi} &= (\mathbb{R}(i + J_{n-m}) \oplus \{\varphi(A) + A \mid A \in \mathfrak{u}\}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}); \\
\mathfrak{h}^{m,\mathfrak{u},\lambda} &= (\mathbb{R}(1 + \lambda(i + J_{n-m})) \oplus \mathfrak{u}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}), \quad \text{where } \lambda \in \mathbb{R}; \\
\mathfrak{h}^{n,\mathfrak{u},\psi,k,l} &= \{A + \psi(A) \mid A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^k \oplus \mathbb{R}^{n-l}) \ltimes \mathbb{R}),
\end{aligned}$$

where k and l are integers such that $0 < k \leq l \leq n$, we have the decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{l-k} \oplus \mathbb{C}^{n-l}$, $\mathfrak{u} \subset \mathfrak{u}(k)$ is a subalgebra with $\dim \mathfrak{z}(\mathfrak{u}) \geq n + l - 2k$

and $\psi : \mathfrak{u} \to \mathbb{C}^{l-k} \oplus i\mathbb{R}^{n-l}$ *is a surjective linear map with* $\psi|_{\mathfrak{u}'} = 0$*;*

$$\mathfrak{h}^{m,\mathfrak{u},\psi,k,l,r} = \{A + \psi(A) \mid A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^k \oplus \mathbb{R}^{m-l} \oplus \mathbb{R}^{r-m}) \ltimes \mathbb{R}),$$

where k*,* l*,* r *and* m *are integers such that* $0 < k \leq l \leq m \leq r \leq n$ *and* $m < n$*, we have the decomposition* $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{l-k} \oplus \mathbb{C}^{m-l} \oplus \mathbb{C}^{r-m} \oplus \mathbb{C}^{n-r}$*,* $\mathfrak{u} \subset \mathfrak{u}(k)$ *is a subalgebra with* $\dim \mathfrak{z}(\mathfrak{u}) \geq n + m + l - 2k - r$ *and* $\psi : \mathfrak{u} \to \mathbb{C}^{l-k} \oplus i\mathbb{R}^{m-l} \oplus \mathbb{R}^{n-r}$ *is a surjective linear map with* $\psi|_{\mathfrak{u}'} = 0$*.*

Note that $\mathfrak{h}_{n=0}^{\gamma_1=1,\gamma_2=0} = \mathfrak{su}(1,1)_{\mathcal{I}}$. As an example for the Lie algebras in the theorem, an element in the Lie algebra $\mathfrak{h}^{m,\mathfrak{u},\varphi,\phi}$ is given by

$$\begin{pmatrix}
\varphi(A) & -\phi(A) & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c \\
\phi(A) & \varphi(A) & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0 \\
0 & 0 & B & 0 & -C & 0 & z_1 & -z_2 \\
0 & 0 & 0 & 0 & 0 & -\phi(A)\mathbb{I}_{n-m} & z_1' & 0 \\
0 & 0 & C & 0 & B & 0 & z_2 & z_1 \\
0 & 0 & 0 & \phi(A)\mathbb{I}_{n-m} & 0 & 0 & 0 & z_1' \\
0 & 0 & 0 & 0 & 0 & 0 & -\varphi(A) & -\phi(A) \\
0 & 0 & 0 & 0 & 0 & 0 & \phi(A) & -\varphi(A)
\end{pmatrix}$$

with $c \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^m$, $z_1' \in \mathbb{R}^{n-m}$, $A = \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}$.

As a corollary, we get the classification of indecomposable, non-irreducible holonomy algebras contained in $\mathfrak{su}(1, n+1)$, i.e., of the holonomy algebras of special pseudo-Kählerian manifolds (these manifolds are pseudo-Kählerian and Ricci-flat).

Corollary 4.3. *Let* (M, h, J) *be a indecomposable, non-irreducible pseudo-Kählerian manifold of signature* $(2, n+2)$ *which is Ricci-flat. Then its holonomy algebra is given by*

$$\mathfrak{h}_{n=0}^{\gamma_1=0,\gamma_2=0} \quad \text{or} \quad \mathfrak{su}(1,1)_{\mathcal{I}} = \mathfrak{h}_{n=0}^{\gamma_1=1,\gamma_2=0}$$

in the case $n = 0$*, and by*

$$\mathfrak{h}^{m,\mathfrak{u},\phi},\ \mathfrak{h}^{m,\mathfrak{u},\varphi,\phi} \quad \text{with } \mathfrak{u} \subset \mathfrak{su}(n) \text{ and } \phi(A) = -\frac{1}{n-m+2}\operatorname{tr}_{\mathbb{C}} A,$$

and

$$\mathfrak{h}^{n,\mathfrak{u},\psi,k,l},\ \mathfrak{h}^{m,\mathfrak{u},\psi,k,l,r} \quad \text{with } \mathfrak{u} \subset \mathfrak{su}(k)$$

for $n > 0$*.*

As in the first case, we consider all subalgebras of $\mathfrak{u}(1,1)_{\mathcal{I}}$ and show which of these subalgebras are indecomposable Berger subalgebras. We realise the Lie algebras of Part 1) of Theorem 4.2 as the holonomy algebras of pseudo-Kählerian metrics on $\mathbb{R}^{2,2}$, see Section 5.2.

By analogy to the case of Lorentzian manifolds, the proof of Part 2) of Theorem 4.2 consists of the following 3 steps:

Step 1. Classification of indecomposable, non-irreducible subalgebras of $\mathfrak{su}(1,n+1)$.

Step 2. Classification of indecomposable, non-irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)$.

Step 3. Construction of a pseudo-Riemannian manifold with the holonomy algebra $\mathfrak{h}$ for each indecomposable, non-irreducible Berger subalgebra $\mathfrak{h} \subset \mathfrak{u}(1,n+1)$.

Step 1. First we classify all connected subgroups of $\mathrm{SU}(1,n+1)$ that act indecomposably and non-irreducibly on $\mathbb{R}^{2,2n+2}$, that is equivalent to the classification of indecomposable, non-irreducible subalgebras of $\mathfrak{su}(1,n+1)$. Any such subgroup preserves a 2-dimensional isotropic J-invariant subspace of $\mathbb{R}^{2,2n+2}$. We use a generalization of the method from [47] (see Section 3.2).

We denote by $\mathbb{C}^{1,n+1}$ the $(n+2)$-dimensional complex vector space given by $(\mathbb{R}^{2,2n+2}, J, \eta)$. Let g be the pseudo-Hermitian metric on $\mathbb{C}^{1,n+1}$ of signature $(1,n+1)$ corresponding to η. If a subgroup $G \subset \mathrm{U}(1,n+1)$ acts indecomposably on $\mathbb{R}^{2,2n+2}$, then G acts indecomposably on $\mathbb{C}^{1,n+1}$, i.e., does not preserve any proper g-non-degenerate complex vector subspace.

We consider the boundary $\partial\mathbb{H}^{n+1}_{\mathbb{C}}$ of the complex hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{C}}$. The boundary $\partial\mathbb{H}^{n+1}_{\mathbb{C}}$ consists of complex isotropic lines of $\mathbb{C}^{1,n+1}$. We identify $\partial\mathbb{H}^{n+1}_{\mathbb{C}}$ with the $(2n+1)$-dimensional sphere S^{2n+1}. Consider the complex isotropic line $\mathcal{I} = \mathbb{C}X_1 = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathbb{C}^{1,n+1}$ and denote by $\mathrm{U}(1,n+1)_{\mathcal{I}} \subset \mathrm{U}(1,n+1)$ the connected Lie subgroup that preserves the line $\mathcal{I}$. The Lie algebra of the Lie group $\mathrm{U}(1,n+1)_{\mathcal{I}}$ is $\mathfrak{u}(1,n+1)_{\mathcal{I}}$. Any connected subgroup $G \subset \mathrm{U}(1,n+1)$ that acts on $\mathbb{C}^{1,n+1}$ indecomposably and non-irreducibly is conjugated to a subgroup of $\mathrm{U}(1,n+1)_{\mathcal{I}}$. From the above decomposition of the Lie algebra $\mathfrak{u}(1,n+1)_{\mathcal{I}}$ we obtain the decomposition

$$U(1,n+1)_{\mathcal{I}} = (\mathbb{C}^* \times \mathrm{U}(n)) \ltimes (\mathbb{C}^n \ltimes \mathbb{R}).$$

We identify the set $\partial\mathbb{H}^{n+1}_{\mathbb{C}}\backslash\{\mathcal{I}\} = S^{2n+1}\backslash\{\text{point}\}$ with the Heisenberg space $\mathcal{H}_n = \mathbb{C}^n \oplus \mathbb{R}$. Any element $f \in \mathrm{U}(1,n+1)_{\mathcal{I}}$ induces a transformation $\Gamma(f)$ of $\mathcal{H}_n$, moreover, $\Gamma(f) \in \mathrm{Sim}\,\mathcal{H}_n$, where $\mathrm{Sim}\,\mathcal{H}_n$ is the group of the Heisenberg similarity transformations of $\mathcal{H}_n$. For the Lie group $\mathrm{Sim}\,\mathcal{H}_n$ we have the decomposition

$$\mathrm{Sim}\,\mathcal{H}_n = (\mathbb{R}_+ \times \mathrm{U}(n)) \ltimes (\mathbb{C}^n \ltimes \mathbb{R}).$$

The elements $\lambda \in \mathbb{R}_+$, $A \in \mathrm{U}(n)$ and $(z,u) \in \mathbb{C}^n \ltimes \mathbb{R}$ act on $\mathcal{H}_n = \mathbb{C}^n \oplus \mathbb{R}$ in the following way:

$$\lambda\colon (z,u) \mapsto (\lambda z, \lambda^2 u) \quad \text{(real Heisenberg dilation about the origin)},$$
$$A\colon (z,u) \mapsto (Az, u) \quad \text{(Heisenberg rotation about the vertical axis)},$$
$$(w,v)\colon (z,u) \mapsto (w+z, v+u+2\,\mathrm{Im}\,g(w,z)) \quad \text{(Heisenberg translations)}.$$

We show that $\Gamma\colon \mathrm{U}(1,n+1)_{\mathcal{I}} \to \mathrm{Sim}\,\mathcal{H}_n$ is a surjective Lie group homomorphism with the kernel $\mathbb{T}$, where $\mathbb{T}$ is the 1-dimensional subgroup generated by the complex structure $J \in \mathrm{U}(1,n+1)_{\mathcal{I}}$. In particular, $\mathbb{T}$ is the center of $\mathrm{U}(1,n+1)_{\mathcal{I}}$. Let $\mathrm{SU}(1,n+1)_{\mathcal{I}} = \mathrm{U}(1,n+1)_{\mathcal{I}} \cap \mathrm{SU}(1,n+1)$. Then $\mathrm{U}(1,n+1)_{\mathcal{I}} = \mathrm{SU}(1,n+1)_{\mathcal{I}} \cdot \mathbb{T}$

and the restriction

$$\Gamma|_{\mathrm{SU}(1,n+1)_{\mathcal{I}}}\colon \mathrm{SU}(1,n+1)_{\mathcal{I}} \longrightarrow \mathrm{Sim}\,\mathcal{H}_n$$

is a Lie group isomorphism.

We consider the natural projection $\pi\colon \mathrm{Sim}\,\mathcal{H}_n \to \mathrm{Sim}\,\mathbb{C}^n$, where

$$\mathrm{Sim}\,\mathbb{C}^n = (\mathbb{R}_+ \times \mathrm{U}(n)) \ltimes \mathbb{C}^n$$

is the group of similarity transformations of $\mathbb{C}^n$. The homomorphism π is surjective and its kernel is 1-dimensional.

We prove that *if a subgroup $G \subset \mathrm{U}(1,n+1)_{\mathcal{I}}$ acts indecomposably on $\mathbb{C}^{1,n+1}$, then*

(1) *the subgroup $\pi(\Gamma(G)) \subset \mathrm{Sim}\,\mathbb{C}^n$ does not preserve any proper complex affine subspace of $\mathbb{C}^n$;*

(2) *if $\pi(\Gamma(G)) \subset \mathrm{Sim}\,\mathbb{C}^n$ preserves a proper non-complex affine subspace $L \subset \mathbb{C}^n$, then the minimal complex affine subspace of $\mathbb{C}^n$ containing L is $\mathbb{C}^n$.*

This is the key statement for our classification.

Since we are interested in connected Lie groups, it is enough to classify the corresponding Lie algebras. The classification is done in the following way:

- First we describe non-complex vector subspaces $L \subset \mathbb{C}^n$ with $\mathrm{span}_{\mathbb{C}} L = \mathbb{C}^n$ (it is enough to consider only vector subspaces, since we do the classification up to conjugacy). Any such non-complex vector subspace has the form $L = \mathbb{C}^m \oplus \mathbb{R}^{n-m}$, where $0 \le m \le n$. Here we have 3 types of subspaces: 1) $m = 0$ (L is a real form of $\mathbb{C}^n$); 2) $0 < m < n$; 3) $m = n$ ($L = \mathbb{C}^n$).

- We describe the Lie algebras $\mathfrak{f}$ of the connected Lie subgroups $F \subset \mathrm{Sim}\,\mathbb{C}^n$ preserving L. Without loss of generality, we can assume that each Lie group F does not preserve any proper affine subspace of L. This means that F acts irreducibly on L. By a theorem of D. V. Alekseevsky [2], [3], F acts transitively on L. In Theorem 3.4 we divided transitive similarity transformation groups of Euclidean spaces into 4 types. Here we unify type 2 and 3. The group F is contained in $(\mathbb{R}_+ \times \mathrm{SO}(L) \times \mathrm{SO}(L^{\perp_\eta})) \ltimes L$, where $\mathbb{R}_+$ is the group of real dilations of $\mathbb{C}^n$ about the origin and L is the group of all translations in $\mathbb{C}^n$ by vectors of L. In general situation we know only the projection of F on $\mathrm{Sim}\,L = (\mathbb{R}_+ \times \mathrm{SO}(L)) \ltimes L$, but in our case the projection of F on $\mathrm{SO}(L) \times \mathrm{SO}(L^{\perp_\eta})$ is also contained in $\mathrm{U}(n)$ and we know the full information about F. On this step we obtain 9 types of Lie algebras.

- Then we describe subalgebras $\mathfrak{a} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ with $\pi(\mathfrak{a}) = \mathfrak{f}$. For each $\mathfrak{f}$ we have 2 possibilities: $\mathfrak{a} = \mathfrak{f} + \ker\pi$ or $\mathfrak{a} = \{x + \zeta(x) \mid x \in \mathfrak{f}\}$, where $\zeta\colon \mathfrak{f} \to \ker\pi$ is a linear map. Using the isomorphism $(\Gamma|_{\mathfrak{su}(1,n+1)_{\mathcal{I}}})^{-1}$ we obtain a list of subalgebras $\mathfrak{h} \subset \mathfrak{su}(1,n+1)_{\mathcal{I}}$. This gives us 12 types of Lie algebras.

- Finally we check which of the so obtained subalgebras of $\mathfrak{su}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(2,2n+2)$ are indecomposable. It turns out that some of the types contain Lie algebras that are not indecomposable. Giving new definitions to these types we obtain 12 types of indecomposable Lie algebras. Unifying some of the types we obtain 8 types of indecomposable subalgebras of $\mathfrak{su}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(2,2n+2)$.

Example 4.4. Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a subalgebra with non-trivial center and $\zeta\colon \mathfrak{g} \to \mathbb{R}$ be a non-zero linear map with $\zeta|_{\mathfrak{g}'} = 0$. Then the subalgebra $\{(0, B, z_1, \zeta(B)) \mid B \in \mathfrak{g},\ z_1 \in \mathbb{R}^n\} \subset \mathfrak{su}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(2,2n+2)$ is indecomposable.

Note that the Lie algebra of the above example was not considered in [56].

Step 2. In this step we classify indecomposable Berger subalgebras of $\mathfrak{u}(1,n+1)_{\mathcal{I}}$.

First we obtain a list of candidates for the indecomposable subalgebras of $\mathfrak{u}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(2,2n+2)$. For each $\mathfrak{f} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ as above and for each $\mathfrak{h} \subset \mathfrak{su}(1,n+1)_{\mathcal{I}}$ with $\pi(\Gamma(\mathfrak{h})) = \mathfrak{f}$ we consider the Lie algebras $\mathfrak{h}^J = \mathfrak{h} \oplus \mathbb{R}J$ and $\mathfrak{h}^\xi = \{x + \xi(x) \mid x \in \mathfrak{h}\}$, where $\xi\colon \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map. As we claimed above, any indecomposable subalgebra of $\mathfrak{u}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(2,2n+2)$ is of the form $\mathfrak{h}$, $\mathfrak{h}^J$ or $\mathfrak{h}^\xi$. These subalgebras are candidates for the indecomposable subalgebras of $\mathfrak{u}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(2,2n+2)$. We associate with each of these subalgebras an integer $0 \le m \le n$. If $m > 0$, then the subalgebras of the form $\mathfrak{h}$, $\mathfrak{h}^J$ and $\mathfrak{h}^\xi \subset \mathfrak{u}(1,n+1)_{\mathcal{I}}$ are indecomposable. We have inclusions $\mathfrak{u}(m) \subset \mathfrak{u}(n) \subset \mathfrak{u}(1,n+1)_{\mathcal{I}}$ and projection maps $\mathrm{pr}_{\mathfrak{u}(m)}\colon \mathfrak{u}(1,n+1)_{\mathcal{I}} \to \mathfrak{u}(m)$, $\mathrm{pr}_{\mathfrak{u}(n)}\colon \mathfrak{u}(1,n+1)_{\mathcal{I}} \to \mathfrak{u}(n)$.

For any integer $0 \le m \le n$ and any subalgebra $\mathfrak{u} \subset \mathfrak{u}(m) \oplus ((\mathfrak{so}(n-m) \oplus \mathfrak{so}(n-m)) \cap \mathfrak{u}(n-m))$ we consider a subalgebra $\mathfrak{h}_0^{m,\mathfrak{u}} \subset \mathfrak{u}(1,n+1)_{\mathcal{I}}$ and describe the space $\mathcal{K}(\mathfrak{h}_0^{m,\mathfrak{u}})$. The Lie algebras of the form $\mathfrak{h}_0^{m,\mathfrak{u}}$ contain all candidates for the indecomposable subalgebras of $\mathfrak{u}(1,n+1)_{\mathcal{I}}$. For any subalgebra $\mathfrak{h} \subset \mathfrak{h}_0^{m,\mathfrak{u}}$ the space $\mathcal{K}(\mathfrak{h})$ can be found from the following condition:

$$R \in \mathcal{K}(\mathfrak{h}) \text{ if and only if } R \in \mathcal{K}(\mathfrak{h}_0^{m,\mathfrak{u}}) \text{ and } R(\mathbb{R}^{2,2n+2}, \mathbb{R}^{2,2n+2}) \subset \mathfrak{h}.$$

Using this, we easily find all indecomposable, non-irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)_{\mathcal{I}}$.

Step 3. As the last step of the classification, we construct metrics on $\mathbb{R}^{2n+4}$ that realise all Berger algebras obtained above as holonomy algebras. Idea of constructions of the metrics is similar to the one in Section 4.1. The coefficients of the metrics are polynomial functions, hence the corresponding Levi-Civita connections are analytic and in each case the holonomy algebra at the point $0 \in \mathbb{R}^{2n+4}$ is generated by the operators images of the curvature tensor and of all its derivatives. We explicitly compute for each metric the components of the curvature tensor and its derivatives. Then using the induction, we find the holonomy algebra for each of the metrics.

4.3 Examples of 4-dimensional Lie groups with left-invariant pseudo-Kählerian metrics

Let G be a Lie group endowed with a left-invariant metric g and let $\mathfrak{g}$ be the Lie algebra of G. We will consider $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields on G and as the tangent space at the identity $e \in G$. Let $X, Y, Z \in \mathfrak{g}$. Since X, Y, Z and g are left-invariant, from the Koszul formula it follows that the Levi-Civita connection on (G, g) is given by

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y]), \tag{2}$$

where $X, Y, Z \in \mathfrak{g}$. In particular, we see that the vector field $\nabla_X Y$ is also left-invariant. Hence ∇_X can be considered as the linear operator $\nabla_X : \mathfrak{g} \to \mathfrak{g}$. Obviously, $\nabla_X \in \mathfrak{so}(\mathfrak{g}, g)$. For the curvature tensor $\mathcal{R}$ of (G, g) at the point $e \in G$ we have

$$\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \tag{3}$$

where $X, Y \in \mathfrak{g}$. The holonomy algebra $\mathfrak{h}_e$ at the point $e \in G$ is given by

$$\mathfrak{h}_e = \mathfrak{m}_0 + [\mathfrak{m}_1, \mathfrak{m}_0] + [\mathfrak{m}_1, [\mathfrak{m}_1, \mathfrak{m}_0]] + \cdots, \tag{4}$$

where

$$\mathfrak{m}_0 = \operatorname{span}\{\mathcal{R}(X, Y) \mid X, Y \in \mathfrak{g}\} \quad \text{and} \quad \mathfrak{m}_1 = \operatorname{span}\{\nabla_X \mid X \in \mathfrak{g}\}.$$

Now we consider 4-dimensional Lie algebras with the basis X_1, X_2, Z_1, Z_2 and with the metric that has the Gram matrix $\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}$ with respect to this basis. Define the following Lie algebras by giving their non-zero brackets:

$$\begin{aligned} \mathfrak{g}_1\colon\quad & [X_1, Z_1] = X_1 + Z_2, \quad [X_1, Z_2] = -X_2 - Z_1,\\ & [X_2, Z_1] = X_2 + Z_1, \quad [X_2, Z_2] = X_1 + Z_2; \end{aligned}$$

$$\mathfrak{g}_2\colon\quad [X_1, Z_2] = X_1, \quad [X_2, Z_1] = -X_1, \quad [Z_1, Z_2] = X_1 + Z_1;$$

Example 4.5. The holonomy algebras of the Levi-Civita connections on the simply connected Lie groups corresponding to the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\mathfrak{h}^2_{n=0}$ and $\mathfrak{h}^{\gamma_1=0,\gamma_2=1}_{n=0}$, respectively.

5 Holonomy in neutral signature

The Berger list in Theorem 2.4 shows that neutral signature (n, n) has the largest variety of irreducible, non-symmetric holonomy groups of all signatures. Apart from $\mathrm{SO}(n, n)$ we have the unitary groups $\mathrm{U}(p, p)$ and $\mathrm{SU}(p, p)$ and the symplectic groups $\mathrm{Sp}(q, q)$ and $\mathrm{Sp}(1) \cdot \mathrm{Sp}(q, q)$, but also

$$\mathrm{SO}(r, \mathbb{C}), \quad \mathrm{Sp}(p, \mathbb{R}) \cdot \mathrm{SL}(2, \mathbb{R}), \quad \text{and} \quad \mathrm{Sp}(p, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C}),$$

and finally the exceptional groups

$$\mathrm{G}_2^{\mathbb{C}} \subset \mathrm{SO}(7,7), \quad \mathrm{Spin}(4,3) \subset \mathrm{SO}(4,4), \quad \text{and} \quad \mathrm{Spin}(7)^{\mathbb{C}} \subset \mathrm{SO}(8,8).$$

Nevertheless, for a complete classification of holonomy groups of neutral signature one also has to consider indecomposable, non-irreducible ones. In [12] some partial results for the holonomy algebras of pseudo-Riemannian manifolds of signature (n,n) were obtained. In particular, a complete classification for $n = 2$ can be given.

For a point p in a pseudo-Riemannian manifold of signature (n,n) we always fix a basis in T_pM such that the metric h at p is of the form

$$\begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}, \tag{1}$$

where $\mathbb{I}_n$ is the identity on $\mathbb{R}^n$. If V is a degenerate subspace which is invariant under the holonomy, again we can form the totally isotropic subspace $V \cap V^\perp$, but in contrary to the previous cases this subspace can have any dimension from 1 up to n. In the case in which this dimension is n we have that $V^\perp = V$, i.e., the orthogonal complement gives no further information, but this case contains a very special situation which will be described first.

5.1 Para-Kähler structures

Recall that we have defined a holonomy representation to be *indecomposable* if any proper invariant subspace is degenerate. Neutral signature (n,n) is the only case where this property does not prevent the holonomy representation from *decomposing*, i.e., to split into two invariant subspaces which are complementary. This is the case if we have two invariant totally isotropic and complementary subspaces V^+ and V^- of dimension n, i.e.,

$$T_pM = V^+ \oplus V^-, \tag{2}$$

with V^+ and V^- totally isotropic and holonomy invariant. This property is equivalent to the existence of a *para Kähler structure* on (M,h) (see [35] and [34]).

Definition 5.1. A *para-Kähler manifold* (M,J,h) is a pseudo-Riemannian manifold (M,h) equipped with a non-trivial section J in the endomorphism bundle such that:

(1) $\nabla J = 0$ where ∇ is the Levi-Civita connection of h,

(2) $J^*h = -h$, and

(3) $J^2 = \mathrm{Id}$ and the eigendistributions $V^\pm := \ker(\mathrm{Id} \mp J)$ have the same rank.

The second condition forces the metric to be of neutral signature (n,n), the third condition ensures that the eigendistributions are totally isotropic of dimension n. That J is parallel is equivalent to the fact that $V^\pm$ are holonomy invariant. It can be shown that the local coefficients of the metric of a para-Kähler metric can be expressed as

second derivatives of a *para-Kähler potential* Ω (for a proof see [81], [12], or [35] in terms of para-Kähler manifolds). Regarding the holonomy groups one gets the following result:

Theorem 5.2 (Bérard-Bergery, Ikemakhen, [12]). *Let (M,h) be a pseudo-Riemannian manifold of signature (n,n) and let $H := \mathrm{Hol}_p(M,h)$ be its holonomy group at a point $p \in M$. If H leaves invariant two totally isotropic complementary subspaces of dimension n, then*

$$H \subset G := \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \;\middle|\; U \in \mathrm{GL}(n,\mathbb{R}) \right\}. \tag{3}$$

Moreover, there exists coordinates $(x_1,\dots x_n,y_1,\dots,y_n)$ around p sending p to $0 \in \mathbb{R}^{2n}$, and a smooth function Ω on a neighborhood of $0 \in \mathbb{R}^{2n}$ such that:

(1) $h = \sum_{i,j=1}^{n} D_{ij}\,dx_i dy_j$ *with* $D_{ij} = \frac{\partial^2 \Omega}{\partial x_i \partial y_j}$.

(2) *The Taylor series of Ω in 0 starts with $x_1y_1 + \cdots + x_ny_n$ and continues with terms which are at least quadratic in the x_i's and quadratic in the y_j's.*

(3) *H is the smallest connected subgroup of G which contains the element*

$$D := \begin{pmatrix} D & 0 \\ 0 & (D^t)^{-1} \end{pmatrix} \in G.$$

Here Ω is the para-Kähler potential. In [5] it is shown that cones over para-Sasakian manifolds are examples of manifolds with the property (2).

5.2 Neutral metrics in dimension four

Now we consider the case where the dimension of the manifold M is four, i.e., the signature is $(2,2)$. Let H be the indecomposable, non-irreducible holonomy group of (M,h). In this case the invariant totally isotropic subspace is a null-line or a totally isotropic plane. The first case is contained in the second: If $(e_1,\dots,e_4)$ is a basis of T_pM such that the metric has the form (1) with e_1 spanning the invariant null-line, then the plane spanned by e_1 and e_2 is invariant as well because Ae_2 is orthogonal to e_1 for A in H. Hence, in both case H leaves invariant a totally isotropic plane $\mathcal{I}$, i.e., the Lie algebra $\mathfrak{h}$ of H is contained in

$$\mathfrak{so}(2,2)_{\mathcal{I}} = \left\{ \begin{pmatrix} U & aJ \\ 0 & -U^t \end{pmatrix} \;\middle|\; U \in \mathfrak{gl}(2,\mathbb{R}), a \in \mathbb{R} \right\}, \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\mathfrak{so}(2,2)_{\mathcal{I}}$ is a semi-direct sum, $\mathfrak{so}(2,2)_{\mathcal{I}} = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathcal{A}$, where $\mathcal{A}$ is the ideal spanned by $\left(\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right)$, with the commutator

$$[(U,a),(V,b)] = ([U,V], b \cdot \mathrm{trace}(U) - a \cdot \mathrm{trace}(V)).$$

Then one can prove:

Theorem 5.3 (Bérard-Bergery, Ikemakhen, [12]). *Let $\mathfrak{h}$ be an indecomposable subalgebra of $\mathfrak{so}(2,2)_{\mathcal{I}} \subset \mathfrak{so}(2,2)$. Then either $\mathfrak{h}$ contains an ideal which is conjugated in $\mathrm{SO}_0(2,2)$ to $\mathcal{A}$, or it is conjugated to one of the three exceptions*

$$\mathfrak{h}_1 = \mathfrak{so}(2) \oplus \mathbb{R} = \left\{ \begin{pmatrix} r\mathbb{I}_2 + sJ & 0 \\ 0 & -r\mathbb{I}_2 + sJ \end{pmatrix} \,\middle|\, r, s \in \mathbb{R} \right\},$$

$\mathfrak{h}_2 = \mathbb{R} \left(\begin{smallmatrix} L & J \\ 0 & L \end{smallmatrix} \right)$, *or* $\mathfrak{h}_3 = \mathbb{R} \left(\begin{smallmatrix} J & J \\ 0 & J \end{smallmatrix} \right)$, *with J as above and* $L = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$.

Note that the other cases of subalgebras contained in the Lie algebra $\mathfrak{g}$ of G defined in Theorem 5.2 contain an ideal which is conjugated to $\mathcal{A}$, namely the ideals spanned by matrices $U \in \mathrm{GL}(2,\mathbb{R})$ which are nonzero only at one entry. Note also that $\mathfrak{h}_1 = \mathfrak{h}^2_{n=0}$ defined in Theorem 4.2. One can show that the algebras $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are not Berger algebras and therefore cannot be holonomy algebras.

Corollary 5.4. *Let $\mathfrak{h}$ be an indecomposable Berger algebra in $\mathfrak{so}(2,2)_{\mathcal{I}} \subset \mathfrak{so}(2,2)$. Then $\mathfrak{h}$ is conjugated to a subalgebra which is either contained in $\mathfrak{g}$ or contains the ideal $\mathcal{A}$.*

Based on this in [12] a complete classification of indecomposable, non-irreducible holonomy groups of pseudo-Riemannian manifolds of signature $(2,2)$ is obtained.

Theorem 5.5. *Let $H \subset \mathrm{SO}(2,2)$ be the indecomposable, non-irreducible acting holonomy group of a 4-dimensional pseudo-Riemannian manifold of signature $(2,2)$ and $\mathfrak{h}$ its Lie algebra. Then H leaves invariant a totally isotropic plane $\mathcal{I}$ and it holds one of the following:*

(A) *H leaves invariant another totally isotropic plane complementary to $\mathcal{I}$, in which case $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}$ is conjugated in $\mathrm{SO}_0(2,2)$ to one of the following: $\mathfrak{gl}(2,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{R})$, the strictly upper triangular, the upper triangular matrices,*

$$\mathfrak{k}_\lambda = \left\{ \begin{pmatrix} a & b \\ 0 & \lambda a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} \quad \text{for } \lambda \in \mathbb{R},$$

or

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\},$$

under use of the identification $\mathfrak{g} \simeq \mathfrak{gl}(2,\mathbb{R})$.

(B) *H does not leave invariant a complementary totally isotropic plane. In this case the Lie algebra $\mathfrak{h}$ of H is conjugated to a semi-direct sum*

$$\mathfrak{h}' \ltimes \mathcal{A},$$

where $\mathfrak{h}'$ is conjugated to one of the subalgebras of $\mathfrak{g}$ listed in (A)*, to $\mathbb{R} \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, or to $\mathfrak{u}_\mu := \mathbb{R} \cdot \left(\begin{smallmatrix} \mu & 1 \\ -1 & \mu \end{smallmatrix} \right)$.*

Remark 5.6. From this Theorem we can also recover the classification of unitary holonomy algebras in signature $(2,2)$ given in Theorem 4.2. First we get the obvious subalgebras of $\mathfrak{u}(1,1)$:

$$\mathcal{A} = \mathfrak{h}_{n=0}^{\gamma_1=0,\gamma_2=0},\quad \mathfrak{h}_1 = \mathfrak{h}_{n=0}^{2},\quad \mathfrak{h}_1 \ltimes \mathcal{A} = \mathfrak{u}(1,1)_{\mathcal{I}},\ \text{and}\ \mathfrak{u}_\mu \ltimes \mathcal{A} = \mathfrak{h}_{n=0}^{\mu\gamma_2,\gamma_2\neq 0}.$$

For the remaining subalgebra $\mathbb{R}\cdot\left(\begin{smallmatrix}\mathbb{I}_2 & 0\\ 0 & -\mathbb{I}_2\end{smallmatrix}\right)\oplus\mathcal{A} = \mathfrak{h}_{n=0}^{\gamma_1=1,\gamma_2=0} = \mathfrak{su}(1,1)_{\mathcal{I}}$ one has to pay attention to the fact that in order to classify subalgebras of $\mathfrak{u}(1,1)$ one needs to classify them up to conjugation in $\mathrm{U}(1,1)$. In fact, $\mathfrak{h}_{n=0}^{\gamma_1=1}$ is conjugated to $\left\{\left(\begin{smallmatrix}a & b\\ 0 & a\end{smallmatrix}\right) \mid a,b\in\mathbb{R}\right\} = \mathfrak{k}_1$ of (A), which is obviously not a subalgebra of $\mathfrak{u}(1,1)$ because the conjugation lies in $\mathrm{O}(2,2)$ but not in $\mathrm{U}(1,1)$.

From Theorem 5.5 we get a conclusion about the closedness of holonomy groups.

Corollary 5.7. *The holonomy group of a* 4*-dimensional pseudo-Riemannian manifold of neutral signature* $(2,2)$ *is closed.*

Finally we want to address the question, which of the algebras obtained can be realised as holonomy algebras. First we recall results in [12], where it is shown that all the possible holonomy groups that leave invariant a pair of complementary totally isotropic planes listed in (A) of Theorem 5.5 can be realised as holonomy groups of metrics which are not locally symmetric. As seen in Theorem 5.2, for these manifolds the metric can be written in terms of the para-Kähler potential Ω, which is given by

$$\Omega = x_1y_1 + x_2y_2 + f(x_1,x_2,y_1,y_2),$$

where f is a smooth function. In the generic case $\mathfrak{h} = \mathfrak{gl}(2,\mathbb{R})$, the Taylor series of f in 0 is at least quadratic in the x_i's and quadratic in the y_i's, $f = x_1x_2y_1y_2$ is an example. In [12] the other algebras are realised by specifying f:

$\mathfrak{h}$	f
$\left\{\left(\begin{smallmatrix}a & b\\ 0 & c\end{smallmatrix}\right)\right\}_{a,b,c\in\mathbb{R}}$	$f = y_1^2 f_1(x_1,x_2,y_1) + x_2^2 f_2(x_2,y_1,y_2)$, where the Taylor series of f_1 and f_2 in 0 start with polynomials of degree 2 in (x_1,x_2) and (y_1,y_2) respectively, e.g., $fx_2y_1(x_1y_1+x_2y_2)$.
$\mathfrak{k}_{\lambda\neq 0}$	$f = x_1\int_0^{y_1} x_2y_1\hat{f}(x_2,t)dt + y_2\int_0^{x_2}((1+x_2y_1\hat{f}((t,y_1))^2-1)dt$, with $\hat{f}$ non-zero
$\mathfrak{k}_0$	$f = x_1x_2y_1^2\hat{f}(x_1,x_2,y_1)$, with $\hat{f}$ non-zero,
$\mathbb{R}\cdot\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$	$f = x_2^2y_1^2\hat{f}(x_2,y_1)$, with $\hat{f}$ non-zero.

A metric with holonomy $\mathfrak{h}_1 \subset \mathfrak{u}(1,1)_{\mathcal{I}}$ is given in terms of complex coordinates. For the existence of a metric with holonomy $\mathfrak{sl}(2,\mathbb{R})$ as listed in (A) of Theorem 5.5, in [12] is argued on general grounds referring to [23] and [24]. [12] leaves open all the cases in (B) of Theorem 5.5.

We will solve this problem partially, first by giving metrics with holonomy $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathcal{A}$. This metric has the form

$$g = 2dx_1dy_1 + 2dx_2dy_2 + f_1dy_1^2 + f_2dy_2^2 + 2f_3dy_1dy_2 \tag{4}$$

with the functions $f_1 = x_1^2$, $f_2 = x_2^2$, $f_3 = -2x_1x_2$.

To all the subalgebras of $\mathfrak{u}(1,1)_{\mathcal{I}}$ obtained in Theorem 4.2 our method of constructing metrics as sketched in Section 4.2 applies and ensures that all of them can be realised as holonomy algebras. The metrics again have the form (4) with the following functions:

$\mathfrak{h}$	f_1, f_2, f_3
$\mathfrak{u}(1,1)_{\mathcal{I}}$	$f_1 = -2x_2y_1 - x_1y_1^2, \quad f_2 = -f_1, \quad f_3 = 2x_1y_1 - x_2y_1^2$
$\mathfrak{h}^2_{n=0}$	$f_1 = x_1^2 - x_2^2, \quad f_2 = -f_1, \quad f_3 = 2x_1x_2$
$\mathfrak{h}^{\gamma_1,\gamma_2}_{n=0}$ $(\gamma_1^2+\gamma_2^2 \neq 0)$	$f_1 = -2\gamma_1x_2y_1 - 2\gamma_2x_1y_1, \quad f_2 = -f_1,$ $f_3 = 2\gamma_1x_1y_1 - 2\gamma_2x_2y_1$
$\mathfrak{h}^{\gamma_1=0,\gamma_2=0}_{n=0}$	$f_1 = y_2^2, \quad f_2 = f_3 = 0$

These metrics disprove a claim in [49], that the Lie algebras

$$\mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathcal{A}, \quad \mathfrak{u}_\mu \ltimes \mathcal{A} \ \text{ for } \mu \neq 0, \quad \text{and} \quad \mathfrak{h}_1 \ltimes \mathcal{A} = \mathfrak{u}(1,1)_{\mathcal{I}}$$

– in [49] denoted by A_{21}, A_{29}, A_{12}, and A_{24} – cannot be realised as holonomy algebras in $\mathfrak{so}(2,2)$. The only case we have to leave undecided, and which is left undecided in [49] as A_{13}, is the one of

$$\mathfrak{h} = \mathbb{R} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ltimes \mathcal{A}. \tag{5}$$

However, very recently L. Bérard-Bergery and T. Krantz developed a general construction to build connections on vector bundles which give new holonomies [13]. In particular, by a construction on the cotangent bundle of a surface they can show that all the algebras listed in Theorem 5.5, even the last one in Equation (5), are holonomy algebras of a metric of signature $(2,2)$.

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Chapter 18

Geometric applications of irreducible representations of Lie groups

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1 Background, results and applications

This chapter is motivated by three algebraic questions which are related to problems in holonomy theory of affine or semi-Riemannian manifolds and in the theory of homogeneous spaces.

Two of these questions are known for most experts in differential geometry but, besides special cases, general proofs are not easy to find in the literature. Thus, one goal of this chapter is to supply such proofs. Indeed, we think that these results are not so well-known for experts in other areas of mathematics or physics. So, we hope this exposition will be useful for non experts in differential geometry. We will give some applications in order to illustrate the use of it. We also state some open problems in the form of conjectures.

The three motivating problems are the following: Are holonomy groups closed? What are special holonomy groups of Lorentzian manifolds? And finally, how many G-invariant bilinear forms exist on a homogeneous space G/H?

Regarding the first question, the first thing to be observed is that the answer should not depend on the mere fact of being a holonomy group, because due to a result of [HO65] any linear Lie group can be realised as a holonomy group of a linear connection (usually with non trivial torsion). Hence, one may try to find algebraic conditions to ensure closedness. Secondly one finds that the answer is 'yes' if one restricts it to holonomy groups of Riemannian manifolds, because any *irreducibly acting*, connected subgroup of $\mathrm{O}(n)$ is closed (see for example [KN63, Appendix 5]),

and thus, by the de Rham decomposition theorem [dR52] the holonomy group is a direct product of closed ones. But for a general connection the answer is 'no', there are a lot of examples of non-closed holonomy groups, see [HO65] for affine, *torsion-free* connections, [Wu67] for pseudo-Riemannian manifolds and [BI93] for Lorentzian manifolds. In all of these examples the holonomy group does not act irreducibly. So it arises the question if all irreducibly acting subgroups of $\mathrm{GL}(n,\mathbb{R})$ are closed? There is a note in [Bes87, Note 10.50, p. 290] that the following theorem was proved in [Wak71].

Theorem 1. *Any irreducibly acting connected Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$ *is closed in* $\mathrm{GL}(n,\mathbb{R})$.

In this note we will prove this theorem (see Section 3) independently of the proof in [Wak71]. Regarding holonomy groups of linear connections, this has the following consequence.

Corollary 1. *If the restricted holonomy group of a linear connection acts irreducibly, then it is closed. Furthermore, the restricted holonomy group of a semi-Riemannian manifold is closed if it acts completely reducibly.*

This statement follows from the de Rham/Wu decomposition theorem [Wu64] and another theorem in [Wu67] (see below, last paragraph in Section 3). Our proof of Theorem 1 uses a theorem of Yosida [Yos37] and Malcev [Mal45] and a very explicit description of the centre of G. This description gives us two corollaries, the first of which will be useful in the proof of Theorem 2.

Corollary 2. *Let* $G \subset \mathrm{O}_0(p,q)$ *be a connected Lie subgroup of* $\mathrm{O}_0(p,q)$ *which acts irreducibly. If* G *is not semisimple, then* p *and* q *are even and* G *is a subgroup of* $\mathrm{U}(p/2,q/2)$ *with centre* $\mathrm{U}(1)$.

Applying this to the spin representations of the orthogonal algebras we get:

Corollary 3. *Let* $G \subset \mathrm{SO}_0(p,q)$ *be a connected Lie subgroup which acts irreducibly and* $\widetilde{G} \subset \mathrm{Spin}(p,q)$ *the pre-image of the covering* $\mathrm{Spin}(p,q) \to \mathrm{SO}_0(p,q)$. *If the spin representation of* $\widetilde{G}$ *admits a trivial subrepresentation, then* G *is semisimple, or* $G = \mathrm{U}(1)\cdot G'$ *with* $G' \subsetneqq \mathrm{SU}(p/2,q/2)$ *and* $(p+q)/2$ *is even.*

Unfortunately, one cannot show that the existence of a trivial subrepresentation of the spin representation implies semisimplicity of G as the following example shows: Let $G = \mathrm{U}(1)\cdot \mathrm{Sp}(p,q)$ with $(p+q)$ even, where $\mathrm{Sp}(p,q)$ is the symplectic or quaternionic unitary group (for a definition see Section 5). Then the intersection of the spaces of spinors which are annihilated by $\mathrm{U}(1)$ with the ones which are annihilated by $\mathrm{Sp}(p,q)$ is a one-dimensional space (for details see [BK99]) .

Nevertheless, this corollary has applications to geometric problems. The first application is a well known fact. If the holonomy group of a semi-Riemannian manifold

acts irreducibly and has a centre, the manifold cannot admit parallel spinors. It was obtained by the classification of irreducible holonomy groups of semi-Riemannian manifolds with parallel spinors ([Wan89] for Riemannian manifolds, and [BK99] for pseudo-Riemannian manifolds). But furthermore it gives results in any case where the holonomy group has a irreducibly acting component on which the existence of parallel spinors depends, as it is the case for indecomposable, non-irreducible Lorentzian manifolds, see [Lei02b].

Regarding the second question which special Lorentzian holonomy groups might exist, one distinguishes between the irreducible and the indecomposable, non-irreducible case. While in the latter case there are several possibilities (for a classification see [BI93], [Lei02a], [Lei03a], [Lei03b], [Lei07], and [Gal05]), for the irreducible case the situation is very limited. The irreducible holonomy groups of semi-Riemannian manifolds were determined by M. Berger in [Ber55] and [Ber57]. For definite metrics and many other signatures the list depends essentially on the property of being a *holonomy group*, whereas in the Lorentzian case it turns out that *irreducibility* is sufficient to determine the group.

Theorem 2. $\mathrm{SO}_0(1,n)$ *is the only connected Lie subgroup of* $\mathrm{O}(1,n)$ *which acts irreducibly.*

The consequence for irreducible Lorentzian holonomy groups follows immediately.

Corollary 4. *If the restricted holonomy group of a Lorentzian manifold acts irreducibly, then it is equal to* $\mathrm{SO}_0(1,n)$.

Here is an application to isotropy irreducibly Lorentzian homogeneous spaces.

Corollary 5. *Isotropy irreducibly Lorentzian homogeneous spaces have constant sectional curvature, i.e., they are flat, de-Sitter or anti de-Sitter spaces.*

A direct and geometric proof of Theorem 2 was given in [DSO01]. An almost algebra-free proof which uses dynamical methods, can be found in [BZ04]. Finally, a purely algebraic proof was given in [BdlH04]. A nice and direct proof of Corollary 5, based on dynamical methods, can be found in [Zeg04]. In Section 4 we will give a short proof of Theorem 2 based on a theorem of Karpelevich [Kar53] and Mostow [Mos55] (i.e., Theorems 7 and 8). Theorem 2 also follows from Corollary 4.5.1 in [ChGr74] which, in turn, depends upon Karpelevich–Mostow's theorem (notice that Lemma 4.4.3 in [ChGr74] is actually Theorem 8).

Theorem 2 seems to be a well-known fact to experts in hyperbolic geometry but widely unknown to mathematicians working in representation theory and differential geometry. For this reason it would be nice to have a geometric and almost algebra-free proof of Karpelevich–Mostow's Theorems 7 and 8. Unfortunately, the proofs in both papers [DSO01] and [BZ04] make use that the boundary at the infinity is homogenous (i.e., the isometry group acts transitively on it or equivalently the space has rank one).

We would like to thank the referee to call our attention to the paper of S. S. Chen and L. Greenberg [ChGr74].

The result of the last Section 5 is motivated by the geometric problem of describing the space of metrics or symplectic forms on a homogeneous space G/H which are invariant under G. Any G-invariant metric or symplectic form corresponds to a non-degenerate bilinear form on $\mathfrak{g}/\mathfrak{h}$ which is invariant under the linear isotropy representation $\mathrm{Ad}_G(H) \subset \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. In our context $\mathrm{Ad}_G(H)$ is assumed to act irreducibly. This is a special case of the following algebraic problem: Given an irreducibly acting Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$, what is the dimension of the space of G-invariant bilinear forms on $\mathbb{R}^n$. We prove the following statement.

Theorem 3. *Let G be an irreducibly acting subgroup of $\mathrm{GL}(n, \mathbb{R})$. The the space of G-invariant symmetric bilinear forms which are not of neutral signature (p, p) is at most one-dimensional. Moreover, the space of invariant symmetric bilinear forms is at most three-dimensional.*

We will describe all possible cases for the dimension of the space of (skew-) symmetric bilinear forms and determine the maximal subgroup which fixes these bilinear forms.

We should point out that many results in this chapter rely on the classification of G-invariant endomorphisms for $G \subset \mathrm{GL}(n, \mathbb{R})$. This classification follows from Schur's lemma and the classification of associative division algebras by Frobenius, but we will give an elementary proof of it in Section 2.

2 The algebra of invariant endomorphisms

If G is a Lie group and V and W two (real or complex) G-modules the algebra of invariant homomorphism is defined as

$$\mathrm{Hom}_G(V, W) := \{X \in \mathrm{Hom}(V, W) \mid A \circ X = X \circ A \text{ for all } A \in G\}.$$

Now, Schur's lemma says that $\mathrm{Hom}_G(V, W) \subset \mathrm{Iso}(V, W) \cup \{0\}$, and furthermore, if $V = W$ is complex, then $\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{Id}$. In any case, it implies that $\mathrm{Hom}_G(V, W)$ is a real associative division algebra, and thus by their classification of Frobenius (1878), it is isomorphic to the algebra of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$ (see e.g. [Pal68]). We are interested in the description of $\mathrm{End}_G(V)$ where V is a real vector space, and in this section we will recall some facts about real irreducible representations which provide an elementary proof of this result.

Suppose that G is a real Lie group and V a real irreducible module. Then there are two cases which can occur for the complexified G-module $V^{\mathbb{C}}$. The first case is that $W := V^{\mathbb{C}}$ is still irreducible. In this case V or W is called *of real type*. One should remark that, if W is a complex irreducible G-module then one can consider W as a real vector space which we denote by $W_{\mathbb{R}}$. This is a reducible real G-module

with invariant real subspace V if and only if W is the complexification of the real irreducible G-module V.

In the other and more complicated case, regarding the application of Schur's lemma, $V^{\mathbb{C}}$ is a reducible G-module. In this case $V^{\mathbb{C}}$ splits into two irreducible G-modules,

$$V^{\mathbb{C}} = W \oplus \overline{W}.$$

In fact, if W is an invariant complex subspace of $V^{\mathbb{C}}$ then the conjugate module $\overline{W}$ defined by the conjugation with respect to $V \subset V^{\mathbb{C}}$ is invariant as well. Furthermore, the spaces $W + \overline{W}$ and $W \cap \overline{W}$ are invariant and equal to their conjugation. Hence they are complexifications of real vector spaces, i.e., $W + \overline{W} = V_1^{\mathbb{C}}$ and $W \cap \overline{W} = V_2^{\mathbb{C}}$. Of course V_1 and V_2 are invariant subspaces of V and thus $V_1 = V$ and $V_2 = \{0\}$. The same argument ensures the irreducibility of W.

Now, since $W \cap \overline{W} = \{0\}$, the mapping $\psi\colon W_{\mathbb{R}} \ni v \mapsto \frac{1}{2}(v + \overline{v}) \in V$ is an isomorphism of real vector spaces yielding the identification

$$W_{\mathbb{R}} \simeq V \simeq \overline{W}_{\mathbb{R}} \tag{1}$$

of real G-modules. In this case V, respectively W, are called *of complex type*, and again we have that a complex module W is irreducible as real module $V = W_{\mathbb{R}}$ if and only if $V^{\mathbb{C}} = W \oplus \overline{W}$ is reducible.

Now we are able to describe the algebra of invariant endomorphisms of a real irreducible G-module V.

Proposition 1. *Let G be a Lie group and V a real irreducible G-module. Then* $\mathrm{End}_G(V)$ *is isomorphic to one of the real algebras* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$.

Proof. As above, we consider two cases. First, assume that $V^{\mathbb{C}}$ is irreducible which ensures that $\mathrm{End}_G(V^{\mathbb{C}}) = \mathbb{C} \cdot \mathrm{Id}$ by Schur's lemma. Hence, if $A \in \mathrm{End}_G(V)$, its complexification $A^{\mathbb{C}} \in \mathrm{End}_G(V^{\mathbb{C}})$ is given by $A^{\mathbb{C}} = \lambda \cdot \mathrm{Id}$ with $\lambda \in \mathbb{C}$. Since $A^{\mathbb{C}}$ leaves V invariant and V is invariant under conjugation we get for $v = \bar{v} \in V$ that

$$\bar{\lambda}\, v = \bar{\lambda}\, \bar{v} = \overline{A^{\mathbb{C}} v} = A^{\mathbb{C}} v = \lambda v,$$

i.e., $\lambda \in \mathbb{R}$. $A = A^{\mathbb{C}}|_V$ gives that $\mathrm{End}_G(V) = \mathbb{R} \cdot \mathrm{Id}$.

For the second case we have to assume that $V^{\mathbb{C}}$ is reducible, i.e., by the above $V^{\mathbb{C}} = W \oplus \overline{W}$, $V \simeq W_{\mathbb{R}}$ and thus $\mathrm{End}_G(V) = \mathrm{End}_G(W_{\mathbb{R}})$. Now any real endomorphism on $W_{\mathbb{R}}$ decomposes uniquely into a complex linear and complex anti-linear part:

$$\mathrm{End}(W_{\mathbb{R}}) \simeq \mathrm{End}(W) \oplus \mathrm{Hom}(W, \overline{W}),$$
$$A = \frac{1}{2}(A + iAi) + \frac{1}{2}(A - iAi).$$

This decomposition descends to $\mathrm{End}_G(W_{\mathbb{R}})$:

$$\mathrm{End}_G(W_{\mathbb{R}}) \simeq \mathrm{End}_G(W) \oplus \mathrm{Hom}_G(W, \overline{W}).$$

Now Schur's lemma implies that $\mathrm{End}_G(W) = \mathbb{C} \cdot \mathrm{Id}$ and, since both, W and $\overline{W}$ are irreducible, that $\mathrm{Hom}_G(W, \overline{W}) \subset \mathrm{Iso}(W, \overline{W}) \cup \{0\}$.

If $\mathrm{Hom}_G(W, \overline{W}) = \{0\}$ we get immediately

$$\mathbb{C} \cdot \mathrm{Id} = \mathrm{End}_G(W_{\mathbb{R}}) \simeq \mathrm{End}_G(V) = \mathrm{span}_{\mathbb{R}}\{\mathrm{Id}, I\},$$

with the complex structure $I := \psi \circ (i \cdot \mathrm{Id}) \circ \psi^{-1}$ where ψ is defined in (1).

Otherwise consider a non-zero $j \in \mathrm{Hom}_G(W, \overline{W})$ which is an isomorphism by Schur's lemma. Then $j^2 \in \mathrm{End}_G(W)$, hence $j^2 = \lambda \cdot \mathrm{Id}$ with $0 \neq \lambda \in \mathbb{C}$. In fact, $\lambda \in \mathbb{R}$ since

$$\bar{\lambda}\, j(w) = j(\lambda\, w) = j\left(j^2(w)\right) = j^2\left(j(w)\right) = \lambda\, j(w)$$

for all $w \in W$. Finally $\lambda < 0$ since otherwise $W_{\mathbb{R}}$ would decompose into the G-invariant $\pm\sqrt{\lambda}$-eigenspaces of $j_{\mathbb{R}}$. Thus, we may assume $j^2 = -1$. For another $A \in \mathrm{Hom}_G(W, \overline{W})$ we get $j \circ A \in \mathrm{End}_G(W)$ and therefore $j \circ A = c \cdot \mathrm{Id}$ for some $c \in \mathbb{C}$. On the other hand $j \circ (-\bar{c} j) = c \cdot \mathrm{Id}$ and thus $A = -\bar{c} j$. Hence, we obtain

$$\mathrm{End}_G(W_{\mathbb{R}}) \simeq \mathrm{End}_G(W) \oplus \mathrm{Hom}_G(W, \overline{W}) = \mathbb{C} \cdot \mathrm{Id} \oplus \mathbb{C} \cdot j,$$

which gives finally

$$\mathrm{End}_G(V) = \mathrm{span}_{\mathbb{R}}\{\mathrm{Id}, I, J, I \circ J\} \simeq \mathbb{H},$$

with $I := \psi \circ i \circ \psi^{-1}$ and $J := \psi \circ j \circ \psi^{-1}$ anti-commuting complex structures. □

Corresponding to the structure of $\mathrm{End}_G(V)$, the real irreducible G-module V is said to be of *real*, *complex* or *quaternionic* type. This corresponds to the convention to call a complex irreducible G-module W of *real type* if it is self-conjugated with respect to an anti-linear bijection J with $J^2 = \mathrm{Id}$, of *quaternionic type* if it is self-conjugated with $J^2 = -\mathrm{Id}$ and of *complex type* if it is not self-conjugated. Here is a useful consequence of the preceding proposition.

Corollary 6. *For a real irreducible G-module V any $A \in \mathrm{End}_G(V)$ is of the form $A = \alpha\,\mathrm{Id} + \beta J$ with $\alpha, \beta \in \mathbb{R}$ and J a G-invariant complex structure (depending on A).*

Proof. Although this follows directly from Proposition 1 we will give another proof which will be useful later on. Applying the Schur-lemma we see that the minimal polynomial $\mu_A(x)$ of A is irreducible over $\mathbb{R}$ (cf. [KN63, Appendix 5, Lemma 1]). If $\mu_A(x) = x - \alpha$ is of degree one $0 = \mu_A(A) = A - \alpha \cdot \mathrm{Id}$. Otherwise $\mu_A(x) = (x-\alpha)^2 + \beta^2$ is a polynomial of degree 2 with strictly positive quadratic supplement, since μ_A is irreducible. Thus $J := (A - \alpha \cdot \mathrm{Id})/\beta$ defines a complex structure on V. □

Finally, we describe the maximal representations of different types, i.e., any other irreducible representation occurs as a subrepresentation of them.

Proposition 2. *Let $G \subset \mathrm{GL}(n,\mathbb{R})$ be an irreducibly acting subgroup. Then up to conjugation G is contained in one of the following subgroups $L \subset \mathrm{GL}(n,\mathbb{R})$.*

$\mathrm{End}_G(\mathbb{R}^n)$	$L \subset \mathrm{GL}(n,\mathbb{R})$
$\mathbb{R}$	$\mathrm{GL}(n,\mathbb{R})$
$\mathbb{C}$	$\mathrm{GL}(n/2,\mathbb{C})$
$\mathbb{H}$	$\mathrm{GL}(n/4,\mathbb{H})$

Proof. We set $V := \mathbb{R}^n$ and $\mathbb{K} := \mathrm{End}_G(V)$. Thus V becomes a *left* $\mathbb{K}$-vector space in a natural way. In order to make it a *right* $\mathbb{K}$-vector space we choose an anti-automorphism $\lambda \mapsto \bar{\lambda}$ of $\mathbb{K}$ (i.e., $\overline{\lambda+\mu} = \bar{\lambda} + \bar{\mu}$ and $\overline{\lambda\cdot\mu} = \bar{\mu}\cdot\bar{\lambda}$). Then $v \cdot \lambda := \bar{\lambda}(v)$ defines a right-multiplication on V with respect to the scalar field $\mathbb{K}$ (This is essential only in case of the non-commutative field $\mathbb{K} = \mathbb{H}$). The group $\mathrm{GL}(V,\mathbb{K})$ of $\mathbb{K}$-linear invertible maps from V into itself is by definition the centralizer of the homothety group $H_{\mathbb{K}} := \{\{v \mapsto v\lambda\} \mid \lambda \in \mathbb{K}^*\}$. By choosing a $\mathbb{K}$-basis $\{b_i\}_{i=1}^{n/d}$, where $d = \dim_{\mathbb{R}} \mathbb{K}$, we get $\mathbb{K}^{n/d} \simeq V$. Under this identification $\mathrm{GL}(V,\mathbb{K})$ corresponds to the group $\mathrm{GL}(n/d,\mathbb{K})$ of invertible $(n/d \times n/d)$-matrices acting on $\mathbb{K}^{n/d}$ from the left. By definition $H_{\mathbb{K}}$ is the centralizer of G and thus G is contained in $L := \mathrm{GL}(V,\mathbb{K})$. As explained, this yields an inclusion $G \subset \mathrm{GL}(n/d,\mathbb{K})$. Conversely it is known that the centralizer of $\mathrm{GL}(n/d,\mathbb{K})$ equals $H_{\mathbb{K}}$, hence $\mathrm{End}_L(\mathbb{R}^n) = H_{\mathbb{K}}$. Finally, the embedding $\mathrm{GL}(n/d,\mathbb{K}) \subset \mathrm{GL}(n,\mathbb{R})$ is obtained by associating to the $\mathbb{K}$-basis $\{b_i\}$ the real basis $\{b_i\lambda_k\}_{\substack{i=1,\dots,n/d\\ k=1,\dots,d}}$, where $\{\lambda_k\}_{k=1,\dots,d}$ is a basis of $\mathbb{K}$. □

Remark 1. In the proof of this proposition we see that if the action of a group $G \subset \mathrm{GL}(n,\mathbb{R})$ is defined by scalar multiplication from the right, the invariant endomorphism have to act from the left. Of course, this becomes only relevant in case of $\mathrm{End}_G(\mathbb{R}^n) = \mathbb{H}$, and we can see this in the example of $G := \mathrm{GL}(1,\mathbb{H})$: It is

$$\mathrm{GL}(1,\mathbb{H}) = \{R_q \colon \mathbb{H} \to \mathbb{H} \mid q \in \mathbb{H}^* \text{ and } R_q(p) := p\cdot q\} = \mathbb{H}^*,$$

whereas

$$\begin{aligned}\mathrm{End}_{\mathrm{GL}(1,\mathbb{H})}(\mathbb{R}^4) &= \{A \in \mathrm{GL}(4,\mathbb{R}) \mid A(R_q(p)) = R_q(A(p)\}\\ &= \{L_q \in \mathrm{GL}(4,\mathbb{R}) \mid L_q(p) := q\cdot p\}\\ &= \mathbb{H}\end{aligned}$$

since $L_q \circ R_p = R_p \circ L_q$ but $R_p \circ R_q \neq R_q \circ R_p$. This gives the seemingly paradoxical situation where both, the centraliser $Z_{\mathrm{GL}(4,\mathbb{R})}(G)$ and the group G itself are equal to $\mathbb{H}^*$, but its centre $Z(G)$ which is the intersection of G with its centraliser is commutative and thus equal to $\mathbb{C}^*$.

3 Irreducibly acting, connected subgroups of $\mathrm{GL}(n,\mathbb{R})$

In this section we shall give a proof of Theorem 1 by using the results of the first section and two general results from Lie theory. First we describe the identity component of the centre of an irreducibly acting Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. We should remark that we use 'Lie subgroup' always in the weaker sense of being a submanifold but not necessarily an immersion in order to make the statement of Theorem 1 non-trivial.

Proposition 3. *Let $G \subset \mathrm{GL}(n,\mathbb{R})$ be an irreducibly acting, connected Lie subgroup, $Z(G)$ its centre and $Z(G)_0$ the identity component of the centre. If $Z(G)_0$ is non-trivial, then $Z(G)_0$ is either*

(a) *equal to $\mathbb{R}^+\mathrm{Id}$, or*

(b) *isomorphic to $\mathbb{C}^* = \mathbb{R}^+ \times S^1$, or*

(c) *isomorphic to a one-parameter subgroup of $\mathbb{C}^*$.*

Cases (b) and (c) can only occur if n is even.

Proof. Let $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ be the Lie algebra of G and suppose that the centre $\mathfrak{z}$ of $\mathfrak{g}$ is non trivial. Considering the three cases of Proposition 1 we first assume that the representation is of real type, i.e., that $\mathrm{End}_G(\mathbb{R}^n) = \mathbb{R}\,\mathrm{Id}$. Since $\mathfrak{z} \subset \mathrm{End}_G(\mathbb{R}^n)$ we obtain in this case that $\mathfrak{z} = \mathbb{R}\,\mathrm{Id}$ and therefore $Z(G)_0 = \exp(\mathfrak{z}) = \mathbb{R}^+\,\mathrm{Id}$.

Now suppose that $\mathbb{R}^n$ is a G-module of non-real type, i.e., $\mathrm{End}_G(\mathbb{R}^{2n})$ isomorphic to $\mathbb{C}$ or $\mathbb{H}$. Again $\mathfrak{z}$ is an Abelian subalgebra of $\mathrm{End}_G(\mathbb{R}^{2n})$. In case $\mathrm{End}_G(\mathbb{R}^{2n}) \simeq \mathbb{H} \simeq \mathfrak{u}(2)$ any maximal Abelian subalgebra is isomorphic to $\mathbb{C}$. Hence $\mathfrak{z}$ is isomorphic to a subalgebra of $\mathbb{C} = \mathrm{span}_{\mathbb{R}}(\mathrm{Id}, J)$ where J is a complex structure on $\mathbb{R}^{2n}$. But $\exp tJ = (\cos t)\,\mathrm{Id} + (\sin t)J$, i.e., $\exp(\mathbb{R}J) \simeq S^1$. But this implies that either isomorphic to $\mathbb{C}^*$, i.e.,

$$Z(G)_0 = \mathbb{R}^+\,\mathrm{Id} \times \{(\cos t)\,\mathrm{Id} + (\sin t)J \mid t \in \mathbb{R}\} \simeq \mathbb{R}^+ \times S^1 = \mathbb{C}^*,$$

or to a one-parameter subgroup of it, i.e.,

$$\begin{aligned} Z(G)_0 &= \exp\left(\mathbb{R} \cdot (a\,\mathrm{Id} + bJ)\right) \\ &= \left\{\left(\mathrm{e}^{at} \cdot \mathrm{Id}\right) \circ ((\cos bt)\,\mathrm{Id} + (\sin bt)J) \mid t \in \mathbb{R}\right\}, \end{aligned}$$

for some real constants a and b. Of course if a or b are zero this is either $\mathbb{R}^+$ or S^1, if not this is a logarithmic spiral in $\mathbb{C}^*$. □

Proposition 3 will be the main ingredient in our proof of Theorem 1 but it implies also Corollaries 2 and 3 given in the introduction. But before we can prove these we have to recall that for a completely reducibly acting Lie subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$ the centre decides whether the Lie algebra is semisimple or not. This is due to the well known fact from the theory of Lie algebras, that a Lie algebra $\mathfrak{g}$ which admits a completely reducible representation is reductive. Hence $\mathfrak{g}$ admits a Lie algebra decomposition into its centre and its derived Lie algebra,

$$\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}], \tag{1}$$

the derived Lie algebra being semisimple. A proof of this fact can be found in [Che47], see also [Bou71]. This means that the irreducibly acting, connected Lie subgroup in question is semisimple if the identity component of its centre is trivial.

Remark 2. In this context we should remark that the centre of a semisimple subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$ is finite (see e.g. [Got48]): if G is semisimple, due to Weyl's theorem it acts completely reducibly, and furthermore its elements are of determinant 1, hence by Schur's lemma the centre of G corresponds to the n_k-th roots of 1 where n_k are the dimensions of the irreducible subspaces.

For verifying Corollary 2 we assume that $G \subset \mathrm{O}(p, q)$ is connected and acts irreducibly. If G is not semisimple, its Lie algebra $\mathfrak{g}$ has a non trivial centre $\mathfrak{z}$, but the orthogonality of the representation implies that projection of the centre on $\mathbb{R}\,\mathrm{Id}$ is trivial. Hence, the representation is not of real type, i.e., $n = p + q$ is even, and $\mathfrak{z} = \mathbb{R}J$ where J is the complex structure which commutes with $\mathfrak{g}$. But on the other hand, $J \in \mathfrak{so}(p, q)$, i.e., J is compatible with the inner product, which gives that p and q are even as well. Thus, by Proposition 2,

$$\mathfrak{g} \subset \mathfrak{so}(p, q) \cap \mathfrak{gl}(n/2, \mathbb{C}) = \mathfrak{u}(p/2, q/2).$$

which is the statement of Corollary 2.

This also implies Corollary 3: If $\mathfrak{g}$ is not semisimple, then $\mathfrak{g} = \mathbb{R} \cdot J \oplus \mathfrak{g}'$ where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. Hence, $\mathfrak{g}' = [\mathfrak{g}', \mathfrak{g}'] \subset \mathfrak{su}(r, s)$, with $\mathfrak{su}(r, s)$ defined by the complex structure J, $r = p/2$ and $s = q/2$. The complex structure J is given by

$$J = \sum_{k}^{r+s} \kappa_{2k} E_{2k-1\ 2k},$$

where $\kappa_i = \pm 1$ is defined by $\langle e_i, e_j \rangle = \kappa_i \delta_{ij}$ with e_i an orthonormal basis of $\mathbb{R}^{p+q}$, and E_{ij} is the corresponding basis of $\mathfrak{so}(p, q)$. Let $u(\varepsilon_{r+s}, \ldots, \varepsilon_1)$ be the basis of the spinor module $\Delta_{(p,q)}$ as defined in [BK99]. This is an eigenbasis for the spin representation of J:

$$\begin{aligned} J u(\varepsilon_{r+s}, \ldots, \varepsilon_1) &= i\Big(\sum_{k}^{r+s} \kappa_{2k} \tau_{2k-1} \tau_{2k} \varepsilon_k \Big) u(\varepsilon_{r+s}, \ldots, \varepsilon_1) \\ &= i\Big(\sum_{k}^{r+s} \varepsilon_k \Big) u(\varepsilon_{r+s}, \ldots, \varepsilon_1), \end{aligned}$$

where $\tau_i = i$ if $\kappa_i = -1$ and 1 otherwise. If $V_{\mathbb{R}J}$ denotes the subspace in the spinor module which is annihilated by $\mathbb{R}J$ under its spin representation, then

$$V_{\mathbb{R}J} = \mathrm{span}\big\{ u(\varepsilon_{r+s}, \ldots, \varepsilon_1) \mid \textstyle\sum_{k}^{r+s} \varepsilon_i = 0 \big\},$$

which implies that J has zero eigenvectors only if $r + s$ is even. This is the statement of Corollary 3. The example given in the introduction of a non-semisimple irreducibly

acting Lie algebra with trivial spin representation is given by $\mathfrak{g} = \mathbb{R}J \oplus \mathfrak{sp}(r,s)$ with $r+s$ even. Again in the notation of [BK99] it follows that

$$\varphi_{\frac{r+s}{2}} = \sum_{\varepsilon_i = -1 \text{ for } \frac{r+s}{2} \text{ times}} u(\varepsilon_{r+s}, \varepsilon_{r+s}, \ldots, \varepsilon_1, \varepsilon_1)$$

is annihilated by $\mathfrak{sp}(r,s)$. But it is also annihilated by J:

$$J\varphi_{\frac{r+s}{2}} = 2 \sum_{\substack{\varepsilon_i = -1 \text{ for} \\ \frac{r+s}{2} \text{ many } \varepsilon_i\text{'s}}} \underbrace{(\varepsilon_1 + \ldots + \varepsilon_{r+s})}_{=0} u(\varepsilon_{r+s}, \varepsilon_{r+s}, \ldots, \varepsilon_1, \varepsilon_1).$$

Hence, $V_{\mathfrak{g}} = V_{\mathbb{R}J} \cap V_{\mathfrak{sp}(r,s)} = \mathbb{R}\varphi_{\frac{r+s}{2}}$ is one-dimensional.

Example. An example for an irreducible real representation of a Lie group with 2-dimensional centre is the representation of $S^1 \times \mathrm{CO}(n,\mathbb{R})$ on $\mathbb{C}^n$ considered as real vector space $\mathbb{R}^{2n}$. Here $\mathrm{CO}(n,\mathbb{R})$ denotes the conformal group of $\mathbb{R}^n$. The Lie algebra consists of the matrices $\left(\begin{smallmatrix} A & aI_n \\ -aI_n & A \end{smallmatrix}\right) \in \mathfrak{gl}(2n,\mathbb{R})$ with $a \in \mathbb{R}$ and $A \in \mathfrak{co}(n,\mathbb{R})$, where I_n denotes the n-dimensional unit matrix. The centre of the identity component of this group is $S^1 \times \mathbb{R}^+$, the semisimple part is $\mathrm{SO}(n)$. In the same manner we can built an example where the centre is a spiral in $\mathbb{C}^*$ by taking as Lie algebra

$$\mathfrak{g} := \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right) \middle| A \in \mathfrak{so}(n) \right\} \oplus \mathbb{R} \cdot \left(\begin{smallmatrix} I_n & I_n \\ -I_n & I_n \end{smallmatrix}\right) \subset \mathfrak{gl}(2n,\mathbb{R}),$$

and as group G the connected subgroup in $\mathrm{GL}(2n,\mathbb{R})$ with this Lie algebra. Both groups do not act orthogonally.

Now we can go ahead with the proof of Theorem 1. Let G be a connected, irreducibly acting Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$, and $\mathfrak{g}$ be its Lie algebra. Our proof now relies on the following result of [Yos37] and [Mal45] (see also [Got48] where it is a corollary to a deeper result).

Theorem 4 ([Yos37], [Mal45], [Got48]). *A connected Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$ *is closed in* $\mathrm{GL}(n,\mathbb{R})$ *if and only if its radical is closed. In particular, if it is semisimple, it is closed.*

Recall that the radical of G is the connected Lie subgroup of G which corresponds to maximal solvable ideal in the Lie algebra $\mathfrak{g}$. Thus we have to show, that the radical of G is closed in $\mathrm{GL}(n,\mathbb{R})$. But by the remarks above, the Lie algebra of G is reductive, and thus the radical of G is equal to the identity component of its centre, denoted by $Z(G)_0$. Now the closure $\bar{G}$ of G is still connected, acts irreducibly and has a reductive Lie algebra. By Theorem 4 the identity component of its centre $Z\left(\bar{G}\right)_0$ is closed in $\mathrm{GL}(n,\mathbb{R})$. But $Z(G) \subset Z(\bar{G})$ because for $z \in Z(G)$ and $g = \lim g_n \in \bar{G}$ it is

$$z \cdot g = z \cdot \lim g_n = \lim(z \cdot g_n) = \lim(g_n \cdot z) = \lim g_n \cdot z = g \cdot z.$$

If we now assume that G is not closed we get by Theorem 4 that $Z(G)_0$ is not closed in $\mathrm{GL}(n,\mathbb{R})$, i.e.,

$$Z(G)_0 \subsetneqq \overline{Z(G)}_0 \subset Z(\bar{G})_0.$$

Now, since $\bar{G}$ is irreducible and connected, Proposition 3 leaves us only with the possibility that $Z(\bar{G})_0$ is isomorphic to $\mathbb{C}^*$ and $Z(G)_0$ is a one-parameter subgroup of $\mathbb{C}^*$. But these are closed in $\mathbb{C}^*$. This is a contradiction which completes the proof of Theorem 1.

Since holonomy groups are Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$, the first point of Corollary 1 is a direct consequence of the Theorem 1. The second can be obtained by a theorem in [Wu67] which contains several results with different algebraic conditions for subgroups of the pseudo-orthogonal group, having consequences for holonomy groups.

Theorem 5 ([Wu67]). *The following subgroups of* $\mathrm{GL}(p+q)$ *are closed:*

(1) *reductive, indecomposable subgroups of* $\mathrm{O}(p,q)$,

(2) *indecomposable subgroups of* $\mathrm{O}(p,q)$ *if* $p+q<6$,

(3) *holonomy groups of affine symmetric spaces.*

Here 'indecomposable' means 'no proper non-degenerate invariant subspace'. One should remark that the restriction to the dimension in the second point is sharp: In [Wu67] is constructed a 6-dimensional Kähler manifold whose reduced holonomy group is non-closed in $\mathrm{SO}(4,2)$; also the Lorentzian examples in [BI93] are constructed in dimension 6. Also in [Wu67] is constructed an example of a symmetric space with solvable, non-Abelian holonomy group which shows that the third point does not follow from the first. Some of these examples are obtained by constructing subgroups containing a torus, which has non-closed 1-parameter subgroups. Our proof shows that such a situation can be excluded if the group acts irreducibly.

In order to obtain the second statement of Corollary 1, note that the first point of Theorem 5 implies that semi-Riemannian holonomy groups which act completely reducibly are closed: by the de Rham/Wu decomposition theorem [Wu64] any semi-Riemannian holonomy group is a product of indecomposably acting holonomy groups, but if the group is assumed to act completely reducibly it is reductive and hence closed by the first point of Theorem 5.

Since the dense line on the Clifford torus provides an example of a completely reducibly acting group which is not closed in $\mathrm{GL}(2,\mathbb{C})$, such a result cannot be true for holonomy groups of an arbitrary affine connection due to the result in [HO65], that any connected linear Lie group can be obtained as the holonomy group of an affine connection. It is not difficult to check that the connection of this example is not torsion-free. But such a result might be true for torsion-free connections.

Conjecture. *Let* (M,∇) *be an affine manifold where* ∇ *is a torsion-free connection. Assume that the restricted holonomy group* $\mathrm{Hol}_p^*(\nabla)$ *acts completely reducible on* T_pM. *Then,* $\mathrm{Hol}_p^*(\nabla)$ *is closed inside* $\mathrm{GL}(T_pM)$.

4 Irreducibly acting, connected subgroups of $\mathrm{O}(1,n)$

In this section we want to give a short proof of Theorem 2, that the only connected subgroup G of $\mathrm{O}(1,n)$ which acts irreducibly on Minkowski space $\mathbb{R}^{1,n}$ is the connected component of the identity of $\mathrm{O}(1,n)$, i.e., $G = \mathrm{SO}_0(1,n)$. This statement was proven in [DSO01] where the main goal was to generalize to real hyperbolic space the following result about minimal homogeneous submanifolds, i.e., orbits of isometry subgroups, in the Euclidean space.

Theorem 6 ([DS02]). *A (extrinsically) homogeneous minimal submanifold of the Euclidean space must be totally geodesic.*

It turns out that such result also holds in the real hyperbolic space (see [DSO01] for details). It is interesting to remark that further investigations of minimal homogeneous submanifolds were done in several directions [ADS03], [DS03]. In particular, the following conjecture was posed in [DS03].

Conjecture. *Let M be a Riemannian manifold that is either locally homogeneous or Einstein. Then, any minimal isometric immersion $f : M \to \mathbb{R}^n$ must be totally geodesic.*

Now, in order to prove Theorem 2 we assume that $G \subset \mathrm{O}(1,n)$ acts irreducibly and is connected. By Corollary 2 it is semisimple and closed by Theorem 1. Our proof requires the following Karpelevich's theorem.

Theorem 7 ([Kar53], [Mos55]). *Let $M = \mathrm{Iso}(M)/K$ be a Riemannian symmetric space of non-compact type. Then any connected and semisimple subgroup G of the full isometry group $\mathrm{Iso}(M)$ has a totally geodesic orbit $G \cdot p \subset M$.*

The above theorem can also be stated in a purely algebraic way as follows.

Theorem 8. *Let $\mathfrak{g}'$ be a real semisimple Lie algebra of non compact type and let $\mathfrak{g} \subset \mathfrak{g}'$ be a semisimple Lie subalgebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$. Then there exists a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ for $\mathfrak{g}'$ such that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$.*

The original proof of Theorem 7 is as a corollary of Theorem 8. The proof of Theorem 8 is not trivial and very algebraic (see [Mos55, Theorem 6] or [Oni04]). Hence, it would be nice to have a proof of Karpelevich's theorem by use of more geometric methods.

Remark 3. Observe that if $\mathfrak{g}' = \mathrm{SL}(n,\mathbb{R})$ then Theorem 8 is a well-known fact related to the Cartan's procedure to pass from a symmetric space to its dual one. Namely, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ then the Lie algebra $\mathfrak{g}_c := \mathfrak{k} \oplus i\mathfrak{p}$ is compact. So there exists an

Hermitian inner product $(\cdot|\cdot)$ on $\mathbb{C}^n$ invariant by $\mathfrak{g}_c$. Now it is not difficult to check that $\langle\cdot,\cdot\rangle := \mathrm{Real}(\cdot|\cdot)$ is an inner product of $\mathbb{R}^n$ such that $\mathfrak{k}$ consist of skew-symmetric matrices w.r.t $\langle\cdot,\cdot\rangle$ and $\mathfrak{p}$ consist of symmetric matrices w.r.t. $\langle\cdot,\cdot\rangle$.

Remark 4. Note that in Theorems 7 and 8 the hypothesis on $\mathfrak{g}'$ of being of "non compact type" can not be deleted. Any irreducible representation of a compact simple Lie group G gives an example. Indeed, the group $G \subset \mathrm{SO}(n+1)$ acts on a sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ without totally geodesic orbits. Notice that if the orbit $G \cdot x$ is totally geodesic then $G \cdot x = S^n \bigcap V$, where V is a linear subspace of $\mathbb{R}^{n+1}$. So V is G-invariant giving a contradiction.

Now let us continue the proof of Theorem 2. Since $G \subset \mathrm{O}(1,n)$ is semisimple, Karpelevich's theorem applies to our situation. It implies that the action of G on the hyperbolic space $H^n = \mathrm{SO}(1,n)/\mathrm{SO}(n) \subset \mathbb{R}^{1,n}$ is transitive. Indeed, if the totally geodesic orbit $G \cdot p$ is not the whole hyperbolic space H^n, then $G \cdot p$ is contained in a Lorentzian subspace $\mathbb{L}$ of $\mathbb{R}^{1,n}$. This is due to the fact that totally geodesic submanifolds of H^n are intersections $H^n \cap \mathbb{L}$ where $\mathbb{L}$ is a Lorentzian subspaces of $\mathbb{R}^{1,n}$. Thus, G can not act irreducibly as we had assumed.

Now, let K be a maximal connected compact subgroup of the semisimple group G. Then by Cartan's fixed point theorem K has a fixed point $p \in H^n$. Since $(G_p)_0$ is compact, we get $K = (G_p)_0$. Thus, (G, K) is a symmetric pair such that $H^N = G/K$. Then, from the uniqueness of such symmetric pairs (see [Hel78, pp. 243]) we get $G = \mathrm{SO}_0(1,n)$ and $K = \mathrm{SO}(n)$. This proves Theorem 2.

Remark 5. Note that Theorem 8 implies the "uniqueness" of the presentation $M = G/K$ with (G, K) as a symmetric pair, where M is a symmetric space of non compact type. Namely, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathrm{Lie}(\mathrm{Iso}(M)) = \mathfrak{k}' \oplus \mathfrak{p}'$ then the dimensions of $\mathfrak{p}$ and $\mathfrak{p}'$ are the same since both are identified with the tangent space at some point of M. Thus, $\mathfrak{g} = \mathrm{Lie}(\mathrm{Iso}(M))$. In geometric words, the "uniqueness" of such pairs follows from the so-called Cartan's construction of the Lie algebra of the group of isometries $\mathrm{Iso}(M)$ of M. Namely, the curvature tensor R of M belong to $\mathfrak{k}$. Since $\mathfrak{k}$ is generated by R (i.e., $\mathfrak{k}$ is the holonomy algebra) and R gives the Lie bracket on $\mathfrak{p}$ it follows that $\mathrm{Iso}(M)_0 = G$. This is the geometric explanation of the "uniqueness" quoted above (see [Hel78, pp. 243]).

A different, almost algebra-free proof of Theorem 2 which uses dynamical methods, can be found in [BZ04]. A purely algebraic proof was given in [BdlH04]. Theorem 2 also follows from Corollary 4.5.1 in [ChGr74] which in turn depends upon Karpelevich–Mostow's theorem (cf. Lemma 4.4.3 in [ChGr74] with Theorem 8).

Let M be a (locally) indecomposable Lorentzian manifold, i.e., the restricted holonomy group Φ_p^* acts indecomposably on T_pM. In [DSO01] was proved that either Φ_p acts transitively on hyperbolic spaces $H_r := \{v \in T_pM \mid \langle v, v\rangle = -r^2\}$ or transitively on horospheres $Q_z \subset H_r$ centered at a point $z \in H_r(\infty)$. Observe that in the Riemannian case the non-transitivity (on a sphere) of the holonomy group implies

the locally symmetry of the Riemannian manifold, i.e., the Berger–Simons Theorem (see [Olm05] for a direct and geometric proof). It is interesting to note that the holonomy group of a non irreducible indecomposable Lorentzian symmetric space is abelian [CaWa70]. Thus, such holonomy group must act transitively on horospheres $Q_z \subset H_r$ centered at a point $z \in H_r(\infty)$, i.e., z is associated to the (unique) light-like direction invariant by Φ_p^*.

There are also non symmetric global (i.e., geodesically complete) examples of (non irreducible and indecomposable) homogeneous Lorentzian spaces such that the restricted holonomy group is abelian, i.e., the so called *homogeneous plane waves* [BlLo]. Notice that such spaces had a rich family of totally geodesic Lorentzian surfaces through each point. Namely, the exponential of the normal space to each horosphere $Q_z \subset T_pM$. Anyway, it should be desirable to know if some type of Berger–Simons theorem holds in the Lorentzian setting under suitable assumptions.

Finally, here is another application, i.e., Lemma 3.1 of [Sen06], see also [Hal91].

Proposition 4. *Let (M, g) be a simply connected Lorentzian manifold. Assume that there exist a non-zero covariantly constant symmetric tensor field $h_{\mu\nu}$ not proportional to the metric. Then, (M, g) is reducible, and further it is indecomposable only if there exists a unique (up to multiples) light-like parallel vector field.*

Indeed, if (M, g) is irreducible then the holonomy group Φ_p is $\mathrm{SO}_0(1, n)$. Thus, $h_{\mu\nu}$ being $\mathrm{SO}_0(1, n)$-invariant should be proportional to the metric g (See the first line in the table of Proposition 6 of the next section). The second part follows from the fact that if Φ_p leaves invariant two non proportional light-like vectors then Φ_p must leave invariant the two dimensional Lorentzian space generated by these vectors. Thus, (M, g) is decomposable. Notice that the "uniqueness" of a light-like parallel vector field does not imply indecomposability of the Lorentzian manifold, e.g. the product of a Riemannian manifold and an indecomposable Lorentzian manifold.

A modern exposition of Karpelevich's theory is [Oni04]. A step towards the generalization of Theorem 2 to arbitrary signatures has recently been made in [DSL10], where irreducible subalgebras of $\mathfrak{so}(2, n)$ are classified.

5 Invariant bilinear forms of irreducible representations

As in the second section we consider an irreducibly acting subgroup $G \subset \mathrm{GL}(V)$ of a real vector space V and denote by $B_G(V)$ the vector space of G-invariant bilinear forms. If $B_G(V)$ is non-trivial it is intimately related to $\mathrm{End}_G(V)$. By Schur's lemma a non-zero $a \in B_G(V)$ is non-degenerate since its kernel is G-invariant and not equal to V. Thus Riesz' theorem provides a one-to-one correspondence between $B_G(V)$

and $\mathrm{End}_G(V)$ via

$$\begin{aligned} \mathrm{End}_G(V) &\xrightarrow{\sim} B_G(V) \\ B &\longmapsto b = a(B(\cdot),\cdot). \end{aligned} \tag{1}$$

In particular, this map endows $B_G(V)$ with the structure of an associative algebra and by Proposition 1, $B_G(V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The unique decomposition of a bilinear form into symmetric and skew-symmetric parts applies also to G-invariant bilinear forms, since the (skew-)symmetrization of a G-invariant form inherits this property. Thus,

$$B_G(V) = S_G(V) \oplus \Lambda_G(V). \tag{2}$$

This induces a decomposition of $\mathrm{End}_G(V)$ into a-selfadjoint and a-skew-adjoint operators,

$$\mathrm{End}_G(V) = S^a_G(V) \oplus \Lambda^a_G(V). \tag{3}$$

If a is symmetric, $S_G(V)$ corresponds to $S^a_G(V)$ under (1) and if a is skew-symmetric, $S_G(V)$ corresponds to $\Lambda^a_G(V)$. The main question is what are the possible dimensions of $S_G(V)$ and what are the occurring signatures. A first step towards an answer is given by the following statement.

Proposition 5. *Let $a, b \in B_G(V)$ be linearly independent, $b = a(B(\cdot),\cdot)$ by (1) and $B = \alpha\,\mathrm{Id} + \beta J$ according to Corollary 6.*

(i) *If a and b are both symmetric (or skew-symmetric), $J \in S^a_G(V)$ and thus J is an anti-isometry with respect to both a and b. In particular $\mathrm{Sig}(a) = \mathrm{Sig}(b) = (n/2, n/2)$ where $n = \dim V$.*

(ii) *If a is symmetric and b is skew-symmetric, $B = \beta J \in \Lambda^a_G(V)$ and thus J is an isometry with respect to both a and b.*

Proof. (i) Since B and Id are a-selfadjoint, the same holds for J. Using $[B, J] = 0$ we obtain

$$\begin{aligned} a(J(x), J(y)) = a(J^2(x), y) = -a(x, y), \quad \text{and} \\ b(J(x), J(y)) = a(B \circ J(x), J(y)) = -a(B(x), y) = -b(x, y). \end{aligned}$$

(ii) Here B is a-skew-adjoint. This implies for its minimal polynomial, $\mu_B(x) = \mu_{-B}(x) = \mu_B(-x)$, hence $\mu_B(x) = x^2 + \beta^2$ (cf. Corollary 6), i.e., $B = \beta J$. The remaining part is analogous to (i). □

Next we determine all possible pairs $\big(\dim S_G(V), \dim \Lambda_G(V)\big)$ by describing their maximal representations analogous to Proposition 2 of Section 2. Recall that a representation is self-dual if and only if the space of non-degenerate invariant bilinear forms is non-trivial.

Proposition 6. *Let $\kappa\colon G \to \mathrm{GL}(n, \mathbb{R})$ be an irreducible self-dual representation on $\mathbb{R}^n$. Then up to conjugation $\kappa(G)$ is contained in one of the following subgroups $L \subset \mathrm{GL}(n, \mathbb{R})$ with $p + q = n$:*

$\mathrm{End}_{\kappa(G)}(\mathbb{R}^n)$	$\dim S_{\kappa(G)}(\mathbb{R}^n)$	$\dim \Lambda_{\kappa(G)}(\mathbb{R}^n)$	$L \subset \mathrm{GL}(n,\mathbb{R})$
$\mathbb{R}$	1	0	$\mathrm{O}(p,q)$
	0	1	$\mathrm{Sp}(n/2,\mathbb{R})$
$\mathbb{C}$	2	0	$\mathrm{O}(n/2,\mathbb{C}),\ (n \geq 4)$
	0	2	$\mathrm{Sp}(n/4,\mathbb{C})$
	1	1	$\mathrm{U}(p/2,q/2)$
$\mathbb{H}$	1	3	$\mathrm{Sp}(p/4,q/4),\ (n \geq 8)$
	3	1	$\mathrm{O}^*(n/4),\ (n \geq 8)$

Moreover we have the following isomorphisms:[1]

$$\begin{aligned} \mathrm{O}(n/2,\mathbb{C}) &\simeq \mathrm{O}(n/2,n/2) \cap \mathrm{GL}(n/2,\mathbb{C}),\\ \mathrm{Sp}(n/2,\mathbb{C}) &\simeq \mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{GL}(n/2,\mathbb{C}),\\ \mathrm{U}(p/2,q/2) &\simeq \mathrm{O}(p,q) \cap \mathrm{Sp}(n,\mathbb{R}),\\ \mathrm{Sp}(p/4,q/4) &\simeq \mathrm{U}(p/2,q/2) \cap \mathrm{Sp}(n/4,\mathbb{C}),\\ \mathrm{O}^*(n/4) &\simeq \mathrm{U}(n/4,n/4) \cap \mathrm{O}(n/2,\mathbb{C}). \end{aligned} \tag{4}$$

Remark 6. This proposition raises the question if there are proper subrepresentations of the different groups $L \subset \mathrm{GL}(n,\mathbb{R})$. The answer depends very much on the group L in question. To illustrate this, note that any compact simple Lie group admits irreducible representations in arbitrary high dimensions. All these representations are contained in $\mathrm{O}(n)$ due to Weyl's trick. Considered the other way around, $\mathrm{O}(n)$ has in general a lot of irreducible subrepresentations. In contrast, there are no proper subgroups of $\mathrm{SO}_0(1,n)$ which act irreducibly, see Section 4.

Proof. For general considerations set $V := \mathbb{R}^n$; we will return at the end to $\mathbb{R}^n$ by choosing an appropriate basis. First note that (κ, V) is self-dual if and only if $B_{\kappa(G)}(V) \neq \{0\}$. In particular $\mathrm{End}_{\kappa(G)}(V) \simeq B_{\kappa(G)}(V)$ according to (1) and we may distinguish between the various types of (κ, V). We determine in each case the maximal subgroup $L \subset \mathrm{GL}(V)$ fixing every element of $B_{\kappa(G)}(V)$ and thus $\kappa(G) \subset L$.

(κ, V) *of real type*: This is the simplest case, since $B_{\kappa(G)}(V)$ is 1-dimensional and thus spanned either by a symmetric or skew-symmetric bilinear form a (cf. (2)). In

[1] For details how the groups are embedded into $\mathrm{GL}(n,\mathbb{R})$ resp. $\mathrm{GL}(n,\mathbb{C})$ we refer to the proof.

the symmetric case we can find a (pseudo-)orthonormal basis, i.e.,

$$(a_{ij}) = \mathbb{I}_{p,q} := \begin{pmatrix} -\mathbb{I}_p & \\ & \mathbb{I}_q \end{pmatrix}.$$

Its isometry group is the (pseudo-)orthogonal group

$$\mathrm{O}(p,q) := \{\, A \in \mathrm{GL}(n,\mathbb{R}) \mid A^t \cdot \mathbb{I}_{p,q} \cdot A = \mathbb{I}_{p,q} \}.$$

In the skew-symmetric case we can find a symplectic basis, i.e.,

$$(a_{ij}) = \mathbb{J}_{n/2} := \begin{pmatrix} & -\mathbb{I}_{n/2} \\ \mathbb{I}_{n/2} & \end{pmatrix}.$$

Its isometry group is the real symplectic group

$$\mathrm{Sp}(n/2,\mathbb{R}) := \{\, A \in \mathrm{GL}(n,\mathbb{R}) \mid A^t \cdot \mathbb{J}_{n/2} \cdot A = \mathbb{J}_{n/2} \}.$$

(κ, V) *of complex type*: First we fix a (skew-)symmetric $a \in B_G(V)$ and consider its bilinear extension $a_{\mathbb{C}}$ as well as its sesquilinear extension $\bar{a}_{\mathbb{C}}$ to the complexification $V_{\mathbb{C}}$:

$$\begin{aligned} a_{\mathbb{C}}(x+iy, u+iz) &:= a(x,u) - a(y,z) + i\big(a(x,z) + a(y,u)\big), \\ \bar{a}_{\mathbb{C}}(x+iy, u+iz) &:= a(x,u) + a(y,z) + i\big(a(x,z) - a(y,u)\big). \end{aligned}$$

$a_{\mathbb{C}}$ is (skew-)symmetric, $\bar{a}_{\mathbb{C}}$ is (skew-)Hermitian and they are linked by the formula $a_{\mathbb{C}}(\bar{v}, w) = \bar{a}_{\mathbb{C}}(v, w)$. From this it follows

$$a_{\mathbb{C}}(\bar{v}, \bar{w}) = \overline{a_{\mathbb{C}}(v,w)} \quad \text{and} \quad \bar{a}_{\mathbb{C}}(\bar{v}, \bar{w}) = \overline{\bar{a}_{\mathbb{C}}(v,w)}. \tag{5}$$

Exactly one of them has to vanish on the $\kappa_{\mathbb{C}}$-irreducible subspace W. Indeed, if we suppose both to be non-zero and set $a_{\mathbb{C}}\big|_W = \bar{a}_{\mathbb{C}}\big|_W(J(\cdot),\cdot)$ one shows that $J \in \overline{\mathrm{End}_\rho(W)}$, hence $J = 0$. On the other hand $a_{\mathbb{C}}\big|_{W\times W} = 0$ together with (5) implies $a_{\mathbb{C}}\big|_{\overline{W}\times\overline{W}} = 0$. Thus, $a_{\mathbb{C}}\big|_{\overline{W}\times W} = \bar{a}_{\mathbb{C}}\big|_{W\times W}$ has to be non-degenerate and vice versa. Lets denote by $\tilde{a}$ the non-vanishing form on W. Since $\tilde{a}$ is ρ-invariant, it induces the κ-invariant $\mathbb{C}$-valued $\mathbb{R}$-bilinear form $\psi_*\tilde{a}$ on V via (1). In the following we will suppress the isomorphism ψ. Real and imaginary part of $\tilde{a}$ are related by

$$\mathrm{Im}\,(\tilde{a})(x,y) = -\mathrm{Re}\,(\tilde{a})(x, I(y)).$$

In particular they are linear independent and thus $B_{\kappa(G)}(V)$ is spanned by these two forms. So the isometry group of $\tilde{a}$ is isomorphic to the maximal subgroup of $L \subset \mathrm{GL}(V)$ which fixes any element of $B_{\kappa(G)}(V)$. Note that any element of L commutes with I and thus it is complex linear.

If $\tilde{a}$ is symmetric, the same holds for its real and imaginary part and their signature has to be $(n/2, n/2)$ (cf. Proposition 5 (i)). We can find a complex orthonormal basis, i.e., $(\tilde{a}_{ij}) = \mathbb{I}_{n/2}$ and the isometry group is the complex orthogonal group

$$\mathrm{O}(n/2,\mathbb{C}) = \{\, A \in \mathrm{GL}(n/2,\mathbb{C}) \mid A^t \cdot A = \mathbb{I}_{n/2} \}.$$

If $\tilde{a}$ is skew-symmetric, the same holds for its real and imaginary part. We can find a complex symplectic basis, i.e., $(\tilde{a}_{ij}) = \mathbb{J}_{n/4}$ and the isometry group is the complex symplectic group

$$\mathrm{Sp}(n/2,\mathbb{C}) = \{\, A \in \mathrm{GL}(n/2,\mathbb{C}) \mid A^t \cdot \mathbb{J}_{n/4} \cdot A = \mathbb{J}_{n/4}\}.$$

Finally, $\tilde{a}$ might be Hermitian (a complex skew-Hermitian form turns into a Hermitian one by multiplication with i). We can find a complex (pseudo-)orthonormal basis, i.e., $(\tilde{a}_{ij}) = \mathbb{I}_{p/2,q/2}$ and the isometry group is the unitary group

$$\mathrm{U}(p/2,q/2) := \{\, A \in \mathrm{GL}(n/2,\mathbb{C}) \mid \bar{A}^t \cdot \mathbb{I}_{p/2,q/2} \cdot A = \mathbb{I}_{p/2,q/2}\}.$$

In this case the real part is symmetric and has signature (p,q) and the imaginary part is skew-symmetric.

As mentioned in Proposition 2, the complex basis $\{b_i\}_{i=1}^{n/2}$ of V (with respect to I) induces the real basis $\{b_i, I(b_i)\}_{i=1}^{n/2}$. Thus, $I = \mathbb{J}_{n/2}$ and we obtain the embedding

$$\begin{aligned} \mathrm{GL}(n/2,\mathbb{C}) &\xrightarrow{\sim} \{\, C \in \mathrm{GL}(n,\mathbb{R}) \mid C \circ \mathbb{J}_{n/2} = \mathbb{J}_{n/2} \circ C\}, \\ A + iB &\longmapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \end{aligned}$$

Now the real part of the symmetric form $\tilde{a} = \mathbb{I}_{n/2}$ is given in the associated real basis by $\mathrm{Re}(\tilde{a}) = \begin{pmatrix} \mathbb{I}_{n/2} & \\ & -\mathbb{I}_{n/2} \end{pmatrix}$. Its isometry group is $\mathrm{O}(n/2,n/2)$, thus we get the first identity of (4). Analogously, the real part of the symplectic form $\tilde{a} = \mathbb{J}_{n/4}$ is given by $\mathrm{Re}(\tilde{a}) = \begin{pmatrix} \mathbb{J}_{n/4} & \\ & -\mathbb{J}_{n/4} \end{pmatrix}$. Its isometry group is conjugated to $\mathrm{O}(n/2,n/2)$ which yields the second identity of (4). The real part of the Hermitian form $\tilde{a} = \mathbb{I}_{p/2,q/2}$ is given by $\mathrm{Re}(\tilde{a}) = \begin{pmatrix} \mathbb{I}_{p/2,q/2} & \\ & \mathbb{I}_{p/2,q/2} \end{pmatrix}$. Its isometry group is conjugate to $\mathrm{O}(p,q)$. Instead of taking the intersection with the centralizer of $\mathbb{J}_{n/2}$ as above we take here the isometry group of the imaginary part $\mathrm{Im}(\tilde{a}) = \mathbb{J}_{n/2}$ which is $\mathrm{Sp}(n/2,\mathbb{R})$, hence the third identity of (4).

(κ, V) *of quaternionic type*: For representations of real or complex type all possible dimensions for the subspaces $S_{\kappa(G)}(V)$ and $\Lambda_{\kappa(G)}(V)$ occurred. This is no longer true in the quaternionic case.

Lemma 1. *If (κ, V) is self-dual and of quaternionic type then $S_{\kappa(G)}(V)$ and $\Lambda_{\kappa(G)}(V)$ are odd-dimensional, i.e., the dimension of one space is 1 and of the other is 3. In particular, κ is both, orthogonal and symplectic. If the 1-dimensional subspace is spanned by $\{a\}$ then under the identification $\mathrm{End}_{\kappa(G)}(V) \simeq \mathbb{H}$ the decomposition (3) is given by*

$$\mathrm{Re}(\mathbb{H}) = S^a_{\kappa(G)}(V) \quad \textit{and} \quad \mathrm{Im}(\mathbb{H}) = \Lambda^a_{\kappa(G)}(V).$$

Proof. Clearly $\mathrm{Re}(\mathbb{H}) = \mathbb{R} \cdot \mathrm{Id} \subset S^a_{\kappa(G)}(V)$. On the other hand $\Lambda^a_{\kappa(G)}(V) \subset \mathrm{Im}(\mathbb{H})$ by Proposition 5, hence $\mathrm{Im}(\mathbb{H}) = \big(S^a_{\kappa(G)}(V) \cap \mathrm{Im}(\mathbb{H})\big) \oplus \Lambda^a_{\kappa(G)}(V)$. One of the

subspaces has dimension greater than or equal to two and is spanned by anti-commuting complex structures I, J. Irrespective of whether I, J are self- or skew-adjoint with respect to a, their product is skew-adjoint: $(I \circ J)^* = J^* \circ I^* = J \circ I = -I \circ J$. $\square$

As in Proposition 2 we consider V as right $\mathbb{H}$-vector space via $x \cdot \lambda = \bar{\lambda}(x)$. Then an element $a \in B_{\kappa(G)}(V)$ as in the preceding lemma yields the following quaternionic sesquilinear form on V:

$$a_{\mathbb{H}}(x, y) := a(x, y) + i \cdot a(xi, y) + j \cdot a(xj, y) + k \cdot a(xk, y).$$

Recall that one has to check $a_{\mathbb{H}}(x\lambda, y) = \bar{\lambda} a_{\mathbb{H}}(x, y)$ and $a_{\mathbb{H}}(x, y\lambda) = a_{\mathbb{H}}(x, y)\lambda$. Since multiplication (from the right) with imaginary quaternions is an a-skew-adjoint operation according to Lemma 1, $a_{\mathbb{H}}$ is Hermitian if a is symmetric and skew-Hermitian otherwise.

By construction, $B_{\kappa(G)}(V)$ is spanned by the real and imaginary parts of $a_{\mathbb{H}}$, hence the group L which fixes all elements of $B_{\kappa(G)}(V)$ is the isometry group of $a_{\mathbb{H}}$ (again $\mathbb{H}$-linearity is ensured already by leaving $a_{\mathbb{H}}$ invariant).

Now, for any (skew-)Hermitian form one can find an orthogonal basis [Die71, Chapter I, §8]. In the Hermitian case the basis can be normed to the length ± 1. Thus the isometry group is the quaternionic unitary group

$$\mathrm{Sp}(p/4, q/4) = \{\, A \in \mathrm{GL}(n/4, \mathbb{H}) \mid \bar{A}^t \cdot \mathbb{I}_{p/4,q/4} \cdot A = \mathbb{I}_{p/4,q/4} \}.$$

In the skew-Hermitian case the basis can be normed to the length i. Thus the isometry group is

$$\mathrm{O}^*(n/4) = \{\, A \in \mathrm{GL}(n/4, \mathbb{H}) \mid \bar{A}^t \cdot i\,\mathbb{I}_{n/4} \cdot A = i\,\mathbb{I}_{n/4} \}.$$

The embedding $L \subset \mathrm{GL}(n, \mathbb{R})$ follows from the embedding $\mathrm{GL}(n/4, \mathbb{H}) \subset \mathrm{GL}(n, \mathbb{R})$ as in Proposition 2. In order to obtain the remaining identities of (4), we fix i as complex structure and thus represent any quaternionic matrix by two complex matrices: $A + iB + jC + kD = (A + iB) + (C + iD)j = U + Vj$. This yields the algebra isomorphism

$$\begin{aligned} \mathrm{GL}(n/4, \mathbb{H}) &\xrightarrow{\sim} \{\, C \in \mathrm{GL}(n/2, \mathbb{C}) \mid \bar{C} \circ \mathbb{J}_{n/4} = \mathbb{J}_{n/4} \circ C \}, \\ U + Vj &\longmapsto \begin{pmatrix} U & -V \\ \bar{V} & \bar{U} \end{pmatrix}. \end{aligned}$$

Since under this identification the operation $C \mapsto \bar{C}^t$ is the same in $\mathrm{GL}(n/4, \mathbb{H})$ and $\mathrm{GL}(n/2, \mathbb{C})$ it is easily seen, that $\mathrm{Sp}(p/4, q/4)$ is equal to the intersection of the isometry group $\mathrm{U}(p/2, q/2)$ of the Hermitian form $\begin{pmatrix} \mathbb{I}_{p/4,q/4} & \\ & \mathbb{I}_{p/4,q/4} \end{pmatrix}$ with the isometry group $\mathrm{Sp}(n/4, \mathbb{C})$ of the symplectic form $\begin{pmatrix} & -\mathbb{I}_{p/4,q/4} \\ \mathbb{I}_{p/4,q/4} & \end{pmatrix}$. Analogously, we obtain $\mathrm{O}^*(n/4)$ as intersection of the isometry group $\mathrm{U}(n/4, n/4)$ of the skew-Hermitian form $\begin{pmatrix} i\mathbb{I}_{n/4} & \\ & -i\mathbb{I}_{n/4} \end{pmatrix}$ with the isometry group $\mathrm{O}(n/2, \mathbb{C})$ of the symmetric form $\begin{pmatrix} & \mathbb{I}_{n/4} \\ \mathbb{I}_{n/4} & \end{pmatrix}$. This yields the remaining identities of (4).

We conclude the proof by showing that the maximal groups $L \subset \mathrm{GL}(n,\mathbb{R})$ are acting irreducibly on $\mathbb{R}^n$. One knows even more: For any subgroup $L \subset \mathrm{GL}(n/d,\mathbb{K}) \subset \mathrm{GL}(n,\mathbb{R})$ occurring in the list of the proposition its centralizer coincides with the corresponding homothety group $H_{\mathbb{K}}$:

$$\mathrm{End}_L(\mathbb{R}^n) = H_{\mathbb{K}}, \quad L \subset \mathrm{GL}(\mathbb{K}).$$

For the symplectic groups this can be easily verified. For the unitary groups this is true beginning with $n/d \geq 2$ and for the orthogonal groups it is true for $n/d \geq 3$ (see [Die71, Chapter II, §3]). In this context the quaternionic unitary groups $\mathrm{Sp}(p/4,q/4)$ and $\mathrm{O}^*(n/4)$ are understood as unitary groups. Since any homothety is invertible, the above groups act irreducibly, otherwise the projection onto an invariant subspace would be an element of the centralizer which is certainly not invertible. It remains to discuss irreducibility in the excluded small dimensions. □

Remark 7. A quaternionic vector space does not admit any symmetric or skew-symmetric bilinear form. This is reflected in the fact that the space of symmetric or skew-symmetric bilinear forms is never 4-dimensional (cf. Lemma 1).

Remark 8. Changing the complex basis by the matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$ we obtain the embedding $\mathrm{O}^*(n/4) \subset \mathrm{GL}(n/2,\mathbb{C})$ as given in [Hel01, Chapter X, §2,1]. There it is explained how $\mathrm{O}^*(n/4)$ occurs as dual of the symmetric space $\mathrm{O}(n/2)\big/\mathrm{U}(n/4)$ which justifies the notation.

The considerations above can be generalized to non-irreducible representations of Lie groups or Lie algebras. This has been done in [MR93]. Of course the structure of the algebra $\mathrm{End}_G(V)$ becomes more involved. On the other hand we may restrict our attention to special representations as such as the adjoint representation $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$ of a Lie group G. To ask for an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ becomes interesting from a geometrical point of view, since any such bilinear form induces a pseudo-Riemannian metric on G which is invariant under left- and right-multiplication. In particular G becomes a symmetric space. Hereafter we cite some results in this direction.

As shown above there are representations which are symplectic but not orthogonal. This fails for adjoint representations:

Proposition 7 ([MR93], Theorem 1.4). *A Lie algebra* $\mathfrak{g}$ *admits an* $\mathfrak{aut}(\mathfrak{g})$*-invariant non-degenerate symmetric bilinear form if and only if it is self-dual.*

On the other hand it has been shown:

Proposition 8 ([MR93], Corollary 1.7). *A Lie algebra* $\mathfrak{g}$ *admits an* $\mathfrak{aut}(\mathfrak{g})$*-invariant skew-symmetric bilinear form if and only if* $\mathrm{codim}_{\mathfrak{g}}[\mathfrak{g},\mathfrak{g}] \geq 2$.

In particular, for simple Lie algebras the adjoint representation is irreducible and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Thus, they cannot be symplectic which excludes many cases of Proposition 6.

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Chapter 19

Surface holonomy

Konrad Waldorf

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1 Introduction

From the theory of hermitian line bundles with connection we single out one interesting aspect: such a line bundle $L \to M$ assigns to each loop $\gamma\colon S^1 \to M$ a certain complex number $\mathrm{hol}_L(\gamma)$ in $\mathrm{U}(1)$ which is called the holonomy of L around γ. This assignment has many properties, and we will gradually encounter some of them during this chapter. It is worthwhile to specify three of them here, which will turn out to be analogous for surface holonomy. To start with, the holonomy of L measures the curvature of L in the following way: for an embedded two-dimensional submanifold D of M with parameterized boundary ∂D one finds

$$\exp\left(\int_D \mathrm{curv}(L)\right) = \mathrm{hol}_L(\partial D), \tag{1}$$

where we have normalized the exponential function by a factor of $2\pi\mathrm{i}$ such that it is 1-periodic.

Equation (1) can be seen as an improvement of Stokes' Theorem in two dimensions,

$$\int_D F = \int_{\partial D} \rho$$

which expresses the integral of the closed 2-form F by something on the boundary, just like the above formula for the integral of the curvature of the line bundle L. However, Stokes' Theorem is restricted to exact 2-forms $F = \mathrm{d}\rho$ whose cohomology class

$[F] \in \mathrm{H}^2(M, \mathbb{R})$ vanishes. Our improvement allows us at the price of exponentiation not only to take 2-forms with trivial cohomology class, but also – and more general – 2-forms whose class lies in the image of the homomorphism $\mathrm{H}^k(M, \mathbb{Z}) \to \mathrm{H}^k(M, \mathbb{R})$. Those 2-forms F – which we shall call forms with integral class – arise as the curvature of a hermitian line bundle L with connection. Then,

$$\int_D F = \ln\left(\mathrm{hol}_L(\partial D)\right) \mod \mathbb{Z}$$

generalizes Stokes' Theorem.

A second reason why holonomy is interesting lies in understanding the assignment $\gamma \mapsto \mathrm{hol}_L(\gamma)$ as a $\mathrm{U}(1)$-valued function $\mathrm{hol}_L \colon LM \to \mathrm{U}(1)$ on the loop space LM that consists of all smooth loops $S^1 \to M$. The loop space can canonically be endowed with the structure of an infinite dimensional manifold, so that the map hol_L becomes smooth [Bry93]. One can consider going in the other direction, when one is concerned with a smooth $\mathrm{U}(1)$-valued function on the loop space and is able to express it as the holonomy of a hermitian line bundle with connection over the – finite dimensional – manifold M.

Also of importance is the application of line bundles in physics: a hermitian line bundle $L \to M$ with connection offers a natural description of a $\mathrm{U}(1)$-gauge theory. On the quantum level, such a theory can be defined by assigning a complex number to each particle moving (on a closed line) through the target space M; this number acts as an amplitude in some path integral. If a particle moves on a circle $\gamma \colon S^1 \to M$, its amplitude is given by

$$\mathcal{A}(\gamma) = \mathrm{e}^{S_{\mathrm{kin}}(\gamma)} \cdot \mathrm{hol}_L(\gamma),$$

where $S_{\mathrm{kin}}(\gamma)$ is a kinetic term, and the holonomy expresses the coupling to the gauge field. The line bundle L comes up with all features you would expect from a gauge field: its curvature is a 2-form F and may be called the field strength of the gauge field. The second Bianchi identity $\mathrm{d}F = 0$ implies two of the four Maxwell equations, and – last but not least – the fact that F has an integral class is nothing but Dirac's quantization condition for the electric charge.

The question for an appropriate concept of surface holonomy has similar origins:

- it could provide a generalization Stokes' Theorem in three dimensions similar to that in two dimensions indicated above.
- it could be used in string theory to couple strings to non-trivial background fluxes, analogous to the coupling of a point particle to a gauge field.
- it could also provide actions for 3-form gauge fields in certain (classical) gauge theories.
- it could provide a way to describe structure on the loop space by structure on a finite-dimensional space, very much in the same way like smooth $\mathrm{U}(1)$-valued functions may be interpreted as the holonomy of a hermitian line bundle with connection over M.

Let us pick out the first two points and describe how they are related to surface holonomy. The generalization of Stokes' Theorem is obvious: if H is a 3-form on a three-dimensional manifold B, which may be not exact but with integral class, its integral over B could be expressed as (the logarithm of) the holonomy around the surface $\Sigma = \partial B$.

In fact, this is exactly a question which arises in two-dimensional conformal field theory, when studying non-linear sigma models on a Lie group G. Such a model can be defined by amplitudes $\mathcal{A}(\phi)$ for some path integral, where ϕ is a map from a closed complex surface Σ – the worldsheet – into the target space G of the model. In [Wit84], Witten gives the following definition for $G = \mathrm{SU}(2)$. Σ is the boundary of a three dimensional manifold B, and because the homotopy groups $\pi_i(\mathrm{SU}(2))$ vanish for $i = 1, 2$, every map $\phi\colon \Sigma \to M$ can be extended into the interior B to a map $\Phi\colon B \to G$. From the theory of compact, simple, connected and simply connected Lie groups it is known that the Ad-invariant trilinear form $k\langle -, [-, -]\rangle$ on the Lie algebra $\mathfrak{su}(2)$ induces a closed, bi-invariant 3-form H on SU(2) with integral class, provided k is an integer. Witten showed that – due to the integrality of H –

$$\mathcal{A}(\phi) := \exp\left(S_{\mathrm{kin}}(\phi) + \int_B \Phi^* H\right)$$

neither depends on the choice of B nor on the choice of the extension Φ, so that he obtained a well-defined amplitude. Here $S_{\mathrm{kin}}(\phi)$ is a kinetic term, and with a certain relative normalization this model is called the Wess–Zumino–Witten model on G at level k.

Now, if we could express the integral of $\Phi^* H$ over B by something on the boundary Σ – for instance via a generalization of Stokes' Theorem – the definition of the amplitude $\mathcal{A}(\phi)$ would be independent of conditions on the existence of the extension Φ, in particular of the simply-connectedness of G. It would hence provide a proper definition of Wess–Zumino–Witten models on arbitrary simple and compact Lie groups.

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2 Gerbes

To define surface holonomy, we first need a mathematical object which plays the role of the hermitian line bundle with connection. Such an object is collectively called gerbe with connective structure.

What makes gerbes a bit mysterious is that there are numerous definitions which look outmost different. To give an impression we show some examples of the different manifestations of gerbes and their connective structures.

Gerbes as stacks. This is the original definition given in 1971 by J. Giraud with a view to non-abelian cohomology [Gir71]. A stack is a fibred category satisfying a gluing (or descent) axiom. According to Giraud, a gerbe is a stack, whose fibres are groupoids and satisfy a certain transitivity and non-emptiness condition, also see [Moe02]. As an important characterization of a gerbe, Giraud defines the band of a gerbe, a certain sheaf of groups.

The definition of a connective structure on a gerbe in this sense was given thirty years later by L. Breen and W. Messing [BM05], who considered gerbes with arbitrary bands living over a scheme.

Gerbes as cohomology classes. Around 1972, P. Deligne invented a cohomology theory which is now called Deligne cohomology [Del91], [Bry93]. It is build up on cochain complexes

$$0 \longrightarrow \mathcal{D}^0(n) \xrightarrow{\mathrm{D}} \mathcal{D}^1(n) \xrightarrow{\mathrm{D}} \cdots \xrightarrow{\mathrm{D}} \mathcal{D}^k(n)\ ,$$

one for each natural number n. Deligne was originally interested in algebraic geometry, and realized that the cohomology group $\mathrm{H}^1(M, \mathcal{D}(1))$ classifies hermitian line bundles with connection [Del91].

A class in the cohomology group $\mathrm{H}^2(M, \mathcal{D}(2))$ can be seen as a $\mathrm{U}(1)$-banded gerbe with connective structure in the sense that it provides a definition of holonomy around surfaces. This was shown by K. Gawędzki and applied to topological field theory [Gaw88]. Gawędzki also showed that gerbes are related to a certain structure on the loop space, namely to a line bundle, which is obtained by a so-called transgression procedure.

Deligne cohomology provides a natural way to see gerbes with connective structure in a hierarchy of objects, starting with $\mathrm{U}(1)$-valued functions, hermitian line bundles with connection and gerbes with connective structure, and which is continued by n-gerbes which are classified by $\mathrm{H}^{n+1}(M, \mathcal{D}(n+1))$.

Gerbes as sheaves of groupoids. This concept is strongly related to the original one of a certain kind of stack, and defines a gerbe over a manifold M by an assignment

$$U \mapsto \mathcal{G}(U)$$

of a groupoid $\mathcal{G}(U)$ to any open subset U of M. It is developed in great detail in J.-L. Brylinski's book [Bry93]. Analogous to the gluing axiom of a sheaf, gluing axioms for such a gerbe a given. For $\mathbb{C}^\times$-banded sheaves of groupoids, Brylinski develops the definition and properties of a connective structure. He gives precise relations and classification result between sheaves of groupoids and Deligne cohomology. Furthermore he constructs a line bundle over the loop space and shows that it coincides with the one Gawędzki constructed from Deligne cohomology.

Gerbes as bundle gerbes. The concept of a sheaf of groupoids and even more the one of a connective structure on it is quite general but also quite complex. For some purposes, e.g. for the definition of holonomy and for applications in conformal field

theory, it is sufficient to use a simplified version – simplified in the sense that it just uses well-known geometric structure like line bundles and differential forms. For U(1)- or $\mathbb{C}^\times$-banded gerbes, M. K. Murray invented bundle gerbes [Mur96], which we will use in this chapter to develop surface holonomy. As we will learn, bundle gerbes admit a very simple and natural definition of a connective structure. To make contact to other concepts of gerbe, we will see, that every bundle gerbe induces a sheaf of groupoids, and that their isomorphism classes are in bijection to the Deligne cohomology group $\mathrm{H}^2(M, \mathcal{D}(2))$. Bundle gerbes have also been used in two-dimensional conformal field theory [GR02], [Gaw05], [SSW05].

Gerbes defined on open covers. Closely related to bundle gerbes are gerbes defined on open covers [Hit01], although bundle gerbes are a bit more general.

Gerbes as 2-bundles. A somewhat different approach to surface holonomy is by categorification of a vector bundle with connection. This leads to the concept of a 2-bundle with certain additional structure [BS04]. This approach also covers gerbes with non-abelian bands. Furthermore it realizes consequently the 2-categorial nature of gerbes, which we will also discover during this chapter.

For the purposes of this exposition we consider gerbes with band U(1), also called abelian (hermitian) gerbes. Accordingly we drop the qualifier hermitian for gerbes and for line bundles to improve the readability.

3 From line bundles to bundle gerbes

As indicated before, bundle gerbes are built up of line bundles and differential forms. One of basic features of a line bundle $L \to M$ is that it is locally trivializable. This is usually stated with respect to an open cover, but here we state it with respect to a covering $\pi : Y \to M$. From any open cover $\{V_i\}_{i\in I}$ of M one can produce such a covering by defining Y as the disjoint union of the sets,

$$Y = \bigsqcup_{i\in I} V_i,$$

and π as patched together from the inclusions $V_i \hookrightarrow M$. Locally trivializable means that there is a covering $\pi : Y \to M$ and a commutative diagram

$$\begin{array}{ccc} Y \times \mathbb{C} & \longrightarrow & L \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\pi]{} & M. \end{array}$$

A local trivialization defines a transition function $g : Y^{[2]} \to \mathrm{U}(1)$, where $Y^{[k]}$ denotes the k-fold fibre product of Y with itself. The transition function satisfies the cocycle

condition

$$\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g \tag{1}$$

over $Y^{[3]}$, where $\pi_{12} \colon Y^{[3]} \to Y^{[2]}$ are the projections on the respective components. If Y comes from on open cover, $Y^{[k]}$ is the disjoint union of all k-fold intersections of the open sets V_i. Accordingly, the transition function g decomposes in functions $g_{ij} \colon V_i \cap V_j \to \mathrm{U}(1)$, and the cocycle condition becomes $g_{ij} \cdot g_{jk} = g_{ik}$ as functions on $V_i \cap V_j \cap V_k$.

Bundle gerbes do not have a total space as line bundles do. For the definition of a bundle gerbe we step in after having locally trivialized, i.e. after having chosen a covering $\pi \colon Y \to M$,

$$\begin{array}{ccc} & & ? \\ & & \downarrow \\ Y & \xrightarrow[\pi]{} & M. \end{array}$$

Now we define the bundle gerbe analogous to what remains of the locally trivialized line bundle. According to our remarks about Deligne cohomology, U(1)-valued functions, hermitian line bundles with connection and gerbes with connective structure form a certain series of objects. Now we move up one step: instead of a transition function on $Y^{[2]}$, we take a line bundle $L \to Y^{[2]}$. The next steps are predicted: because we cannot multiply line bundles like the pullbacks of the transition function g in (1), the cocycle condition has to be relaxed to an isomorphism

$$\mu \colon \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$$

of line bundles over $Y^{[3]}$ – called the groupoid multiplication. To capture an essential aspect of the multiplication of functions, we demand that this isomorphism is associative.

It is straightforward to define a connective structure on a bundle gerbe. Consider first a connection on a line bundle $L \to M$. In a local trivialization $\pi \colon Y \to M$, this connection defines a 1-form $A \in \Omega^1(Y)$, which is related to the transition function g by

$$g^{-1} \mathrm{d} g = \pi_2^* A - \pi_1^* A.$$

For the bundle gerbe, we take a connection on the line bundle $L \to Y^{[2]}$ and impose that the isomorphism μ respects connections. Additionally, we take a 2-form $C \in \Omega^2(Y)$ – called the curving – which has to be related to the connection on L by

$$\mathrm{curv}(L) = \pi_2^* C - \pi_1^* C.$$

The connection on L together with the curving C form the connective structure. It is shown in [Mur96] that every bundle gerbe admits a connective structure.

In the rest of this chapter, we will only be concerned with bundle gerbes with connective structure, and hence drop the last suffix. We will also understand a line bundle as a (hermitian) line bundle with connection. Accordingly, all isomorphisms

of line bundles will be isomorphisms of line bundles which preserve the connections. With these conventions, we are arrived at the following definition:

Definition 1. A bundle gerbe $\mathcal{G}$ over M consists of a covering $\pi : Y \to M$, a 2-form $C \in \Omega^2(Y)$, a line bundle $L \to Y^{[2]}$ and an isomorphism

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$$

of line bundles over $Y^{[3]}$. Two axioms have to be satisfied:

(G1) The curvature of L is related to the curving by

$$\operatorname{curv}(L) = \pi_2^* C - \pi_1^* C.$$

(G2) The groupoid multiplication is associative in the sense that the diagram

$$\begin{array}{ccc} \pi_{12}^* L \otimes \pi_{23}^* L \otimes \pi_{34}^* L & \xrightarrow{\pi_{123}^*\mu \otimes 1} & \pi_{13}^* L \otimes \pi_{34}^* L \\ \downarrow{\scriptstyle 1 \otimes \pi_{234}^*\mu} & & \downarrow{\scriptstyle \pi_{134}^*\mu} \\ \pi_{12}^* L \otimes \pi_{24}^* L & \xrightarrow[\pi_{124}^*\mu]{} & \pi_{14}^* L \end{array}$$

of isomorphisms of line bundles over $Y^{[4]}$ is commutative.

In the following we will specify some properties of this definition and show several constructions what one can do with it. Similar to line bundles (with connection), each bundle gerbe $\mathcal{G}$ determines a closed 3-form $\operatorname{curv}(\mathcal{G})$ on M, called the curvature of $\mathcal{G}$: the derivative of axiom (G1) gives $\pi_1^* \mathrm{d}C = \pi_2^* \mathrm{d}C$, since the curvature of the line bundle L is a closed form. This means that $\mathrm{d}C$ descends along $\pi : Y \to M$ to a 3-form on M – the curvature of the bundle gerbe $\mathcal{G}$. It is obviously a closed form, and it will turn out later that it has an integral class.

To give an example of a bundle gerbe, we introduce trivial bundle gerbes. Just as for every 1-form $A \in \Omega^1(M)$ there is a trivial line bundle over M having this 1-form as its connection, we find a trivial bundle gerbe for every 2-form $\rho \in \Omega^2(M)$. The construction of this bundle gerbe is quite easy: for the covering we take the identity $\mathrm{id} : M \to M$, and the curving is the given 2-form ρ. The line bundle L is the trivial line bundle with the trivial connection, and the groupoid multiplication is the identity isomorphism between trivial line bundles. Now, axiom (G1) is satisfied since $\operatorname{curv}(L) = 0$ and $\pi_1 = \pi_2 = \mathrm{id}_M$. The associativity axiom (G2) is surely satisfied by the identity isomorphism. Thus we have defined a bundle gerbe, which we denote by $\mathcal{I}_\rho$. The curvature of a trivial gerbe is $\operatorname{curv}(\mathcal{I}_\rho) = \mathrm{d}\rho$.

Less elementary examples of bundle gerbes have been constructed in [GR02], [Mei02], [GR03], namely all (bi-invariant) bundle gerbes over all simple compact Lie groups. The availability of concrete examples in such non-trivial cases is an important advantage of bundle gerbes. Their constructions were possible by an explicit use of the geometric nature of bundle gerbes. For example, to construct a bundle gerbe

over $\mathrm{SU}(N)$, the line bundle L occurs as the canonical line bundle on the coadjoint orbit through a simple weight of some representation of $\mathrm{SU}(N)$. Another remarkable aspect is that the Lie-theoretic construction of bundle gerbes over Lie groups apart from $\mathrm{SU}(N)$ or $\mathrm{Sp}(4n)$ makes use of the fact that a covering $\pi\colon Y \to M$ is more general that having an open cover of M: the dimension of Y may be greater than the dimension of M.

Let us again assume that the covering $\pi\colon Y \to M$ of a bundle gerbe $\mathcal{G}$ comes from an open cover $\{V_i\}_{i\in I}$ of M (in fact this is the kind of gerbe which was called before a gerbe defined on open covers [Hit01]). Remember that we introduced the gerbe data – namely the line bundle L, the groupoid multiplication μ and the curving C – as analogues of the data of a trivialized line bundle. Surely the curving restricts to a 2-form B_i on each open set V_i. To get similar local expressions for the line bundle and the groupoid multiplication, we can trivialize once more: if the open sets V_i are chosen such that every double intersection $V_i \cap V_j$ is contractible, we are able to choose sections

$$\sigma_{ij}\colon V_i \cap V_j \to L$$

of unit length. Then, the connection on L pulls back to 1-forms A_{ij} on each double intersection $V_i \cap V_j$. Furthermore, over a three-fold intersection $V_i \cap V_j \cap V_k$, we can multiply two sections using the groupoid multiplication, and compare the result to a third section,

$$\mu(\sigma_{ij} \otimes \sigma_{jk}) = g_{ijk} \cdot \sigma_{ik},$$

via a function $g_{ijk}\colon V_i \cap V_j \cap V_k \to \mathrm{U}(1)$. Summarizing, we have extracted $\mathrm{U}(1)$-valued functions g_{ijk} on three-fold intersections, 1-forms A_{ij} on two-fold intersections, and 2-forms B_i on each open set. One can deduce the following relations among this local data:

$$\begin{aligned} g_{ijk} \cdot g_{ikl} &= g_{jkl} \cdot g_{ijl}, \\ g_{ijk}^{-1}\mathrm{d}g_{ijk} &= A_{jk} - A_{ik} + A_{ij}, \\ \mathrm{d}A_{ij} &= B_j - B_i. \end{aligned}$$

The first one is a consequence of the associativity of μ from axiom (G2), the second is the fact that μ preserves connections, and the third is the curvature condition (G1). These equations look like analogues of the two conditions for local data of a line bundle – the transition functions $g_{ij}\colon V_i \cap V_j \to \mathrm{U}(1)$ and the connection 1-forms A_i on V_i – namely

$$\begin{aligned} g_{ij} \cdot g_{jk} &= g_{ik}, \\ g_{ij}^{-1}\mathrm{d}g_{ij} &= A_j - A_i. \end{aligned}$$

A precise meaning of this analogy is given by Deligne cohomology. We already have indicated that Deligne cohomology comes from a cochain complex with cochain groups $\mathcal{D}^k(n)$ and a coboundary operator D. Their definition is such that the collection of local data (g, A, B) of the bundle gerbe $\mathcal{G}$ defines an element in $\mathcal{D}^2(2)$. The three

relations among this local data give the cocycle condition $\mathrm{D}(g, A, B) = 0$, so that we see, that the bundle gerbe $\mathcal{G}$ defines a class in the cohomology group $\mathrm{H}^2(M, \mathcal{D}(2))$. Similarly, the collection of local data (g, A) of a line bundle forms an element in $\mathcal{D}^1(1)$, and the two relations above give the cocycle condition $\mathrm{D}(g, A) = 0$. So, a line bundle defines a class in $\mathrm{H}^1(M, \mathcal{D}(1))$. In the next section we will see, that with the correct definition of isomorphisms between bundle gerbes, the correspondence of isomorphism classes of bundle gerbes with classes in Deligne cohomology is one-to-one.

To continue with specifying properties of a bundle gerbe, we come to the definition of characteristic classes. The transition function $g_{ij} : V_i \cap V_j \to \mathrm{U}(1)$ of a line bundle $L \to M$ with respect to an open cover $\{V_i\}_{i \in I}$ defines a class $[g]$ in the Čech cohomology of the sheaf $\underline{\mathrm{U}(1)}$ with respect to the chosen open cover. Via the exponential sequence, it hence gives rise to a class

$$[g] \in \mathrm{H}^2(M, \mathbb{Z}),$$

which is independent of the cover and of the sections chosen to get the transition function. It is thus an intrinsic quantity of the line bundle L, called the (first) Chern class and denoted by $c_1(L)$. The image of the Chern class in de Rham cohomology is equal to the class of the curvature of L,

$$c_1(L) = [\mathrm{curv}(L)],$$

which proves that the curvature of a line bundle with connection is a 2-form with integral class.

In the same way, the function $g_{ijk} : V_i \cap V_j \cap V_k \to \mathrm{U}(1)$ defined by the groupoid multiplication of a bundle gerbe $\mathcal{G}$ trivialized on an open cover, defines a class

$$[g] \in \mathrm{H}^3(M, \mathbb{Z})$$

which is also independent of the open cover and of the sections σ_{ij} which were chosen to extract the local data. This class is called the Dixmier–Douady class of the bundle gerbe $\mathcal{G}$ and denoted by $\mathrm{dd}(\mathcal{G})$. The Dixmier–Douady class and the curvature of $\mathcal{G}$ obey the same relation as the Chern class and the curvature of a line bundle [Mur96]:

$$\mathrm{dd}(\mathcal{G}) = [\mathrm{curv}(\mathcal{G})].$$

This shows that the curvature of a bundle gerbe is a 3-form with integral class.

As a little example, we may compute the Dixmier–Douady class of a trivial bundle gerbe $\mathcal{I}_\rho$. It is easy to see that – for any open cover – the functions g_{ijk} can be chosen to be constantly 1, so that the local connection 1-forms vanish, $A_{ij} = 0$. Solely the curving ρ restricts to non-trivial 2-forms $\rho|_{V_i}$ on V_i. Now it is clear that the Deligne class of $\mathcal{I}_\rho$ is represented by $(1, 0, \rho)$, and that the Dixmier–Douady class vanishes. Indeed, the curvature is an exact form, whose class also vanishes.

Finally, let us remark that there are three standard constructions one can do with bundle gerbes: tensor products, duals and pullbacks. Without giving the details of these constructions, one observes that they behave naturally under the correspondence

with classes in Deligne cohomology: the tensor product of bundle gerbes corresponds to the sum of classes, taking the dual bundle gerbe corresponds to taking opposite sign, and the pullback of bundle gerbes corresponds to the pullback of classes.

4 Morphisms of bundle gerbes

To find the appropriate definition of a morphism between two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$, assume for a moment that the coverings $\pi_i : Y_i \to M$ come from open covers for $i = 1, 2$. To compare the gerbe data, it would be natural to go to a common refinement of these covers. On the double intersections of this common refinement the line bundles L_1 and L_2 could be compared. For general coverings π_1 and π_2 the common refinement amounts to consider the fibre product

$$Z := Y_1 \times_M Y_2$$

thought of as a new covering $\zeta : Z \to M$ sending (y_1, y_2) to the point $\pi_1(y_1) = \pi_2(y_2)$. The two-fold intersections amount to consider $Z^{[2]}$. The restriction of the line bundle L_i to $Z^{[2]}$ is implemented by the pullback along the canonical map $y_i : Z^{[2]} \to Y_i^{[2]}$. A first idea is to require that the line bundles $y_1^* L_1$ and $y_2^* L_2$ are isomorphic. In fact, this was the original definition of a morphism between bundle gerbes [Mur96]. However, it turned out that this definition was too restrictive, in other words: the isomorphism classes were too small, and there were many non-isomorphic bundle gerbes having the same Deligne class and hence the same surface holonomy.

A solution to this was presented in [MS00]: the line bundles should not be isomorphic but stably isomorphic in the sense that there is a line bundle $A \to Z$ with an isomorphism

$$y_1^* L_1 \otimes \zeta_2^* A \cong \zeta_1^* A \otimes y_2^* L_2$$

of line bundles over $Z^{[2]}$. Here ζ_1 and ζ_2 are two natural projections from $Z^{[2]}$ to Z. It is natural to demand that the data of a morphism of bundle gerbes – the line bundle $A \to Z$ and an isomorphism α as above – is compatible with the rest of the structure of the bundle gerbes, namely the curvings and the groupoid multiplications.

Summarizing, with the additional generalization that A may be a vector bundle (of rank higher than 1), the correct definition of a morphism between bundle gerbes is

Definition 2. A morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ of bundle gerbes is a vector bundle $A \to Z$ together with an isomorphism

$$\alpha : y_1^* L_1 \otimes \zeta_2^* A \to \zeta_1^* A \otimes y_2^* L_2$$

of vector bundles over $Z^{[2]}$. Two axioms have to be satisfied:

(M1) The curvature of A is a real 2-form and fixed by

$$\mathrm{curv}(A) = y_2^* C_2 - y_1^* C_1.$$

(M2) The isomorphism α commutes with the groupoid multiplications μ_1 and μ_2 of the bundle gerbes in the sense that the diagram

$$\begin{array}{ccc}
y_1^*\pi_{12}^*L_1 \otimes y_1^*\pi_{23}^*L_1 \otimes \zeta_3^*A & \xrightarrow{y_1^*\mu_1 \otimes \mathrm{id}} & y_1^*\pi_{13}^*L_1 \otimes \zeta_3^*A \\
\downarrow{\scriptstyle 1\otimes\zeta_{23}^*\alpha} & & \\
\zeta_{12}^*y_1^*L_1 \otimes \zeta_2^*A \otimes \zeta_{23}^*y_2^*L_2 & & \downarrow{\scriptstyle \zeta_{13}^*\alpha} \\
\downarrow{\scriptstyle \zeta_{12}^*\alpha\otimes 1} & & \\
\zeta_1^*A \otimes y_2^*\pi_{12}^*L_2 \otimes y_2^*\pi_{23}^*L_2 & \xrightarrow[1\otimes y_2^*\mu_2]{} & \zeta_1^*A \otimes y_2^*\pi_{13}^*L_2
\end{array}$$

of isomorphisms of vector bundles over $Z^{[3]}$ is commutative.

There is one important point to notice from this definition of a morphism. Given two such morphisms

$$\mathcal{A}\colon \mathcal{G}_1 \to \mathcal{G}_2 \quad \text{and} \quad \mathcal{A}'\colon \mathcal{G}_1 \to \mathcal{G}_2$$

each providing a vector bundle A, A' over Z, to compare both morphisms it does not make sense to state that they are equal or not: to compare vector bundles one needs isomorphisms between them. This leads us forthright to the fact that bundle gerbes form a 2-category [Ste00]: we have bundle gerbes as objects, morphisms between the objects as defined above, and 2-morphisms between the morphisms. Before we discuss the 2-categorial aspects of bundle gerbes, we give the precise definition of a 2-morphism.

Definition 3. Let $\mathcal{A}\colon \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}'\colon \mathcal{G}_1 \to \mathcal{G}_2$ be two morphisms. A 2-morphism

$$\beta : \mathcal{A} \Rightarrow \mathcal{A}'$$

is an isomorphism $\beta\colon A \to A'$ of vector bundles over Z, which is compatible with the isomorphisms α and α' in the sense that the diagram

$$\begin{array}{ccc}
y_1^*L_1 \otimes \zeta_2^*A & \xrightarrow{\alpha} & \zeta_1^*A \otimes y_2^*L_2 \\
\downarrow{\scriptstyle 1\otimes\zeta_2^*\beta} & & \downarrow{\scriptstyle \zeta_1^*\beta\otimes 1} \\
y_1^*L_1 \otimes \zeta_2^*A' & \xrightarrow{\alpha'} & \zeta_1^*A' \otimes y_2^*L_2
\end{array}$$

of isomorphisms of vector bundles over $Z^{[2]}$ is commutative.

Realizing that bundle gerbes form a 2-category is not a fault of the theory, it is a feature. To give an example, notice that – as in every 2-category – the set of all morphisms between two fixed bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$, together with the set of all the 2-morphisms between such morphisms, forms a category. To be a bit more precise:

for bundle gerbes they form a groupoid, since all 2-morphisms are invertible. In particular, we have a groupoid of endomorphisms from a bundle gerbe $\mathcal{G}$ to itself. This groupoid may be considered as the groupoid of gauge transformations of $\mathcal{G}$. So we get a clear understanding what the gauge symmetry of a gauge theory for strings is: it is a groupoid rather than a group.

Another feature of this 2-categorial point of view is the following. In any 2-category, a morphism $\mathcal{A}\colon \mathcal{G}_1 \to \mathcal{G}_2$ is called isomorphism, if it is invertible in the sense that there is another morphism $\mathcal{B}\colon \mathcal{G}_2 \to \mathcal{G}_1$ in the opposite direction, such that there are 2-isomorphisms

$$\mathcal{B} \circ \mathcal{A} \Rightarrow \mathrm{id}_{\mathcal{G}_1} \quad \text{and} \quad \mathcal{A} \circ \mathcal{B} \Rightarrow \mathrm{id}_{\mathcal{G}_2}.$$

Now we ask, what this general definition means for morphisms of bundle gerbes from Definition 2. Of course one has to say two things: what are the identity morphisms $\mathrm{id}_{\mathcal{G}_1}$ and $\mathrm{id}_{\mathcal{G}_2}$, and how the composition $\circ$ is defined. The last point is quite involved, however it can be done [Ste00]. We only present the result here.

Proposition 1. *A morphism $\mathcal{A}\colon \mathcal{G}_1 \to \mathcal{G}_2$ of bundle gerbes is an isomorphism, if and only if the vector bundle $A \to Z$ is a line bundle, i.e. has rank* 1.

The standard literature about bundle gerbes, e.g. [CJM02], [GR02], takes the last proposition as the definition of morphisms between bundle gerbes (so-called stable isomorphisms), and neglects that there are morphisms with vector bundles of higher rank. As a consequence, in the standard literature bundle gerbes form a 2-groupoid rather than a 2-category. The advantages of our definition become apparent later on.

For the rest of this section, we restrict ourselves to isomorphisms between bundle gerbes. We would like to get an overview over the set of possible isomorphisms between $\mathcal{G}_1$ and $\mathcal{G}_2$. First we get rid of the 2-morphisms by going to equivalence classes: we call two such isomorphisms equivalent, if there is a 2-morphism between them (which is automatically a 2-isomorphism). In other words, we consider the skeleton of the groupoid of isomorphisms between $\mathcal{G}_1$ and $\mathcal{G}_2$.

Proposition 2 ([CJM02], [SSW05]). *The set of equivalence classes of isomorphisms between two fixed bundle gerbes is a torsor over the group* $\mathrm{Pic}_0(M)$ *of isomorphism classes of flat line bundles over M.*

Supposed we are able to compose morphisms as indicated above, being isomorphic is an equivalence relation on the set of bundle gerbes. We present a result which classifies these isomorphism classes of bundle gerbes, in other words: we are looking for the skeleton of the 2-category of bundle gerbes. Recall that a bundle gerbe defines by its local data a cocycle in the Deligne cochain group $\mathcal{D}^2(2)$. Similarly it can be shown, that an isomorphism $\mathcal{A}\colon \mathcal{G}_1 \to \mathcal{G}_2$ defines a cochain (t, W) in the cochain group $\mathcal{D}^1(2)$. It relates the cocycles of the two bundle gerbes by its coboundary,

$$(g_2, A_2, B_2) = (g_1, A_1, B_1) + \mathrm{D}(t, W).$$

This equation means that two isomorphic bundle gerbes define the same class in Deligne cohomology. Even stronger is the following theorem.

Theorem 1 ([MS00]). *The set of isomorphism classes of bundle gerbes is in bijection to the Deligne cohomology group* $\mathrm{H}^2(M, \mathcal{D}(2))$.

This theorem makes contact to one of the other definitions of gerbes with connective structure and gives the precise relation between line bundles and bundle gerbes: line bundles are classified by the Deligne cohomology group $\mathrm{H}^1(M, \mathcal{D}(1))$, and bundle gerbes by $\mathrm{H}^2(M, \mathcal{D}(2))$.

In the remainder of this section, we are going to sketch the relation between bundle gerbes and yet another realization of gerbes, namely sheaves of groupoids. To this purpose we need the trivial bundle gerbes $\mathcal{I}_\rho$ defined in the last section as an example for bundle gerbes. In the same way as a trivialization of a line bundle is an isomorphism to a trivial line bundle, we say

Definition 4. A trivialization of a bundle gerbe $\mathcal{G}$ is an isomorphism

$$\mathcal{T} : \mathcal{G} \to \mathcal{I}_\rho.$$

Let us briefly exhibit the details of a trivialization, which follow from the definitions of an isomorphism and of the trivial bundle gerbe $\mathcal{I}_\rho$. The isomorphism $\mathcal{T}$ consists of a line bundle T over the space $Z = Y \times_M M$ which we identify canonically with Y itself. Under this identification, the two projections to the coverings of the bundle gerbes become the identity $\mathrm{id} : Y \to Y$ and the covering $\pi : Y \to M$ itself, so that axiom (M1) becomes

$$\mathrm{curv}(T) = \pi^*\rho - C.$$

Further, the isomorphism $\mathcal{T}$ consists of an isomorphism $\tau : L \otimes \pi_2^* T \to \pi_1^* T$ of line bundles over $Z^{[2]} = Y^{[2]}$. Because the groupoid multiplication of the trivial bundle gerbe $\mathcal{I}_\rho$ is the identity, axiom (M2) for τ reduces to the equation

$$\pi_{13}^*\tau \circ \mu = \pi_{12}^*\tau \circ \pi_{23}^*\tau$$

of isomorphisms of line bundles over $Z^{[3]} = Y^{[3]}$.

Of course not every bundle gerbe admits a trivialization. In the same way as for line bundles the obstruction to the existence of a trivialization is given by the first Chern class, a bundle gerbe $\mathcal{G}$ admits a trivialization if and only if its Dixmier–Douady class vanishes [CJM02]. In this case, the curvature of the bundle gerbe $\mathcal{G}$ is an exact form, and

$$\mathrm{curv}(\mathcal{G}) = \mathrm{d}\rho$$

for any trivialization $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\rho$.

We define a sheaf of groupoids in the following way: for an open subset U of M consider the set of trivializations $\mathcal{G}|_U \to \mathcal{I}_\rho$ for all 2-forms ρ (the set may be empty). Since trivializations are nothing but isomorphisms of bundle gerbes, together with the

2-morphisms they form naturally a groupoid $\mathcal{G}(U)$. Furthermore, trivializations can clearly be restricted to smaller subsets, so that the assignment

$$U \mapsto \mathcal{G}(U)$$

is a presheaf of groupoids. The gluing axiom can be shown by gluing together two trivializations over U_1 and U_2, if there is a 2-morphisms between their restrictions to the intersection $U_1 \cap U_2$. This way, every bundle gerbe defines a sheaf of groupoids in Brylinski's sense [MS00].

5 Holonomy around closed oriented surfaces

The holonomy of a bundle gerbe around a closed oriented surface should be analogous to the holonomy of a line bundle around a loop $\gamma\colon S^1 \to M$, since S^1 is also closed and oriented.

So, it is worthwhile to recall how the holonomy of a line bundle L over M around a loop $\gamma\colon S^1 \to M$ can be defined. The pullback of L along γ gives a line bundle over the circle, whose first Chern class vanishes for dimensional reasons. Hence, it becomes isomorphic to a trivial line bundle with some connection 1-form $A \in \Omega^1(S^1)$. Then,

$$\mathrm{hol}_L(\gamma) := \exp\left(\int_{S^1} A\right)$$

is a number in U(1) which is in fact independent of the choice of the trivialization. We also write out this definition in terms of local data (g, A) of the line bundle L with respect to an open cover $\{V_i\}_{i\in I}$. Choose a triangulation Δ of S^1 that is subordinated to the open cover by a map $i\colon \Delta \to I$, such that $\gamma(e) \subset V_{i(e)}$ for any edge e and $\gamma(v) \in V_{i(v)}$ for any vertex v. Then, by splitting the integral over A with respect to the triangulation and using Stokes' Theorem, one can derive the formula

$$\mathrm{hol}_N(\gamma) := \prod_{i\in I} \exp\left(\int_e \gamma^* A_{i(e)}\right) \cdot \prod_{v\in\partial e} g_{i(e)i(v)}^{\epsilon(e,v)}(\gamma(v))$$

where $\epsilon(e, v) \in \{-1, 1\}$ is positive, if v is the endpoint of e end negative otherwise. The meaning of this formula is that one has to integrate the local connection 1-forms along the edges, and to use the transition functions to intermediate at the vertices between two edges.

For the definition of the holonomy of a bundle gerbe $\mathcal{G}$ we start with a configuration like in Figure 1 and mimic the same procedure as for line bundles.

Definition 5 ([CJM02]). Let $\mathcal{G}$ be a bundle gerbe over M. For a closed oriented surface Σ and a smooth map $\phi\colon \Sigma \to M$, let

$$\mathcal{T}\colon \phi^*\mathcal{G} \to \mathcal{I}_\rho$$

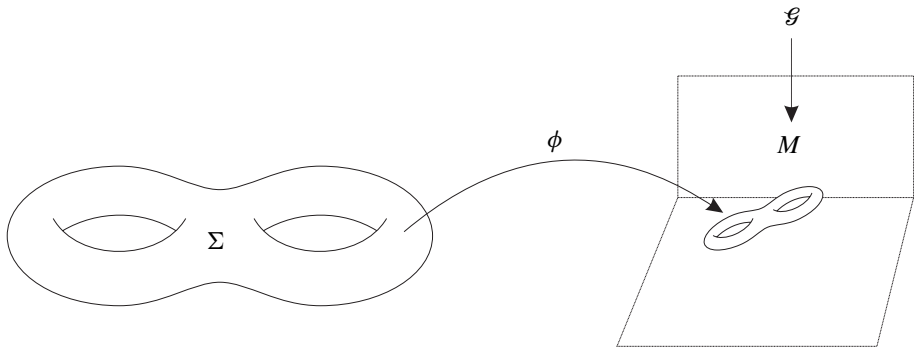

Figure 1. A surface is mapped into some space M with bundle gerbe $\mathcal{G}$.

be a trivialization of the pullback of the bundle gerbe $\mathcal{G}$ along ϕ. Then we define

$$\mathrm{hol}_{\mathcal{G}}(\phi) := \exp\Big(\int_\Sigma \rho\Big)$$

to be the holonomy of the bundle gerbe $\mathcal{G}$ around $\phi\colon \Sigma \to M$.

This sounds easy but we have to assure that the number $\mathrm{hol}_{\mathcal{G}}(\phi)$ is independent of the choice of the trivialization, which exists for dimensional reasons. Different trivializations may have different 2-forms ρ, however, in the next lemma we show that the difference $\rho_2 - \rho_1$ between to such 2-forms is the curvature of some line bundle over M, in particular: it is a closed form with integral class. Then, the calculation

$$\exp\Big(\int_\Sigma \rho_2\Big) = \exp\Big(\int_\Sigma \rho_2 - \rho_1\Big)\cdot \exp\Big(\int_\Sigma \rho_1\Big) = \exp\Big(\int_\Sigma \rho_1\Big)$$

shows that the definition $\mathrm{hol}_{\mathcal{G}}(\phi)$ is independent of the choice of the trivialization.

Lemma 1. *Two trivializations*

$$\mathcal{T}_1\colon \mathcal{G} \to \mathcal{I}_{\rho_1} \quad \textit{and} \quad \mathcal{T}_2\colon \mathcal{G} \to \mathcal{I}_{\rho_2}$$

of the same bundle gerbe $\mathcal{G}$ over M determine a line bundle over M with curvature $\rho_2 - \rho_1$.

Proof. Using the features of the 2-category of bundle gerbes, we can give a very relaxed proof: by taking the inverse and composition, we obtain an isomorphism

$$\mathcal{T}_2 \circ \mathcal{T}_1^{-1}\colon \mathcal{I}_{\rho_1} \to \mathcal{I}_{\rho_2}$$

of trivial bundle gerbes. From the definitions of isomorphisms and trivial bundle gerbes it follows immediately, that $\mathcal{T}_2 \circ \mathcal{T}_1^{-1}$ is a line bundle with curvature $\rho_2 - \rho_1$. But since we have not defined inverses and composition of isomorphisms, let us also give a more concrete proof. Recall that the two trivializations provide line bundles T_1

and T_2 over Y. We show that the line bundle $T_2 \otimes T_1^*$ descends along $\pi\colon Y \to M$. To do so we have to specify descent data, namely an isomorphism

$$\chi\colon \pi_2^*(T_2 \otimes T_1^*) \to \pi_1^*(T_1 \otimes T_2^*)$$

of line bundles over $Y^{[2]}$ which satisfies the cocycle condition

$$\pi_{13}^*\chi = \pi_{12}^*\chi \circ \pi_{23}^*\chi$$

over $Y^{[3]}$. For the definition of χ recall that the two trivializations provide isomorphisms $\tau_i\colon L \otimes \pi_2^*T_i \to \pi_1^*T_i$ for $i = 1, 2$. Then we declare χ to be the following isomorphism:

$$\begin{aligned}\chi : \pi_2^*(T_2 \otimes T_1^*) &\cong (L \otimes \pi_2^*T_2) \otimes (\pi_2^*T_1^* \otimes L^*) \\ &\qquad\qquad \big\downarrow{\scriptstyle \tau_1 \otimes \tau_2^{*-1}} \\ &\qquad \pi_1^*T_2 \otimes \pi_1^*T_1^* \cong \pi_1^*(T_2 \otimes T_1^*)\end{aligned}$$

Using axiom (M2) for τ_1 and τ_2 one can now show that χ satisfies the cocycle condition. Hence, $T_2 \otimes T_1^*$ descends to a line bundle N over M with the property

$$\pi^*N \cong T_2 \otimes T_1^*.$$

To finish the proof we have to compute the curvature of N. Notice that by axiom (M1) we find

$$\operatorname{curv}(T_2 \otimes T_1^*) = \pi^*\rho_2 - C - (\pi^*\rho_1 - C) = \pi^*(\rho_2 - \rho_1).$$

This shows that the curvature of N is $\rho_2 - \rho_1$. □

Now that we have the definition of holonomy around a closed oriented surface we address the question if it provides the desired generalization of Stokes' Theorem, which is important for the application in Wess–Zumino–Witten models.

Proposition 3. *Let $\mathcal{G}$ be a bundle gerbe over M with curvature H. For a three-dimensional oriented manifold B with boundary and a map $\Phi\colon B \to M$, we find*

$$\operatorname{hol}_{\mathcal{G}}(\Phi|_{\partial B}) = \exp\Big(\int_B \Phi^*H\Big).$$

Proof. Remember that for any trivialization $\mathcal{T}\colon \Phi^*\mathcal{G}|_{\partial B} \to \mathcal{I}_\rho$ we have $\Phi^*H|_{\partial B} = \mathrm{d}\rho$. Then, by definition,

$$\operatorname{hol}_{\mathcal{G}}(\Phi|_{\partial B}) = \exp\Big(\int_{\partial B} \rho\Big) = \exp\Big(\int_B \Phi^*H\Big)$$

It is important to recognize that the last step does not just follow from Stokes' Theorem (because Φ^*H is not exact over B). In fact one can triangulate B, choose local trivializations with 2-forms ρ_i differing by closed 2-forms with integral class by Lemma 1, and then use Stokes' Theorem on each 3-face. The remaining differences between the

2-forms disappear after exponentiation. □

This way we reproduce the amplitude of the coupling term of the Wess–Zumino–Witten model by

$$\mathcal{A}(\phi) = \exp\left(S_{\text{kin}}(\phi)\right) \cdot \text{hol}_{\mathcal{G}}(\phi).$$

Notice that we did not impose any condition on the topology of the target space M. While in Witten's definition the background field (apart from the metric) is just the three 3-form H, in the approach using bundle gerbes the background field is the bundle gerbe $\mathcal{G}$. This is no contradiction since all bundle gerbes over SU(2) with curvature $\text{curv}(\mathcal{G}) = H$ are isomorphic [GR02]. However, for general target spaces there may be bundle gerbes with same curvature, which are not isomorphic. This occurs for instance on the Lie group $\text{Spin}(4n)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then, the bundle gerbe contains more information as just its curvature, and it becomes essential to recognize the bundle gerbe itself as the background field which defines the theory. Just as little as by its curvature, the bundle gerbe can be replaced by a 2-form, sometimes called the B-field, or Kalb–Ramond field. Such a 2-form ρ exists in general only locally, namely when there is a trivialization $\mathcal{G}|_U \to \mathcal{I}_\rho$. Even then, it is not unique.

This situation is analogous in every detail to what is called the Aharonov–Bohm effect in electrodynamics: in a (first) quantized theory of charged particles moving through a certain electric field of field strength F, not only the field strength, but also the gauge potential A is an observable quantity which is necessary to describe the theory – opposed to a classical theory of electrodynamics where everything is determined by the field strength and Maxwell's equations. This Aharonov–Bohm effect (which in fact was predicted 10 years before by W. Ehrenberg and R. E. Siday) has been measured. Now one might think that the gauge potential A, a 1-form, is the object which describes the theory. But the gauge potential with $\mathrm{d}A = F$ is only defined locally, and if it is, it is not unique. This is exactly the behaviour of a line bundle and shows that a line bundle provides the correct description for this situation.

To close this section, we reformulate the holonomy in terms of local data of the bundle gerbe, analogous the formula for the holonomy of a line bundle. Recall that a trivialization

$$\mathcal{T} : \phi^*\mathcal{G} \to \mathcal{I}_\rho$$

chosen in Definition 5 implies the following relation between the local data of the bundle gerbe and the local data of the trivial gerbe,

$$(1, 0, \rho) = \phi^*(g, A, B) + \mathrm{D}(t, W).$$

Now, following the strategy of the local expression for the line bundle, we choose a triangulation Δ of the surface Σ, consisting of faces f, edges e and vertices v. It should be chosen subordinated to the same open cover $\{V_i\}_{i\in I}$ of M which was used to extract the local data (g, A, B) of the bundle gerbe $\mathcal{G}$. So there is a map $i : \Delta \to I$, assigning to each face, edge or vertex f an index $i(f)$ so that $\phi(f) \subset V_{i(f)}$. Now the integral of the 2-form ρ over Σ which defines the holonomy may be split up with

respect to the triangulation. By a subsequent use of Stokes' Theorem and the above formula for the local data [CJM02], one ends up with the following formula

$$\begin{aligned}\mathrm{hol}_{\mathcal{G}}(\phi) = \prod_{f\in\Delta} \exp\Big(\int_f \phi^* B_{i(f)}\Big) \\ \cdot \prod_{e\in\partial f} \exp\Big(\int_e \phi^* A_{i(f),i(e)}\Big) \cdot \prod_{v\in\partial e} g_{i(f),i(e),i(v)}^{\epsilon(f,e,v)}(\phi(v)).\end{aligned} \tag{1}$$

This formula shows explicitly what's going on: the surface integral is expressed by local integrals over the local 2-forms B_i, the local Kalb–Ramond fields with $\mathrm{d}B_i = H$. But on the edges and vertices, the sum has to be corrected by the rest of the local data of the bundle gerbe. This way, the triangulated surface gets decorated like shown in Figure 2.

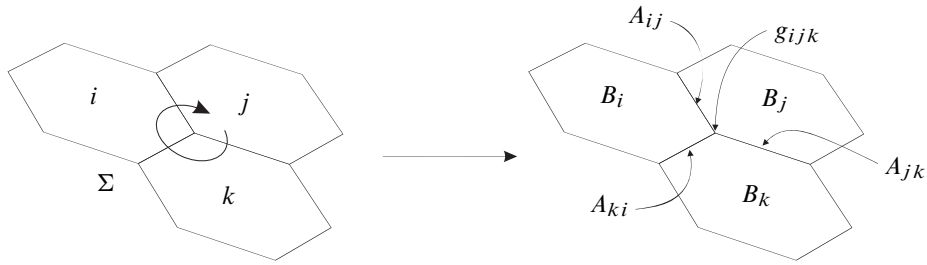

Figure 2. The triangulation of the surface Σ is decorated by the local data: the faces with 2-forms B_i, the edges with 1-forms A_{ij} and the vertices with the functions g_{ijk}.

Of course one can define the last expression without knowing bundle gerbes just by starting with a class in Deligne cohomology represented by local data (g, A, B). In fact, surface holonomy appeared first in this form and was recognized to be useful for string theory [Alv85], [Gaw88].

6 The line bundle over the loop space

As pointed out in the introduction, the holonomy of a line bundle L over a manifold M can be seen as a U(1)-valued function

$$\mathrm{hol}_L : LM \to \mathrm{U}(1)$$

on its loop space. Recall that such a function is the first object in a series of mathematical objects and is followed by a line bundle. So one might guess that, when starting with a bundle gerbe instead of a line bundle, the function on the loop space is replaced by a line bundle over the loop space. This correspondence in question between gerbes

over M and line bundles over LM was first discovered in terms of Deligne cohomology [Gaw88]. It was later redefined and extended to sheaves of groupoids [Bry93]. The associated homomorphism of Deligne cohomology groups

$$\mathrm{H}^k(M, \mathcal{D}(k)) \to \mathrm{H}^{k-1}(LM, \mathcal{D}(k-1))$$

is called transgression, see also [GT01]. In particular, it reproduces for $k = 1$ the correspondence between line bundles over M and $\mathrm{U}(1)$-valued functions on LM.

Let us describe briefly how one can construct the line bundle over LM from a bundle gerbe $\mathcal{G}$ over M. The construction we present here is an adaption of Brylinski's construction to bundle gerbes. Another construction starting from bundle gerbes is proposed in [GR02]. Let the fibre over a loop $\gamma\colon S^1 \to M$ consist of all trivializations

$$\mathcal{T}\colon {}^*\mathcal{G} \to \mathcal{I}_0.$$

Such trivializations exist, because the Dixmier–Douady class of the pullback bundle gerbe lives in $\mathrm{H}^3(S^1, \mathbb{Z}) = 0$. We identify two trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$, if there is a 2-morphism

$$\mathcal{T}_1 \Rightarrow \mathcal{T}_2.$$

After that, the fibre over γ consists of equivalence classes of isomorphisms between $\gamma^*\mathcal{G}$ and $\mathcal{I}_0$. This set is by Proposition 2 a torsor over the group $\mathrm{Pic}_0(S^1)$ of isomorphism classes of flat line bundles over the circle. It is well-known that this group is canonically isomorphic to $\mathrm{U}(1)$. Under this identification, the fibre over each loop γ is in a natural way a $\mathrm{U}(1)$-torsor. One can further show that the union of the fibres carries a canonical smooth structure, which makes it into a principal $\mathrm{U}(1)$-bundle over LM.

On the associated line bundle L the bundle gerbe defines a connection. The Deligne cohomology class of this line bundle (with connection) is the image of the class of the bundle gerbe $\mathcal{G}$ we started with under the transgression homomorphism of Deligne cohomology groups. There is an interesting relation between the holonomy of the line bundle L over LM and the holonomy of the bundle gerbe $\mathcal{G}$ over M: if $\gamma\colon S^1 \to LM$ is a loop in the loop space we can naturally identify it with a map $\phi\colon S^1 \times S^1 \to M$. One can now consider both the holonomy of the line bundle L around the loop γ as well as the holonomy of the bundle gerbe $\mathcal{G}$ around ϕ. Both coincide [Bry93]:

$$\mathrm{hol}_L(\gamma) = \mathrm{hol}_{\mathcal{G}}(\phi).$$

The line bundle L over the loop space plays an important role for Wess–Zumino–Witten models on Lie groups G. Its total space can be endowed with a group structure in a way that it becomes a central extension of the loop group LG. It can also be completed with respect to an appropriate scalar product, so that the space of holomorphic sections forms a Hilbert space, which acts as the space of states for the quantized Wess–Zumino–Witten model [Gaw99].

7 D-branes and surfaces with boundary

Some string theories involve not only loops (closed strings) but also open strings, whose worldsheets are surfaces with boundary. For those, we are not able to apply Definition 5 of the holonomy of a bundle gerbe $\mathcal{G}$: the integral of the closed form $\rho_2 - \rho_1$ with integral class is not an integer anymore. More precisely, a boundary term emerges which has to be compensated to achieve a holonomy independent of the choice of the trivialization.

Unfortunately, there is no analogous situation for the holonomy of line bundles. Therefore, we adopt the concept of D-branes from string theory [Pol96]. D-branes restrict the endpoints of open strings to submanifolds Q of the target space, and hence impose (generalized) Dirichlet boundary conditions to the motion of open strings. In the former definition of the Wess–Zumino–Witten model on a Lie group G by a 3-form H, a typical choice of the submanifold Q is a (twisted) conjugacy class of G. Here there is an additional condition, namely that the 3-form is fixed on the D-brane to $H|_Q = \mathrm{d}\omega$ for some 2-form ω on Q. This 2-form can be interpreted as the field strength of a twisted U(1)-gauge field on Q. Let us for simplicity assume that the surface Σ has only one boundary component. Accordingly, there is one D-brane (Q, ω) chosen, and we consider a map $\phi\colon \Sigma \to G$ with

$$\phi(\partial\Sigma) \subset Q.$$

Then, the following definition of the amplitudes is given [Gaw99]. Let D^2 be a disk acting as a cap for the surface Σ, so that there exists a three-dimensional manifold B whose boundary is $\partial B = \Sigma \cup D^2$. We again have to assume that there is an extension $\Phi\colon B \to M$ of the map ϕ, which now has to send the cap D^2 into the D-brane Q. Then, the amplitude is defined by

$$\mathcal{A}(\phi) := \exp\left(S_{\mathrm{kin}}(\phi) + \int_B \Phi^* H - \int_{D^2} \Phi^*\omega\right).$$

There is a condition on the well-definedness of this amplitude; here it is not sufficient that H is a closed 3-form with integral class. The equations $\mathrm{d}H = 0$ and $H|_Q = \mathrm{d}!$ mean that the pair (H, ω) defines a class in the relative cohomology $\mathrm{H}^3(G, Q, \mathbb{R})$, and the condition is, that this class is integral in the sense that it lies in the image of the relative cohomology with integer coefficients. There are explicit expressions for ω in the case that Q is a (twisted) conjugacy class, so that this integrality condition is satisfied.

We have learned before that the theory of bundle gerbes extends the former 3-form approach to the Wess–Zumino–Witten-model. Accordingly, we have to adjust the definition of a D-brane. It is still build up on a submanifold Q. In the first attempt [GR02], the 2-form ω was replaced by a trivialization

$$\mathcal{E}\colon \mathcal{G}|_Q \to \mathcal{I}_\omega$$

of the bundle gerbe $\mathcal{G}$ restricted to Q. Notice that this reproduces in particular the old condition $H|_Q = \mathrm{d}\omega$ for the curvature H of the bundle gerbe $\mathcal{G}$. Later it was recognized [Gaw05] that having a trivialization, i.e. an isomorphism, was too strong. In fact a D-brane for a certain bundle gerbe over $\mathrm{SO}(4n)/\mathbb{Z}_2$ was found which does not admit an isomorphism but a weaker structure – a morphism from $\mathcal{G}|_Q$ to the trivial bundle gerbe $\mathcal{I}_\omega$.

Such a morphism $\mathcal{G} \to \mathcal{I}_\omega$ is also called a $\mathcal{G}$-module or bundle gerbe module. A slightly more general version was also considered in a purely mathematical context to obtain a geometric realization of twisted K-theory [BCM+02].

Definition 6. Let $\mathcal{G}$ be a bundle gerbe over M. A D-brane for $\mathcal{G}$ is a submanifold Q of M together with a $\mathcal{G}|_Q$-module

$$\mathcal{E} : \mathcal{G}|_Q \to \mathcal{I}_\omega.$$

The 2-form ω on Q is called the curvature of the D-brane.

Recall that Lemma 1 implies that two trivializations $\mathcal{G} \to \mathcal{I}_{\rho_1}$ and $\mathcal{G} \to \mathcal{I}_{\rho_2}$ lead to a line bundle over M of curvature $\rho_2 - \rho_1$. The same statement holds for a gerbe module $\mathcal{G} \to \mathcal{I}_\omega$ and an trivialization $\mathcal{G} \to \mathcal{I}_\rho$: they define a vector bundle $E \to M$ of curvature $\omega - \rho$. To prove this, recall that a gerbe module is a trivialization with a vector bundle instead of a line bundle. To this situation, the proof of Lemma 1 extends without changes. The holonomy of this vector bundle will be the term which compensates the changes of ρ on the boundary.

Now consider a configuration like shown in Figure 3. We give the following definition of holonomy.

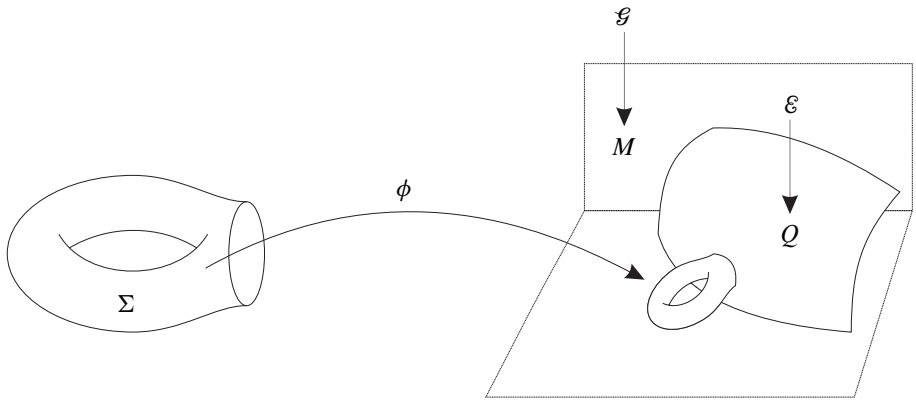

Figure 3. A surface is mapped into a target space with bundle gerbe $\mathcal{G}$, so that its boundary is mapped into the submanifold Q with bundle gerbe module $\mathcal{E}$.

Definition 7 ([CJM02]). Let $\mathcal{G}$ be a bundle gerbe over M, and let $(Q, \mathcal{E})$ be a D-brane for $\mathcal{G}$. For an oriented surface Σ and a map $\phi\colon \Sigma \to M$ which maps the boundary of Σ into Q, choose a trivialization

$$\mathcal{T}\colon \phi^*\mathcal{G} \to \mathcal{I}_\rho$$

of the pullback of $\mathcal{G}$ along ϕ. Its restriction to $\partial\Sigma$ determines in combination with the bundle gerbe module

$$\phi^*\mathcal{E}\colon \phi^*\mathcal{G}|_{\partial\Sigma} \to \mathcal{I}_{\phi^*\omega}$$

a vector bundle $E \to \partial\Sigma$ of curvature $\phi^*\omega - \rho$. Then, we define

$$\mathrm{hol}_{\mathcal{G},\mathcal{E}}(\phi) = \exp\left(\int_\Sigma \rho\right) \cdot \mathrm{tr}(\mathrm{hol}_E(\partial\Sigma))$$

to be the holonomy of the bundle gerbe $\mathcal{G}$ with D-brane $(Q, \mathcal{E})$ around ϕ.

In fact it is easy to see that this definition does not depend on the choice of the trivialization $\mathcal{T}$: for another trivialization $\mathcal{T}'$ recall that by Lemma 1 we obtain a line bundle $N \to \partial\Sigma$ with curvature $\rho' - \rho$. The following change in the first factor of $\mathrm{hol}_{\mathcal{G},\mathcal{E}}(\phi)$ emerges:

$$\begin{aligned}\exp\left(\int_\Sigma \rho\right) &= \exp\left(\int_\Sigma \rho'\right) \cdot \exp\left(\int_\Sigma -\mathrm{curv}(N)\right)\\ &= \exp\left(\int_\Sigma \rho'\right) \cdot (\mathrm{hol}_N(\partial\Sigma))^{-1}.\end{aligned}$$

This change has to be compensated by the second factor. Indeed, the second trivialization determines another vector bundle $E' \to \partial\Sigma$ of curvature $\phi^*\omega - \rho'$. From the construction of these bundles in the proof of Lemma 1 it becomes clear that they satisfy

$$E \cong N \otimes E'.$$

This means for the second factor

$$\mathrm{tr}\,(\mathrm{hol}_E(\partial\Sigma)) = \mathrm{tr}\,(\mathrm{hol}_{E'\otimes N}(\partial\Sigma)) = \mathrm{hol}_N(\partial\Sigma) \cdot \mathrm{tr}\,(\mathrm{hol}_{E'}(\partial\Sigma))$$

Thus we have shown that the holonomy defined in Definition 7 does not depend on the choice of the trivialization.

With this definition of holonomy around a surface with boundary, we have to check that it reproduces the amplitude given above in terms of the 3-form H on M and the 2-form ω on Q.

Proposition 4. *Let $\mathcal{G}$ be a bundle gerbe over M with curvature H and let $(Q, \mathcal{E})$ be a D-brane with curvature ω. For an oriented three-dimensional manifold B, whose boundary decomposes in two parts Σ and D^2, and a map $\Phi\colon B \to M$ with $\Phi(D^2) \subset Q$ we find*

$$\mathrm{hol}_{\mathcal{G},\mathcal{E}}(\Phi|_\Sigma) = \exp\left(\int_B \Phi^*H - \int_{D^2} \Phi^*\omega\right).$$

This proposition can be proven similarly to Proposition 3. It shows that the theory of bundle gerbes and bundle gerbe modules is helpful for open string theories and allows for a proper definition of the amplitudes of worldsheets with boundary in the Wess–Zumino–Witten model – even in topologically non-trivial situations.

As a last point, we have a look on the holonomy in terms of local data. Let (g, A, B) be local data of the bundle gerbe $\mathcal{G}$ with respect to an open cover $\{V_i\}_{i\in I}$ of M. Similarly to the local data of an isomorphism it is possible to extract local data of a morphism, in particular of a bundle gerbe module [Gaw05]. It consists of functions

$$G_{ij} : V_i \cap V_j \to \mathrm{U}(n)$$

on double intersections, where n is the rank of the vector bundle, which is part of the structure of a morphism. It consists further of $\mathfrak{u}(n)$-valued 1-forms P_i on each open set V_i. Like an isomorphism, a morphism

$$\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$$

relates the local data (g, A, B) of the bundle gerbe $\mathcal{G}$ to that of the trivial bundle gerbe $\mathcal{I}_\omega$ by the following equations:

$$\begin{aligned} G_{ij} \cdot G_{jk} \cdot G_{ik}^{-1} \cdot g_{ijk} &= 1, \\ P_j - \mathrm{Ad}_{G_{ij}}(P_i) - G_{ij}^{-1}\mathrm{d}G_{ij} + A_{ij} &= 0, \\ \mathrm{d}P_i + B_i &= \omega. \end{aligned}$$

In these equations, we identify $\mathrm{U}(1)$ with the diagonal subgroup of $\mathrm{U}(n)$, and correspondingly the Lie algebra of $\mathrm{U}(1) - \mathbb{R}$ – with a subalgebra of $\mathfrak{u}(n)$. Notice that if the local data of the bundle gerbe is trivial, i.e. $(g, A, B) = (1, 0, 0)$, the three equations would be the usual cocycle conditions for a $\mathrm{U}(n)$-vector bundle with connection of curvature ω. With non-trivial local data, these cocycle conditions become twisted – for this reason, gerbe modules are also known as twisted vector bundles.

We use a triangulation of Σ which is subordinated to the open cover of M just like we did to derive the formula (1) of the local expression of the holonomy around a closed surface. Splitting of the integral of the 2-form ρ over Σ, which build the first factor of $\mathrm{hol}_{\mathcal{G},\mathcal{E}}(\phi)$, leads exactly to formula (1). It has to be amended by the local expression for the second factor, which is the holonomy of the vector bundle E around the boundary of Σ. The ladder is similar to the local expression for the holonomy of a line bundle around a loop, namely

$$\mathrm{tr}\,(\mathrm{hol}_E(\partial\Sigma)) = \mathrm{tr}\,\mathrm{P}\Big\{ \prod_{e\in\Delta\cap\partial\Sigma} \exp\Big(\int_e \phi^* P_{i(e)} \Big) \cdot \prod_{v\in\partial e} G_{i(e),i(v)}^{\epsilon(e,v)} \Big\}.$$

The only difference is that the terms now live in the non-abelian group $\mathrm{U}(n)$ and have to be ordered with respect to the induced orientation on $\partial\Sigma$, which is indicated by the path-ordering operator P. The cyclic property of the trace assures that it does not depend on a specific point from where one starts multiplying terms. The complete picture where the local data is used is shown in Figure 4.

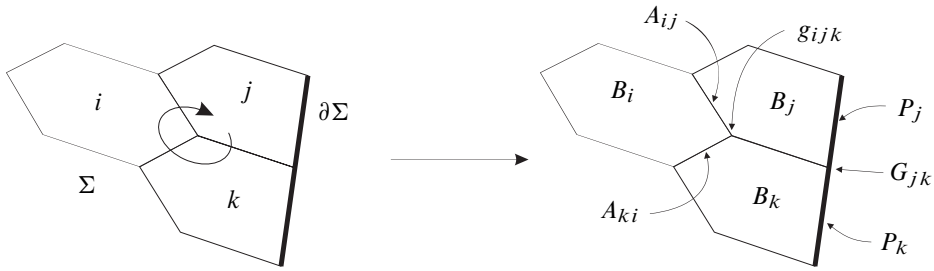

Figure 4. The triangulation of the surface Σ is decorated by the local data as in Figure 2, completed by the local 1-forms P_i and the functions G_{ij} coming from the bundle gerbe module, which are placed on the boundary $\partial\Sigma$.

8 Unoriented closed surfaces

From the preceding section we learn two things. To incorporate boundaries we first had to choose structure – the D-branes – additional to the given bundle gerbe over M. Secondly, we restricted the possible maps $\phi\colon \Sigma \to M$ to those which respect this additional structure.

To incorporate unoriented surfaces Σ we also have to do these two steps. The additional structure has been defined in [SSW05] and called a Jandl structure on the bundle gerbe $\mathcal{G}$. It consists of an involution of M – i.e. a diffeomorphism $k\colon M \to M$ with $k \circ k = \mathrm{id}_M$ – and of a certain isomorphism between the pullback bundle gerbe $k^*\mathcal{G}$ and the dual bundle gerbe $\mathcal{G}^*$. In the second step, we have to specify the space of maps we want to consider. As we will see, they have to be compatible in a certain sense with the involution k.

For any (unoriented) closed surface Σ there is an oriented two-fold covering $\mathrm{pr}\colon \widehat{\Sigma} \to \Sigma$. It is unique up to orientation-preserving diffeomorphisms and it is connected if and only if Σ is not orientable. It has a canonical, orientation-reversing involution σ, which permutes the sheets and preserves the fibres. We call this two-fold covering the orientation covering of Σ.

Given a closed surface Σ, we consider maps $\hat{\phi}\colon \widehat{\Sigma} \to M$ starting from the orientation covering $\widehat{\Sigma}$, which are equivariant with respect to the two involutions on $\widehat{\Sigma}$ and M, i.e. the diagram

$$\begin{array}{ccc} \widehat{\Sigma} & \xrightarrow{\hat{\phi}} & M \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle k} \\ \widehat{\Sigma} & \xrightarrow[\hat{\phi}]{} & M \end{array}$$

has to be commutative.

Before we come to the details of a Jandl structure, let us briefly describe the idea behind its definition. If we pullback the bundle gerbe $\mathcal{G}$ along an equivariant map $\hat{\phi}$, we obtain a bundle gerbe on the orientation covering $\widehat{\Sigma}$, and could in principle compute the holonomy of this bundle gerbe around $\widehat{\Sigma}$. If we do so, we would get the square of what we originally wanted, since each point of Σ is twice covered. To reveal this, we are going to establish a descent procedure – not for the bundle gerbe but rather for its holonomy. In a first attempt we assume that there is an isomorphism

$$\mathcal{A}\colon k^*\mathcal{G} \to \mathcal{G}^*.$$

Due to the equivariance of $\hat{\phi}$, it induces an isomorphism between $\sigma^*\hat{\phi}^*\mathcal{G}$ and $\hat{\phi}^*\mathcal{G}^*$. This isomorphism says: changing the sheet by σ goes hand in hand with replacing the bundle gerbe with its dual. With the precise definition of the dual bundle gerbe $\mathcal{G}^*$ it becomes easy to see from Definition 7 that both processes – changing the orientation and taking the dual – give each a sign in the holonomy. So the isomorphism $\mathcal{A}$ implies that the holonomy of $\hat{\phi}^*\mathcal{G}$ takes locally the same value on both sheets, which is the initial condition for a descent from $\widehat{\Sigma}$ to Σ.

A detailed calculation shows that it is not enough to choose any isomorphism like above as additional structure. It shows that the isomorphism $\mathcal{A}$ itself has to be equivariant in a certain sense. To give a complete definition of the Jandl structure it is convenient to use the 2-categorial language.

Definition 8. A Jandl structure $\mathcal{J}$ on a bundle gerbe $\mathcal{G}$ over M is an involution k of M together with an isomorphism

$$\mathcal{A}\colon k^*\mathcal{G} \to \mathcal{G}^*$$

and a 2-morphism

$$\varphi : k^*\mathcal{A} \Rightarrow \mathcal{A}^*$$

which satisfies the equivariance condition

$$k^*\varphi = \varphi^*.$$

Notice the remarkable symmetry of the three lines. Of course we have not developed the full theory of pullbacks and duals of morphisms and 2-morphisms in this chapter (although they turn out to be quite canonical constructions). For instance, $k^*\mathcal{A}$ as well as $\mathcal{A}^*$ are both isomorphisms from $\mathcal{G}$ to $k^*\mathcal{G}^*$, so it makes sense to have a 2-morphism φ between them. Similarly, both $k^*\varphi$ and φ^* are 2-morphisms from $\mathcal{A}$ to $k^*\mathcal{A}^*$, so it makes sense to demand that they are equal.

To give an impression of the details of a Jandl structure, recall that an isomorphism such as $\mathcal{A}$ consists of a line bundle A over the space Z which is build up from the two coverings of the bundle gerbes $k^*\mathcal{G}$ and $\mathcal{G}^*$. In this particular situation, there is a canonical lift $\tilde{k}$ of the involution k into the space Z, and it is in fact easy to work out that the 2-morphism φ defines a $\tilde{k}$-equivariant structure on the line bundle A.

Summarizing, a Jandl structure $\mathcal{J}$ on $\mathcal{G}$ is an isomorphism

$$\mathcal{A}\colon k^*\mathcal{G} \to \mathcal{G}^*$$

whose line bundle A is equivariant with respect to the involution $\tilde{k}$ on Z.

As always when there is an additional structure to choose, one would like to know how many inequivalent choices there are. To say what equivalent Jandl structures are amounts to define a morphism between two of them.

Definition 9. A morphism $\beta\colon \mathcal{J} \to \mathcal{J}'$ between Jandl structures $\mathcal{J} = (k, \mathcal{A}, \varphi)$ and $\mathcal{J}' = (k, \mathcal{A}', \varphi')$ on the same bundle gerbe $\mathcal{G}$ over M with the same involution k is a 2-morphism

$$\beta : \mathcal{A} \Rightarrow \mathcal{A}'$$

which commutes with φ and φ' in the sense that the diagram

$$\begin{array}{ccc} k^*\mathcal{A} & \xRightarrow{\varphi} & \mathcal{A}^* \\ {\scriptstyle k^*\beta}\Downarrow & & \Downarrow{\scriptstyle \beta^*} \\ k^*\mathcal{A}' & \xRightarrow[\varphi']{} & \mathcal{A}'^* \end{array}$$

is commutative.

Recall that 2-morphisms are certain isomorphisms of vector bundles, so that the diagram is in fact a diagram of isomorphisms of line bundles over Z. The definition of a morphism between Jandl structures allows us to consider the set of equivalence classes of Jandl structures on $\mathcal{G}$ with involution k. Recall that by Proposition 2 the set of equivalence classes of isomorphisms is a torsor over the group of isomorphism classes of flat line bundles over M. The following theorem is a refinement for Jandl structures.

Theorem 2 ([SSW05]). *The set of equivalence classes of Jandl structures on a bundle gerbe $\mathcal{G}$ with involution k is a torsor over the group of isomorphism classes of flat k-equivariant line bundles over M.*

This theorem can be used to determine the number of inequivalent choices of a Jandl structure, for instance for manifolds which have been considered before in unoriented string theories – so called orientifolds, e.g. [BPS92], [BCW01], [HSS02]. The following known results are reproduced.

- For $M = \mathrm{SU}(2)$ with involution $k(g) := zg^{-1}$ for any element z in the center of G, and for a bundle gerbe $\mathcal{G}$ over M whose curvature is an integral multiple of the 3-form H introduced in the introduction, there are two inequivalent Jandl structures on $\mathcal{G}$ with respect to k.
- For $M = \mathrm{SO}(3)$ with involution $k(g) = g^{-1}$ and a bundle gerbe $\mathcal{G}$ which is isomorphic to $k^*\mathcal{G}^*$, there are four inequivalent Jandl structures on $\mathcal{G}$ with respect to k.

- For the 2-torus $T = S^1 \times S^1$ with involution $k = \mathrm{id}_T$ and for a bundle gerbe $\mathcal{G}$ which admits Jandl structures, their equivalence classes are in bijection to the group $\mathbb{Z}_2 \times \mathrm{U}(1) \times \mathrm{U}(1)$.

Let us now use a Jandl structure on a bundle gerbe $\mathcal{G}$. We are going to pursuit the idea of descent holonomy. The existence of the isomorphism $\mathcal{A}$ assures that the holonomy locally coincides on both sheets – now we have to make local selections of one of them. This is exactly what a choice of a fundamental domain of Σ in $\hat{\Sigma}$ does. It can be constructed locally as shown in Figure 5 as a submanifold F of $\hat{\Sigma}$ with

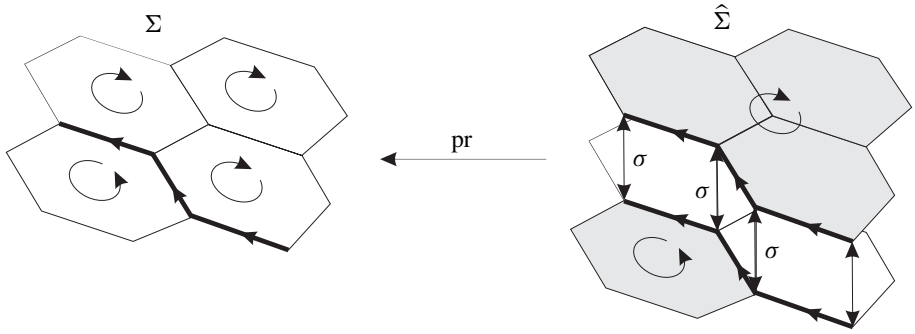

Figure 5. On the left hand side we show a dual triangulation, i.e. every vertex has three edges. According to a choice of local orientations for every face of the triangulation – which is always possible – select one of the two preimages of this face under the covering $\mathrm{pr}\colon \hat{\Sigma} \to \Sigma$. This defines a fundamental domain as the grey-shaded surface on the right hand side.

(piecewise smooth) boundary. A key observation, which can be heuristically seen from Figure 5, is that the involution σ restricts to an orientation-*preserving* involution on $\partial F \subset \hat{\Sigma}$. Accordingly, the quotient $\overline{\partial F}$ is an oriented closed submanifold of Σ.

Remember the following two situations: two trivializations of the same bundle gerbe define a line bundle over M with a certain curvature (Lemma 1). A trivialization together with a D-brane gives a vector bundle with a certain curvature. A similar situation appears for a trivialization together with a Jandl structure.

Lemma 2 ([SSW05]). *A trivialization $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\rho$ and a Jandl structure $\mathcal{J}$ on $\mathcal{G}$ with involution k determine a k-equivariant line bundle $R \to M$ with curvature $k^*\rho + \rho$.*

Now we are ready to put the pieces together:

Definition 10 ([SSW05]). Let $\mathcal{G}$ be a bundle gerbe over M with Jandl structure $\mathcal{J}$ with involution k. Let Σ be a closed surface with orientation covering $\hat{\Sigma}$ and let $\hat{\phi}\colon \hat{\Sigma} \to M$ be an equivariant map. The pullback along $\hat{\phi}$ gives a bundle gerbe $\phi^*\mathcal{G}$ over $\hat{\Sigma}$ with Jandl structure $\hat{\phi}^*\mathcal{J}$ with involution σ. A choice of a trivialization

$$\mathcal{T} : \hat{\phi}^*\mathcal{G} \to \mathcal{I}_\rho$$

determines in combination with the Jandl structure a σ-equivariant line bundle $R \to \hat{\Sigma}$. In turn, this equivariant line bundle determines a quotient line bundle $\overline{R} \to \Sigma$. Let F be a fundamental domain. Then we define

$$\mathrm{hol}_{\mathcal{G},\mathcal{E},\mathcal{J}}(\hat{\phi}) := \exp\left(\int_F \rho\right) \cdot \mathrm{hol}_{\overline{R}}(\overline{\partial F})$$

to be the holonomy of the bundle gerbe $\mathcal{G}$ with Jandl structure $\mathcal{J}$ around $\hat{\phi}$.

In [SSW05] we show that this definition depends neither on the choice of the trivialization nor on the choice of the fundamental domain.

To close, let us remark that also a Jandl structure can be understood in terms of local data. Recall that an isomorphism $\mathcal{A}$ relates local data of bundle gerbes, here

$$-(g, A, B) = k^*(g, A, B) + \mathrm{D}(t, W).$$

It is also possible to extract a function $j_i : V_i \to \mathrm{U}(1)$ from the 2-morphism φ, which relates the local data (t, W) of the isomorphism $\mathcal{A}$ to their pullback,

$$(t, W) = k^*(t, W) + \mathrm{D}(j).$$

Finally, the condition on the 2-morphism φ leads to

$$j^{-1} = k^* j.$$

In [SSW05] an expression for the holonomy $\mathrm{hol}_{\mathcal{G},\mathcal{E},\mathcal{J}}(\hat{\phi})$ is derived in terms of the local data of the bundle gerbe and of the Jandl structure, analogous to (1). We do not give the full expression here, but indicate how the local data should be placed on a triangulated surface in Figure 6.

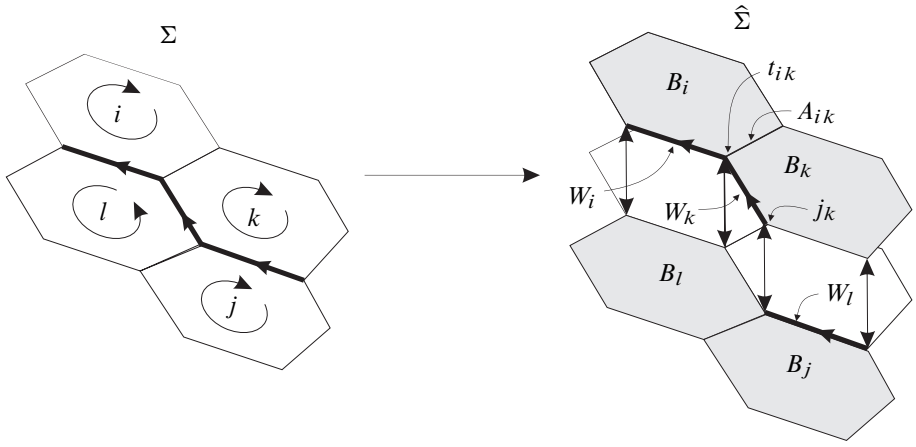

Figure 6. The oriented triangulation of the closed surface Σ determines a fundamental domain, which is decorated by the local data of the bundle gerbe in its interior and by the local data of the Jandl structure along the orientation reversing edges.

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Part F

Symmetric spaces and spaces of constant curvature

Chapter 20

Classification results for pseudo-Riemannian symmetric spaces

Ines Kath

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1 Introduction

The aim of this chapter is to give an introduction to the classification problem for pseudo-Riemannian symmetric spaces. We will present the main ideas and an overview on several attempts to tackle the problem rather than the various classification results in full detail. Symmetric spaces constitute an important class of (pseudo-)Riemannian manifolds. On the one hand they are sufficiently complicated to serve as examples for many geometric phenomena and on the other hand they are simple enough to calculate various geometric quantities explicitly. The study of symmetric spaces began with the work of É. Cartan who succeeded to classify Riemannian symmetric spaces completely. Here we will be interested mainly in pseudo-Riemannian symmetric spaces, i.e., in the case where the metric is non-degenerate but not necessarily definite. In this case it is impossible to give a full classification in the sense of a list. Therefore the aim is to find at least a suitable description of the structure of pseudo-Riemannian symmetric spaces that allows a systematic study of these manifolds.

The classification problem for symmetric spaces is closely related to the holonomy problem. Holonomy groups were also introduced by É. Cartan. Besides some low-dimensional manifolds the first large class of manifolds for which the holonomy group could be computed was that of symmetric spaces. By de Rham's Theorem the study of holonomy groups and representations of pseudo-Riemannian manifolds is reduced to the case where the manifold is (locally) indecomposable, i.e., where it is not locally the non-trivial product of two pseudo-Riemannian manifolds. The

holonomy group of a pseudo-Riemannian manifold (M, g) has a natural orthogonal representation on the tangent space at a point of M called holonomy representation. (M, g) is indecomposable if this representation does not have proper invariant subspaces which are non-degenerate with respect to the metric. The easiest case of an indecomposable pseudo-Riemannian manifold is that of an irreducible one. These are manifolds whose holonomy representation is irreducible. Lie groups which can appear as holonomy group of an irreducible pseudo-Riemannian manifold are classified. For locally symmetric manifolds this classification follows from Berger's classification of irreducible symmetric spaces, for the non-locally symmetric case it is known as Berger's list. In the Riemannian case each indecomposable manifold is irreducible. Of course, this is not true in the general pseudo-Riemannian case since here we can have degenerate invariant subspaces. While Berger's list gives an answer to the question which groups can be holonomy groups of Riemannian manifolds this question is widely open for pseudo-Riemannian manifolds. Holonomy groups of Lorentzian manifolds were classified by Th. Leistner. However, except of some partial results there is no such classification known if the index of the metric is at least two. And even partial results are involved and hard to find. We will see that all this is reflected in the study of pseudo-Riemannian symmetric spaces. Irreducible symmetric spaces are easy to handle whereas the general case of indecomposable ones is very complicated.

2 Pseudo-Riemannian symmetric spaces and symmetric triples

A connected pseudo-Riemannian manifold (M, g) is called *pseudo-Riemannian symmetric space* if for all $x \in M$ there is an involutive isometry θ_x of (M, g) (called reflection) for which x is an isolated fixed point. This means that for any $x \in M$ the geodesic reflection at x extends to a globally defined isometry. The covariant derivative ∇R of the curvature tensor R of a pseudo-Riemannian symmetric space must vanish. In other words, R is parallel. Indeed, ∇R is invariant with respect to all isometries of (M, g). In particular, we have $\theta_x^*(\nabla R)_x = (\nabla R)_x$ for all $x \in M$. On the other hand, $\theta_x^*(\nabla R)_x = -(\nabla R)_x$, since ∇R depends on an odd number of tangent vectors and $d\theta_x = -\mathrm{Id}\colon T_xM \to T_xM$. Conversely one can show, that every complete and simply connected pseudo-Riemannian manifold with parallel curvature tensor is a symmetric space.

Before we will start with the general pseudo-Riemannian case let us say a few words about the above mentioned classification of simply connected Riemannian symmetric spaces due to É. Cartan. Each of these symmetric spaces decomposes into so-called irreducible ones, i.e., into Riemannian symmetric spaces that have an irreducible holonomy representation. Hence it suffices to classify irreducible symmetric spaces. Obviously, a simply connected irreducible Riemannian symmetric space is flat if and only if it is isometric to $(\mathbb{R}, dt)$. Now assume that (M, g) is a non-flat irreducible Riemannian symmetric space. Then we have four cases:

(I) $M = G/H$, where G is a compact simple simply connected Lie group and H is the unity component of the group of fixed points of some non-trivial involution of G,

(II) M itself is a compact simple simply connected Lie group endowed with a bi-invariant metric, i.e., a metric that is invariant with respect to all left and right translations,

(III) $M = G/H$, where G is a non-compact simple Lie group with trivial centre, H is a maximal compact subgroup of G and, moreover, the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra of G is simple as a complex Lie algebra,

(IV) as in (III), but with non-simple $\mathfrak{g}^{\mathbb{C}}$ (in which case G is a simple complex Lie group).

Now the classification of Riemannian symmetric spaces essentially reduces to the classification of simple Lie groups, which is also due to Cartan. In each of the cases (I) and (III) one obtains seven infinite series and twelve exceptional spaces and in each of the cases (II) and (IV) four infinite series and five exceptional spaces.

All these Riemannian symmetric spaces have pseudo-Riemannian analoga, e.g., instead of the Riemannian sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ we can consider pseudo-Riemannian spheres $S^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x\rangle = 1\} = \mathrm{SO}(p, q+1)/\mathrm{SO}(p, q)$ and pseudo-Riemannian hyperbolic spaces $H^{p,q} = \{x \in \mathbb{R}^{p+1,q} \mid \langle x, x\rangle = -1\} = \mathrm{SO}(p+1, q)/\mathrm{SO}(p, q)$. Similarly, we have pseudo-Riemannian Grassmannians etc. However, there are much more pseudo-Riemannian symmetric spaces than these analoga of Riemannian ones. Let us consider for example the following series of symmetric spaces constructed by Cahen and Wallach [14]. Take $M = \mathbb{R}^{n+2}$ with co-ordinates $(z, x_1, \ldots, x_n, l)$ and define a metric on M by

$$g_{\lambda_1,\ldots,\lambda_n} = \sum_{i=1}^{n} dx_i^2 + 2dz\,dl + \sum_{i=1}^{n} \lambda_i x_i^2 dl^2,$$

where $\lambda_1, \ldots, \lambda_n$ are real parameters. Then one can prove that the manifolds $M^{1,n+1}_{\lambda_1,\ldots,\lambda_n}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, are Lorentzian symmetric spaces. We will call them Cahen–Wallach spaces. We do not want to give the reflections explicitly here since later on we will learn about an easier way how to see that these manifolds are symmetric spaces. Note, that some of these spaces are isometric. If we assume that the parameters are ordered by magnitude, then $M^{1,n+1}_{\lambda_1,\ldots,\lambda_n}$ and $M^{1,n+1}_{\lambda'_1,\ldots,\lambda'_n}$ are isometric if and only if $(\lambda_1, \ldots, \lambda_n) = c(\lambda'_1, \ldots, \lambda'_n)$ for some real number $c > 0$.

As another class of examples of pseudo-Riemannian symmetric spaces let us consider (finite-dimensional) Lie groups G with a bi-invariant metric. The reflection θ_e at the unity $e \in G$ is just the inversion and we obtain any other reflection θ_g by conjugation of θ_e by the left translation L_g. Any simply connected Lie group G with a bi-invariant metric is uniquely determined by its Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-invariant (non-degenerate) inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. We will call such a pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ *metric Lie algebra*. Of course, any finite-dimensional abelian Lie algebra with an arbitrary inner

product is a metric Lie algebra. Moreover, each semi-simple Lie algebra $\mathfrak{g}$ together with its Killing form is a metric Lie algebra. However, there are much more examples than these. E.g., take an arbitrary finite-dimensional Lie algebra $\mathfrak{l}$ and define an inner product on $\mathfrak{l} \ltimes_{\mathrm{ad}^*} \mathfrak{l}^*$ by the dual pairing. More exactly, put

$$\langle L_1 + Z_1, L_2 + Z_2 \rangle := Z_1(L_2) + Z_2(L_2)$$

for $L_1, L_2 \in \mathfrak{l}$ and $Z_1, Z_2 \in \mathfrak{l}^*$. Then $(\mathfrak{l} \ltimes_{\mathrm{ad}^*} \mathfrak{l}^*, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra. In particular, if L is a Lie group, then we can consider its cotangent bundle $T^*L = L \ltimes_{\mathrm{Ad}^*} \mathfrak{l}^*$ as a Lie group with bi-invariant metric. Generalising this example Medina and Revoy [31] considered the semi-direct sum $\mathfrak{g} \,{}_{\pi}\!\rtimes \mathfrak{l}$ of a metric Lie algebra by an arbitrary Lie algebra $\mathfrak{l}$, where π is a homomorphism from $\mathfrak{l}$ into the Lie algebra of antisymmetric derivations of $\mathfrak{g}$. Then they constructed metric Lie algebras as suitable extensions of $\mathfrak{g} \,{}_{\pi}\!\rtimes \mathfrak{l}$ by the abelian Lie algebra $\mathfrak{l}^*$. The inner product on these extensions is given by the inner product on $\mathfrak{g}$ and the dual pairing between $\mathfrak{l}$ and $\mathfrak{l}^*$. Such a metric Lie algebra is called double extension of $\mathfrak{g}$ by $\mathfrak{l}$. If (p, q) is the signature of (the inner product on) $\mathfrak{g}$, then the arising metric Lie algebra has signature $(p + \dim \mathfrak{l}, q + \dim \mathfrak{l})$. Medina and Revoy [31] could prove that one can inductively produce all metric Lie algebras from simple and one-dimensional ones by taking direct sums and double extensions. More exactly, let $\mathfrak{g}$ be an indecomposable metric Lie algebra, i.e., a metric Lie algebra that is not the direct sum of two non-trivial metric Lie algebras. Then either $\mathfrak{g}$ is simple or $\mathfrak{g}$ is one-dimensional or $\mathfrak{g}$ is a double extension of a metric Lie algebra $\tilde{\mathfrak{g}}$ by a one-dimensional or a simple Lie algebra $\mathfrak{l}$. Using this it is not hard to see that any indecomposable non-simple Lorentzian metric Lie algebra of dimension > 1 is the double extension of an abelian Euclidean metric Lie algebra by a one-dimensional Lie algebra. This allows the classification of Lorentzian metric Lie algebras up to isomorphisms [30]. However, for the classification of metric Lie algebras of higher index (of the metric) the method of double extensions is less suitable since, in general, a given metric Lie algebra can be obtained in many different ways by a series of subsequent double extensions. For another construction of metric Lie algebras, which avoids this problem see Section 3.

Already the examples described above show that there is a huge variety of pseudo-Riemannian symmetric spaces. Is it possible to get a grasp of the structure of these spaces? Is it possible to classify them? Here we will study only simply connected symmetric spaces. This restriction is not essential but it makes the presentation easier. For the non-simply connected case see e.g. [28]. A simply connected pseudo-Riemannian symmetric space is called indecomposable if it is not the direct product of two pseudo-Riemannian symmetric spaces of dimension > 0. As far as the classification problem is concerned we may obviously concentrate on indecomposable pseudo-Riemannian symmetric spaces. Note, however, that in the pseudo-Riemannian case most of indecomposable symmetric spaces are not irreducible since the holonomy representation can have degenerate invariant subspaces.

As in the case of Lie groups with bi-invariant metric it is much easier to study the infinitesimal objects associated with symmetric spaces than the symmetric spaces

themselves. Because this is discussed also in the following chapter in this book ([1]) we will give only a very short description of the correspondence between pseudo-Riemannian symmetric spaces and suitable algebraic objects, which we will call symmetric triples. Since involutions on Lie algebras will play a role in this description we will introduce the following notation. If θ is an involution on a Lie algebra $\mathfrak{g}$, we will denote by $\mathfrak{g}_+$ and $\mathfrak{g}_-$ the eigenspaces with eigenvalues 1 and -1, respectively. A *symmetric triple* $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ consists of a metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ and an isometric involution θ on $\mathfrak{g}$ whose eigenspaces satisfy $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$.

Proposition 2.1. *Isometry classes of simply-connected pseudo-Riemannian symmetric spaces correspond bijectively to isomorphism classes of symmetric triples.*

The correspondence is constructed as follows. Let (M, g) be a simply-connected pseudo-Riemannian symmetric space. Then we can associate with (M, g) a distinguished subgroup of the isometry group $\mathrm{Iso}(M, g)$ of (M, g), namely the so-called transvection group

$$G = \langle \theta_x \circ \theta_y \mid x, y \in M \rangle.$$

If G is semisimple, then its Lie algebra $\mathfrak{g}$ equals the Lie algebra of $\mathrm{Iso}(M, g)$. However, in general it is smaller. The transvection group G acts transitively on M. In particular, M is a homogeneous space and can be identified with G/G_+, where G_+ is the stabiliser of a point $x_0 \in M$. Then θ_{x_0} acts by conjugation on G. Let us denote this action as well as the induced involution on the Lie algebra $\mathfrak{g}$ of G by θ. Then it is easy to see, that $\mathfrak{g}_+$ is the Lie algebra of G_+ and $\mathfrak{g}_-$ can be identified with $T_{x_0}M$. Moreover, one can prove that $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$ holds. The metric g on M induces a scalar product $\langle\cdot,\cdot\rangle_-$ on $\mathfrak{g}_- \cong T_{x_0}M$, which is $\mathfrak{g}_+$-invariant. Cahen and Parker [13] proved that $\langle\cdot,\cdot\rangle_-$ can be uniquely extended to a scalar product on $\mathfrak{g}$ that is invariant with respect to $\mathfrak{g}$ and to θ. Hence we obtain a symmetric triple $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$.

The subgroup G_+ of G is the holonomy group of (M, g). The holonomy representation equals the adjoint representation of G_+ on $\mathfrak{g}_-$. The pseudo-Riemannian symmetric space (M, g) is indecomposable if and only if this representation does not contain a proper non-degenerate invariant subspace. Moreover, (M, g) is indecomposable if and only if the associated symmetric triple is indecomposable, i.e., is not the direct sum of two non-trivial symmetric triples.

By Proposition 2.1 the classification problem for pseudo-Riemannian symmetric spaces is equivalent to the classification problem for symmetric triples. We will call a symmetric triple $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ and the associated symmetric space semisimple, reductive etc., if $\mathfrak{g}$ is semisimple, reductive etc. As explained above Riemannian symmetric spaces were classified by Cartan. We have seen that symmetric triples that are associated with these spaces are necessarily reductive and that their classification relies on the classification of semisimple Lie algebras. This result was generalised by Berger [9] to the case of arbitrary reductive symmetric triples. Similarly to the case of Riemannian symmetric spaces Berger obtained a complete list of reductive pseudo-Riemannian symmetric spaces.

Contrary to that the non-reductive case appears to be much more involved. All of what we know of non-reductive pseudo-Riemannian symmetric spaces suggests that it is impossible to get a complete classification. However, there are results for metrics with small index. Lorentzian symmetric spaces were classified by Cahen and Wallach. In the language of double extensions introduced by Medina and Revoy (see above) we can formulate the result as follows. Any indecomposable symmetric triple $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ that is associated with such a space the following holds. Either $\mathfrak{g}$ is one-dimensional or semisimple or $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is the double extension of a pseudo-Euclidean abelian Lie algebra by $\mathbb{R}$. In the semisimple case only symmetric spaces of constant curvature can have Lorentzian signature. In the third case we find the associated symmetric space by integration. This gives the following result.

Theorem 2.2 (Cahen–Wallach [14]). *Any indecomposable simply connected Lorentzian symmetric space is isometric to exactly one of the following manifolds*

(1) $(\mathbb{R}, -dt^2)$,

(2) *a complete simply-connected Lorentzian manifold with non-vanishing constant curvature,*

(3) *one of the above defined Cahen–Wallach spaces* $M^{1,n+1}_{\lambda_1,\dots,\lambda_n}$ *with* $\lambda_1,\dots,\lambda_n \in \mathbb{R}\setminus\{0\}$, *where* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ *and* $\lambda_1 = \pm 1$.

Pseudo-Riemannian symmetric spaces with a metric of index 2 were studied by Cahen and Parker [12], [13]. Unfortunately, in their classification result an essential case is missing, namely, speaking in the language of [14], that of symmetric triples with maximal centre that are neither nilpotent nor 'at least nilpotent'. This was noticed by Th. Neukirchner, who gave a revised classification in [32]. Because of the lack of a general theory he had to do a lot of calculations 'by hand', which had caused a few errors. We will discuss this further in the next section.

All these attempts of getting lists for pseudo-Riemannian symmetric spaces show that even for small index of the metric this is a rather involved task. It seems to be a more realistic aim to find a 'good description' of the set of all isometry/isomorphism classes of symmetric spaces/triples. To reach this aim it is necessary to develop a structure theory for symmetric triples.

The first result in this direction is due to M. Cahen and M. Parker. In [13] they study Levi decompositions of indecomposable symmetric triples. In particular, they describe the structure of the solvable radical as representation of the Levi factor.

3 Quadratic cohomology and quadratic extensions

In this section we will present an approach to the classification problem for general pseudo-Riemannian symmetric spaces. We will give a description of the set of all isomorphism classes of symmetric triples by an algebraic object that is constructed

from cohomology sets. We will call this description classification scheme. It will not be a classification in the true sense since, e.g., in general the appearing cohomology sets cannot be calculated explicitly. However, this classification scheme allows a systematic construction of symmetric triples and it gives a 'recipe' how to get an explicit classification under suitable additional conditions, e.g., for small index of the inner product. This approach uses the following construction due to Bérard-Bergery, which associates an isotropic ideal with each metric Lie algebra in a canonical way. This ideal will allow to consider any symmetric triple as the result of a certain extension procedure.

Let $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ be a symmetric triple. There is a chain of θ-invariant ideals

$$\mathfrak{g} = R_0(\mathfrak{g}) \supset R_1(\mathfrak{g}) \supset R_2(\mathfrak{g}) \supset \cdots \supset R_k(\mathfrak{g}) = 0$$

defined by the condition that $R_j(\mathfrak{g})$ is the smallest ideal of $\mathfrak{g}$ contained in $R_{j-1}(\mathfrak{g})$ such that the $\mathfrak{g}$-module $R_{j-1}(\mathfrak{g})/R_j(\mathfrak{g})$ is semisimple. The ideal

$$\mathfrak{i}(\mathfrak{g}) := \sum_{j=1}^{k-1} R_j(\mathfrak{g}) \cap R_j(\mathfrak{g})^{\perp} \subset \mathfrak{g}$$

is called *canonical isotropic ideal* of $\mathfrak{g}$.

Proposition 3.1 ([6], [7]; [25], Lemma 3.4). *If $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ is a symmetric triple, then $\mathfrak{i}(\mathfrak{g})$ is a θ-invariant (totally) isotropic ideal and the $\mathfrak{g}$-module $\mathfrak{i}(\mathfrak{g})^{\perp}/\mathfrak{i}(\mathfrak{g})$ is semisimple. If $\mathfrak{g}$ does not contain simple ideals, then the Lie algebra $\mathfrak{i}(\mathfrak{g})^{\perp}/\mathfrak{i}(\mathfrak{g})$ is abelian.*

Here we may assume that our symmetric triple $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ does not contain simple ideals. Indeed, any symmetric triple that contains simple ideals can be decomposed into the direct sum of a semisimple symmetric triple and one without simple ideals. Since semisimple symmetric Lie algebras are completely known we needn't to consider them here. Hence, here $\mathfrak{a} := \mathfrak{i}(\mathfrak{g})^{\perp}/\mathfrak{i}(\mathfrak{g})$ is always abelian. Obviously, the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ induces a (non-degenerate) inner product $\langle\cdot,\cdot\rangle_{\mathfrak{a}}$ on $\mathfrak{a}$ and θ induces an involution $\theta_{\mathfrak{a}}$ on $\mathfrak{a}$. Moreover, θ induces an involution $\theta_{\mathfrak{l}}$ on $\mathfrak{l} := \mathfrak{g}/\mathfrak{i}(\mathfrak{g})^{\perp}$. This means that the symmetric triple $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ is a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by $\mathfrak{a}$ in the following sense.

Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a finite-dimensional Lie algebra with involution. An *orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$-module* $(\rho, \mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$ (often abbreviated to $\mathfrak{a}$) consists of a (finite-dimensional) pseudo-Euclidean vector space $(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}})$, an isometric involution $\theta_{\mathfrak{a}}$ on $\mathfrak{a}$ and an orthogonal representation ρ of $\mathfrak{l}$ on $\mathfrak{a}$ satisfying $\rho(\theta_{\mathfrak{l}}(L)) = \theta_{\mathfrak{a}} \circ \rho(L) \circ \theta_{\mathfrak{a}}$.

Definition 3.2. Let $(\rho, \mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$ be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$-module. A quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by $\mathfrak{a}$ is given by a quadruple $(\mathfrak{g}, \mathfrak{i}, i, p)$, where

(i) $\mathfrak{g}$ is a metric Lie algebra with involution (see below),

(ii) $\mathfrak{i}$ is an isotropic θ-invariant ideal of $\mathfrak{g}$,

(iii) i and p are homomorphisms of Lie algebras with involution constituting an exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{l} \longrightarrow 0$$

that is consistent with the representation ρ of $\mathfrak{l}$ on $\mathfrak{a}$ and has the property that i is an isometry from $\mathfrak{a}$ to $\mathfrak{i}^{\perp}/\mathfrak{i}$.

Here a metric Lie algebra with involution is a triple $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$, where $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a metric Lie algebra and θ is an involutive isometry on $\mathfrak{g}$. This is less than to be a symmetric triple, which has to satisfy in addition $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$.

This kind of extension is called quadratic, since $\mathfrak{g}$ is the result of two subsequent (ordinary) extensions. First we extend $\mathfrak{l}$ by $\mathfrak{a}$ and then we extend the resulting Lie algebra by $\mathfrak{i} \cong \mathfrak{l}^*$. Similarly to ordinary extensions there is a natural notion of equivalence of quadratic extensions. And in the same way as one classifies (ordinary) extensions of a Lie algebra $\mathfrak{l}$ by an $\mathfrak{l}$-module $\mathfrak{a}$ by the second Lie algebra cohomology group $H^2(\mathfrak{l}, \mathfrak{a})$ we can describe quadratic extensions of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$-module by a suitable second cohomology set. This cohomology is a special case of non-linear cohomology sets introduced by Grishkov [21]. Let us describe this cohomology now. We start with the standard Lie algebra cochain complex $C^p(\mathfrak{l}, \mathfrak{a}) := \mathrm{Hom}(\bigwedge^p \mathfrak{l}, \mathfrak{a})$ with differential $d : C^p(\mathfrak{l}, \mathfrak{a}) \to C^{p+1}(\mathfrak{l}, \mathfrak{a})$ given by

$$\begin{aligned} d\tau(L_1, \ldots, L_{p+1}) = &\sum_{i=1}^{p+1} (-1)^{i-1} \rho(L_i)\tau(L_1, \ldots, \hat{L}_i, \ldots, L_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \tau([L_i, L_j], L_1, \ldots, \hat{L}_i, \ldots, \hat{L}_j, \ldots, L_{p+1}). \end{aligned}$$

In the special case where $\mathfrak{a}$ is the one-dimensional trivial representation of $\mathfrak{l}$ we denote the standard cochain complex also by $(C^*(\mathfrak{l}), d)$. The pair $(\theta_{\mathfrak{l}}, \theta_{\mathfrak{a}})$ defines an involution on $C^p(\mathfrak{l}, \mathfrak{a})$ by $\tau \mapsto \theta_{\mathfrak{a}} \circ \theta_{\mathfrak{l}}{}^* \tau$. As usual, let us denote the space of invariant elements in $C^p(\mathfrak{l}, \mathfrak{a})$ by $C^p(\mathfrak{l}, \mathfrak{a})_+$. We define a product $C^p(\mathfrak{l}, \mathfrak{a})_+ \times C^q(\mathfrak{l}, \mathfrak{a})_+ \to C^{p+q}(\mathfrak{l})_+$ by

$$\langle\cdot\wedge\cdot\rangle : C^p(\mathfrak{l}, \mathfrak{a})_+ \times C^q(\mathfrak{l}, \mathfrak{a})_+ \xrightarrow{\wedge} C^{p+q}(\mathfrak{l}, \mathfrak{a} \otimes \mathfrak{a})_+ \xrightarrow{\langle\cdot,\cdot\rangle_{\mathfrak{a}}} C^{p+q}(\mathfrak{l})_+.$$

Now we define the set of quadratic 1-cochains as

$$\mathcal{C}^1_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) = C^1(\mathfrak{l}, \mathfrak{a})_+ \oplus C^2(\mathfrak{l})_+.$$

This set is a group with group operation defined by

$$(\tau_1, \sigma_1) * (\tau_2, \sigma_2) = (\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \tfrac{1}{2}\langle\tau_1 \wedge \tau_2\rangle).$$

Let us consider the set

$$\mathcal{Z}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) = \{(\alpha, \gamma) \in C^2(\mathfrak{l}, \mathfrak{a})_+ \oplus C^3(\mathfrak{l})_+ \mid d\alpha = 0,\ d\gamma = \tfrac{1}{2}\langle\alpha \wedge \alpha\rangle\}$$

whose elements are called quadratic 2-cocycles. Then the group $\mathcal{C}^1_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ acts from the right on $\mathcal{Z}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ by

$$(\alpha,\gamma)(\tau,\sigma)=(\alpha+d\tau,\gamma+d\sigma+\langle(\alpha+\tfrac{1}{2}d\tau)\wedge\tau\rangle). \tag{1}$$

We define the second quadratic cohomology set as the orbit space

$$\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a}):=\mathcal{Z}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})/\mathcal{C}^1_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a}).$$

The equivalence class of $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ in $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ is denoted by $[\alpha,\gamma]$.

H.-Ch. Herbig suggested to relate this cohomology set to another construction, which is well known to people who work with differential graded algebras. Namely, the elements of $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ can be regarded as equivalence classes of so-called twisting elements. This can be seen in the following way. We define a differential graded algebra $A=\bigoplus_{p\in\mathbb{N}}A^p$ by

$$A^p=C^{p+1}(\mathfrak{l},\mathfrak{a})_+\oplus C^{p+2}(\mathfrak{l})_+$$

with differential $d\colon A^p\to A^{p+1}$, $d(\sigma,\tau)=(d\sigma,d\tau)$ and product

$$(\sigma_1,\tau_1)\cdot(\sigma_2,\tau_2)=(-1)^q(0,\tfrac{1}{2}\langle\sigma_1\wedge\sigma_2\rangle)$$

for $(\sigma_1,\tau_1)\in A^p$ and $(\sigma_2,\tau_2)\in A^q$. Now we adjoin to A an element 1 of degree 0 satisfying $d1=0$, $1\cdot1=1$ and $1\cdot a=a\cdot1=a$ for all $a\in A$. We denote the resulting differential graded algebra by $\hat{A}$. Then $\mathcal{Z}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})\subset A^1=\hat{A}^1$ equals the set

$$\{a\in\hat{A}\mid da+a\cdot a=0\}\subset\hat{A}^1$$

of so-called twisting elements of $\hat{A}$. Now we consider the group $(\hat{A}^0)^*$ of invertible elements in $\hat{A}^0$. This group consists of all elements $r\cdot1+c$ with $r\in\mathbb{R}$, $r\neq0$ and $c\in A^0=\mathcal{C}^1_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$. It acts from the right on $\hat{A}^1$ by

$$(a,g)\longmapsto g^{-1}ag+g^{-1}dg \tag{2}$$

and this action leaves invariant the set of twisting elements. Moreover, $\mathbb{R}=\mathbb{R}\cdot1\subset(\hat{A}^0)^*$ acts trivially. Comparing (2) with the action of $\mathcal{C}^1_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ on $\mathcal{Z}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ defined in (1) we see that $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ is just the set of twisting elements in $\hat{A}$ modulo the group of invertible elements of $\hat{A}^0$.

Now we want to formulate a description of quadratic extensions by cohomology.

Theorem 3.3. *Let $(\mathfrak{l},\theta_\mathfrak{l})$ be a finite-dimensional Lie algebra with involution and let $\mathfrak{a}$ be an orthogonal $(\mathfrak{l},\theta_\mathfrak{l})$-module. There is a bijection from the set of equivalence classes of quadratic extensions of $(\mathfrak{l},\theta)$ by $\mathfrak{a}$ to $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$.*

We want to describe the inverse of this bijection explicitly since this description gives us a construction for all metric Lie algebras with involution, hence, in particular, for all symmetric triples. Take $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$. We consider the vector space

$\mathfrak{d} := \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ and define an inner product $\langle \cdot, \cdot \rangle$ and an isometry θ on $\mathfrak{d}$ by

$$\begin{aligned} \langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2 \rangle &:= \langle A_1, A_2 \rangle_{\mathfrak{a}} + Z_1(L_2) + Z_2(L_1), \\ \theta(Z + A + L) &:= \theta_{\mathfrak{l}}^*(Z) + \theta_{\mathfrak{a}}(A) + \theta_{\mathfrak{l}}(L) \end{aligned}$$

for $Z, Z_1, Z_2 \in \mathfrak{l}^*$, $A, A_1, A_2 \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$. Moreover, we define an antisymmetric bilinear map $[\cdot, \cdot] \colon \mathfrak{d} \times \mathfrak{d} \to \mathfrak{d}$ by $[\mathfrak{l}^*, \mathfrak{l}^* \oplus \mathfrak{a}] = 0$ and

$$\begin{aligned} [L_1, L_2] &= \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}}, \\ [L, A] &= -\langle A, \alpha(L, \cdot) \rangle + L(A), \\ [A_1, A_2] &= \langle \rho(\cdot) A_1, A_2 \rangle, \\ [L, Z] &= \mathrm{ad}^*(L)(Z) \end{aligned}$$

for $L, L_1, L_2 \in \mathfrak{l}$, $A, A_1, A_2 \in \mathfrak{a}$ and $Z \in \mathfrak{l}^*$. Because of the cocycle condition for (α, γ) the bracket $[\cdot, \cdot]$ satisfies the Jacobi identity. Moreover, θ is an involution and $\langle \cdot, \cdot \rangle$ is $\mathfrak{d}$-invariant. Hence $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) := (\mathfrak{d}, \theta, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra with involution.

This construction is related to other known extension procedures. In the special case where $\alpha = \gamma = 0$ the metric Lie algebra $(\mathfrak{d}, \langle \cdot, \cdot \rangle)$ is a double extension of the abelian metric Lie algebra $\mathfrak{a}$ by $\mathfrak{l}$ in the sense of Medina and Revoy. For $\mathfrak{a} = 0$ it coincides with the T^*-extension introduced by Bordemann [11].

Now let us turn to the promised classification scheme for symmetric triples, more exactly for symmetric triples without simple ideals. Up to now we know that every such triple has the structure of a quadratic extension in a canonical way. Moreover, we can classify quadratic extensions of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$-module $\mathfrak{a}$ by $\mathcal{H}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$. Now the following problem occurs. In general we can write a metric Lie algebra with involution in different ways as quadratic extension and only one of these possibilities is the canonical one. Hence we must find those quadratic extensions that come from our canonical construction. These are exactly those quadratic extensions $(\mathfrak{g}, \mathfrak{i}, i, p)$ for which $\mathfrak{i}(\mathfrak{g}) = \mathfrak{i}$ holds. We will call them balanced. In [25] we managed to describe the subset $\mathcal{H}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_b \subset \mathcal{H}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ of cohomology classes that correspond to balanced quadratic extensions. A further problem is caused by the fact that many quadratic extensions $(\mathfrak{g}, \mathfrak{i}, i, p)$ do not satisfy $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$, hence they are not symmetric triples. Therefore, in a second step, we have to determine the subset $\mathcal{H}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_\sharp \subset \mathcal{H}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_b$ of cohomology classes that correspond to balanced quadratic extensions $(\mathfrak{g}, \mathfrak{i}, i, p)$ satisfying $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$. This is an easy task, see [26]. We will call these cohomology classes and the corresponding extensions admissible.

There is a natural notion of a direct sum of cohomology classes $a_1 \in \mathcal{H}^2_Q(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$ and $a_2 \in \mathcal{H}^2_Q(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$. The direct sum $a_1 \oplus a_2$ is contained in $\mathcal{H}^2_Q(\mathfrak{l}_1 \oplus \mathfrak{l}_2, \theta_{\mathfrak{l}_1} \oplus \theta_{\mathfrak{l}_2}, \mathfrak{a}_1 \oplus \mathfrak{a}_2)$. A cohomology class is called indecomposable if it cannot be written as such a sum, where at least one of the pairs $(\mathfrak{l}_1, \mathfrak{a}_1)$, $(\mathfrak{l}_2, \mathfrak{a}_2)$ is non-zero. We will denote the subset of indecomposable cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_\sharp$ by

$\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})_0$. It is not hard to prove that admissible quadratic extensions with indecomposable underlying symmetric triple correspond to indecomposable cohomology classes [26].

Finally we have to take into consideration that non-equivalent quadratic extensions of $(\mathfrak{l},\theta_\mathfrak{l})$ by $\mathfrak{a}$ can have isomorphic underlying symmetric spaces, hence we have to divide out automorphisms of $(\mathfrak{l},\theta_\mathfrak{l})$ and $\mathfrak{a}$. We obtain:

Theorem 3.4. *Let $\mathfrak{L}$ be a complete set of representatives of isomorphism classes of finite-dimensional Lie algebras with involution, and let $\mathfrak{A}$ be a complete set of representatives of isometry classes of pseudo-Euclidean vector spaces with involution. For $(\mathfrak{l},\theta_\mathfrak{l})\in\mathfrak{L}$ and $\mathfrak{a}=(\mathfrak{a},\langle\cdot,\cdot\rangle_\mathfrak{a},\theta_\mathfrak{a})\in\mathfrak{A}$ we denote by $\mathfrak{R}_{\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a}}$ the set of all semisimple representations of $(\mathfrak{l},\theta_\mathfrak{l})$ on $\mathfrak{a}$ such that $\mathfrak{a}_\rho=(\rho,\mathfrak{a},\langle\cdot,\cdot\rangle_\mathfrak{a},\theta_\mathfrak{a})$ is an orthogonal $(\mathfrak{l},\theta_\mathfrak{l})$-module. Then there is a bijection from the set of isomorphism classes of indecomposable non-semisimple symmetric triples to*

$$\coprod_{(\mathfrak{l},\theta_\mathfrak{l})\in\mathfrak{L}}\coprod_{\mathfrak{a}\in\mathfrak{A}}\Big(\coprod_{\rho\in\mathfrak{R}_{\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a}}}\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a}_\rho)_0\Big)\Big/\operatorname{Aut}(\mathfrak{l},\theta_\mathfrak{l})\times\operatorname{Aut}(\mathfrak{a},\langle\cdot,\cdot\rangle_\mathfrak{a},\theta_\mathfrak{a})$$

The inverse of this map assigns the isomorphism class of $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})$ to the orbit of $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a}_\rho)_0$.

Note that in the above formula one does not really have to consider all $(\mathfrak{l},\theta_\mathfrak{l})\in\mathfrak{L}$. A contribution to the coproduct comes only from those $(\mathfrak{l},\theta_\mathfrak{l})$ for which there exists an orthogonal $(\mathfrak{l},\theta_\mathfrak{l})$-module $\mathfrak{a}$ such that $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})_0$ is not empty. This a rather restrictive condition for $(\mathfrak{l},\theta_\mathfrak{l})$, see e.g. [22]. However, it does not appear to be strong enough to allow a classification for arbitrary dimension. Note that by definition of admissibility $\emptyset\neq\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})_0\subset\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})_\sharp$ can hold only if $[\mathfrak{l}_-,\mathfrak{l}_-]=\mathfrak{l}_+$.

This formula gives us a classification scheme, which is especially suitable for the study of symmetric spaces with small index of the metric. Here the existence of additional geometric structures allows to increase the index of the metric. The general procedure is as follows. Suppose we are given a pseudo-Riemannian symmetric space (M,g) such that the associated symmetric triple $(\mathfrak{g},\theta,\langle\cdot,\cdot\rangle)$ does not have simple ideals. Then $(\mathfrak{g},\theta,\langle\cdot,\cdot\rangle)$ can be written as an admissible quadratic extension of a Lie algebra with involution $(\mathfrak{l},\theta_\mathfrak{l})$ by an orthogonal $(\mathfrak{l},\theta_\mathfrak{l})$-module $\mathfrak{a}$. By definition of quadratic extension the index of g is an upper bound for $\dim\mathfrak{l}_-$, which gives an upper bound for the dimension of $\mathfrak{l}$. Lie algebras $(\mathfrak{l},\theta_\mathfrak{l})$ of small dimension that satisfy $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})_0\neq\emptyset$ for some semisimple orthogonal representation $\mathfrak{a}$ can be classified. All such semisimple orthogonal $(\mathfrak{l},\theta_\mathfrak{l})$-modules $\mathfrak{a}$ can be determined and we can calculate $\mathcal{H}^2_Q(\mathfrak{l},\theta_\mathfrak{l},\mathfrak{a})_0$ explicitly. This yields explicit classification results for pseudo-Riemannian symmetric spaces of small index.

If, e.g., (M,g) is an indecomposable, non-semisimple Lorentzian symmetric space, then $(\mathfrak{l},\theta_\mathfrak{l})$ must be isomorphic to $(\mathbb{R},-\mathrm{Id})$. Moreover, $\mathfrak{a}$ is isometric to $\mathbb{R}^{p,p}\oplus\mathbb{R}^{2q}$ and the representation of $\mathfrak{l}$ on $\mathfrak{a}$ is given by non-zero weights $\mu_1,\dots,\mu_{p+q}\in\mathfrak{l}^*\cong\mathbb{R}$.

Obviously, in this case we have $\mathcal{H}^2_Q(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0 = \{[0,0]\}$. In this way it is easy to recover Cahen–Wallach's classification of Lorentzian symmetric spaces.

If (M, g) is an indecomposable, non-semisimple symmetric space with metric of index 2, then $\mathfrak{l}$ is isomorphic to one of the Lie algebras $\mathbb{R}$, $\mathbb{R}^2$, $\mathfrak{so}(2) \ltimes \mathbb{R}^2$, $\mathfrak{so}(1,1) \ltimes \mathbb{R}^{1,1}$, $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$ or the 3-dimensional Heisenberg algebra $\mathfrak{h}(1)$. All these Lie algebras have a unique (up to isomorphism) involution such that $[\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+$ except for $\mathfrak{sl}(2,\mathbb{R})$, which has two. For each of these $(\mathfrak{l}, \theta_\mathfrak{l})$ it is easy to determine all semisimple orthogonal $(\mathfrak{l}, \theta_\mathfrak{l})$-modules and to calculate the quadratic cohomology sets. This gives a list of all pseudo-Riemannian symmetric spaces of index 2 (see [26]), which corrects the results by Cahen, Parker and Neukirchner discussed in Section 2.

For index 3 up to now only Lie groups with biinvariant metric are classified [25]. More results on Lie groups with biinvariant metric or, equivalently, on metric Lie algebras can be found in [22], where in particular nilpotent metric Lie algebras of dimension ≤ 10 are classified.

4 Pseudo-Riemannian symmetric spaces with special geometric structures

In this section we want to consider pseudo-Riemannian symmetric spaces that have additional structures coming from complex or quaternionic geometry. Let us recall some basic notions related to these geometries. Let (M, g) be a pseudo-Riemannian manifold. Objects on (M, g) that are covariant constant with respect to the Levi-Civita connection are called parallel. A Kähler structure on (M, g) is a parallel section I of $\mathfrak{so}(TM)$ that satisfies $I^2 = -\mathrm{Id}$. A hyper-Kähler structure on (M, g) is a pair (I, J) of Kähler structures satisfying $IJ = -JI$. If there is a hyper-Kähler structure on (M, g), then (M, g) is Ricci flat. Its signature equals $(4p, 4q)$, $p, q \in \mathbb{N}$. If we weaken the parallelity condition for I and J we are lead to the notion of a quaternionic Kähler structure. It consists of a 3-dimensional parallel subbundle of $\mathfrak{so}(TM)$ that can be locally spanned by sections I, J and K satisfying $I^2 = J^2 = -\mathrm{Id}$ and $K = IJ = -JI$. Here we want to exclude the case, where this parallel subbundle is spanned by parallel sections, i.e., that it even defines a hyper-Kähler structure. If there exists a quaternionic Kähler structure on (M, g), then g is an Einstein metric of non-vanishing scalar curvature. Moreover, (M, g) is not locally the product of two non-trivial pseudo-Riemannian manifolds. It is easy to see, that (M, g) admits a quaternionic Kähler structure if and only if the holonomy group of (M, g) is contained in $\mathrm{Sp}(p, q) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4p, 4q)$ but not in $\mathrm{Sp}(p, q)$. Moreover, the holonomy group of a manifold admitting a quaternionic Kähler structure always contains the whole $\mathrm{Sp}(1)$-factor of $\mathrm{Sp}(p, q) \cdot \mathrm{Sp}(1)$. For these general facts we refer to [10] and note that all arguments given there for the Riemannian situation remain valid in the indefinite case.

All these structures have 'para'-analogs. A para-Kähler structure on (M, g) is a parallel section I of $\mathfrak{so}(TM)$ that satisfies $I^2 = \mathrm{Id}$. It can exist only if g has neutral signature (q, q). A *para-hyper-Kähler structure* is a pair (I, J), where I is a Kähler structure and J is a para-Kähler structure such that $IJ = -JI$. Para-quaternionic Kähler structures are defined analogously.

A *pseudo-Hermitian symmetric space* (M, g, I) is a pseudo-Riemannian symmetric space (M, g) together with a Kähler structure I on (M, g). Analogously, *hyper-Kähler symmetric spaces* and *quaternionic Kähler symmetric spaces* are pseudo-Riemannian symmetric spaces together with a fixed hyper-Kähler and quaternionic Kähler structure, resp. Similarly, we have *para-Hermitian*, *para-hyper-Kähler* (also called *hypersymplectic*) and *para-quaternionic Kähler symmetric spaces*.

Let us start with the pseudo-/para-Hermitian case. The pseudo-/para-Kähler structure I on (M, g) defines an additional structure on the symmetric triple $(\mathfrak{g}, \langle\cdot,\cdot\rangle, \theta)$ associated with (M, g), namely a $\mathfrak{g}_+$-invariant map $\hat{I} \in \mathfrak{so}(\mathfrak{g}_-)$ satisfying $\hat{I}^2 = -\mathrm{Id}$. We will call $(\mathfrak{g}, \langle\cdot,\cdot\rangle, \theta, \hat{I})$ a pseudo-Hermitian symmetric triple. For semisimple $\mathfrak{g}$ such triples were classified by Berger [9]. If $\mathfrak{g}$ does not contain simple ideals, then we can consider $\mathfrak{g}$ again as a quadratic extension of a Lie algebra with involution $(\mathfrak{l}, \theta_\mathfrak{l})$ by an orthogonal $(\mathfrak{l}, \theta_\mathfrak{l})$-module. It is not hard to see that $\hat{I}$ induces maps $\hat{I}_\mathfrak{l}$ on $\mathfrak{l}_-$ and $\hat{I}_\mathfrak{a}$ on $\mathfrak{a}_-$ with properties analogous to those of $\hat{I}$. One can prove that every solvable pseudo- or para-Hermitian triple is nilpotent [28]. If we apply this to a pseudo-Hermitian symmetric space $\mathfrak{g}$ of index 2 which has the structure of a quadratic extension, then either $\mathfrak{g}$ has a non-trivial Levi factor and then $\mathfrak{l} \in \{\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R})\}$ or $\mathfrak{g}$ and therefore also $\mathfrak{l}$ is nilpotent and then $\mathfrak{l} \in \{\mathbb{R}, \mathfrak{h}(1)\}$. Moreover, if $\mathfrak{l}$ is nilpotent, then the representation of $\mathfrak{l}$ on $\mathfrak{a}$ must be trivial. In this way we easily find all pseudo-Hermitian symmetric spaces of index 2 without checking the whole (rather long) list of all symmetric spaces of index 2 [26], [28]. In [28] we also classify para-Hermitian symmetric spaces of index 2. For another approach to the classification of pseudo- and para-Hermitian symmetric spaces see [1] in this book, where D. Alekseevsky describes such spaces that have a special kind of abelian holonomy.

Now let us turn to quaternionic Kähler symmetric spaces. Riemannian quaternionic Kähler symmetric spaces were classified by J. Wolf [33]. Let us explain the main idea on which this classification is based. Since in the Riemannian situation compact spaces and non-compact ones are in duality we will restrict ourselves to the compact case. Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. We consider the corresponding roots and choose a set of positive ones. Let β be the highest root and let $\mathfrak{s}_\beta := \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta} \oplus [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \subset \mathfrak{g}^\mathbb{C}$ be the 3-dimensional subalgebra generated by the root spaces of β and $-\beta$. Then $\mathfrak{s}_\beta \cap \mathfrak{g}$ is isomorphic to $\mathfrak{sp}(1)$. We define an involution on $\mathfrak{g}$ by

$$\mathfrak{g}_+ := \mathfrak{s}_\beta \cap \mathfrak{g} \oplus Z_\mathfrak{g}(\mathfrak{s}_\beta \cap \mathfrak{g}), \quad \mathfrak{g}_- = (\mathfrak{g}_+)^\perp,$$

where $Z_\mathfrak{g}$ denotes the centraliser in $\mathfrak{g}$. Then the symmetric triple which is constituted by $\mathfrak{g}$ together with this involution and a non-zero multiple of the Killing form is always associated with a compact Riemannian quaternionic Kähler symmetric space. Con-

versely, every symmetric triple associated with a compact Riemannian quaternionic Kähler symmetric space can be obtained in this way.

Now we turn to the indefinite case. Recall that any quaternionic Kähler manifold is Einstein with non vanishing scalar curvature, hence it has a non-degenerate Ricci tensor. Since the Ricci tensor of a symmetric space is essentially given by the Killing form of the transvection group we see that the transvection group of a quaternionic Kähler symmetric space is semisimple. Since, moreover, such a symmetric space is (locally) irreducible it is either isometric to a simple group G or its transvection group is simple. The first case can be excluded since G would also be the holonomy group of the symmetric space and would therefore be of the form $\mathrm{Sp}(1)\cdot K'$, which contradicts the assumption that G is simple. Hence also in the indefinite case the transvection group of a quaternionic Kähler symmetric spaces is simple. Let us now consider the complex version of Wolf's construction discussed above. We start with a complex simple Lie algebra $\mathfrak{g}$. We fix a Cartan subalgebra in $\mathfrak{g}$ and choose β and $\mathfrak{s}_\beta \subset \mathfrak{g}$ as above. We define an involution on $\mathfrak{g}$ by

$$\begin{aligned}\mathfrak{g}_+ &= \mathfrak{s}_\beta \oplus Z_{\mathfrak{g}}(\mathfrak{s}_\beta),\\ \mathfrak{g}_- &= (\mathfrak{g}_+)^\perp.\end{aligned}$$

Now suppose that σ is an anti-linear involution on $\mathfrak{g}$ that preserves $\mathfrak{s}_\beta$ and for which $\mathfrak{s}_\beta^\sigma$ is isomorphic to $\mathfrak{sp}(1)$. Then σ respects the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, thus we obtain an involution on $\mathfrak{g}^\sigma$. The real Lie algebra $\mathfrak{g}^\sigma$ together with this involution and a non-zero multiple of the Killing form defines a symmetric triple such that the associated symmetric space is quaternionic Kählerian. As in the Riemannian case one proves that, conversely, every such symmetric triple can be obtained in this way. In [4] D.V. Alekseevsky and V. Cortés determine all these anti-linear involutions σ for every complex simple Lie algebra $\mathfrak{g}$. In this way they obtain a classification of indefinite simply connected quaternionic Kähler symmetric spaces avoiding to check Berger's list of all semisimple pseudo-Riemannian symmetric spaces for quaternionic Kählerian ones.

The para-quaternionic case was studied by D.V. Alekseevsky, N. Blažić, V. Cortés and S. Vukmirović in [2].

Next let us consider hyper-Kähler symmetric spaces. Since such spaces are Ricci-flat the Killing form of the transvection group vanishes. In particular, there are no non-flat Riemannian hyper-Kähler symmetric spaces. In analogy to the Kählerian case hyper-Kähler symmetric spaces correspond to so-called hyper-Kähler symmetric triples. These are triples $(\mathfrak{g}, \Phi, \langle\cdot,\cdot\rangle)$, $\Phi = (\theta, \hat{I}, \hat{J})$, where $(\mathfrak{g}, \theta, \langle\cdot,\cdot\rangle)$ is a symmetric triple and $\hat{I}$ and $\hat{J}$ are $\mathfrak{g}_+$-invariant elements of $\mathfrak{so}(\mathfrak{g}_-)$ such that $\hat{I}^2 = \hat{J}^2 = -\mathrm{Id}$ and $\hat{K} := \hat{I}\hat{J} = -\hat{J}\hat{I}$.

First we want to explain a description of hyper-Kähler symmetric triples by certain quartic polynomials due to Alekseevsky and Cortés. This description will lead to a classification in the special case of abelian holonomy. Let (E, ω) be a complex symplectic vector space. Take $S \in S^4E$. For $v, w \in E$ let $S_{v,w}$ be the contraction of

S with v and w via the symplectic form ω. Now let us consider the subspace

$$\mathfrak{h}_S = \operatorname{span}\{S_{v,w} \in S^2E \mid v, w \in E\}$$

of $S^2E \cong \mathfrak{sp}(E,\omega)$. If

$$S \in (S^4E)^{\mathfrak{h}_S}, \tag{3}$$

then $\mathfrak{h}_S \subset \mathfrak{sp}(E,\omega)$ is a subalgebra. In this case there is a Lie bracket on

$$\mathfrak{g}_S = \mathfrak{h}_S \oplus (\mathbb{H} \otimes_{\mathbb{C}} E)$$

such that $\mathfrak{h}_S$ is a subalgebra, $\mathfrak{h}_S \subset \mathfrak{sp}(E,\omega)$ acts on $\mathbb{H} \otimes_{\mathbb{C}} E$ by the natural action on the second factor and the commutator of two elements of $\mathbb{H} \otimes_{\mathbb{C}} E$ is in $\mathfrak{h}_S$ and defined by

$$[p \otimes v, q \otimes w] = \omega_{\mathbb{H}}(p,q)S_{v,w}, \quad p,q \in \mathbb{H},\ v,w \in E.$$

Here $\omega_{\mathbb{H}}$ denotes the complex bilinear 2-form on $\mathbb{H}$ with $\omega_{\mathbb{H}}(1,j) = 1$.

Now assume that E admits a quaternionic structure J such that $J^*\omega = \bar{\omega}$. Then J induces a real structure on S^4E and on $S^2E \cong \mathfrak{sp}(E,\omega)$. Moreover, $j \otimes J$ is a real structure on $\mathbb{H} \otimes_{\mathbb{C}} E$. Let us denote all these real structures by the same symbol τ. If $S \in (S^4E)^{\tau}$ satisfies (3), then $\mathfrak{g}_{J,S} := (\mathfrak{h}_S)^{\tau} \oplus (\mathbb{H} \otimes_{\mathbb{C}} E)^{\tau}$ is a real Lie algebra. We can define an involution θ on $\mathfrak{g}_{J,S}$ by $(\mathfrak{g}_{J,S})_+ = (\mathfrak{h}_S)^{\tau}$ and $(\mathfrak{g}_{J,S})_- = (\mathbb{H} \otimes_{\mathbb{C}} E)^{\tau}$. Then $(\mathfrak{g}_{J,S})_+ = [(\mathfrak{g}_{J,S})_-, (\mathfrak{g}_{J,S})_-]$ holds. In particular, the inner product $\omega_{\mathbb{H}} \otimes \omega$ on $(\mathfrak{g}_{J,S})_- = (\mathbb{H} \otimes_{\mathbb{C}} E)^{\tau}$ extends to an invariant inner product on $\mathfrak{g}_{J,S}$. If we let, moreover, $\hat{I}$ and $\hat{J}$ be the left multiplication by i and j on the first factor of $(\mathbb{H} \otimes_{\mathbb{C}} E)^{\tau}$, then we obtain the structure of a hyper-Kähler symmetric triple on $\mathfrak{g}_{J,S}$.

Theorem 4.1 (Alekseevsky–Cortés [3]). *Every hyper-Kähler symmetric triple $(\mathfrak{g}, \Phi, \langle\cdot,\cdot\rangle)$ is isomorphic to some hyper-Kähler symmetric triple $\mathfrak{g}_{J,S}$ for suitable data (E,ω,J,S), where $S \in S^4E$ is a τ-invariant solution of (3). The quadruple (E,ω,J,S) is uniquely determined by $(\mathfrak{g},\Phi,\langle\cdot,\cdot\rangle)$ up to complex linear isomorphisms.*

Hence the classification of hyper-Kähler symmetric triples is equivalent to the classification of all τ-invariant solutions of (3). Up to now it is not known how to find all these solutions. However, there is a large family of special solutions that can easily be described. Indeed, if $E_+ \subset E$ is a Lagrangian subspace, then all $S \in S^4E_+ \subset S^4E$ satisfy (3). We will call these solutions tame. If S is tame, then $\mathfrak{h}_S$ is abelian. Cortés proved that also the converse is true. This yields the following classification of hyper-Kähler symmetric triples with abelian holonomy.

Proposition 4.2 (Cortés [15]). *In the description of hyper-Kähler symmetric spaces given in Theorem 4.1 those with abelian holonomy group correspond exactly to tame solutions of (3).*

In [27] we prove that there exist indeed hyper-Kähler symmetric spaces that have non-abelian holonomy and we give several examples. All examples were constructed

by our method of quadratic extensions. In particular this shows that there are solutions of (3) that are not tame. Up to now it is not clear how to find such solutions in a more direct way. Let us sketch the 'simplest' example of a hyper-Kähler symmetric space with non-abelian holonomy. We start with the Lie algebra with involution $(\mathfrak{l}, \theta_\mathfrak{l})$ defined by $\mathfrak{l} = \mathfrak{l}_- \oplus \mathfrak{l}_+ = \mathbb{H} \oplus \operatorname{Im}\mathbb{H}$, where $\mathfrak{l}_+ = \operatorname{Im}\mathbb{H}$ is the centre of $\mathfrak{l}$ and $[q_1, q_2] = \operatorname{Im}\bar{q}_1 q_2 \in \mathfrak{l}_+ = \operatorname{Im}\mathbb{H}$ for $q_1, q_2 \in \mathfrak{l}_- = \mathbb{H}$. We consider $\mathfrak{a} = \mathfrak{a}_- := \mathbb{H}^2$ with the standard scalar product as a trivial $(\mathfrak{l}, \theta_\mathfrak{l})$-module. Let $\hat{I}_\mathfrak{a}$, $\hat{I}_\mathfrak{l}$, $\hat{J}_\mathfrak{a}$, $\hat{J}_\mathfrak{l}$ be the left-multiplication by i and j on $\mathfrak{a} = \mathfrak{a}_- = \mathbb{H}^2$ and $\mathfrak{l}_- = \mathbb{H}$, respectively. Then it is not hard to find an element $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})$ which satisfies

$$\gamma|_{\mathfrak{l}_+ \wedge \mathfrak{l}_+ \wedge \mathfrak{l}_+} \neq 0 \tag{4}$$

and which is in a certain sense invariant with respect to the multiplication by i and j on $\mathfrak{a}_-$ and $\mathfrak{l}_-$. Now we consider the quadratic extension $\mathfrak{d} := \mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})$. By the mentioned invariance property the maps

$$\hat{I} = -\hat{I}^*_\mathfrak{l} \oplus \hat{I}_\mathfrak{a} \oplus \hat{I}_\mathfrak{l}, \quad \hat{J} = -\hat{J}^*_\mathfrak{l} \oplus \hat{J}_\mathfrak{a} \oplus \hat{J}_\mathfrak{l}$$

define $\mathfrak{d}_+$-invariant elements of $\mathfrak{so}(\mathfrak{d}_-)$ and $\mathfrak{d}$ together with $\hat{I}$ and $\hat{J}$ constitutes a hyper-Kähler symmetric triple. The associated hyper-Kähler symmetric space has signature $(4, 12)$. The holonomy group of this symmetric space cannot be abelian. Indeed, its Lie algebra $\mathfrak{d}_+$ is not abelian because of (4). For more detailed explanations see [27].

In [27] one can also find a classification of indecomposable simply connected hyper-Kähler symmetric spaces of signature $(4, 4q)$. Besides the example of signature $(4, 12)$ discussed above the flat space $\mathbb{H}$ and a one-parameter family of spaces of signature $(4, 4)$ with abelian holonomy occur.

Para-hyper-Kähler symmetric spaces are considered in [16]. Of course, the above discussed hyper-Kähler symmetric space with non-abelian holonomy has a para-hyper-Kählerian analogon.

Finally let me draw your attention to manifolds that are symmetric from an extrinsic point of view. A non-degenerate connected submanifold $M \subset \mathbb{R}^{p,q}$ is called *extrinsic symmetric* if it is invariant under the reflection at the (affine) normal space $T^\perp_x M$ for every $x \in M$. Extrinsic symmetric spaces in $\mathbb{R}^{p,q}$ are exactly those complete submanifolds whose second fundamental form is parallel. Obviously, these spaces are pseudo-Riemannian symmetric spaces in the ordinary sense. Extrinsic symmetric spaces in the Euclidean space are well understood. Any such space is the product of a Euclidean space and a compact extrinsic symmetric space, which lies in a round sphere. A classification of compact Euclidean extrinsic symmetric spaces follows from a nice construction due to Ferus. This construction shows that compact extrinsic symmetric spaces in the Euclidean space are orbits of isotropy representations of semi-simple symmetric spaces, namely they are exactly the standard embedded symmetric R-spaces [18], [19], [20]. A classification of Euclidean symmetric spaces now follows from the classification of symmetric R-spaces due to Kobayashi and Nagano. J. R. Kim [29] showed that Ferus' construction can be carried out also for

pseudo-Riemannian extrinsic symmetric spaces that satisfy additional conditions. As in the Riemannian case we see that theses extrinsic symmetric spaces are orbits of the isotropy representation of symmetric spaces. But now these symmetric spaces are in general non-reductive and a full classification becomes impossible. However, it seems to be a realistic aim to classify Lorentzian extrinsic symmetric spaces by the method of quadratic extensions [23].

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Chapter 21

Pseudo-Kähler and para-Kähler symmetric spaces

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Introduction

Pseudo-Riemannian symmetric spaces form a big class of pseudo-Riemannian manifolds. Riemannian symmetric spaces were classified by E. Cartan. The classification reduces to the description of involutive automorphisms of semisimple Lie algebras, since an irreducible Riemannian symmetric space has semisimple isometry group. This approach leads also to a classification of pseudo-Riemannian symmetric spaces with semisimple isometry group. However, the isometry group of a pseudo-Riemannian (in particular Kähler and para-Kähler) symmetric space is non-semisimple in general. There is no hope to get a classification of all pseudo-Riemannian symmetric spaces, since the cotangent bundle TG of *any* Lie group G has a natural structure of pseudo-Riemannian symmetric space of neutral signature (n,n). Nevertheless, many interesting special classes of pseudo-Riemannian symmetric spaces have been studied and classified, for example, Lorentzian symmetric spaces [CW] and symmetric spaces of signature $(2,n)$ [CP], [KO]. In [KO1], I. Kath and M. Olbrich developed a general approach for description and classification of pseudo-Riemannian symmetric spaces, based on a construction of quadratic extension of Lie algebras, and got many

interesting results. In the paper [AC1], the classification of (pseudo-Riemannian) quaternionic Kähler symmetric space with non-zero scalar curvature is given. In the papers [AC] and [ABCV], pseudo-Riemannian hyper-Kähler and para-hyper-Kähler symmetric spaces were studied and their classification was reduced to the description of some homogenous polynomials of degree 4. In particular, a classification of such symmetric spaces with commutative holonomy group was given, see [Cor]. First examples of hyper-Kähler symmetric spaces with non-commutative holonomy were constructed in the paper [KO2], where also a classification of hyper-Kähler symmetric spaces of signature $(4, n)$ was obtained.

The present chapter is devoted to the investigation of the structure of pseudo-Kähler and para-Kähler symmetric spaces, that is, pseudo-Riemannian symmetric spaces (M, g) together with a parallel field J of skew-symmetric complex, or, respectively, para-complex structures (such that $J^2 = -\mathrm{Id}$ or, respectively, $J^2 = \mathrm{Id}$). A para-Kähler structure (g, J) on a manifold M is also called a bi-Lagrangian structure since it is equivalent to a symplectic structure together with a decomposition of the tangent bundle into a direct sum of two integrable Lagrangian distributions.

The classification of simply connected para-Kähler symmetric spaces reduces to the algebraic problem of classification of $\mathbb{Z}$-graded Lie algebras of the form $\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1$ (called 3-graded Lie algebras) such that the adjoint representations of $\mathfrak{g}^0$ on $\mathfrak{g}^{-1}$ and $\mathfrak{g}^1$ are exact and mutually contragredient (dual). Note that the structure of a 3-graded Lie algebra on a vector space $\mathfrak{g}$ is equivalent to the structure of a Jordan pair, that is, a pair $(V^- = \mathfrak{g}^{-1}, V^+ = \mathfrak{g}^1)$ of vector spaces together with two 3-linear maps

$$V^- \times V^+ \times V_- \to V^-, \quad V^- \times V^- \times V_+ \to V^+$$

which satisfies some identities, see [B]. Jordan pairs and 3-graded Lie algebras are closely related also with Jordan algebras. For example, if the grading derivation D of a 3-graded algebra $\mathfrak{g}$ (defined by $D|_{\mathfrak{g}^i} = i\,\mathrm{Id}$) is inner and the corresponding element d (such that $\mathrm{ad}_d = D$) can be imbedded into a 3-dimensional subalgebra $\mathfrak{sl}_2$ with the standard basis $(e \in \mathfrak{g}^{-1}, h, f)$, then the multiplication $(x, y) \mapsto x \circ y := [[x, e], y]$ defines the structure of a Jordan algebra on $V = \mathfrak{g}^{-1}$. Jordan pairs are related with many algebraic and differential-geometric objects and they have been intensively studied, see, for example, [B], [KO3], [N], [T], [Tan]. In particular, in [S] it is shown that one can associate with a Jordan pair a completely integrable equation of Schrödinger type. The classification of 3-graded semisimple Lie algebras is well known, see [KN], [Kac]. However, it seems that only a few constructions of non-semisimple 3-graded Lie algebras have been proposed in the literature, see for example, [As], [deO], [Tan]. Note that Ricci-flat para-Kähler manifold has the holonomy group $H \subset \mathrm{SU}(p, q)$ and such a manifold is a pseudo-Riemannian analogue of a Calabi–Yau manifold.

The aim of this chapter is to describe a class of Ricci-flat para-Kähler and pseudo-Kähler symmetric spaces. In particular, we classify all Ricci-flat para-Kähler and pseudo-Kähler symmetric spaces of dimension $n \leq 6$.

In Section 1, we recall some basic facts about pseudo-Riemannian symmetric spaces. We prove that a symmetric space has semisimple isometry group if and only if its Ricci form ric is non-degenerate and that ric $= 0$ if and only if the Lie algebra $\mathfrak{g}$ generated by transvections has zero Killing form and, in particular, is solvable.

In Section 2, we reduce the classification of simply connected para-Kähler symmetric spaces to the description of 3-graded Lie algebras $\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1$ such that the $\mathfrak{g}^0$-modules $V = \mathfrak{g}^{-1}$ and $V^* = \mathfrak{g}^1$ are dual to each other and the representation of $\mathfrak{h} = \mathfrak{g}^0$ on V is exact. Then $\mathfrak{h}$ is identified with a subalgebra $\mathfrak{h}$ of the linear algebra

$$V \wedge V^* = \mathfrak{gl}(V) \subset \mathfrak{so}(V + V^*).$$

We prove that the gradation of $\mathfrak{g}$ is consistent with an appropriate Levi–Malcev decomposition of $\mathfrak{g}$.

If a subalgebra $\mathfrak{h} \subset \mathfrak{gl}(V)$ is given, the structure of an admissible graded Lie algebra on $\mathfrak{g} = V + \mathfrak{h} + V^*$ is determined by an $\mathfrak{h}$-invariant bi-form $R \in (S^2V \otimes S^2V^*)^{\mathfrak{h}}$ such that the associated endomorphism

$$R\colon V \wedge V^* \to \mathfrak{gl}(V) = V \wedge V^*, \quad X \wedge \xi \mapsto R(X \wedge \xi),$$

defined by the contraction take values in $\mathfrak{h}$: $R(V \wedge V^*) \subset \mathfrak{h}$. Using this, we describe all admissible graded Lie algebras of the form

$$\mathfrak{g} = V + V_+ \wedge V_-^* + V^*,$$

where $V = V_+ + V_-$ is a direct sum decomposition of a vector space V and $V^* = V_+^* + V_-^*$ the dual decomposition. Such a gradation is determined by a bi-form $R \in S^2V_+ \wedge S^2V_-^*$. The associated para-Kähler symmetric space is Ricci flat and has commutative holonomy algebra $\mathfrak{h} \subset V_+ \wedge V_-^*$.

Section 3 deals with pseudo-Kähler symmetric spaces (M, g, J). We prove that a pseudo-Kähler symmetric space is determined by a complex admissible graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = V + \mathfrak{h} + V^*,$$

and by its admissible anti-involution, that is, by an antilinear involutive automorphism of $\mathfrak{g}$ (considered as a real Lie algebra) which satisfies certain conditions. We describe all admissible anti-involutions of a complex graded Lie algebra of the form $\mathfrak{g} = V + V_+ \wedge V_-^* + V^*$, where $V = V_+ + V_-$ is a decomposition of a complex vector space into the direct sum of two isomorphic subspaces and $V^* = V_+^* + V_-^*$ is the dual decomposition. This gives a big class of pseudo-Calabi–Yau symmetric spaces, i.e., Ricci-flat pseudo-Kähler symmetric spaces. We also classify all pseudo-Kähler and para-Kähler Ricci-flat symmetric spaces of dimension $n \leq 6$.

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1 Pseudo-Riemannian symmetric spaces

1.1 Basic properties of symmetric spaces

A *pseudo-Riemannian symmetric space* is a pseudo-Riemannian manifold (M, g) such that any point x is an isolated fixed point of an isometric involution S_x called the *central symmetry* with the center at x. The connected component G of the group generated by central symmetries is called the *group of transvections.* The group G acts transitively on M and M can be identified with the coset space G/H, where $H = G_o$ is the stabilizer of a point $o \in M$. The central symmetry S_0 defines an involutive automorphism

$$\sigma : G \ni a \mapsto S_0 \circ a \circ S_0$$

of the Lie group G and its Lie algebra $\mathfrak{g}$. The σ-eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \tag{1.1}$$

of the Lie algebra $\mathfrak{g}$ is a *symmetric decomposition*, that is,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

Moreover, the symmetric decomposition satisfies the following two properties:

It is *effective*, that is, $\mathfrak{h}$ does not contain a non-trivial ideal of the Lie algebra $\mathfrak{g}$ and it is *minimal*, that is, $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$.

We will identify $\mathfrak{m}$ with the tangent space T_0M and we denote by $\langle \cdot, \cdot \rangle = g_0$ a pseudo-Euclidean metric in $\mathfrak{m}$ defined by the pseudo-Riemannian metric g. Then the isotropy representation of the stability subgroup H (respectively, stability subalgebra $\mathfrak{h} = \mathrm{Lie} H$) is identified with the adjoint representation $\mathrm{Ad}_H|_{\mathfrak{m}}$ (respectively, $\mathrm{ad}_{\mathfrak{h}}|_{\mathfrak{m}}$). Moreover, the isotropy group $\mathrm{Ad}_H|_{\mathfrak{m}}$ coincides with the holonomy group of $(M = G/H, g)$.

Conversely, let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a symmetric decomposition of a Lie algebra $\mathfrak{g}$ and $g_0 = \langle \cdot, \cdot \rangle$ an $\mathrm{ad}_{\mathfrak{h}}$-invariant pseudo-Euclidean metric in $\mathfrak{m}$. They define a simply connected pseudo-Riemannian symmetric space $(M = G/H, g)$, where G is the simply connected Lie group with the Lie algebra $\mathfrak{g}$, H connected subgroup of G, generated by the subalgebra $\mathfrak{h}$ and g a G-invariant pseudo-Riemannian metric on M defined by the pseudo-Euclidean metric g_0 in $\mathfrak{m} = T_o(G/H)$, $o = eH \in G/H$. The central symmetry S_o with center at o is given by

$$S_o(aH) = \sigma(a)H, \quad aH \in G/H,$$

where σ is the involutive automorphism of G which corresponds to the involutive automorphism σ of the Lie algebra $\mathfrak{g}$ defined by $\sigma|_{\mathfrak{h}} = \mathrm{Id}$, $\sigma|_{\mathfrak{m}} = -\mathrm{Id}$. If the symmetric decomposition is effective, then the action of G on $M = G/H$ is almost effective, that is, the kernel of effectivity $N = \{g \in G,\ gx = x \text{ for all } x \in M\} \subset H$ is a discrete central subgroup of G and the group $\bar{G} = G/N$ acts on M effectively and transitively as a group of isometries and $M = \bar{G}/\bar{H}$, where $\bar{H} = H/N$. Moreover, $\bar{G}$ is the

group of transvections if and only if the symmetric decomposition is minimal. The following known theorem reduces the classification of simply connected Riemannian symmetric spaces to the classification of effective minimal symmetric decompositions (1.1) of Lie algebras with pseudo-orthogonal isotropy representation of $\mathfrak{h}$ in $\mathfrak{m}$.

Theorem 1. *There exists a natural* 1-1 *correspondence between effective minimal symmetric decompositions* (1.1) *of a Lie algebra* $\mathfrak{g}$ *together with an* $\mathrm{ad}_{\mathfrak{h}}$*-invariant pseudo-Euclidean metric* g_o *on* $\mathfrak{m}$ *and simply connected pseudo-Riemannian symmetric spaces* $(M = G/H, g)$*, where* G *is the simply connected Lie group with* $\mathrm{Lie}(G) = \mathfrak{g}$ *and* H *the connected subgroup generated by* $\mathfrak{h}$ *as described above. Moreover, any pseudo-Riemannian with the symmetric decomposition* (1.1) *has the form* $(G/\widehat{H}, g)$*, where* $\widehat{H}$ *is a subgroup such that* $H \subset \widehat{H} \subset G^{\sigma}$*. Here* G^{σ} *is the fixed point set of the involutive automorphism* σ *of* G *associated with the symmetric decomposition.*

In the future, by a symmetric space (M, g) we will understand a simply connected pseudo-Riemannian associated with a minimal effective symmetric decomposition of a Lie algebra $\mathfrak{g}$ and a pseudo-Euclidean $\mathrm{ad}_{\mathfrak{h}}$-invariant metric g_0 on $\mathfrak{m}$ and we will denote by G the simply connected Lie group with the Lie algebra $\mathfrak{g}$ which acts on $M = G/H$ almost effectively.

Proposition 1. *Let* $(M = G/H, g)$ *be a simply connected pseudo-Riemannian manifold associated with an effective minimal symmetric decomposition* (1.1) *and an* $\mathrm{ad}_{\mathfrak{h}}$*-invariant metric* $g_o = \langle \cdot, \cdot \rangle$ *on* $\mathfrak{m}$.

Then the following hold:

i) *The curvature tensor* R *and the Ricci tensor* ric *of* (M, g) *at the point* $o = eH$ *is given by*

$$R(X, Y) = -\mathrm{ad}_{[X,Y]}|_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m} = T_o M,$$
$$\mathrm{ric} = -\frac{1}{2} B|_{\mathfrak{m}}, \qquad \text{where } B \text{ is the Killing form of } \mathfrak{g}.$$

ii) *The holonomy algebra at* o *is spanned by the curvature operators* $R(X, Y)$*,* $X, Y \in \mathfrak{m}$.

iii) *The Ricci tensor* ric *is non-degenerate if and only if the Lie algebra* $\mathfrak{g}$ *is semisimple.*

iv) *The Ricci tensor vanishes if and only if the Killing form* B *of* $\mathfrak{g}$ *is zero. This implies that* $\mathfrak{g}$ *is solvable and* $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ *is nilpotent.*

v) *Assume that* (M, g) *has no flat factor. Then the following conditions are equivalent:*

   (a) *The holonomy algebra* $\mathrm{ad}_{\mathfrak{h}}|_{\mathfrak{m}}$ *is totally reducible.*

   (b) $\mathfrak{g}$ *is semisimple.*

   (c) ric *is non-degenerate.*

Note that the Killing form $B = 0$ for a nilpotent Lie algebra. The converse statement is not true. For example, the solvable non-nilpotent Lie algebra $\mathfrak{g} = \mathbb{R}A + \mathbb{C}^2$ which is a semidirect sum of the commutative ideal $\mathbb{C}^2$ and 1-dimensional subalgebra spanned by the derivation $A = \operatorname{diag}(i, 1)$ of $\mathbb{C}^2$ has $B = 0$.

Proof. Claims i) and ii) are well known, see [Bess]. Since $B(\mathfrak{h}, \mathfrak{m}) = 0$, the kernel of $B|_{\mathfrak{m}}$ belongs to the kernel of B in $\mathfrak{g}$. So if $\mathfrak{g}$ is semisimple, then B is non-degenerate by Cartan criterion and $\mathrm{ric} = -\frac{1}{2}B|_{\mathfrak{m}}$ is non-degenerate. Conversely, if ric is non-degenerate, then $\ker B \subset \mathfrak{h}$ is an ideal of $\mathfrak{g}$ which is zero by effectivity. Hence, $\mathfrak{g}$ is semisimple. This proves iii).

If $B = 0$ then $\mathrm{ric} = 0$ by i). Conversely, if $\mathrm{ric} = 0$, the kernel of B is an ideal of $\mathfrak{g}$ which contains $\mathfrak{m}$. Since $\mathfrak{g} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$, we get $\ker B \subset \mathfrak{m}^{\perp} \subset \mathfrak{g}$ and $B = 0$. This proves iv).

v) Suppose that the holonomy algebra is totally reducible. We may also assume that (M, g) is not a product of two pseudo-Riemannian manifolds. Then, by the De Rham–Wu decomposition theorem, either $\mathfrak{m}$ is an irreducible non-trivial $\mathfrak{h}$-module or a direct sum of two totally isotropic irreducible modules $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Note that $\mathrm{ric} \neq 0$. Indeed, in the opposite case, $\mathfrak{g}$ is solvable by iv) and $\mathrm{ad}_{\mathfrak{h}} = \mathrm{ad}_{[\mathfrak{m},\mathfrak{m}]}$ consists of nilpotent endomorphisms and cannot be totally reducible. Then the Schur Lemma shows that ric is non-degenerate in the first case. We prove that this is also true in the second case. First of all, we prove that $[\mathfrak{m}_1, \mathfrak{m}_1] = R(\mathfrak{m}_1, \mathfrak{m}_1) = 0$ and similar $[\mathfrak{m}_2, \mathfrak{m}_2] = 0$. Since $R(\mathfrak{m}, \mathfrak{m}) \in \mathrm{ad}_{\mathfrak{h}}$, the Bianchi identity implies that $R(\mathfrak{m}_1, \mathfrak{m}_1)\mathfrak{m}_2 = 0$. Using this and the fact that the curvature operators are skew-symmetric, we get $\langle R(\mathfrak{m}_1, \mathfrak{m}_1)\mathfrak{m}_1, \mathfrak{m}_2\rangle = \langle \mathfrak{m}_1, R(\mathfrak{m}_1, \mathfrak{m}_1)\mathfrak{m}_2\rangle = 0$. This implies that $R(\mathfrak{m}_1, \mathfrak{m}_1) = [\mathfrak{m}_1, \mathfrak{m}_1] = 0$. With respect to a basis which is consistent with the decomposition $\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{h}$, the matrix of operators $\mathrm{ad}_{\mathfrak{m}_1}$ has the form

$$g = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix}.$$

Hence,

$$B(X, Y) = \operatorname{tr} \mathrm{ad}_X \mathrm{ad}_Y = 0 \quad \text{for all } X, Y \in \mathfrak{m}_1. \tag{1.2}$$

Suppose now that ric is degenerate. Without loss of generality, we may assume that $\ker \mathrm{ric} = \ker B|_{\mathfrak{m}} \supset \mathfrak{m}_2$. This together with (1.2) implies that $B(\mathfrak{m}, \mathfrak{m}) = \mathrm{ric}(\mathfrak{m}, \mathfrak{m}) = 0$ and we get a contradiction. Hence, the Ricci tensor is non-degenerate and $\mathfrak{g}$ is semisimple. Conversely, if $\mathfrak{g}$ is a semisimple Lie algebra, then it is well known that the fixed point set $\mathfrak{h} = \mathfrak{g}^{\sigma}$ of an involutive automorphism σ is a reductive subalgebra. □

Let $(\mathfrak{m}, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space and $\mathfrak{h}$ a Lie subalgebra of the pseudo-orthogonal Lie algebra $\mathfrak{so}(\mathfrak{m})$. We denote by

$$\mathcal{R}(\mathfrak{h}) = \left\{R \in \mathfrak{h} \otimes \textstyle\bigwedge^2(\mathfrak{m}^*),\ \beta(R) = 0\right\}$$

the space of curvature tensor of type $\mathfrak{h}$, that is, of $\mathfrak{h}$-valued 2-forms which satisfy the Bianchi identity

$$\beta(R)(X,Y,Z) = R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \quad \text{for all } X,Y,Z \in \mathfrak{m}.$$

For $R \in \mathcal{R}(\mathfrak{h})$, we denote by $\mathfrak{h}(R)$ the subalgebra of $\mathfrak{h}$ generated by the curvature operators $R(X,Y)$, $X,Y \in \mathfrak{m}$. A tensor $R \in \mathcal{R}(\mathfrak{h})$ is a curvature tensor of a pseudo-Riemannian symmetric space if and only if it is invariant under the Lie algebra $\mathfrak{h}(R)$. Such a tensor defines a Lie algebra $\mathfrak{g}$ with a symmetric decomposition $\mathfrak{g} = \mathfrak{h}(R) + \mathfrak{m}$ defined by

$$[A,B] = AB - BA, \quad [A,X] = AX, \quad [X,Y] = -R(X,Y), \quad A,B \in \mathfrak{h},\ X,Y \in \mathfrak{m}.$$

Note that the associated simply connected pseudo-Riemannian symmetric space $(M = G/H, g)$ has holonomy algebra $\mathfrak{h}$. We have

Proposition 2. *There is natural* 1-1 *correspondence between simply connected pseudo-Riemannian symmetric spaces with holonomy algebra* $\mathfrak{h} \subset \mathfrak{so}(\mathfrak{m})$ (*considered up to an isometry*) *and elements* R *of the space* $\mathcal{R}(\mathfrak{h})^{\mathfrak{h}}$ *of* $\mathfrak{h}$*-invariants curvature tensors from* $\mathcal{R}(\mathfrak{h})$ *which satisfy the condition*

$$\mathfrak{h} = \mathfrak{h}(R)$$

(*considered up to a transformation from the normalizer* $N_{\mathrm{O}(\mathfrak{m})}(\mathfrak{h})$.)

1.2 Pseudo-Kähler and para-Kähler symmetric spaces

Recall that a *pseudo-Kähler* (respectively, *para-Kähler*) *manifold* is a $2n$-dimensional pseudo-Riemannian manifold (M,g) together with a parallel filed J of skew-symmetric endomorphisms such that $J^2 = -\mathrm{Id}$ (respectively, $J^2 = \mathrm{Id}$). A pseudo-Kähler manifold is called also a *bi-Lagrangian manifold.*

The existence of parallel complex structure J (such that $J^2 = -\mathrm{Id}$) (respectively, parallel *para-complex structure* J (such that $J^2 = \mathrm{Id}$)) on a pseudo-Riemannian manifold (M,g) means that the holonomy group $\mathrm{Hol}\,(M,g)$ is a subgroup of the pseudo-unitary group $\mathrm{U}(p,q)$, $p+q=n$ (respectively, the general linear group $\mathrm{GL}(n,\mathbb{R}) \subset \mathrm{SO}(n,n)$).

The curvature tensor R of a pseudo-Kähler manifold (respectively, para-Kähler manifold) (M,g) belongs to the space $\mathcal{R}(\mathfrak{u}(p,q))$ (respectively, $\mathcal{R}(\mathfrak{gl}(n,\mathbb{R}))$).

Lemma 1. *The Ricci tensor* ric(R) *of a para-Kähler or pseudo-Kähler manifold* (M,g,J) *with the curvature tensor* R *is given by*

$$\mathrm{ric}(X,JY) = -\frac{1}{2}\,\mathrm{tr}\,JR(X,JY).$$

In particular, the manifold (M,g) *is Ricci-flat if and only if the holonomy algebra is a subalgebra of* $\mathfrak{sl}(n,\mathbb{R})$ *or, respectively, of* $\mathfrak{su}(p,q)$.

Proof. Let e_i be an orthonormal basis of the tangent space T_xM such that $\langle e_i, e_i\rangle = \varepsilon_i \in \{\pm 1\}$. Then $\mathrm{ric}(X,Y) = \varepsilon_i \langle R(X,e_i)e_i, Y\rangle$ (where we omit the sign $\sum$ of summation over repeated indices). Using the Bianchi identity and the fact that J is a skew-symmetric endomorphism with $J^2 = \varepsilon\,\mathrm{Id}$ commuting with the curvature operator $R(X,Y)$, where $\varepsilon = -1$ in Kähler case and $\varepsilon = 1$ in para-Kähler case, we have

$$\begin{aligned}
\mathrm{tr}\, JR(X,Y) &= \varepsilon_i \langle JR(X,Y)e_i, e_i\rangle \\
&= -\varepsilon_i \langle R(X,Y)e_i, Je_i\rangle \\
&= -\varepsilon_i \langle R(e_i, Je_i)X, Y\rangle \\
&= \varepsilon_i \langle R(Je_i, X)e_i, Y\rangle + \varepsilon_i \langle R(X,e_i)Je_i, Y\rangle \\
&= -\varepsilon\varepsilon_i \langle R(Je_i, X)Je_i, JY\rangle + \varepsilon_i \langle R(X,e_i)Je_i, Y\rangle \\
&= \varepsilon\varepsilon_i \langle R(X, Je_i)Je_i, JY\rangle - \varepsilon_i \langle R(X,e_i)e_i, JY\rangle \\
&= -\mathrm{ric}(X, JY) - \mathrm{ric}(X, JY) \\
&= -2\,\mathrm{ric}(X, JY).
\end{aligned}$$

Here we use the identity $\langle Je_i, Je_i\rangle = -\varepsilon\varepsilon_i$. □

Lemma 2. *The space* $\mathcal{R}(\mathfrak{u}(p,q))$ (*respectively,* $\mathcal{R}(\mathfrak{gl}(n,\mathbb{R}))$) *of curvature tensors of type* $\mathfrak{u}(p,q)$ (*respectively, of type* $\mathfrak{gl}(n,\mathbb{R})$) *consists of tensors which are invariant under the complex* (*respectively, the para-complex*) *structure* J.

Proof. Using the pseudo-Euclidean metric, one can identify the spaces $\mathcal{R}(\mathfrak{u}(p,q))$ and $\mathcal{R}(\mathfrak{gl}(n,\mathbb{R}))$ with subspaces of $S^2(\mathfrak{u}(p,q))$ and $S^2(\mathfrak{gl}(n,\mathbb{R}))$, respectively, which obviously consist of J-invariant tensors. □

The following proposition, which gives an infinitesimal description of pseudo-Kähler and para-Kähler symmetric spaces, is a specialization of Theorem 1.

Proposition 3. *A simply connected symmetric space* $(M = G/K, g)$ *associated with an effective symmetric decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ *of a Lie algebra* $\mathfrak{g}$ *and an* $\mathrm{ad}_{\mathfrak{h}}$*-invariant pseudo-Euclidean metric* g_0 *on* $\mathfrak{m}$ *is a pseudo-Kähler symmetric space* (*respectively, a para-Kähler symmetric space*) *if there exists a skew-symmetric* $\mathrm{ad}_{\mathfrak{h}}$*-invariant endomorphism* J *on* $\mathfrak{m}$ *with* $J^2 = -\mathrm{Id}$ (*respectively,* $J^2 = \mathrm{Id}$).

We will call a triple $(\mathfrak{g} = \mathfrak{h}+\mathfrak{m}, g_0, J)$ described in the proposition an *infinitesimal pseudo-Kähler symmetric space* (respectively, an *infinitesimal para-Kähler symmetric space*) if $J^2 = -\mathrm{Id}$ (respectively, $J^2 = \mathrm{Id}$).

So the classification of simply connected pseudo-Kähler symmetric spaces (respectively, para-Kähler symmetric spaces) reduces to the classification of infinitesimal pseudo-Kähler symmetric spaces and (respectively, infinitesimal para-Kähler symmetric spaces).

2 Structure of para-Kähler symmetric spaces

2.1 Curvature bi-form of a para-Kähler symmetric space

Now we describe infinitesimal para-Kähler symmetric spaces in terms of admissible 3-graded Lie algebras.

Definition 1. A 3-graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1, \quad [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j} \tag{2.1}$$

(and its underlying gradation) is called *admissible* if the subalgebra $\mathfrak{g}^0$ has no non-zero ideal of $\mathfrak{g}$ and $\mathfrak{g}^0$-modules $\mathfrak{g}^{-1}$ and $\mathfrak{g}^1$ are conjugate, i.e., there is a $\mathfrak{g}^0$-invariant non-degenerate bilinear pairing

$$\mathfrak{g}^{-1} \times \mathfrak{g}^1 \to \mathbb{R}, \quad (X, \xi) \mapsto \xi(X)$$

which allows to identify $\mathfrak{g}^1$ with the dual module $(\mathfrak{g}^{-1})^*$.

We put $\mathfrak{h} = \mathfrak{g}^0$, $V = \mathfrak{g}^{-1}$ and $V^* = \mathfrak{g}^1$. An admissible graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = V + \mathfrak{h} + V^* \tag{2.2}$$

is called *minimal* if $[V, V^*] = \mathfrak{h}$.

Proposition 4. *There is a natural* 1-1 *correspondence between minimal admissible graded Lie algebras* (2.2) (*up to an isomorphism*) *and simply connected para-Kähler symmetric spaces* $(M = G/H, g, J)$ (*up to an isometry which preserves the para-complex structure*). *A minimal admissible graded Lie algebra defines the para-Kähler symmetric space* $(M = G/H, g, J)$ *associated with the minimal admissible symmetric decomposition*

$$\mathfrak{g} = \mathfrak{h} + (V + V^*) = \mathfrak{g}^0 + (\mathfrak{g}^{-1} + \mathfrak{g}^1)$$

and the natural pseudo-Euclidean metric g_0 *on* $\mathfrak{m} = V + V^*$, *defined by the pairing*

$$g_0(X + \xi, Y + \eta) = \xi(Y) + \eta(X).$$

The para-complex structure J *in* (M, g) *is obtained by the parallel extension of the* $\mathrm{ad}_{\mathfrak{h}}$*-invariant para-complex structure* $J_0 \in \mathfrak{so}(\mathfrak{m})$ *defined by* $J_0|_{\mathfrak{g}^{\pm 1}} = \pm\mathrm{Id}$. *The holonomy algebra of* (M, g) *is equal to* $\mathrm{ad}_{\mathfrak{h}}|_{\mathfrak{m}}$.

The symmetric space $(M = G/H, g)$ *is Ricci-flat* (*or, equivalently, has holonomy algebra* $\mathfrak{h} \subset \mathfrak{sl}(n)$) *if and only if* $\mathfrak{g}$ *is a solvable Lie algebra. Then* $\mathrm{ad}_{\mathfrak{h}}$ *is a nilpotent linear Lie algebra, which consists of nilpotent endomorphisms.*

Hence the problem of the classification of para-Kähler symmetric spaces reduces to the description of minimal admissible graded Lie algebras.

Proof. The proof of the first part follows from Proposition 3. Ricci-flatness is equivalent to the vanishing of the Killing form B of $\mathfrak{g}$. This implies that the Lie algebra $\mathfrak{g}$

is solvable. Conversely, assume that $\mathfrak{g}$ is solvable. Then $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ is a nilpotent subalgebra such that $\mathrm{ad}_{\mathfrak{h}}$ consists of nilpotent endomorphisms. Then Lemma 1 shows that the Ricci tensor vanishes. □

Let $\mathfrak{g} = V + \mathfrak{h} + V^*$ be an admissible graded Lie algebra and $(M = G/H, g, J)$ the associated para-Kähler symmetric space. We identify $\mathfrak{m} = V + V^*$ with the tangent space T_oM. Then

$$\mathrm{ad}_{\mathfrak{h}}|_{\mathfrak{m}} \subset V \wedge V^* \simeq \mathfrak{gl}(V) \subset \mathfrak{so}(\mathfrak{m}) \simeq \bigwedge^2 \mathfrak{m}$$

is the holonomy algebra of (M, g). We identify $\mathfrak{h}$ with a subalgebra of $\mathfrak{gl}(V) = V \wedge V^* \subset \mathfrak{so}(\mathfrak{m})$. Then the space $\mathcal{R}(\mathfrak{gl}(V))$ can be identified with the space of symmetric endomorphisms $R \in S^2(\mathfrak{gl}(V)) \subset \mathrm{End}(\mathfrak{gl}(V))$ which satisfy the following Bianchi identities: $R(X \wedge \xi)Y = R(Y \wedge \xi)X$ and $R(X \wedge \xi)\eta = R(Y \wedge \eta)\xi$ for $X, Y \in V$ and $\xi, \eta \in V^*$.

The Bianchi identities take this form since $R(V, V) = R(V^*, V^*) = 0$.

Lemma 3. *The space $\mathcal{R}(\mathfrak{gl}(V)) = \mathcal{R}(V \wedge V^*)$ of curvature tensors of type $\mathfrak{gl}(V)$ is naturally isomorphic to the space $S^2V \vee S^2V^*$, where $\vee$ is the symmetric product, hence, also to the space $S^2V \otimes S^2V^*$ of bi-forms.*

Proof. The contraction of a bi-form $R \in S^2V \otimes S^2V^*$ with a vector X and a covector ξ defines a symmetric endomorphism

$$R\colon \mathfrak{gl}(V) = V \wedge V^* \to \mathfrak{gl}(V), \quad X \wedge \xi \mapsto R(X \wedge \xi)$$

which satisfies the Bianchi identities, and hence is a curvature tensor. □

For example, if e, f are vectors and α, β covectors, then the bi-form

$$R = (e \vee f) \otimes (\alpha \vee \beta),$$

corresponds to the curvature tensor

$$R = (e \wedge \alpha) \vee (f \wedge \beta) + (e \wedge \beta) \vee (f \wedge \alpha).$$

We will often identify the curvature tensor with the corresponding curvature endomorphism and curvature bi-form.

We choose a basis e_i of the vector space V and the dual basis e^i of the dual vector space V^* and identify elements from V with vector columns, elements from V^* with vector rows and elements from $\mathfrak{gl}(V) = V \otimes V^* \simeq V \wedge V^*$ with matrices. Then the action of an element $A \in \mathfrak{gl}(V) = V \wedge V^*$ is given by

$$A(X) = AX, \quad A(\xi) = -\xi A, \quad X \in V, \ \xi \in V^*.$$

Since V and V^* are commutative subalgebras of $\mathfrak{g}$, the structure of an admissible graded Lie algebra in $\mathfrak{g} = V + \mathfrak{h} + V^*$ is defined by an $\mathfrak{h}$-invariant curvature endomorphism $R \in \mathcal{R}(\mathfrak{h})^{\mathfrak{h}}$ of type $\mathfrak{h}$. We get

Proposition 5. *Let $\mathfrak{h} \subset \mathfrak{gl}(V) = V \wedge V^*$ be a linear Lie algebra. A structure of an admissible graded Lie algebra in $\mathfrak{g} = V + \mathfrak{h} + V^*$ is defined by an $\mathfrak{h}$-invariant bi-form R of type $\mathfrak{h}$, that is, an element $R \in (S^2V \otimes S^2V^*)^{\mathfrak{h}}$ such that $R(V \wedge V^*) \subset \mathfrak{h}$. The Lie bracket in $\mathfrak{g}$ associated with the bi-form R is defined by*

$$V \times V^* \to \mathfrak{h}, \quad [X, \xi] = -[\xi, X] = R(X \wedge \xi).$$

The admissible graded Lie algebra $\mathfrak{g}$ is minimal if and only if the corresponding curvature endomorphism R of type $\mathfrak{h}$ is surjective, i.e. $R(V \wedge V^*) = \mathfrak{h}$.

Two curvature bi-forms R, R' define isomorphic admissible graded Lie algebras if and only if they belong to the same orbit of the normalizer $N_{\mathrm{GL}(V)}\mathfrak{h}$ in the space $(S^2V \otimes S^2V^*)^{\mathfrak{h}}$.

Proof. The last claim follows from the observation that an isomorphism

$$\varphi\colon V + \mathfrak{h} + V^* \to V + \mathfrak{h} + V^*$$

of admissible graded Lie algebras is canonically defined by a linear transformation $C \in \mathrm{GL}(V)$ such that the induced linear transformation of the space $V \otimes V^*$ preserves $\mathfrak{h}$. This means that C belongs to the normalizer of $\mathfrak{h}$ in $\mathrm{GL}(V)$. □

This proposition reduces the classification of simply connected para-Kähler symmetric spaces with holonomy algebra $\mathfrak{h} \subset V \otimes V^* \subset \mathfrak{so}(V + V^*)$ to a description of tensors $R \in (S^2V \otimes S^2V^*)^{\mathfrak{h}}$ such that $\mathrm{span}\{R(X \wedge \xi), X \in V, \xi \in V^*\} = \mathfrak{h}$.

We have the following corollary.

Corollary 1. *A bi-form $R \in S^2V \otimes S^2V^*$ defines a minimal admissible graded Lie algebra $\mathfrak{g} = V + \mathfrak{h} + V^*$ if and only if it is invariant under the linear Lie algebra $\mathfrak{h}$ generated by endomorphisms $R(X \wedge \xi)$ for $X \in V$, $\xi \in V^*$, that is, it satisfies the equations*

$$R(X \wedge \xi) \cdot R = 0.$$

Remark 1. Obviously, the results remain true for complex Lie algebras $\mathfrak{g}$, their complex admissible gradations $\mathfrak{g} = V + \mathfrak{h} + V^*$ (where V, $\mathfrak{h}$ and V^* are complex subspaces) and for complex curvature tensors $R \in S^2V \wedge S^2V^*$.

All admissible gradations of semisimple Lie algebras can be easily described in terms of Satake diagrams. It implies the classification of para-Kähler symmetric spaces of semisimple Lie groups, see [Kan]. Another approach to the classification is based on the fact that the holonomy algebra $\mathfrak{h}$ of a para-Kähler symmetric space has non-zero first prolongation $\mathfrak{h}^{(1)}$ since the contraction $R(\xi)$ of the curvature tensor R with a covector ξ belongs to $\mathfrak{h}^{(1)}$.

Now we prove a version of the Levi decomposition for admissible graded Lie algebras.

Consider a minimal admissible graded Lie algebra (2.2). We denote by D the derivation of $\mathfrak{g}$ defined by $D|_{\mathfrak{g}^i} = i\,\mathrm{Id}$, and by

$$\widehat{\mathfrak{g}} = V + (\mathfrak{h} + \mathbb{R}D) + V^* = V + \widehat{\mathfrak{h}} + V^* \tag{2.3}$$

the extended admissible graded Lie algebra.

Proposition 6. *There exists a Levi decomposition $\widehat{\mathfrak{g}} = \mathfrak{s} + \hat{\mathfrak{r}}$ of an extended graded Lie algebra (2.3) which is consistent with the gradation, that is,*

$$\widehat{\mathfrak{g}} = \mathfrak{s}^{-1} + \mathfrak{s}^0 + \mathfrak{s}^1 + \hat{\mathfrak{r}}^{-1} + \hat{\mathfrak{r}}^0 + \hat{\mathfrak{r}}^1,$$

where $\mathfrak{s}^i = \mathfrak{s} \cap \mathfrak{g}^i$, $\hat{\mathfrak{r}}^i = \hat{\mathfrak{r}} \cap \mathfrak{g}^i$, and the element D can be decomposed as $D = d + D'$, where $d \in \mathfrak{s}^0$, $D' \in \hat{\mathfrak{r}}^0$, $[\mathfrak{s}, D'] = 0$. The derivation $\mathrm{ad}_d|_{\mathfrak{s}}$ defines a minimal admissible gradation of $\mathfrak{s}$ and the derivation ad_D defines an admissible gradation of $\hat{\mathfrak{r}}$. Moreover, the derivation $D|_{\mathfrak{r}}$ defines an admissible gradation $\mathfrak{r} = \mathfrak{r}^{-1} + \mathfrak{r}^0 + \mathfrak{r}^1$ of the radical $\mathfrak{r}$ of $\mathfrak{g}$, where $\hat{\mathfrak{r}}^0 = \mathfrak{r}^0 + \mathbb{R}D$. If the derivation D is inner, we may take $\widehat{\mathfrak{g}} = \mathfrak{g}$, $D' = 0$ and $D = \mathrm{ad}_d$, $d \in \mathfrak{s}$.

Corollary 2. *Any minimal admissible graded Lie algebra $\mathfrak{g}$ is a semidirect sum of a minimal admissible graded semisimple Lie algebra $\mathfrak{s}$ and an admissible graded solvable ideal $\mathfrak{r} = \mathfrak{r}^{-1} + \mathfrak{r}^0 + \mathfrak{r}^1$.*

Proof of the proposition. Let $\mathfrak{a}$ be a Cartan subalgebra of $\widehat{\mathfrak{h}}$, i.e., a maximal nilpotent subalgebra of $\widehat{\mathfrak{h}}$ which coincides with its normalizer in $\widehat{\mathfrak{h}}$. It contains D and is a Cartan subalgebra of $\widehat{\mathfrak{g}}$ since $D|_{\mathfrak{g}^{\pm 1}} = \pm\mathrm{Id}$. It is known [GOV] that there is a Levi decomposition $\widehat{\mathfrak{g}} = \mathfrak{s} + \hat{\mathfrak{r}}$ which is consistent with the Cartan subalgebra $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{a}_S + \mathfrak{a}_R$, where $\mathfrak{a}_S = \mathfrak{a} \cap \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$ and $\mathfrak{a}_R$ is a Cartan subalgebra of the centralizer $Z_{\hat{\mathfrak{r}}}(\mathfrak{a}_S)$. Since the $\mathfrak{s}$-component d of $D = d + D'$ is semisimple and commutes with the $\mathfrak{r}$-component D', the element D' is also semisimple. Using the theory of splitting, see [SLie], we may assume (changing the Levi subalgebra, if needed) that $[\mathfrak{s}, D'] = 0$. Then $\mathrm{ad}_D|_{\mathfrak{s}} = \mathrm{ad}_d|_{\mathfrak{s}}$ defines a minimal admissible gradation of $\mathfrak{s}$ and ad_D defines an admissible gradation of $\hat{\mathfrak{r}}$ and $\mathfrak{r}$. Since $\mathfrak{r}^0 \subset [\mathfrak{g}, \mathfrak{r}]$, it is in the nilradical of $\mathfrak{g}$. If the derivation D is inner, D' is a semisimple element of $\mathfrak{r}^0$, hence is equal to 0. □

2.2 Ricci-flat para-Kähler symmetric spaces

Now we will study Ricci-flat para-Kähler symmetric spaces. According to Proposition 4, they correspond to minimal admissible graded solvable Lie algebras $\mathfrak{g} = V + \mathfrak{h} + V^*$, where the subalgebra $\mathfrak{h} = [V, V^*]$ is identified with a nilpotent subalgebra of $\mathfrak{gl}(V) = V \wedge V^*$ which consists of nilpotent elements. If $\mathfrak{h} \subset \mathfrak{gl}(V)$ is fixed, the structure of the graded Lie algebra in $\mathfrak{g}$ is defined by an $\mathfrak{h}$-invariant bi-form $R \in (S^2V \otimes S^2V^*)^{\mathfrak{h}}$ with $R(V \wedge V^*) \subset \mathfrak{h}$. We choose a basis $e_1, \dots, e_n$ of V such that elements of $\mathfrak{h}$ are represented by upper triangular matrices with zero diagonal

and denote by $e^1, \dots, e^n$ the dual basis of V^*. We will identify $\mathfrak{h}$ with a subalgebra of the Lie algebra $\mathfrak{n}(n)$ of upper triangular nilpotent matrices. Then a curvature endomorphism $R\colon \mathfrak{gl}(V) \to \mathfrak{h} \subset \mathfrak{gl}(V)$ of type $\mathfrak{h}$ can be written as

$$R(X \wedge \xi) = \begin{pmatrix} 0 & a_2^1(X,\xi) & a_3^1(X,\xi) & a_4^1(X,\xi) & \cdots & a_n^1(X,\xi) \\ 0 & 0 & a_3^2(X,\xi) & a_4^2(X,\xi) & \cdots & a_n^2(X,\xi) \\ 0 & 0 & 0 & a_4^3(X,\xi) & \cdots & a_n^3(X,\xi) \\ 0 & 0 & 0 & 0 & \cdots & a_n^4(X,\xi) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n^{n-1}(X,\xi) \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.4}$$

where $a_j^i(X,\xi)$ are bilinear functions of $X \in V$ and $\xi \in V^*$.

Lemma 4. *Bilinear functions $a_j^i(X,\xi)$ define a curvature endomorphism R of type $\mathfrak{h} = \operatorname{span}\{R(X,\xi),\ X \in V,\ \xi \in V^*\}$ given by (2.4) if and only if*

$$[A, R(X \wedge \xi)] = R(AX,\xi) - R(X, \xi A) \tag{2.5}$$

for any $A = R(Y \wedge \eta)$, $X, Y \in V$, $\xi, \eta \in V^*$, the expression

$$\begin{aligned} R(X \wedge \xi)Y = {} & \Big(\sum_{j>1} a_j^1(X,\xi)Y^j\Big)e_1 + \Big(\sum_{j>2} a_j^2(X,\xi)Y^j\Big)e_2 \\ & + \Big(\sum_{j>3} a_j^3(X,\xi)Y^j\Big)e_3 + \cdots + a_n^{n-1}(X,\xi)Y^n e_{n-1}. \end{aligned} \tag{2.6}$$

is symmetric with respect to any $X = X^i e_i$, $Y = Y^i e_i \in V$ for any $\xi \in V^*$ and the expression

$$\begin{aligned} R(X \wedge \eta)\xi = {} & \xi_1 a_2^1(X,\eta)e^2 + (\xi_1 a_3^1(X,\eta) + \xi_2 a_3^2(X,\eta))e^3 \\ & + \cdots + \sum_{i=1}^{n-2} (\xi_i a_{n-1}^i(X,\eta))e^{n-1} + \sum_{i=1}^{n-1} \xi_i a_n^i(X,\eta))e^n \end{aligned} \tag{2.7}$$

is symmetric with respect to $\xi = \xi_i e^i, \eta = \eta_i e^i \in V^*$ for any $X = X^i e_i \in V$.

Proof. Follows from Proposition 5. □

Note that any upper triangular nilpotent matrix $A = \|a_j^i\| \in \mathfrak{n}(n)$ defines an obvious solution of equations (2.6) and (2.7) given by

$$a_j^i(X,\xi) = a_j^i X^j \xi_i \quad \text{(no summation)}. \tag{2.8}$$

The following lemma describes when the corresponding map R is equivariant and hence is a curvature endomorphism.

Lemma 5. *The map $R\colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ associated with an upper triangular nilpotent matrix $A = \|a^i_j\|$, $a^i_j = 0$ for $i \geq j$, by formulas (2.8), (2.4) is a curvature endomorphism R of type $\mathfrak{n}(n)$ if and only if the following conditions hold:*

$$a^\ell_k \neq 0 \implies a^k_i = a^j_\ell = 0 \quad \text{for all } i,\ j. \tag{2.9}$$

In this case the image

$$\mathfrak{h} = R(\mathfrak{gl}(V)) = \operatorname{span}\{E^k_\ell = e_\ell \wedge e^k,\ a^\ell_k \neq 0\} \tag{2.10}$$

of the endomorphism R is a subalgebra of the commutative algebra $V_+ \wedge V_-^* \subset \mathfrak{n}(n)$, where $V = V_+ + V_-$ is a direct sum decomposition of V and $V^* = V_+^* + V_-^*$ is the dual decomposition of V^*.

Proof. It is clear that the image of the linear map R defined by a matrix A is given by (2.10). The map R is a curvature endomorphism if it is $R(\mathfrak{gl}(V))$-equivariant, that is,

$$[E^k_\ell, R(X \wedge \xi)] = R([E^k_\ell, X \wedge \xi])$$

for $X \in V$, $\xi \in V^*$ and for all (k, ℓ) such that $a^\ell_k \neq 0$. It is sufficient to write these conditions for basic vectors $X = e_i$ and $\xi = e^j$. Since $R(E^j_i) = a^j_i E^i_j$, we get the following conditions:

$i = k$:

$$[E^k_\ell, R(E^j_k)] = [E^k_\ell, a^j_k E^k_j] = 0 = R([E^k_\ell, E^j_k]) = R(E^j_\ell) = a^\ell_j E^j_\ell \implies a^\ell_j = 0;$$

$i = \ell$:

$$[E^k_\ell, R(E^j_\ell)] = [E^k_\ell, a^j_\ell E^\ell_j] = a^j_\ell E^k_j = R([E^k_\ell, E^j_\ell)]) = 0 \implies a^j_\ell = 0;$$

$j = k$:

$$[E^k_\ell, R(E^k_i)] = [E^k_\ell, a^k_i E^i_k] = a^k_i E^i_\ell = R([E^k_\ell, E^k_i]) = 0 \implies a^k_i = 0;$$

$j = \ell$:

$$[E^k_\ell, R(E^\ell_i)] = [E^k_\ell, a^\ell_i E^i_\ell] = 0 = R([E^k_\ell, E^\ell_i)]) = R(E^k_i) = a^k_i E^i_k \implies a^k_i = 0.$$

These are exactly conditions (2.9). They imply $A^2 = 0$. We put $V_+ = AV$ and denote by V_- a complementary subspace. Then $\mathfrak{h}V = V_+$, $\mathfrak{h}V_+ = 0$ and $\mathfrak{h} \subset V_+ \wedge V_-^*$. □

In the next section we describe para-Kähler symmetric spaces with holonomy algebra $\mathfrak{h} \subset V_+ \wedge V_-^*$.

2.3 Para-Kähler symmetric spaces with holonomy algebra $\mathfrak{h} \subset V_+ \wedge V_-^*$

Let $V = V_+ + V_-$ be a direct sum decomposition of a vector space V and $V^* = V_+^* + V_-^*$ the dual decomposition of the dual space V^*.

We will describe all para-Kähler symmetric spaces with holonomy algebra

$$\mathfrak{h} \subset V_+ \wedge V_-^* \subset \mathfrak{so}(\mathfrak{m}) = \Lambda^2(V + V^*).$$

Proposition 7. *A curvature endomorphism R of the type $\mathfrak{h} = V_+ \wedge V_-^*$ is $\mathfrak{h}$-invariant. Such endomorphisms form the vector space $S^2V_+ \vee S^2V_-^* \simeq S^2V_+ \otimes S^2V_-^*$.*

Proof. The proof is straightforward. □

A curvature endomorphism $R \in S^2V_+ \vee S^2V_-^*$ can be written as

$$R = a_1 \vee \alpha_1 + \cdots + a_k \vee \alpha_k,$$

where $a_i \in S^2V_+$ and $\alpha_i \in S^2V_-^*$ are symmetric bilinear forms on V_+^* and V_-, respectively. We will call the minimal possible number k the *rank* of R. The following theorem gives a description of simply connected para-Kähler symmetric spaces with holonomy $\mathfrak{h} \subset V_+ \wedge V_-^*$.

Theorem 2. *There exists a natural* 1-1 *correspondence between curvature endomorphisms $R \in S^2V_+ \vee S^2V_-^*$ (up to a transformation from $\mathrm{GL}(V_+) \times \mathrm{GL}(V_-^*)$), admissible graded Lie algebras of the form $\mathfrak{g} = V + V_+ \wedge V_-^* + V^*$ (up to an isomorphism) and simply connected para-Kähler symmetric spaces with the holonomy algebra $\mathfrak{h} \subset V_+ \wedge V_-^*$ (up to an isometry). The following conditions are equivalent:*

a) *The symmetric space has the holonomy algebra $\mathfrak{h} = V_+ \wedge V_-^*$.*

b) *The admissible graded Lie algebra $\mathfrak{g}$ is minimal.*

c) *The curvature endomorphism $R\colon V \wedge V^* \to \mathfrak{h} = V_+ \wedge V_-^*$ is surjective.*

Proof. The proof follows from Propositions 4, 5 and 7. We put $n_\pm = \dim V_\pm$. □

Corollary 3. *Any rank one surjective curvature endomorphism R of type $\mathfrak{h} = V_+ \wedge V_-^*$ has the form $R = a \vee \alpha$, where a, α are non-degenerate symmetric bilinear forms of signature $(k, n_+ - k)$ and $(\ell, n_- - \ell)$ on V_+^* and V_-, respectively. With respect to an appropriate basis e_i^+ of V_+ and e_-^j of V_-^*, it can be written as*

$$R = \sum \varepsilon_i (e_i^+)^2 \vee \sum \varepsilon^j (e_-^j)^2,$$

where e_i^+ and e_-^j is a basis of V_+ and V_-^*, respectively, and $\varepsilon_i = -1$ for $i \le k$ and $\varepsilon_i = 1$ for $i > k$, $\varepsilon^j = -1$ for $j \le \ell$ and $\varepsilon^j = 1$ for $j > \ell$.

The structure of the associated minimal admissible graded Lie algebra $\mathfrak{g} = V + V_+ \wedge V_-^*$ is given by

$$[e_i^+, e_-^j] = \varepsilon_i \varepsilon^j \delta_i^j.$$

2.4 Ricci-flat para-Kähler symmetric spaces of dimension ≤ 6

Proposition 4 shows that any 2-dimensional Ricci-flat para-Kähler symmetric space is flat. Now we describe all Ricci-flat para-Kähler symmetric spaces of dimension 4 and 6.

According to Proposition 5 and Lemma 4, a 4-dimensional Ricci-flat para-Kähler symmetric space is determined by a curvature endomorphism $R(X,\xi) \subset V \wedge V^*$, V a 2-dimensional space, of the form

$$R(X,\xi) = \begin{pmatrix} 0 & a(X,\xi) \\ 0 & 0 \end{pmatrix}, \tag{2.11}$$

where $a(X,\xi)$ is a bilinear function on $V \times V^*$, such that

$$a(X,\xi)Y^2 = a(Y,\xi)X^2, \quad a(X,\xi)\eta_1 = a(X,\eta)\xi_1$$

for all

$$X = X^1e_1 + X^2e_2, \quad Y = Y^1e_1 + Y^2e_2, \quad \xi = \xi_1e^1 + \xi_2e^2, \quad \eta = \eta_1e^1 + \eta_2e^2.$$

Here e_1, e_2 is a basis of V and e^1, e^2 the dual basis of V^*. The conditions (2.6), (2.7) imply that $a(X,\xi) = aX^2\xi_1$, where a is a constant. The condition (2.5) is satisfied Changing the basis e_1, e_2 we can always attain $a = \varepsilon = \pm 1$. The corresponding curvature bi-form is $R = \varepsilon e_1 \otimes e_1 \otimes e^2 \otimes e^2$. We get

Theorem 3. *Any non-flat Ricci flat para-Kähler symmetric space of dimension* 4 *has the commutative holonomy algebra*

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},\ a \in \mathbb{R} \right\} \subset \mathfrak{gl}(2,\mathbb{R}). \tag{2.12}$$

and is defined (*up to an isometry*) *by the curvature bi-form* $R = \varepsilon e_1 \otimes e_1 \otimes e^2 \otimes e^2$ (*or corresponding curvature tensor* $R = \frac{1}{4}\varepsilon(e_1 \wedge e^2) \vee (e_1 \wedge e^2)$)), *where* $\varepsilon = \pm 1$, e_1, e_2 *is a basis of* V *and* e^1, e^2 *the dual basis of* V^*.

In the 6-dimensional case we have the following

Theorem 4. *Any* 6*-dimensional para-Kähler indecomposable Ricci-flat symmetric space has commutative holonomy algebra* $\mathfrak{h}$ *isomorphic to*

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} 0 & c_1 & c_2 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{h}_2 = \left\{ \begin{pmatrix} 0 & c_1 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \text{or} \quad \mathfrak{h}_3 = \left\{ \begin{pmatrix} 0 & 0 & c_2 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2.13}$$

where $c_1, c_2, \in \mathbb{R}$ *and is determined by the curvature endomorphism* (*or curvature bi-forms and curvature tensor*) *given as follows.*

For $\mathfrak{h}_1$*:*

$$R(X,\xi) = \left\{ \begin{pmatrix} 0 & a_1(X,\xi) & a_2(X,\xi) \\ 0 & 0 & a_1(X,\xi) \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$a_1 = 2\xi_1 X^3, \quad a_2 = 2\xi_1 X^2 + 2\xi_2 X^3,$$

$$R(X,\xi) = 4(\xi_1^2 X^2 X^3 + \xi_1\xi_2 (X^3)^2) \quad (\textit{the curvature bi-form}),$$

$$R = (e_1 \wedge e^3) \vee (e_1 \wedge e^2 + e_2 \wedge e^3) \quad (\textit{the curvature tensor}).$$

For $\mathfrak{h}_2$*:*

$$R(X,\xi) = \left\{ \begin{pmatrix} 0 & a_1(X,\xi) & a_2(X,\xi) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$a_1 = \varepsilon_2 \xi_1 X^2, \quad a_2 = \varepsilon_3 \xi_1 X^3, \quad \textit{where } \varepsilon_2, \varepsilon_3 = \pm 1,$$

$$R = \xi^1 (a_1 X^2 + a_2 X^3) = \xi_1^2 [\varepsilon_2 (X^2)^2 + \varepsilon_3 (X^3)^2] \quad (\textit{the curvature bi-form}),$$

$$R = \frac{1}{4}[\varepsilon_2 (e_1 \wedge e^2) \vee (e_1 \wedge e^2) + \varepsilon_3 (e_1 \wedge e^3) \vee (e_1 \wedge e^3] \quad (\textit{the curvature tensor}).$$

For $\mathfrak{h}_3$*:*

$$R(X,\xi) = \left\{ \begin{pmatrix} 0 & 0 & a_2(X,\xi) \\ 0 & 0 & a_1(X,\xi) \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$a_1 = \varepsilon_2 \xi_2 X^3, \quad a_2 = \varepsilon_1 \xi_1 X^3, \quad \textit{where } \varepsilon_1, \varepsilon_2 = \pm 1,$$

$$R = (\varepsilon_1 \xi_1^2 + \varepsilon_2 \xi_2^2)(X^3)^2 \quad (\textit{the curvature bi-form}),$$

$$R = \frac{1}{4}[\varepsilon_1 (e_1 \wedge e^3) \vee (e_1 \wedge e^3) + \varepsilon_2 (e_2 \wedge e^3) \vee (e_2 \wedge e^3] \quad (\textit{the curvature tensor}).$$

Proof. Lemma 4 reduces the classification of 6-dimensional Ricci-flat para-Kähler symmetric spaces to the description of three bilinear forms $a_3 = a_2^1(X,\xi)$, $a_2 = a_3^1(X,\xi)$, $a_1 = a_3^2(X,\xi) \in V^* \times V$, where V is the 3-dimensional space with a basis e_1, e_2, e_3, which satisfy the conditions (2.5), (2.6), (2.7).

Solving equations (2.6), (2.7), we get

$$a_1 = (\alpha\xi_1 + \alpha'\xi_2)X^3, \ \ a_3 = \xi_1(\gamma X^3 + \gamma' X^2), \ \ a_2 = \gamma\xi_1 X^2 + \beta\xi_1 X^3 + \alpha\xi_2 X^3$$

for some $\alpha, \alpha', \beta, \gamma, \gamma' \in \mathbb{R}$. Let

$$C = \begin{pmatrix} 0 & c_3 & c_2 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.14}$$

be an element from the holonomy algebra $\mathfrak{h} = \mathrm{span}\{R(X,\xi), X \in V, \xi \in V^*\}$. Then the condition (2.5) (with $A = C$) gives

$$\alpha' c_3 = 0, \quad \gamma' c_1 = 0, \quad (\gamma - a_3)c_2 = (\alpha - a_1)c_3. \tag{2.15}$$

Assume that $c_1 \neq 0, c_3 \neq 0$ for some $C \in \mathfrak{h}$. Then $\alpha' = \gamma' = 0$ and the formulas

$$a_1 = \alpha\xi_1 X^3, \quad a_3 = \gamma\xi_1 X^3, \quad a_2 = \gamma\xi_1 X^2 + \beta\xi_1 X^3 + \alpha\xi_2 X^3$$

give a general solution. Using an appropriate change of the basis of the form

$$e_1 \to b_1 e_1, \quad e_2 \to b_2 e_2 + a e_1, \quad e_3 \to b_3 e_3,$$

we can transform the coefficient β into zero and α into 2. Then the curvature bi-form can be written as $R = 4[\xi_1^2 X^2 X^3 + \xi_1\xi_2 (X^3)^2]$.

The coefficients c_1, c_3 cannot be both identically zero, since the manifold is indecomposable. Assume that $c_1 \equiv 0$. Then the equations (2.15) give

$$a_1 = 0, \quad a_2 = \gamma\xi_1 X^2 + \beta\xi_1 X^3, \quad a_3 = \xi_1(\gamma X_3 + \gamma' X^2)$$

and the bi-form R can be written as

$$R = \xi_1(a_3 X^2 + a_2 X^3) = \xi_1^2[\gamma'(X^2)^2 + 2\gamma X^2 X^3 + \beta(X^3)^2].$$

Using the appropriate change of the basis e_i, we may assume that $\gamma = 0, \gamma', \beta \in \pm 1$.

The proof in the case $c_3 \equiv 0$ is similar. □

The following proposition, which we will use later, describes the stability subalgebra of the full algebra of infinitesimal automorphisms of the para-Kähler symmetric space (M, g, J) associated with the curvature tensor $R = (e_1 \wedge e^3) \vee (e_1 \wedge e^2 + e_2 \wedge e^3)$. It consists of all endomorphisms $C \in \mathrm{End}(V)$ (naturally extended to $V + V^*$) which preserve the curvature tensor R.

Proposition 8. *The group* $= \mathrm{Aut}(R)$ *of automorphisms of the space* V *which preserve the curvature tensor* $R = (e_1 \wedge e^3) \vee (e_1 \wedge e^2 + e_2 \wedge e^3)$ *consists of automorphisms of the form*

$$V \ni X \mapsto \lambda C X,$$

where $\lambda \in \mathbb{R}^*$ *and*

$$C = \begin{pmatrix} 1 & c & c' \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad c, c' \in \mathbb{R}. \tag{2.16}$$

Proof. Since the curvature tensor R determines the flag $\mathbb{C}e_1 \subset \mathbb{C}e_1 + \mathbb{C}e_2 \subset V$, the matrix $\|c_j^i\|$ of an automorphism C which preserves R with respect to the basis e_1, e_2, e_3 has the upper triangular form. We denote by $\|\hat{c}_j^i\|$ the inverse matrix and put

$$f_i = Ce_i = c_i^j e_j, \quad f^i = C(e^i) = \hat{c}_j^i e^j.$$

Then

$$\begin{aligned} f_1 &= c_1^1 e_1, & f^1 &= \hat{c}_1^1 e^1 + \hat{c}_2^1 + \hat{c}_3^1 e^3, \\ f_2 &= c_2^1 e_1 + c_2^2 e_2, & f^2 &= \hat{c}_2^2 e^2 + \hat{c}_3^2 e^3, \\ f_3 &= c_3^1 e_1 + c_3^2 e_2 + c_3^3 e_3, & f^3 &= \hat{c}_3^3 e^3. \end{aligned}$$

The condition $C(R) = R$ can be written as

$$\begin{aligned} C(e_1 \wedge e^3) &= f_1 \wedge f^3 = c_1^1 \hat{c}_3^3 e_1 \wedge e^3 = \lambda e_1 \wedge e^3, \\ C(e_1 \wedge e^2 + e_2 \wedge e^3) &= f_1 \wedge f^2 + f_2 \wedge f^3 \\ &= c_1^1 \hat{c}_2^2 e_1 \wedge e^2 + c_2^2 \hat{c}_3^3 e_2 \wedge e^3 + (c_1^1 \hat{c}_3^2 + c_2^1 \hat{c}_3^3) e_1 \wedge e^3 \\ &= \lambda (e_1 \wedge e^2 + e_2 \wedge e^3) \end{aligned}$$

for some $\lambda \in \mathbb{R}^*$. We get the system of equations

$$\lambda = c_1^1 \hat{c}_2^2 = c_2^2 \hat{c}_3^3 = c_1^1 \hat{c}_3^3, \quad 0 = c_1^1 \hat{c}_3^2 + c_2^1 \hat{c}_3^3,$$

which together with the equalities

$$\hat{c}_i^i = (c_i^i)^{-1}, \quad \hat{c}_3^2 = -\frac{c_1^1 c_3^2}{c_1^1 c_2^2 c_3^3}$$

gives $c_1^1 = c_2^2 = c_3^3$, $c_2^1 = c_3^2$. □

Remark 2. Note that the given classification of admissible graded Lie algebras of dimension $n \leq 6$ remains valid also for complex Lie algebras.

3 Structure of pseudo-Kähler symmetric spaces

Now we reduce the classification of infinitesimal pseudo-Kähler symmetric spaces to a description of admissible anti-involutions of a complex admissible graded Lie algebras $\mathfrak{g} = V + \mathfrak{h} + V^*$.

Let $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, g_0 = \langle \cdot, \cdot \rangle, J)$ be an infinitesimal pseudo-Kähler symmetric space. We consider the complexification

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$$

of the symmetric decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and extend the endomorphism J to an endomorphism $\hat{J}$ of $\mathfrak{g}$ putting $\hat{J}\mathfrak{h} = 0$.

Lemma 6. *$\hat{J}$ is a derivation of the Lie algebra $\mathfrak{g}$ with eigenvalues $\pm i$, 0.*

Proof. The result follows from Lemma 2. □

The derivation $D = i\hat{J}$ of $\mathfrak{g}^{\mathbb{C}}$ has eigenvalues ± 1 and 0, and defines an admissible gradation

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1} = V + \mathfrak{h}^{\mathbb{C}} + V^*$$

of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Here $\mathfrak{m}^{\mathbb{C}} = V + V^*$ is a decomposition of the complex Euclidean space into a direct sum of two $g_0^{\mathbb{C}}$-isotropic dual subspaces V, V^* and the complex metric $g = g_0^{\mathbb{C}}$ of $\mathfrak{m}^{\mathbb{C}}$ induced by g_0 is identified with the natural metric defined by the pairing. Like in the real case, the admissible gradation is minimal and it is determined by a surjective curvature endomorphism $R \in (S^2V \otimes S^2V^*)^{\mathfrak{h}^{\mathbb{C}}}$ of type $\mathfrak{h}^{\mathbb{C}}$. Note that the complex conjugation σ of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$ is an antilinear involutive automorphism of $\mathfrak{g}^{\mathbb{C}}$ (considered as a real Lie algebra) which satisfies the following properties:

1) $\sigma : V \to V^*, \sigma : V^* \to V, \sigma : \mathfrak{h}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}$,

2) the restriction of the complex Euclidean metric g to the fixed point space $\mathfrak{m} = (V + V^*)^{\sigma}$ of σ is a (real) pseudo-Euclidean metric.

The real Lie algebra $\mathfrak{g}$ is the fixed point set $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^{\sigma}$ of σ and the symmetric decomposition of $\mathfrak{g}$ is induced by the gradation of $\mathfrak{g}^{\mathbb{C}}$ as follows:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = \mathfrak{g}^{\sigma} = (\mathfrak{h}^{\mathbb{C}})^{\sigma} + (V + V^*)^{\sigma}.$$

Definition 2. Let $\mathfrak{g} = V + \mathfrak{h} + V^*$ be an admissible complex graded Lie algebra and D is the derivation associated with the gradation:

$$D\mathfrak{h} = 0, \quad D|_V = \mathrm{Id}, \quad D|_{V^*} = -\mathrm{Id}.$$

An antilinear involutive automorphism σ of $\mathfrak{g}$ which satisfies properties 1) and 2) is called an *admissible anti-involution* of $\mathfrak{g}$.

Proposition 9. *An admissible anti-involution σ of a complex admissible graded Lie algebra $\mathfrak{g} = V + \mathfrak{h} + V^*$ defines an infinitesimal pseudo-Kähler symmetric space $(\mathfrak{g}^{\sigma} = \mathfrak{h}^{s} + \mathfrak{m} = \mathfrak{h}^{\sigma} + (V + V^*)^{\sigma}), g_0, J)$ where $g_0 = g|_{\mathfrak{m}}$ is the restriction of the complex metric g to $\mathfrak{m}$ and the complex structure $J = -iD|_{\mathfrak{m}}$. Moreover, any infinitesimal pseudo-Kähler symmetric space can be obtained in this way.*

Proof. The proof is straightforward. □

We will show that any admissible anti-involution σ is determined by an antilinear bijective map $\sigma_0 : V \to V^*$ which satisfies certain conditions.

We will identify $\mathfrak{h}^{\mathbb{C}}$ with a complex subalgebra of the Lie algebra

$$\mathfrak{gl}(V) = V \wedge V^* \subset \bigwedge^2 (V + V^*) = \mathfrak{so}(V + V^*).$$

Note that an admissible anti-involution σ is determined by its restriction σ_0 to V_0 since $\sigma|_{V^*} = (s_0)^{-1}|_{V^*}$. The action of σ on $\mathfrak{h}$ is the restriction of the action on $V \wedge V^*$

induced by the action of σ on $V + V^*$. Indeed, since σ is an automorphism of $\mathfrak{g}^{\mathbb{C}}$ as a real Lie algebra, we get

$$\sigma[A, X] = \sigma(AX) = [\sigma(A), \sigma(X)] = \sigma(A)(\sigma(X)) \quad \text{for all } A \in \mathfrak{h}^{\mathbb{C}},\ X \in V.$$

This shows that $\sigma(A) = \sigma \circ A \circ \sigma$.

The following proposition gives necessary and sufficient conditions that an antilinear bijective map $\sigma\colon V \to V^*$ defines an admissible anti-involution of a given complex admissible graded Lie algebra.

With respect to a basis $e_1, \dots, e_n$ of V and the dual basis $e^1, \dots, e^n$ of V^*, an antilinear map $\sigma_0\colon V \to V^*$ is defined by a complex $n \times n$ matrix $B = \|b_{ij}\|$ where

$$\sigma_0(e_i) = b_{ij} e^j.$$

The vectors

$$f_j := e_j + \sigma e_j = e_j + b_{jk} e^k, \quad f'_j := i(e_j - \sigma e_j) = i(e_j - b_{jk} e^k)$$

form a basis of the space $\mathfrak{m} = (V + V^*)^{\sigma}$ where σ is the extension of σ_0 to an antilinear involution of $V + V^*$. The Gram matrix of the induced symmetric bilinear form $g_0 := g|_{\mathfrak{m}}$ with respect to this basis is given by

$$\begin{aligned} g_{ij} &= g(f_i, f_j) = b_{ij} + b_{ji}, \\ g_{i'j'} &= g(f'_i, f'_j) = b_{ij} + b_{ji}, \\ g_{i'j} &= g(f'_i, f_j) = i(b_{ij} - b_{ji}). \end{aligned}$$

In particular, g induces a pseudo-Riemannian metric on $\mathfrak{m}$ if B is a non-degenerate Hermitian matrix. This implies

Proposition 10. *Let $\mathfrak{g} = V + \mathfrak{h} + V^*$ be a complex minimal admissible graded Lie algebra defined by a (complex) curvature endomorphism $R \in (S^2V \vee S^2V^*)^{\mathfrak{h}}$. Let $\sigma_0\colon V \to V^*$ be an antilinear map defined by a matrix $B = \|b_{ij}\|$. It defines an admissible anti-involution σ of $\mathfrak{g}$ if and only if the following conditions hold:*

(1) *The matrix B is Hermitian ($\bar{b}_{ij} = b_{ji}$) and non-degenerate.*

(2) *The induced antilinear involution $\sigma\colon V + V^* \to V + V^*$ (acting naturally on tensors) preserves the tensor $R \in S^2V \vee S^2V^*$ and, hence, the complex subalgebra $\mathfrak{h} \subset V \wedge V^*$.*

Proof. Assume that σ_0 satisfies the conditions. The second condition implies that the induced antilinear involution σ is an involutive automorphism of $\mathfrak{g}$ as a real Lie algebra and the first condition shows that the induced metric on $\mathfrak{m} = (V + V^*)^{\sigma}$ is real and non-degenerate. Hence, σ is an admissible anti-involution. The converse claim is clear. □

3.1 A construction of pseudo-Calabi–Yau symmetric spaces with commutative holonomy

In this section we describe pseudo-Riemannian Calabi–Yau symmetric spaces with commutative holonomy algebra $\mathfrak{h}$ such that the complexification $\mathfrak{g}^{\mathbb{C}} = V + \mathfrak{h}^{\mathbb{C}} + V^*$ of the corresponding infinitesimal symmetric space $(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, g_0, J)$ is an admissible minimal graded complex Lie algebra described in Section 2.3.

According to Proposition 10, the problem reduces to the description of admissible anti-involutions σ of the complex admissible graded Lie algebra of the form

$$\mathfrak{g} = V + \mathfrak{h} + V^* \tag{3.1}$$

defined by a curvature endomorphism $R \in S^2V_+ \vee S^2V_-^*$ where $V = V_+ + V_-$ is a decomposition of a complex vector space into a direct sum of subspaces, $V^* = V_+^* + V_-*$ the dual decomposition and $\mathfrak{h} = R(V \wedge V^*) \subset V_+ \wedge V_-^*$.

Lemma 7. *An admissible anti-involution σ of a complex admissible graded Lie algebra of the form* (3.1) *may exist only when* $\dim V_+ = \dim V_-$.

Proof. Since $\sigma(V) = V^*$, $\sigma(V^*) = V$, the condition $\sigma(\mathfrak{h}) = \sigma(V_+) \wedge \sigma(V_-^*) = V^+ \wedge V_-^*$ implies that $\sigma(V_+) = V_-^*$, $\sigma(V_-^*) = V_+$. Hence

$$\dim V_+ = \dim V_-^* = \dim V_-. \qquad \square$$

Let σ be an admissible anti-involution. Then

$$\sigma(V) = V^*, \quad \sigma(V^*) = V, \quad \sigma(\mathfrak{h}) = \sigma(\mathfrak{h}) = V_+ \wedge V_-^*.$$

Hence, σ interchanges V_+ and V_-^*. Conversely, if an antilinear bijective map $\sigma_0 : V \to V^*$ transforms V_+ into V_-^*, then the induced involutive map σ preserves $V_+ \wedge V_-^*$. Now we find conditions that σ preserves the curvature bi-form $R \in S^2V_+ \vee S^2V_-^*$. We set

$$W = \operatorname{span}\{R(X, Y),\ X, Y \in V_-\} \subset S^2V_+,$$

$$W' = \operatorname{span}\{R(\xi, \eta),\ \xi, \eta \in V_+^*\} \subset S^2V_-^*,$$

where $R(\xi, \eta)$ and $R(X, Y)$ denote the corresponding contractions. It $\sigma(R) = R$, then $\sigma W = W'$. We choose a basis

$$a_1, \dots, a_k$$

of the space W and denote by

$$a_1' = \sigma(a_1), \dots, a_k' = \sigma(a_k)$$

the corresponding basis of W'. Then the curvature bi-form can be written as

$$R = \sum h_{ij} a_i \vee a_j',$$

where $H = \|h_{ij}\|$ is a non-degenerate complex matrix. The condition

$$R = \sigma(R) = \sum \bar{h}_{ji} a_j \vee a'_i$$

means that the matrix C is Hermitian, i.e., $C^* = C$.

Now we can describe admissible anti-involutions of admissible graded Lie algebras of the form (3.1), where $V = V_+ + V_-$ is a decomposition of a complex $2m$-dimensional vector space V into a sum of two m-dimensional subspaces and $V^* = V_+^* + V_-^*$ is the dual decomposition. We denote by e_i a basis of V and by e^j the dual basis of V^*.

Theorem 5. *Let*

$$\sigma_0 \colon V \to V^*, \quad e_i \mapsto b_{ij} e^j,$$

be an antilinear map with $\sigma_0(V_+) = V_-^*$ which is defined by a non-degenerate Hermitian matrix $B = \|b_{ij}\|$ and $\sigma \colon V + V^* \to V + V^*$ is the associated anti-involution of $V + V^*$. Let $a_1, \dots, a_k$ be linear independent elements from $S^2 V_+$ and $a'_1 = \sigma(a_1), \dots, a'_k = \sigma(a_k)$ corresponding linear independent elements from $S^2 V_-^*$. Then any non-degenerate Hermitian $k \times k$ matrix $H = \|h_{ij}\|$ defines a σ-invariant curvature bi-form

$$R = \sum h_{ij} a_i \vee a'_j \tag{3.2}$$

of rank k and hence a structure of complex admissible Lie algebra in

$$\mathfrak{g} = V + V_+ \wedge V_-^* + V^*.$$

Moreover, σ is an admissible anti-involution of this algebra and it defines an infinitesimal pseudo-Kähler symmetric space

$$(\mathfrak{g}^\sigma = \mathfrak{h}^\sigma + (V + V^*)^\sigma, g_0, J = -iD|^\sigma_{V+V^*}),$$

where $\mathfrak{h}^\sigma$ is the Lie subalgebra of $V_+ \wedge V_-$ spanned by curvature operators $R(u, v)$, $u, v \in (V + V^*)^\sigma$, and g_0 is the restriction of the natural complex metric g on $V + V^*$ to $(V + V^*)^\sigma$.

Conversely, any infinitesimal pseudo-Kähler symmetric space whose complexified holonomy algebra is a subalgebra of $V_+ \wedge V_-^* \subset \mathfrak{so}(V + V^*)$ can be obtained by this construction.

Proof. The tensor (3.2) defines the structure of admissible graded Lie algebra in $\mathfrak{g} = V + V_+ \wedge V_-^* + V^*$ and is invariant under the anti-involution σ. Hence σ is an admissible anti-involution and by Proposition 10 it defines an infinitesimal pseudo-Kähler symmetric space. It is clear that any admissible anti-involution of an admissible graded algebra (3.1) can be obtained by this construction. □

3.2 Pseudo-Calabi–Yau symmetric spaces of dimension $n \leq 6$

Here we classify all pseudo-Calabi–Yau symmetric spaces of dimension $n \leq 6$. Since any such space of dimension 2 is flat, it is sufficient to consider the spaces of dimension 4 and 6.

Theorem 6. *There exist two (up to an isometry) 4-dimensional indecomposable pseudo-Kähler symmetric spaces. They correspond to the real forms $\mathfrak{g}^\sigma$ of the complex infinitesimal para-Kähler symmetric space $\mathfrak{g} = V + \mathfrak{h}_1 + V^*$, $V \approx \mathbb{C}^2$ with the curvature tensor $R = (e_1 \wedge e^2) \vee (e_1 \wedge e^2)$ defined by the admissible anti-involution*

$$\sigma = \sigma_\varepsilon \colon e_1 \to \varepsilon e^2, \quad e_2 \to \bar{\varepsilon} e^1, \quad e^1 \to \bar{\varepsilon} e_2, \quad e^2 \to \varepsilon e_1,$$

where $\varepsilon = 1, i$.

Proof. By Proposition 10, admissible anti-involution σ is defined by a Hermitian matrix $B = \|b_{ij}\|$ such that $\sigma e_i = b_{ij} e^j$. Then $\sigma e_j = b^{jk} e_k$ where $B' = \|b^{jk}\| = \bar{B}^{-1}$. The condition $\sigma R = R$ is equivalent to the condition

$$\sigma(e_1 \wedge e^2) = \pm e_1 \wedge e^2. \tag{3.3}$$

We may assume that

$$B = \begin{pmatrix} 0 & \bar{b} \\ b & \gamma \end{pmatrix}.$$

Then

$$B' = \frac{1}{\|b\|^2} \begin{pmatrix} -\gamma & b \\ \bar{b} & 0 \end{pmatrix},$$

where $\gamma \in \mathbb{R}$. The equation (3.3) gives $b^2 = \pm \|b\|^2$. Changing the basis e_1, e_2, we may assume that $\|b\| = 1$ and $\gamma = 0$. Then $b = \pm 1, \pm i$. Moreover, multiple e_1 to -1, if needed, we may assume that $b = 1$ or $b = i$. □

Theorem 7. *There exist two (up to an isometry) 6-dimensional indecomposable pseudo-Kähler symmetric spaces. They correspond to the real forms $\mathfrak{g}^\sigma$ of the complex infinitesimal para-Kähler symmetric space $\mathfrak{g} = V + \mathfrak{h}_1 + V^*$ with the curvature tensor*

$$R = (e_1 \wedge e^3) \vee (e_1 \wedge e^2 + e_2 \wedge e^3) \tag{3.4}$$

defined by admissible anti-involutions

$$\sigma = \sigma_\varepsilon \colon e_1 \to \varepsilon e^3, \quad e_2 \to \varepsilon e^2, \quad e_3 \to \varepsilon e^1,$$

where $\varepsilon = \pm 1$, e_1, e_2, e_3 is a basis of the complex 3-dimensional space V and e^1, e^2, e^3 the dual basis of V^*.

Proof. We have to describe admissible anti-involutions of the complex admissible graded 6-dimensional Lie algebras $\mathfrak{g} = V + \mathfrak{h}_i + V^*$, $i = 1, 2, 3$ described in Theorem 4. Note that $\mathfrak{h}_2 \subset e_1 \otimes V^*$ and $\mathfrak{h}_3 \subset V \otimes e^3$. Then any admissible

anti-involution must map V^* into $\mathbb{C}e_1$, or, respectively, V into $\mathbb{C}e^1$. This implies that the rank of B is one, which is impossible. It remains to describe an admissible anti-involution σ which preserves $R = (e_1 \wedge e^3) \vee (e_1 \wedge e^2 + e_2 \wedge e^3)$. For such an anti-involution σ we have

$$\sigma(e_1 \wedge e^3) = \varepsilon(e_1 \wedge e^3), \sigma(e_1 \wedge e^2 + e_2 \wedge e^3) = \varepsilon(e_1 \wedge e^2 + e_2 \wedge e^3),$$

where $\varepsilon = \pm 1$. Hence σ transforms the flag $\operatorname{span}\{e_1\} \subset \operatorname{span}\{e_1, e_2\} \subset V$ into the flag $\operatorname{span}\{e^3\} \subset \operatorname{span}\{e^3, e^2\} \subset V^*$. In other words, the matrix B of $\sigma|_V : V \to V^*$ and the matrix B' of $\sigma|_{V^*} : V^* \to V$ have the form

$$B = \begin{pmatrix} 0 & 0 & b \\ 0 & \beta & c \\ \bar{b} & \bar{c} & \delta \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} \delta' & c' & b^3 \\ \bar{c}' & \beta' & 0 \\ b^1 & 0 & 0 \end{pmatrix}.$$

Since $\sigma^2 = \mathrm{Id}$, we have $B\bar{B}' = \mathrm{Id}$. This gives

1) $b b' = 1$,
2) $\beta\beta' = 1$,
3) $\beta c' + c\beta' = 0$,
4) $\bar{b}\gamma' + \bar{c}c' + \delta b' = 0$,
5) $\bar{b}\bar{c}' + \bar{c}\beta' = 0$.

The condition that σ preserves R can be written as

1)′ $b\bar{b}' = -\varepsilon$,
2)′ $\bar{b}c' + b'\bar{c} = 0$,
3)′ $\bar{b}\beta' = -\varepsilon$,
4)′ $b'\beta = -\varepsilon$.

The conditions 1), 1′), 3′), 4′) give $\varepsilon = -1$, $b = \bar{b} = \beta = \frac{1}{\beta'} = \frac{1}{b'}$. Then the conditions 3), 2′) show that $c = \bar{c} = \gamma$ where γ is a real number. Hence the matrix B is real and symmetric. Consider a new basis $f_j = C^i_j e_i$ of V defined by a transition matrix of the form (2.16) with real coefficients c, c' and the dual basis $f^j = (C^{-1})^j_i e^i$. By Proposition 8, the curvature tensor R has the same expression with respect to the new basis f_i, f^j. Then $\sigma f_j = \sigma(C^i_j e_i) = C^i_j B_{ik} e^k = C^i_j B_{ik} C^k_l f^l = \widetilde{B}_{jl} f^l$. So the matrix $\widetilde{B}$ of $\sigma|_V$ with respect to the new basis is given by $\widetilde{B} = C^t B C$. The calculations show that the entries of $\widetilde{B}$ are $\tilde{\beta} = \beta$, $\tilde{\gamma} = \gamma + 2\beta c$, $\tilde{\delta} = \delta + 2\gamma c + \beta c^2 + 2\beta c'$. Hence we can choose c, c' such that $\tilde{\chi} = \tilde{\delta} = 0$. Similar, under a scalar change of the basis $f_i \to a f_i$, $f^j \to a^{-1} f^j$, the matrix B of the anti-involution will transform as $B \to a\bar{a} B$. This shows that in an appropriate basis, the matrix B of the anti-involution has the form

$$\begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}. \qquad \square$$

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Chapter 22

Prehomogeneous affine representations and flat pseudo-Riemannian manifolds

Oliver Baues

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1 Introduction

1.1 Flat pseudo-Riemannian and flat affine manifolds

A pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ is a smooth manifold M which is endowed with a possibly indefinite metric $\langle\,,\,\rangle$ on its tangent bundle TM. We let the expression $s(\langle\,,\,\rangle) = (n_+, n_-)$, $n_+ + n_- = \dim M$, denote the signature of $\langle\,,\,\rangle$. A positive definite metric $\langle\,,\,\rangle$ has signature $(n, 0)$ and is traditionally called a *Riemannian metric*. If the signature of $\langle\,,\,\rangle$ is $(n-1, 1)$, $\langle\,,\,\rangle$ is called a *Lorentzian metric*.

Every pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ has a unique torsion-free connection $\nabla_{\langle\,,\,\rangle}$ on its tangent bundle TM which has the property that the metric tensor $\langle\,,\,\rangle$ is parallel for $\nabla_{\langle\,,\,\rangle}$. The connection $\nabla_{\langle\,,\,\rangle}$ is called the Levi-Civita connection for $\langle\,,\,\rangle$. Given vector fields X, Y, Z on M, the tensorial expression,

$$R_\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called the *curvature tensor* for a connection ∇.

The metric $\langle\,,\,\rangle$ is called *flat* if the curvature $R_{\nabla_{\langle\,,\,\rangle}}$ vanishes everywhere on M. Thus, in particular, *flat pseudo-Riemannian manifolds* carry a torsion-free and flat connection on their tangent bundle. A manifold M together with a torsion-free and flat connection ∇ is called a *flat affine manifold*. Naturally, flat affine manifolds (M, ∇) share many of their geometric properties with the more restricted class of flat pseudo-Riemannian manifolds.

1.1.1 Global models and their quotients. There does exist an abundance of examples of simply connected flat pseudo-Riemannian and flat affine manifolds. Indeed, every open subset U of Euclidean space $\mathbb{R}^n$, defines a flat pseudo-Riemannian manifold (of any signature), and, every simply connected flat pseudo-Riemannian manifold $(\widetilde{M}, \langle\,,\,\rangle)$ or flat affine manifold $(\widetilde{M}, \nabla)$ is obtained by pulling back the flat structures along a local diffeomorphism $\Phi\colon \widetilde{M} \to U$ onto an open subset of $\mathbb{R}^n$.

A simply connected pseudo-Riemannian manifold $(\widetilde{M}, \langle\,,\,\rangle)$ will be called a *global model space* for flat pseudo-Riemannian manifolds. By elementary covering theory, every pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ is obtained as a quotient space of a global model $(\widetilde{M}, \langle\,,\,\rangle)$ by a *properly discontinuous* group Γ of isometries of $(\widetilde{M}, \langle\,,\,\rangle)$. Similarly, any flat affine manifold (M, ∇) is a quotient of a simply connected manifold $(\widetilde{M}, \nabla)$ by a group of connection preserving diffeomorphisms.

In principle, the study of flat pseudo-Riemannian manifolds breaks into two parts, namely the determination of *interesting* global models $(\widetilde{M}, \langle\,,\,\rangle)$ (apart from the standard complete model $\mathbb{E}^s$), and the study of the quotient spaces $(M, \langle\,,\,\rangle)$ which are modelled on $(\widetilde{M}, \langle\,,\,\rangle)$.

A particular interesting class of model spaces will be furnished by *homogeneous* flat pseudo-Riemannian manifolds. These are flat spaces which admit a transitive group of isometries. More generally, *homogeneous flat affine manifolds*, that is, flat affine manifolds with a transitive group of affine transformations constitute a natural class of models.

1.1.2 Completeness and pseudo-Euclidean space forms. A pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ is called *complete* if *every* geodesic curve for the connection $\nabla_{\langle\,,\,\rangle}$ can be extended to infinity. If the flat manifold $(M, \langle\,,\,\rangle)$ is complete, the *Killing–Hopf theorem* asserts, that the universal pseudo-Riemannian covering space $(\widetilde{M}, \langle\,,\,\rangle)$ for $(M, \langle\,,\,\rangle)$ is *isometric* to the pseudo-Euclidean space $\mathbb{E}^s = (\mathbb{R}^n, \langle\,,\,\rangle_s)$ of signature $s = s(\langle\,,\,\rangle)$, where $\langle\,,\,\rangle_s$ denotes the standard representative for a scalar product of signature s.

In particular, for fixed signature s, there exists, up to isometry, a unique simply connected *complete model space* $\mathbb{E}^s$ for flat pseudo-Riemannian manifolds of signature s.

As a further consequence, every complete flat pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ of signature s is isometric to a quotient of $\mathbb{E}^s$ by a *properly discontinuous subgroup* Γ of isometries, acting without fixed points on $\mathbb{E}^s$. Such quotient manifolds are also called *pseudo-Euclidean space forms*.

1.2 Flat Riemannian manifolds

By the Hopf–Rinow theorem (which holds solely in Riemannian geometry), the compactness of a Riemannian manifold implies its completeness. Similarly, the equivalence of metric and geodesic completeness for Riemannian manifolds implies that

every homogeneous Riemannian manifold is complete. In fact, every homogeneous flat Riemannian manifold is a quotient of $\mathbb{R}^n$ by a group of translations. In particular, every compact flat Riemannian manifold or homogeneous flat Riemannian manifold is a quotient of Euclidean space by a *discontinuous* group of isometries. Thus, the theory of flat Riemannian manifolds concerns mostly the study of complete space forms, and it is roughly equivalent to the study of discontinuous subgroups of the isometry group $\mathrm{E}(n)$ of Euclidean space.

The structure of discontinuous subgroups of $\mathrm{E}(n)$ is rather well understood, by the famous three *theorems of Bieberbach* [13], [14], dating from around 1910. According to Bieberbach, every discrete subgroup Γ of $\mathrm{E}(n)$ is finitely generated and contains an abelian subgroup of finite index. If Γ acts with compact quotient space, then the subgroup of translations is of finite index in Γ. In particular, every compact Euclidean space form is finitely and isometrically covered by a flat torus $\mathbb{R}^n/\Lambda$, where Λ is a lattice of translations. Moreover, Bieberbach proved, that in every dimension n, there exist only finitely many compact flat Riemannian manifolds up to affine equivalence. Thus, the class of compact flat Riemannian manifolds is rather restricted from a topological point of view.

The determination of compact Riemannian space forms and their geometric properties has a long tradition, and remains a subject of geometric and algebraic (it is related to the study of integral representations of finite groups) interest. Recent contributions concern, for example, isospectrality phenomena [28], [77], [93], and spin structures [27], [86] on flat Riemannian manifolds.

The theory of *almost flat* Riemannian manifolds as developed by Gromov [44], in a sense, extends the theory of Bieberbach to a much more general context. See [44], [23]. Here, the role of flat tori is taken over by compact *nilmanifolds* (these are quotient spaces of nilpotent Lie groups). Incidentally, nilmanifolds also appear (at least conjecturally) as the fundamental building blocks for compact flat pseudo-Riemannian manifolds.

1.3 Flat manifolds of indefinite signature

Much of the theory of flat pseudo-Riemannian manifolds aims to construct an analogy to the theory of Euclidean space forms. But, as is it turned out, many new phenomena and principle difficulties arise. Many of them constitute still open and difficult research questions.

For example, it is widely believed, that, as in the Riemannian case, only the pseudo-Euclidean spaces $\mathbb{E}^s$ admit compact quotient manifolds. This conjecture is so far verified only for Lorentzian manifolds, see [25].

Likewise, the determination of simply connected *homogeneous model spaces* $(\widetilde{M}, \langle\,,\,\rangle)$, and more generally the determination of all *homogeneous flat pseudo-Riemannian manifolds* of a given signature s, is an unsolved problem.

1.3.1 Structure theory of compact flat pseudo-Riemannian manifolds

Compactness and completeness. A first stumbling stone for a structure theory is created by the general lack of understanding about the relationship of compactness and completeness for flat pseudo-Riemannian and affine manifolds. In fact, although there are known examples of compact (non flat) Lorentzian manifolds, which are incomplete, it is expected ("*Conjecture of Markus*") that, as in the Riemannian case, a flat compact pseudo-Riemannian manifold is indeed also complete. This conjecture was proved by Carriére [25] for the case of flat Lorentzian manifolds. But it remains unverified for flat pseudo-Riemannian manifolds of arbitrary signature. More generally, the *Markus conjecture* (attributed to [74]) suggests that an orientable compact affinely flat manifold is complete if and only if it admits a *parallel volume*. Beside the result of Carriére, this conjecture is verified only in a few special cases. For example, *compact homogeneous* affine manifolds satisfy Markus' conjecture.

Note that, by dividing out a cyclic group of linear dilatations on the affine space with the point 0 removed, one constructs a simple example of a compact affinely flat and incomplete manifold. In general, the theory of compact flat affine manifolds is considered as wild and possibly untractable, although some results on the topology of such manifolds have been obtained in low dimensions. See [94] for wild examples.

Group theoretic structure of the fundamental group. Assuming completeness, new obstacles appear to generalise Bieberbach's theory of discontinuous Euclidean groups and Euclidean crystallographic groups to a theory of affine crystallographic groups. These concern the group theoretic structure of the fundamental group of a complete pseudo-Riemannian manifold. Conjecturally, ("*Auslander's conjecture*"), an affine crystallographic group has a solvable subgroup of finite index. The conjecture arose from a paper [4] of Louis Auslander, which contained a flawed proof of the even stronger claim that every finitely generated discontinuous subgroup of affine transformations has a solvable subgroup of finite index.

The solvable group replaces the finite index abelian subgroup in the Euclidean case. In this sense, the *Auslander conjecture* serves as a weak analogue to Bieberbach's first theorem. Auslander's conjecture is verified in low dimensions, in the Lorentzian case, and in some other cases (see [1] for a survey). But it remains one of the main open questions of the subject.

Also, if the assumption of compactness is dropped, examples of flat complete three dimensional Lorentzian manifolds with a free non-abelian fundamental group (cf. [29], [73]) give a counter example to the original claim of Auslander. In this case, the analogy with the Euclidean theory breaks down.

The classification theory of pseudo-Euclidean space forms. Assuming completeness and solvability of the fundamental group, new difficulties and phenomena arise for a possible classification program. For instance, the finiteness part of the Bieberbach theory breaks down, as well. But it admits a weak and rather subtly defined replacement, as is described in [46].

The main achievement of the theory, so far, gives a rather precise and strong link of the theory of pseudo-Euclidean and affine crystallographic groups with the theory of *left-invariant* flat pseudo-Riemannian metrics on Lie groups. (See [36], [46] for an exposition.) For example, a classification theory of compact flat Lorentzian manifolds is developed in [7], [35], [43], [45], [48].

However, despite the achievements in the Lorentzian case, the structure theory for compact complete flat pseudo-Riemannian manifolds remains widely open.

1.3.2 Homogeneous flat pseudo-Riemannian manifolds. Unlike the Riemannian case, a homogeneous pseudo-Riemannian manifold need *not* be complete. The simplest example of a *non-complete* homogeneous flat pseudo-Riemannian manifold is an open orbit of the two-dimensional non-abelian solvable simply connected Lie group in $\mathbb{E}^{1,1}$ (see [103, §11]). In general, the classification of non-complete homogeneous flat pseudo-Riemannian manifolds is not fully understood.

Only if the additional assumption of compactness is made then homogeneity implies completeness for pseudo-Riemannian manifolds (in fact, even without the assumption of flatness), see [51], [75]. Furthermore, in some cases, the group theoretical structure of a homogeneous space is directly linked to its completeness properties. For example, a flat pseudo-Riemannian Lie group, more generally a volume preserving flat affine Lie group, is complete if and only if it is unimodular. In this chapter, we introduce the related new result that a flat affine homogeneous space of a nilpotent Lie group is complete if and only if the action preserves a parallel volume form. In particular, a flat homogeneous pseudo-Riemannian manifold or a flat symplectically homogeneous affine manifold of a nilpotent group is complete. See Section 4 and Section 7 for further discussion of these results.

The structure and classification of *complete* homogeneous flat pseudo-Riemannian manifolds is more accessible than the general case. Every Riemannian homogeneous flat manifold is obtained by dividing out a group of translations in $\mathbb{R}^n$. In the pseudo-Riemannian case interesting phenomena do occur.

For example, contrasting the Riemannian case, there exists a large class of (non-compact) complete homogeneous flat pseudo-Riemannian manifolds with abelian but *non-translational* holonomy groups. See [102], [107], [108] for a detailed study of this examples.

In general, the holonomy group of a homogeneous flat pseudo-Riemannian manifold must be a two-step nilpotent group. (See Section 4.3.1 for details on this.) Non-abelian fundamental groups occur, in particular, as fundamental groups of *compact* homogeneous flat pseudo-Riemannian manifolds. In fact, there exists a large class of compact two-step nilmanifolds, which admit an (essentially unique) homogeneous flat pseudo-Riemannian metric. The first examples, which are not homotopy equivalent to a torus arise in dimension six. See Section 4.2.2 for more details and proofs.

For general homogeneous affinely flat manifolds similar results hold. By [42], the affine holonomy group of a compact affine manifold with parallel volume does not preserve any proper algebraic subsets in affine space $\mathbb{A}^n$. As a corollary, a compact

homogeneous affine manifold with parallel volume is complete. (An independent proof is given in Section 4.2.)

More generally, if a volume preserving homogeneous affine manifold admits a compact Clifford–Klein form then it must be complete (see Section 9.2 for further discussion). In particular, if the universal covering of a compact flat pseudo-Riemannian manifold is homogeneous then it must be complete. It follows that non-complete homogeneous flat pseudo-Riemannian manifolds do not occur as models of compact flat pseudo-Riemannian manifolds.

1.4 Overview

This chapter aims to exhibit several aspects, which link the theory of flat affine and pseudo-Riemannian manifolds with the topic of representations of Lie groups and algebraic groups on affine space $\mathbb{A}^n$.

In Section 1 we started with an overview on some of the main achievements and open problems in the topic of flat affine and pseudo-Riemannian manifolds.

In the following Section 2 we discuss flat manifolds from the point of view of Thurston's theory of locally homogeneous (X, G)-manifolds. Here we introduce our notation, as well as basic definitions and methods.

In Section 3, we describe the structure of the group of affinities and of the Lie algebra of affine vector fields on flat affine manifolds. By the development process, these groups are represented as subgroups of the affine group, and relate to certain associative matrix algebras. Compactness poses strong restrictions on the symmetries of volume preserving, in particular, of pseudo-Riemannian and symplectic affine flat manifolds, which we describe in detail. Similarly, the holonomy groups of flat homogeneous manifolds are determined by the centralisers of prehomogeneous representations. They turn out to be a nilpotent, if the homogeneous manifold is complete.

The following Section 4 is devoted to the basic properties of homogeneous affine manifolds. Their automorphism groups develop to prehomogeneous subgroups of the affine group. We show that the Markus conjecture is satisfied for compact homogeneous affine manifolds, and we describe the structure of *compact* homogeneous flat pseudo-Riemannian, and also of *compact* symplectically homogeneous affine flat manifolds.

We also introduce a result which states that a homogeneous affine manifold of a nilpotent group is complete if and only if the group is volume preserving. In particular, a homogeneous flat pseudo-Riemannian or symplectic affine flat manifold of a nilpotent group is always complete. The proof of these results depends on methods which are developed in Section 7.

In Section 5, we review the relationship between the geometry of a flat affine Lie group and the behaviour of its left and right Haar measures. The main result shows that the completeness of a flat affine Lie group is determined by the interaction of its unimodular character with the volume character which is defined by the affine structure. These results build on the study of étale affine representations of Lie groups.

In Section 6, we discuss the basic properties of affinely homogeneous domains and of prehomogeneous affine representations of Lie groups.

In the following Section 7, we develop a criterion for the transitivity of prehomogeneous affine representations, which extends corresponding results for étale affine representations. The main application shows that every volume preserving nilpotent prehomogeneous group of affine transformations is transitive.

In Section 8, we explain how the geometry of invariant measures on an affine homogeneous space and the transitivity properties of its associated prehomogeneous representation are linked by certain naturally defined *characteristic classes* of the affine representation.

In Section 9, we study properties of the Zariski closure $A(\Gamma)$, where Γ is the holonomy group of a *compact* affine manifold M. One of the main objectives of the subject is to characterise the groups $A(\Gamma)$ in relationship with the geometric properties of M. An important result of Goldman and Hirsch states that $A(\Gamma)$ acts transitively on affine space if M is a volume preserving compact flat affine manifold. This has strong consequences for the geometry of compact volume preserving flat affine manifolds. We explain some of the applications of this result, and we also explain how its proof relates to the methods developed in the previous sections.

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2 Foundations

Flat pseudo-Riemannian and affine manifolds constitute actually a particular class of locally homogenous manifolds. This point of view allows to express many geometric properties of flat manifolds in an elegant and transparent way.

2.1 The development map and holonomy

We start by briefly recalling the fundamental notion of an (X, G)-manifold. For further and more detailed reference on (X, G)-structures, see, for example, [24], [88], [95].

2.1.1 (X, G)-manifolds. Let X be a homogeneous space for the Lie group G. A manifold M is said to be *locally modelled on* (X, G) if M admits an atlas of charts with range in X such that the coordinate changes are restrictions of elements of G. A maximal atlas with this property is then called a (X, G)-*structure on* M, and M is called a (X, G)-*manifold*, or *locally homogeneous space* modelled on (X, G).

A map Φ between two (X, G)-manifolds is called an (X, G)-*map* if it looks like the action of an element of G in the local charts. If the (X, G)-map Φ is a diffeomorphism it is called an (X, G)-*equivalence*.

Every (X, G)-manifold comes equipped with some extra structure, called the development and the holonomy. Let $\pi : \widetilde{M} \to M$ denote the universal covering space of the (X, G)-manifold M. We fix $x_0 \in M$. The *development map* of the (X, G)-structure on M is the local diffeomorphism

$$D\colon \widetilde{M} \to X$$

which is obtained by analytic continuation of a local (X, G)-chart of M in x_0. The development map is an (X, G)-map, and, *for any* (X, G)*-equivalence* Φ *of* $\widetilde{M}$*, there exists an element* $h(\Phi) \in G$ *such that*

$$D \circ \Phi = h(\Phi) \circ D. \tag{2.1}$$

The fundamental group $\pi_1(M) = \pi_1(M, x_0)$ acts on $\widetilde{M}$ via covering transformations, which are (X, G)-equivalences of $\widetilde{M}$. This induces the *holonomy homomorphism* $h\colon \pi_1(M, x_0) \to G$ which satisfies

$$D \circ \gamma = h(\gamma) \circ D, \tag{2.2}$$

for all $\gamma \in \pi_1(M, x_0)$. After the choice of the development map (which corresponds to a choice of a germ of a (X, G)-chart in x_0), the holonomy homomorphism h is well defined. We note that the (X, G)-structure on M determines the conjugacy class of h under the action of G.

Clearly, the development map already determines the (X, G)-structure on $\widetilde{M}$, and specifying a development pair for the action of $\pi_1(M, x_0)$ on $\widetilde{M}$ is equivalent to constructing an (X, G)-structure on M:

Proposition 2.1. *Every local diffeomorphism* $D\colon \widetilde{M} \to X$ *which satisfies* (2.2)*, for some* $h\colon \pi_1(M, x_0) \to G$ *defines a unique* (X, G)*-structure on* M*, and every* (X, G)*-structure on* M *arises in this way.*

Properly discontinuous actions. Let Γ be a group of diffeomorphisms of a manifold M. Then Γ is said to act *properly discontinuously* on M if, for all compact subsets $\mathcal{K} \subset M$, the set

$$\Gamma_{\mathcal{K}} = \{\gamma \in \Gamma \mid \gamma\mathcal{K} \cap \mathcal{K} \neq \emptyset\}$$

is finite. If Γ acts properly discontinuously and freely on M then the quotient space M/Γ is a smooth manifold, and the projection map $\pi\colon M \to M/\Gamma$ is a smooth covering map. Now if Γ acts by (X, G)-equivalences on an (X, G)-manifold M, then the quotient space M/Γ inherits a natural (X, G)-manifold structure from M. In fact, the development maps of M and M/Γ coincide (as well, as their universal coverings.)

Example 2.1 ((X, G)-space forms). Assume that X is simply connected, and Γ is a group of (X, G)-equivalences of X (that is, Γ is a subgroup of G) acting properly

discontinuously and freely on X. Then X/Γ is an (X, G)-manifold, and the identity map of X is a development map for X/Γ.

See [24], [95] for further discussion of (X, G)-geometries and the properties of the development process.

2.1.2 Affine and projectively flat manifolds. The geometry of the pseudo-Euclidean space $\mathbb{E}^s = (\mathbb{R}^n, \mathrm{E}(s))$ is determined by the transitive action of the isometry group $\mathrm{E}(s)$ of the standard scalar product of signature s. Similarly the geometry of affine space $\mathbb{A}^n = (\mathbb{R}^n, \mathrm{Aff}(n))$ is determined by the action of the full affine group $\mathrm{Aff}(n)$.

Coordinate representation of the affine group. We view affine space $\mathbb{A}^n$ as a hyperplane

$$\mathbb{A}^n = \{(x, 1) \mid x \in \mathbb{R}^n\}$$

embedded in $\mathbb{R}^{n+1}$. In this setting, the group of affine transformations $\mathrm{Aff}(n)$ identifies naturally with a group of linear transformations of $\mathbb{R}^{n+1}$. Namely,

$$\mathrm{Aff}(n) = \left\{ A = \begin{pmatrix} g & t \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathrm{GL}(n, \mathbb{R}) \right\},$$

which is a subgroup of $\mathrm{GL}(n+1, \mathbb{R})$. Note that the affine group decomposes as a semi-direct product

$$\mathrm{Aff}(n) = T(n) \rtimes \mathrm{GL}_n(\mathbb{R}),$$

where

$$T(n) = \left\{ A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R}^n \right\}$$

is the group of translations of $\mathbb{R}^n$. The natural quotient homomorphism

$$\ell \colon \mathrm{Aff}(n) \to \mathrm{GL}_n(\mathbb{R}), \quad A \mapsto \ell(A) = g$$

associates to the affine transformation $A \in \mathrm{Aff}(n)$ its *linear part*. The vector $t(A) = t$, is called the *translational part* of A.

The Lie algebra $\mathfrak{aff}(n)$ of $\mathrm{Aff}(n)$ is

$$\mathfrak{aff}(n) = \left\{ X = \begin{pmatrix} \varphi & v \\ 0 & 0 \end{pmatrix} \;\middle|\; \varphi \in \mathfrak{gl}(n, \mathbb{R}) \right\},$$

which is a Lie subalgebra of the matrix algebra $\mathfrak{gl}(n+1, \mathbb{R})$.

Note that the *evaluation map* of the affine action at $x \in \mathbb{A}^n$

$$o_x \colon \mathrm{Aff}(n) \to \mathbb{A}^n, \quad A \mapsto A \cdot x = g(x) + t$$

is expressed by matrix multiplication, and so is its derivative at the identity, which is the map

$$t_x \colon \mathfrak{aff}(n) \to \mathbb{R}^n, \quad X \mapsto X \cdot x = \varphi(X) + v. \tag{2.3}$$

Transitive subgroups of Aff(n). We shall also consider various subgroups of $\mathrm{Aff}(n)$. We let $O(s) = O(\langle , \rangle_s)$ denote the group of linear isometries of the standard scalar product $\langle , \rangle_s$ of signature s. The pseudo-Euclidean isometry group $\mathrm{E}(s)$ is a semidirect product $\mathrm{E}(s) = T(n) \rtimes O(s)$. Thus $\mathrm{E}(s)$ embeds into the affine group as

$$\mathrm{E}(s) = \left\{ A = \begin{pmatrix} g & t \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in O(s) \right\}.$$

The group of volume preserving affine transformations is $\mathrm{SAff}(n) = \{A \in \mathrm{Aff}(n) \mid \ell(A) \in \mathrm{SL}_n(\mathbb{R})\}$. The group of symplectic affine transformations is $\mathrm{Aff}(\omega_n) = \{A \in \mathrm{Aff}(2n) \mid \ell(A) \in \mathrm{Sp}(\omega_n)\}$, where ω_n is a non-degenerate skew bilinear form on $\mathbb{R}^{2n}$.

Fixed points for reductive affine actions. We call a subgroup G of a linear group *reductive* if every G-invariant linear subspace admits an invariant complementary subspace. The following lemma is a particular useful fact:

Lemma 2.2. *Let $G \leq \mathrm{Aff}(n)$ be a reductive subgroup. Then the affine action of G has a fixed point on $\mathbb{A}^n$.*

Proof. Take a complementary line to the G-invariant subspace $\{(v, 0) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$, and intersect with the hyperplane $\mathbb{A}^n$. □

Projective geometry. The space of lines through the origin in $\mathbb{R}^{n+1}$ is called real projective space, and it is denoted by $\mathbb{P}^n\mathbb{R}$. The projective linear group is

$$\mathrm{PGL}(n, \mathbb{R}) = \mathrm{GL}(n + 1, \mathbb{R})/\{\pm E_{n+1}\}$$

acting on the projective space $\mathbb{P}^n\mathbb{R}$. Via the above coordinate representation, affine space $\mathbb{A}^n$ embeds as an open subset in $\mathbb{P}^n\mathbb{R}$, and $\mathrm{Aff}(n)$ embeds as a subgroup of $\mathrm{PGL}(n, \mathbb{R})$. In particular, affine geometry (that is, geometry modelled on $(X, G) = (\mathbb{A}^n, \mathrm{Aff}(n))$), is a subclass of projective $(\mathbb{P}^n\mathbb{R}, \mathrm{PGL}(n, \mathbb{R}))$ geometry. Note also that a cone over any $(\mathbb{P}^n\mathbb{R}, \mathrm{PGL}(n, \mathbb{R}))$-manifold becomes a $(\mathbb{R}^{n+1}, \mathrm{GL}(n + 1, \mathbb{R}))$ manifold. (See [40], for an exposition about projectively flat manifolds, and their relation with affine flat manifolds.)

2.2 Flat manifolds are (X, G)-manifolds

As already remarked, the geometry of the pseudo-Euclidean space $\mathbb{E}^s$ is determined by the transitive action of its isometry group $\mathrm{E}(s)$, and similarly the geometry of affine space $\mathbb{A}^n$ is determined by the action of the affine group $\mathrm{Aff}(n)$.

Theorem 2.3. *A flat pseudo-Riemannian manifold $(M, \langle , \rangle)$ is a locally homogeneous space modelled on the standard pseudo-Euclidean space $\mathbb{E}^s$, $s = s(\langle , \rangle)$. A flat affine manifold (M, ∇) is a locally homogeneous space modelled on affine space $\mathbb{A}^n$.*

Proof. In fact, by flatness of $(M, \langle\,,\,\rangle)$, the exponential map $\exp_p : T_pM \to M$ allows to define an isometry of a neighbourhood of 0 in $\mathbb{E}^s$ to a normal neighbourhood in $(M, \langle\,,\,\rangle)$, for every point $p \in M$. This defines a *compatible atlas* for M, where all charts are local isometries, and with coordinate changes in the pseudo-Euclidean group $\mathrm{E}(s)$. Evidently, the analoguous argument also works for a flat affine manifold (M, ∇). □

Note that the locally homogeneous $\mathbb{E}^s$-structure on $(M, \langle\,,\,\rangle)$ is uniquely determined by the *compatibility condition* that its charts are local isometries. The *compatible $\mathbb{A}^n$-structure* for ∇ is determined by the condition that its charts are affine maps. In particular, we may speak of the development map $D : \widetilde{M} \to \mathbb{A}^n$ and holonomy of a flat affine manifold, where D is then a local locally affine diffeomorphism (respectively, $D : \widetilde{M} \to \mathbb{E}^s$ is a local isometry, for a flat pseudo-Riemannian manifold).

The traditional point of view (pseudo-Riemannian metric and flat connection) and the $\mathbb{E}^s$-manifold point of view are completely equivalent. Namely, every locally homogeneous space modeled on $\mathbb{E}^s$ has a unique flat pseudo-Riemannian metric $\langle\,,\,\rangle$ which turns the local charts of the $\mathbb{E}^s$-structure into local isometries for $\langle\,,\,\rangle$. A map between two flat manifolds $(M, \langle\,,\,\rangle)$ and $(M', \langle\,,\,\rangle')$ is a local isometry if and only if it is a map of $\mathbb{E}^s$-structures, and so on.

Holonomy and parallel transport. Let $h : \pi_1(M) \to \mathrm{E}(s)$ denote the holonomy homomorphism of an $\mathbb{E}^s$-structure on M.

Definition 2.4. The homomorphism h is called the *affine holonomy homomorphism* of the flat manifold $(M, \langle\,,\,\rangle)$. The composition $\mathrm{hol} = \ell \circ h : \pi_1(M) \to O(s)$ is called the *linear holonomy* homomorphism.

Recall that the *parallel transport* of a flat manifold $(M, \langle\,,\,\rangle)$ defines a homomorphism $p_x : \pi_1(M, x) \to O(T_xM, \langle\,,\,\rangle_x)$ of the fundamental group of M into the group of linear isometries of the tangent space. The image $p_x(\pi_1(M, x))$ is called the holonomy group of $(M, \langle\,,\,\rangle)$ (at x). The following is easy to see:

Proposition 2.5. *In a local chart for the induced $\mathbb{E}^s$-structure on M, based at $x \in M$, the parallel transport homomorphism $p_x : \pi_1(M, x) \to O(T_xM, \langle\,,\,\rangle_x)$ corresponds to (that is, it is conjugate to) the linear holonomy homomorphism of the $\mathbb{E}^s$-structure.*

Of course, the analogous result holds for a flat affine manifold (M, ∇) and its parallel transport. Note that also the affine holonomy may be interpreted as a parallel transport in a suitable associated bundle over (M, ∇) (cf. [66]). For yet another interpretation of the affine parallel transport, see [23].

Affine structures of type $\mathcal{A}$. We do not need to restrict our attention to the transitive groups $\mathrm{E}(s)$ or $\mathrm{Aff}(n)$. More generally, we may consider any subgroup $\mathcal{A}$ of the affine group $\mathrm{Aff}(n)$ to define a flat model geometry. This gives rise to the following notion.

Definition 2.6. A locally homogeneous space modelled on $(\mathbb{A}^n, \mathcal{A})$ is called a flat affine manifold of type $\mathcal{A}$.

The holonomy determines if the affine structure group for a flat affine manifold (M, ∇) can be reduced to a subgroup $\mathcal{A}$ of $\mathrm{Aff}(n)$.

Definition 2.7. A flat affine manifold (M, ∇) is called of type $\mathcal{A}$ if (a conjugate) of its affine holonomy homomorphism $h\colon \pi_1(M) \to \mathrm{Aff}(n)$ takes image in $\mathcal{A}$.

If (M, ∇) is of type $\mathcal{A}$ then it admits a compatible structure of an $(\mathbb{A}^n, \mathcal{A})$ manifold. The definition allows to consider various kinds of geometric flatness conditions for (M, ∇). For example, the groups $\mathcal{A} = \mathrm{SAff}(n)$, $\mathrm{Aff}(\omega)$, $\mathrm{Aff}(2n, \mathbb{C})$, $\mathrm{UE}(2n)$ describe the concepts of volume preserving, symplectic, complex or Kähler flat affine manifold, respectively.

2.2.1 Completeness and crystallographic groups. Geodesic completeness of a flat affine manifold is interpreted naturally by its development map.

Theorem 2.8. *A flat affine manifold (M, ∇) is complete if and only if its development map $D\colon \widetilde{M} \to \mathbb{A}^n$ is a diffeomorphism.*

Proof. The development map D is a local affine diffeomorphism. By completeness of $(\widetilde{M}, \nabla)$ it is onto $\mathbb{A}^n$. Moreover, geodesics of $\mathbb{A}^n$ admit a lift along any preimage for D. Therefore, D is a covering map. Hence, it must be a diffeomorphism. □

Question 1. Does there exists a flat affine manifold with development map D, which is onto $\mathbb{A}^n$, but not a diffeomorphism?

The characterisation of completeness via the development implies the following:

Corollary 2.9 (Killing–Hopf theorem). *Let (M, ∇) be a complete flat affine manifold of type $\mathcal{A}$. Let $\Gamma = h(\pi_1(M)) \leq \mathcal{A}$ be its affine holonomy group. Then Γ acts properly discontinuously and freely on $\mathbb{A}^n$, and M is $\mathcal{A}$-equivalent to the $\mathcal{A}$-manifold $\mathbb{A}^n/\Gamma$.*

It follows that every complete pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$ of signature s, is isometric to a quotient of $\mathbb{E}^s$ by a properly discontinuous group $\Gamma \leq \mathrm{E}(s)$. Such a manifold

$$M = \mathbb{E}^s/\Gamma$$

is called a *pseudo-Euclidean space form*.

A complete flat affine manifold (M, ∇) is called an *affine space form*. By Corollary 2.9, the study of affine space forms reduces to the study of properly discontinuous subgroups of $\mathrm{Aff}(n)$.

Example 2.2. Every discrete subgroup Γ of the Euclidean group $\mathrm{E}(n)$ acts properly discontinuously on $\mathbb{R}^n$. If Γ is in addition torsion-free then $\mathbb{E}^n/\Gamma$ is a complete flat Riemannian manifold.

In general, if $\Gamma \leq \mathcal{A}$ is properly discontinuous then it is discrete, but the converse may not hold.

Crystallographic groups. A uniform discrete subgroup of $\mathrm{E}(n)$ is traditionally called a (Euclidean-) crystallographic group, cf. [106]. This motivates the following

Definition 2.10. A properly discontinuous subgroup $\Gamma \leq \mathcal{A}$, $\mathcal{A} \leq \mathrm{Aff}(n)$, is called an *affine crystallographic group of type* $\mathcal{A}$ if the quotient space $\mathbb{A}^n/\Gamma$ is compact.

Question 2. Is an affine crystallographic group (up to finite index) contained in the group of volume preserving affine transformations of $\mathbb{A}^n$? Or equivalently, has an orientable compact complete affine manifold always a parallel volume form? This is one direction of Markus' conjecture, see Section 1.3.1).

Note that a *finite volume* flat complete Riemannian manifold is necessarily compact. This is a consequence of the classification of discrete subgroups of the Euclidean group $\mathrm{E}(n)$, as given by Bieberbach, see [106].

Question 3. Does there exist a non-compact, finite volume complete affine or pseudo-Riemannian manifold? For example, does a complete flat Lorentzian manifold admit finite volume, without being compact?

2.2.2 The development image of a compact affine manifold. Let M be a compact affinely flat manifold, $D\colon \widetilde{M} \to \mathbb{A}^n$ its development map. The development image of M is an open domain

$$U_M = D(\widetilde{M}) \subseteq \mathbb{A}^n.$$

It is conjectured that M is complete, if M admits a parallel volume (Markus' conjecture). In particular, in this case, the conjecture claims that $D(\widetilde{M}) = \mathbb{A}^n$.

Example 2.3. The development image of an affine two-torus U_{T^2} is one of the four affinely homogeneous domains which admit an abelian simply transitive group of affine transformations, see Corollary 6.6. See also [9] for construction methods of affine structures on T^2, and visualisation of the development process.

In general, the development image of a compact affinely flat, or projectively flat, manifold can have a very complicated development image. See [94], for the construction of examples.

3 The group of affinities

Let ∇ be a torsion-free connection on M. We put $\mathrm{Aut}(M,\nabla)$ for the group of connection preserving diffeomorphisms of M. Recall (cf. [65]) that $\mathrm{Aut}(M,\nabla)$ is a Lie group.

3.1 Affine vector fields

A vector field X on (M,∇) is called an *affine vector field* if its local flow preserves the connection ∇. The affine vector fields form a subalgebra $\mathfrak{aut}(M,\nabla)$ of the Lie algebra $\mathrm{Vect}(M)$ of all C^∞-vector fields.

A vector field on M is called *complete* if its local flow generates a one-parameter group of diffeomorphisms of M. The *complete* affine vector fields on M form a subalgebra $\mathfrak{aut}_c(M,\nabla)$ of $\mathfrak{aut}(M,\nabla)$. If M is compact or ∇ is complete then $\mathfrak{aut}_c(M,\nabla) = \mathfrak{aut}(M,\nabla)$ (see [66, VI, §2]).

Note that the Lie algebra of complete affine vector fields is (anti-) isomorphic to the tangent Lie algebra of (left-invariant) vector fields on $\mathrm{Aut}(M,\nabla)$.

3.1.1 The associative algebra of affine vector fields on a flat manifold. Let (M,∇) be a flat affine manifold, and let X be a vector field on M. The covariant derivative of X defines an endomorphism field

$$A_X : Y \mapsto -\nabla_Y X$$

on M. Since ∇ is flat, the vector field X is affine if and only if A_X is parallel, that is, X is affine if and only if, for all vector fields Y, Z,

$$\nabla_Z A_X Y = A_X \nabla_Z Y. \tag{3.1}$$

Moreover, for an affine vector field X on the flat manifold (M,∇) we have, for all $Y \in \mathrm{Vect}(M)$, the relation

$$A_{[X,Y]} = [A_X, A_Y]. \tag{3.2}$$

Induced algebra structure on $\mathrm{Vect}(M)$. We declare the product of two vector fields $X, Y \in \mathrm{Vect}(M)$ by

$$X *_\nabla Y := A_X Y = -\nabla_Y X. \tag{3.3}$$

In terms of the product $*_\nabla$, condition (3.1), that $X \in \mathrm{Vect}(M)$ is affine, is equivalent to

$$X *_\nabla (Y *_\nabla Z) = (X *_\nabla Y) *_\nabla Z, \tag{3.4}$$

for all $Y, Z \in \mathrm{Vect}(M)$.

A short calculation, involving the associativity condition (3.4) shows: if X and X' are affine vector fields then $A_X X' = X *_\nabla X'$ also satisfies (3.4), for all $Y, Z \in$

$\mathrm{Vect}(M)$. Hence, $X *_\nabla X'$ is an affine vector field. In particular, *the Lie algebra of affine vector fields* $\mathfrak{aut}(M,\nabla)$ *is a subalgebra of* $(\mathrm{Vect}(M), *_\nabla)$ *which is associative:*

Proposition 3.1. *The Lie algebra of affine vector fields* $(\mathfrak{aut}(M,\nabla)$ *forms an associative subalgebra of* $(\mathrm{Vect}(M), *_\nabla)$.

Let us furthermore remark that the centralisers of affine vector fields form a subalgebra in $(\mathrm{Vect}(M), *_\nabla)$:

Lemma 3.2. *Let* $Y \in \mathfrak{aut}(M,\nabla)$ *be an affine vector field. Let* $X, X' \in \mathrm{Vect}(M)$ *be vector fields on* M*, which centralise* Y*. Then also* $\nabla_X X'$ *centralises* Y.

Proof. Using (3.1) for the affine field Y, as well as $[X,Y] = [X',Y] = 0$, we calculate

$$\begin{aligned}[\nabla_X X', Y] &= \nabla_{\nabla_X X'} Y - \nabla_Y \nabla_X X' \\ &= \nabla_X \nabla_{X'} Y - \nabla_Y \nabla_X X' \\ &= \nabla_X \nabla_Y X' - \nabla_Y \nabla_X X' \\ &= \nabla_{[X,Y]} X' = 0. \end{aligned}$$

□

Remark. The fact that $\mathfrak{aut}(M,\nabla)$ inherits the structure of an associative matrix algebra, has strong consequences, especially in a situation, where $\mathfrak{aut}(M,\nabla)$ describes the tangent algebra for the group $\mathrm{Aut}(M,\nabla)$. For example, *homogeneous* affine manifolds, which are compact or complete are naturally related to associative matrix algebras. See, for example, [109] for applications. See also some of the results in Section 6 and Section 7 for further exploitation of this principle.

3.1.2 Right- and left-symmetric algebras. A bilinear product "$\cdot$" on a vector space which satisfies the identity

$$X \cdot (Y \cdot Z) - (X \cdot Y) \cdot Z = Y \cdot (X \cdot Z) - (Y \cdot X) \cdot Z \tag{3.5}$$

is called a *left-symmetric algebra*. Complementary, a product is called *right-symmetric* if

$$Z \cdot (Y \cdot X) - (Z \cdot Y) \cdot X = Z \cdot (X \cdot Y) - (Z \cdot X) \cdot Y. \tag{3.6}$$

Both identities naturally generalise the associativity condition for algebras, and, by exchanging factors, right- and left symmetric products are in one-to-one correspondence. Moreover, as in the associative case, there is an associated Lie product which is defined by

$$[X,Y] := X \cdot Y - Y \cdot X.$$

The associative subalgebra of elements X which is defined by equation (3.4) is called the *associative kernel* of the right-symmetric algebra.

Example 3.1. Let (M,∇) be a flat affine manifold. As declared in (3.3), the flat torsion-free connection ∇ induces an algebra product $*_\nabla$ on the Lie algebra $\mathrm{Vect}(M)$

of C^∞- vector fields on M. Note then that $*_\nabla$ is a compatible *right-symmetric algebra* product on the Lie algebra $\mathrm{Vect}(M)$. The associative kernel of $(\mathrm{Vect}(M), *_\nabla)$ is finite dimensional, and it is precisely the subalgebra $(\mathfrak{aut}(M,\nabla), *_\nabla)$ of affine vector fields.

Algebras satisfying (3.5) appeared and a play mayor role in the study of convex homogeneous cones [99]. Naturally finite dimensional LSAs over the real numbers play an important role in the study of left invariant flat affine structures on Lie groups, see Section 5. Left symmetric algebras also appear in several other mathematical and physical contexts, see [22] for a recent survey.

3.2 Development representation of affinities

Let $(\widetilde{M},\nabla)$ be the universal covering manifold of (M,∇). We put $\widetilde{\Gamma} \leq \mathrm{Aut}(\widetilde{M},\nabla)$ for the group of covering transformations, and $\Gamma = h(\widetilde{\Gamma})$ for the affine holonomy group of (M,∇). The development process provides a local representation of the group of affine transformations $\mathrm{Aut}(M,\nabla)$:

We define

$$\widehat{\mathrm{Aut}}(M,\nabla) = \mathrm{Aut}(\widetilde{M},\widetilde{\Gamma},\nabla) = \{\Phi \in \mathrm{Aut}(\widetilde{M},\nabla) \mid \Phi\,\widetilde{\Gamma}\,\Phi^{-1} = \widetilde{\Gamma}\}.$$

The group $\widehat{\mathrm{Aut}}(M,\nabla)$ is a covering group of $\mathrm{Aut}(M,\nabla)$, and the development homomorphism (2.1) induces a homomorphism of Lie groups

$$h\colon \widehat{\mathrm{Aut}}(M,\nabla) \to \mathrm{Aff}(n) \tag{3.7}$$

into the affine group $\mathrm{Aff}(n)$. Note that h has discrete kernel. Moreover, the image $h(\widehat{\mathrm{Aut}}(M,\nabla))$ normalises Γ, and $h(\widehat{\mathrm{Aut}}(M,\nabla)^0)$ centralises Γ.

Representation of affine vector fields. We study now the tangent representation of the development homomorphism.

Proposition 3.3. *The development defines a a natural faithful associative algebra representation*

$$\bar{h}\colon (\mathfrak{aut}(M,\nabla), *_\nabla) \to \mathfrak{aff}(n). \tag{3.8}$$

Proof. We already remarked that $\mathfrak{aut}(M,\nabla)$ forms an associative algebra with respect to $*_\nabla$). Via local affine coordinates, we can identify the tangent space T_pM with $\mathbb{R}^n$. By the formula $A_{X*_\nabla Y} = A_{A_X Y} = A_X A_Y$, which is deduced from (3.4), the map

$$\bar{h}\colon X \mapsto \phi_{X_p} = \begin{pmatrix} A_{X_p} & X_p \\ 0 & 0 \end{pmatrix} \tag{3.9}$$

is easily seen to be a faithful representation of $(\mathfrak{aut}(M,\nabla), *_\nabla)$ into the associative algebra of $\mathfrak{aff}(n)$ (where the associative algebra structure on $\mathfrak{aff}(n)$ is given by the usual product of matrices). □

The development representation

$$\bar{h}\colon \mathfrak{aut}(M,\nabla) \to \mathfrak{aff}(n),$$

as defined in (3.9), is associated to the local representation h of $\mathrm{Aut}(M,\nabla)$. In fact, on the subalgebra $\mathfrak{aut}_c(M,\nabla)$, the development representation $\bar{h}$ corresponds to the derivative of h. We further remark:

Proposition 3.4. *The representation $\bar{h}$ identifies $\mathfrak{aut}(M,\nabla)$ with the subalgebra $\mathfrak{aff}(n)^\Gamma$ of Γ-invariant affine vector fields on $\mathbb{A}^n$.*

Proof. If M is simply connected the map $\bar{h}\colon \mathfrak{aut}(M,\nabla) \to \mathfrak{aff}(n)$ is an isomorphism, since affine vector fields may be extended uniquely to all of M from any coordinate patch. For the general case, note that $\mathfrak{aut}(M,\nabla)$ is isomorphic to the subalgebra $\mathfrak{aut}(\widetilde{M},\nabla)^{\widetilde{\Gamma}}$ of Γ-invariant affine vector fields on $\widetilde{M}$. Since $\bar{h}\colon \mathfrak{aut}(\widetilde{M},\nabla) \to \mathfrak{aff}(n)$ is equivariant with respect to $h\colon \widetilde{\Gamma} \to \Gamma$, $\mathfrak{aut}(M,\nabla)$ is mapped onto $\mathfrak{aff}(n)^\Gamma$. □

3.3 Affinities of compact volume preserving affine manifolds

Recall that a compact volume preserving affine manifold is conjectured to be complete. Now if (M,∇) is a compact complete affine manifold then the centraliser of the affine crystallographic group $\Gamma = \mathrm{hol}(\pi_1(M))$ in $\mathrm{Aff}(n)$ is unipotent (see for example [46], or Section 9). Since the centraliser of $\mathrm{hol}(\pi_1(M))$ is the development of $\widehat{\mathrm{Aut}}(M,\nabla)^0$, the latter group is a simply connected nilpotent Lie group, which is faithfully represented by unipotent matrices. In this section we show that the analogous facts do indeed hold for the affinities of a compact volume preserving affine manifold, without assuming its completeness.

Let $(\mathfrak{aut}(M,\nabla), *_\nabla)$ denote the Lie algebra of affine vector fields on M with the associative algebra structure induced by ∇. We note:

Proposition 3.5. *Let (M,∇) be a flat affine manifold with parallel volume, which is compact (or with finite volume). Then $(\mathfrak{aut}(M,\nabla), *_\nabla)$ is isomorphic to an associative algebra of nilpotent matrices. In particular, $\mathfrak{aut}(M,\nabla)$ is a nilpotent Lie algebra.*

Proof. Define the divergence of $X \in \mathrm{Vect}(M)$ relative to the parallel volume μ on M by the formula

$$\mathrm{div}\, X\mu = L_X\mu.$$

Since M is compact, by Green's theorem (cf. [66, Appendix 6]) we have,

$$\int_M \mathrm{div}\, X d\mu = 0, \tag{3.10}$$

and, moreover, since the volume form is parallel, we also have (again according to [66, Appendix 6]),

$$\mathrm{div}\, X = -\,\mathrm{trace}\, A_X \ (= \mathrm{trace}\, \nabla_\bullet X).$$

In particular, if $X \in \mathfrak{aut}(M, \nabla)$ then $\operatorname{div} X$ is a constant function, which implies $\operatorname{div} X = 0$. Thus, via the linear representation $X \mapsto \phi_{X_p}$ of $(\mathfrak{aut}(M, \nabla), *_\nabla)$ constructed in Proposition 3.3, $\mathfrak{aut}(M, \nabla)$ is isomorphic to an associative algebra of linear operators of trace zero. By Lemma 6.13, ϕ_{X_p} is a nilpotent operator, for all $X \in \mathfrak{aut}(M, \nabla)$.

In the finite volume case, we use an analogous argument, noting that (3.10) holds for an affine vector field X on finite volume M as well. □

Corollary 3.6. *Let (M, ∇) be a flat affine manifold with parallel volume which is compact (or with finite volume). Then the centraliser of the affine holonomy group Γ of M is a connected unipotent group, which is isomorphic to $\widehat{\operatorname{Aut}}(M, \nabla)^0$ under the development.*

Proof. In appropriately chosen coordinates, the homomorphism $\bar{h}$, defined in (3.8), corresponds to the differential of the holonomy representation $h\colon \widehat{\operatorname{Aut}}(M, \nabla) \to \operatorname{Aff}(n)$. By Proposition 3.3, the tangent Lie algebra of the holonomy image is the associative algebra $\mathfrak{aff}(n)^\Gamma$, which is a subalgebra in $\mathfrak{aff}(n)$. By Proposition 3.5, the elements of $\mathfrak{aff}(n)^\Gamma$ are nilpotent. If $A \in Z_{\operatorname{Aff}(n)}(\Gamma)$ is an affine transformation which centralises Γ then $A - E_{n+1} \in \mathfrak{aff}(n)^\Gamma$. Thus, A is a unipotent affine transformation. It follows that $Z_{\operatorname{Aff}(n)}(\Gamma)$ is an algebraic subgroup of $\operatorname{Aff}(n)$ consisting of unipotent elements. In particular, $Z_{\operatorname{Aff}(n)}(\Gamma)$ must be connected, and simply connected. It follows that the development homomorphism (3.7),

$$h\colon \widehat{\operatorname{Aut}}(M, \nabla)^0 \to Z_{\operatorname{Aff}(n)}(\Gamma)$$

is a covering map onto, and, in fact, it is an isomorphism. □

The following is actually an application of Theorem 9.3.

Corollary 3.7. *Let (M, ∇) be a flat affine manifold with parallel volume, which is compact. If an affine vector field on M has a zero, it must be trivial. In particular,* $\dim \operatorname{Aut}(M, \nabla) \leq n = \dim M$.

Proof. In fact, by Theorem 9.3, the Zariski closure of Γ acts transitively on $\mathbb{A}^n$. This implies that the unipotent group $Z_{\operatorname{Aff}(n)}(\Gamma)$ acts without fixed points on $\mathbb{A}^n$. □

In particular, any locally faithful connection preserving action on a compact volume preserving affine manifold is locally free (the stabiliser of any point is discrete).

3.4 Lie groups of isometries

The group of isometries $\operatorname{Isom}(M, \langle\,,\,\rangle)$ of a pseudo-Riemannian manifold is a finite-dimensional Lie group, and it is a subgroup of the group of affine (connection preserving) transformations for the Levi-Civita connection of $\langle\,,\,\rangle$. If M is compact and $\langle\,,\,\rangle$ is Riemannian then $\operatorname{Isom}(M, \langle\,,\,\rangle)$ is compact. The identity components of the

automorphism groups of compact Lorentzian manifolds are known to be of rather restricted type. (Compare [2], [110], [111]). In fact, by [111], non-compact semisimple factors in the isometry group of a compact Lorentzian manifold are locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Moreover, a connected solvable subgroup of isometries of a compact Lorentzian manifold is a product of an abelian group with a 2-step nilpotent group of Heisenberg type (see [110]).

In the presence of a compatible *flat* torsion-free connection, particular strong restrictions hold for the group of isometries of a compact pseudo-Riemannian manifold $(M, \langle\,,\,\rangle)$. Proposition 3.5 already implies that the identity component of $\mathrm{Isom}(M, \langle\,,\,\rangle)$ is a nilpotent Lie group. Here we show, more specifically, that, in case $(M, \langle\,,\,\rangle)$ is a compact flat pseudo-Riemannian manifold, $\mathrm{Isom}(M, \langle\,,\,\rangle)$ is a *two-step nilpotent group* of rather restricted type. If $(M, \langle\,,\,\rangle)$ is a compact flat Lorentzian manifold then the identity component $\mathrm{Isom}(M, \langle\,,\,\rangle)^0$ is abelian and consists of translations only.

3.4.1 Killing vector fields. Let $(M, \langle\,,\,\rangle)$ be a pseudo-Riemannian manifold and ∇ the Levi-Civita connection for $\langle\,,\,\rangle$. A vector field X on M is called a *Killing vector field* if its flow preserves the metric $\langle\,,\,\rangle$. Equivalently, we have $L_X\langle\,,\,\rangle = 0$. Thus, X is Killing if and only if

$$L_X\langle U, V\rangle = \langle [X, U],\ V\rangle + \langle U,\ [X, V]\rangle, \tag{3.11}$$

for all vector fields U, V on M. Since $\langle\,,\,\rangle$ is parallel, (3.11) holds if and only if A_X is skew with respect to $\langle\,,\,\rangle$. Namely, (3.11) is equivalent to

$$\langle \nabla_U X,\ V\rangle + \langle U,\ \nabla_V X\rangle = 0. \tag{3.12}$$

Let $\mathfrak{o}(M, \langle\,,\,\rangle)$ denote the Lie algebra of Killing vector fields for $(M, \langle\,,\,\rangle)$. Since Killing vector fields preserve the Levi-Civita connection, every Killing vector field is affine. We have the inclusion

$$\mathfrak{o}(M, \langle\,,\,\rangle) \subseteq \mathfrak{aut}(M, \nabla).$$

Lemma 3.8. *Let $X, Y \in \mathfrak{o}(M, \langle\,,\,\rangle)$ be Killing vector fields, where Y centralises X. Then*

(1) $\langle \nabla_Y X, X\rangle = 0$.

(2) $\langle \nabla_X X, Y\rangle = 0$.

Proof. Since Y commutes with X, $\nabla_X Y = \nabla_Y X$. Because Y is Killing, A_Y is skew. Hence,

$$\langle \nabla_Y X, X\rangle = \langle \nabla_X Y, X\rangle = -\langle X, \nabla_X Y\rangle.$$

Thus (1) follows. We note next that $L_X\langle Y, X\rangle = 0$, since $[X, Y] = 0$. Since $\langle\,,\,\rangle$ is parallel, this implies

$$0 = \langle \nabla_X Y, X\rangle + \langle Y, \nabla_X X\rangle = \langle \nabla_Y X, X\rangle + \langle Y, \nabla_X X\rangle.$$

By (1), $\langle \nabla_Y X, X\rangle = 0$, and, hence, (2) holds. □

Proposition 3.9. *Let X be a Killing vector field whose flow centralises a group G of isometries of $(M, \langle\,,\,\rangle)$, which has an open orbit on M. Then*

(1) $\nabla_X X = 0$.

If $(M, \langle\,,\,\rangle)$ is also flat then:

(2) $A_X A_X = 0$.

Proof. The Killling fields Y corresponding to the action of G span the tangent spaces on an open subset of M. Thus, $\nabla_X X = 0$ is a consequence of Lemma 3.8.

If $(M, \langle\,,\,\rangle)$ is flat then A_X is parallel. Therefore,

$$A_X A_X Y = -A_X \nabla_Y X = -\nabla_Y \mathrm{A}_X X = \nabla_Y \nabla_X X = 0.$$

Hence, $A_X A_X = 0$. □

We deduce a few consequences:

Proposition 3.10. *Under the assumptions of Proposition 3.9, the following holds for all Killing vector fields X, X', X'', which are commuting with G:*

(1) $[X', X] = -2A_{X'}X = 2A_X X'$.

If $(M, \langle\,,\,\rangle)$ is also flat then:

(2) $A_{X'} A_{X''} X = A_{X''} A_{X'} X$.

(3) $[[X', X''], X] = 0$.

(4) $A_{[X,X']} = [A_X, A_{X'}] = 2A_X A_{X'}$.

(5) $A_X A_{X'} A_{X''} = 0$.

Proof. Note that $X + X'$ is Killing and commutes with G. Therefore, by (1) of Proposition 3.9 , $\nabla_{X+X'} X + X' = \nabla_{X'} X + \nabla_X X' = 0$. It follows that $[X, X'] = \nabla_X X' - \nabla_{X'} X = -2A_X X'$. Therefore (1) holds.

Next (using (1)) we note that $A_{X'} A_{X''} X = -A_{X'} A_X X''$. Since $A_{X'}$ is parallel, $-A_{X'} A_X X'' = -A_{A_{X'} X} X''$. We remark that the Killing vector fields centralising G form a Lie algebra with respect to the bracket of vector fields. Thus, by (1) $A_{X'} X$ is a Killing vector field, and also centralises G. Therefore, $A_{X'} A_{X''} X = A_{X''} A_{X'} X$. Thus, (2) holds.

Now $[[X', X''], X] = -2A_{[X',X'']} X = -2(A_{X'} A_{X''} X - A_{X''} A_{X'} X) = 0$ follows. Thus, (3) holds.

Using polarisation, (2) of Proposition 3.9 implies that $A_X A_{X'} = -A_{X'} A_X$. Hence, $A_{[X,X']} = [A_X, A_{X'}] = 2A_X A_{X'}$. Using these facts, we can compute, $A_X A_{X'} A_{X''} = (A_X A_{X'}) A_{X''} = -A_{X''} A_X A_{X'} = A_X A_{X''} A_{X'} = -A_X A_{X'} A_{X''}$. □

Corollary 3.11. *Let $(M, \langle\,,\,\rangle)$ be a flat pseudo-Riemannian manifold which admits a group G of isometries, which has an open orbit on M. Then:*

(1) *The Lie algebra* $\mathfrak{o}(M, \langle\,,\,\rangle)^G$ *of Killing vector fields on* $(M, \langle\,,\,\rangle)$ *which centralise* G *forms a subalgebra of the associative algebra of affine vector fields* $(\mathfrak{aut}(M, \nabla), *_\nabla)$.

(2) *The Lie algebra* $\mathfrak{o}(M, \langle\,,\,\rangle)^G$ *is (at most) two step nilpotent.*

The result implies that the centraliser of a prehomogeneous group of isometries on $\mathbb{E}^s$ is a unipotent group, see Corollary 6.25. The main applications concern the automorphism groups of compact flat pseudo-Riemannian manifolds (see Section 3.4.2 below) and the holonomy groups of homogeneous flat pseudo-Riemannian manifolds (see Section 4.3).

3.4.2 Isometries of compact flat pseudo-Riemannian manifolds. The following result gives a rough description of the possible connected groups of isometries of compact flat pseudo-Riemannian manifolds. It is another consequence of Theorem 9.3:

Theorem 3.12. *Let* $(M, \langle\,,\,\rangle)$ *be a compact flat pseudo-Riemannian manifold, and* ∇ *the Levi-Civita connection. Let* $X \in \mathfrak{o}(M, \langle\,,\,\rangle)$ *be a Killing vector field. Then the following hold:*

(1) *If* X *has a zero then* $X = 0$.

(2) $A_X A_X = 0$.

(3) *The Lie algebra* $\mathfrak{o}(M, \langle\,,\,\rangle)$ *of Killing vector fields on* $(M, \langle\,,\,\rangle)$ *forms a subalgebra of the associative algebra of affine vector fields* $(\mathfrak{aut}(M, \nabla), *_\nabla)$.

(4) *The Lie algebra* $\mathfrak{o}(M, \langle\,,\,\rangle)$ *is (at most) two step nilpotent.*

Proof. Let Γ be the holonomy group of $(M, \langle\,,\,\rangle)$. By Theorem 9.3, the Zariski closure $A(\Gamma)$ of the holonomy group acts transitively on $\mathbb{A}^n$. This transitive group of isometries commutes with the development of the Killing vector fields on M. Therefore, X cannot have a zero. Moreover, Proposition 3.9 and Proposition 3.10 imply the next three claims. □

Remark. The development homomorphism (3.8), $\bar{h}\colon \mathfrak{aut}(M, \nabla) \to \mathfrak{aff}(n)$, maps the subalgebra $\mathfrak{o}(M, \langle\,,\,\rangle)$ to an associative subalgebra of $\mathfrak{aff}(n)$, which is contained in $\mathfrak{o}(\langle\,,\,\rangle_s)$. By the theorem, the elements $\bar{h}(X) = \phi_{X_p}$, $X \in \mathfrak{o}(M, \langle\,,\,\rangle)$, satisfy the conditions $X_p \notin \operatorname{Im} A_{X_p}$ and $A^2_{X_p} = 0$. Subalgebras of linear maps satisfying both conditions appear in the context of *complete* homogeneous pseudo-Riemannian manifolds, as well. (See Section 4.3.) *Abelian* algebras of this type have been further investigated in [107], [108]. For the construction of non-abelian examples, see Section 5.3.2.

On a compact Riemannian manifold with non-positive Ricci curvature, every Killing vector field must be parallel, see [66]. This also holds in the flat Lorentzian case:

Corollary 3.13. *Let $(M, \langle\,,\,\rangle)$ be a compact flat pseudo-Riemannian manifold. If $\langle\,,\,\rangle$ is Riemannian or $\langle\,,\,\rangle$ has Lorentzian signature then every Killing vector field on M is parallel.*

Proof. Let X be Killing. By Proposition 3.12, the linear operator A_X develops to a two-step nilpotent element contained in $\mathfrak{o}(\langle\,,\,\rangle)$. In the Riemannian case $\mathfrak{o}(\langle\,,\,\rangle)$ has no non-zero nilpotent elements. For the Lorentzian case, note that every two-step nilpotent element of $\mathfrak{o}(\langle\,,\,\rangle)$ is zero. This implies $A_X = 0$, in both cases. □

In Section 5.3.2, we construct a compact flat manifold $(M, \langle\,,\,\rangle)$ of dimension six, with signature $s(\langle\,,\,\rangle) = (3, 3)$, and six-dimensional non-abelian algebra $\mathfrak{o}(M, \langle\,,\,\rangle)$.

3.5 Lie groups of symplectic transformations

Let ω be a non-degenerate 2-form on M which is closed. Then (M, ω) is called a *symplectic manifold*. A diffeomorphism of M which preserves ω is called a *symplectomorphism* of (M, ω). In general, the group of symplectomorphisms $\mathrm{Diff}_\omega(M)$ of a symplectic manifold is very large, and it is not a Lie group, even if M is compact. However, if M is compact, there do exist strong restrictions on the finite-dimensional Lie subgroups of $\mathrm{Diff}_\omega(M)$. For example, Zwart and Boothby [112] proved that a *compact* symplectically homogeneous manifold of a solvable Lie group S is diffeomorphic to a torus T^{2n}. More generally, by Guan's work [47], for any compact symplectic manifold (M, ω), every connected solvable Lie subgroup of $\mathrm{Diff}_\omega(M)$ is 2-step solvable, with a compact adjoint image. Moreover, any finite dimensional connected subgroup of $\mathrm{Diff}_\omega(M)$ is a semi-direct product of a compact group and a group S of the above type.

Let (M, ω) be compact symplectic manifold, which admits a compatible flat affine connection ∇. We show below (cf. Theorem 3.18) that every Lie subgroup of $\mathrm{Diff}_\omega(M)$, which also preserves ∇, is abelian.

Remark. The above mentioned restrictions do *not* apply if M is non-compact. For example, there do exist plenty solvable Lie subgroups of $\mathrm{Diff}(\mathbb{R}^n)$, not 2-step solvable, which preserve the standard symplectic structure on $\mathbb{R}^{2n}$. Such examples may be constructed using solvable Lie groups with left-invariant symplectic structure (symplectic Lie groups). See Section 6.5.2, for a particular construction method for such groups, which also produces non-solvable examples. Further examples are discussed, for example, in [97].

3.5.1 Symplectic vector fields. A vector field X on (M, ω) is called a *symplectic vector field* if its flow preserves ω. Equivalently, X satisfies $L_X\omega = 0$, which means that

$$L_X\omega(U, V) = \omega([X, U], V) + \omega(U, [X, V]),$$

for all vector fields U, V on M.

Lemma 3.14. *Let X, Y, Z be symplectic vector fields on (M, ω). Then*

$$\omega(X, [Y, Z]) + \omega(Z, [X, Y]) + \omega(Y, [Z, X]) = 0. \tag{3.13}$$

Proof. Since $d\omega = 0$, for all vector fields X, Y, Z on M, the relation

$$\begin{aligned} L_X\omega(Y, Z) + L_Z\omega(X, Y) + L_Y\omega(Z, X) \\ + \omega(X, [Y, Z]) + \omega(Z, [X, Y]) + \omega(Y, [Z, X]) = 0 \end{aligned} \tag{3.14}$$

is satisfied. If X, Y, Z are symplectic, $L_X\omega = L_Y\omega = L_Z\omega = 0$. It follows that

$$\begin{aligned} L_X\omega(Y, Z) + L_Y\omega(Z, X) + L_Z\omega(X, Y) \\ = -2\left(\omega(X, [Y, Z]) + \omega(Z, [X, Y]) + \omega(Y, [Z, X])\right). \end{aligned}$$

Substituting the right hand side in (3.14), the lemma follows. □

Proposition 3.15. *Let (M, ω) be a symplectic manifold which admits a Lie group $G \leq \mathrm{Diff}_\omega(M)$, which acts transitively on M. Then every connected Lie subgroup of $\mathrm{Diff}_\omega(M)$, which centralizes G, is abelian.*

Proof. Let X, Y be vector fields on M, whose flows preserve ω, and which centralise G. Let Z be a vector field tangent to G. Since X, Y, Z are symplectic, equation (3.13) holds. Since $[X, Z] = [Y, Z] = 0$, it follows that

$$\omega(Z, [X, Y]) = 0.$$

Since G acts transitively on M, the tangent spaces at every point $x \in M$, are spanned by vectors Z_x, where Z is tangent to G. Hence, $[X, Y] = 0$. This proves the proposition. □

The following special case of Proposition 3.15 is well known:

Corollary 3.16. *Let (G, ω) be a symplectic Lie group, where ω is biinvariant. Then G is abelian.*

Symplectic affine vector fields. Let (M, ∇, ω) be a flat affine manifold with parallel symplectic form ω. As in the metric case, a vector field X is symplectic if and only if A_X is skew with respect to ω. An affine vector field X on (M, ∇) which is also symplectic is called a *symplectic affine vector field*. The symplectic affine vector fields form a Lie subalgebra $\mathfrak{s}(M, \nabla, \omega)$ of $\mathfrak{aut}(M, \nabla)$.

The following gives a useful analogue of Lemma 3.8 for the symplectic case:

Lemma 3.17. *Let X be a symplectic affine vector field on (M, ∇, ω), and let Y, Z be symplectic affine vector fields which commute with X. Then*

$$2\,\omega(A_X A_X Y, Z) = \omega(\nabla_X X, [Y, Z]). \tag{3.15}$$

Proof. The proof is a straightforward computation. See [10, proof of Theorem 9] for a special case. □

3.5.2 Automorphisms of compact symplectic affine manifolds. On a symplectic affine manifold M the group of affine symplectic transformations forms a Lie subgroup of $\mathrm{Diff}_\omega(M)$. If M is compact, we show that this group must be abelian. In fact, the following holds:

Theorem 3.18. *Let (M, ∇) be a compact flat affine manifold with parallel symplectic structure ω. Then the identity component of the group of symplectic affine transformations is an abelian group which develops to a unipotent subgroup of symplectic affine transformations of $\mathbb{R}^n$.*

Proof. Let $\Gamma \leq \mathrm{Aff}(\omega_n)$ be the holonomy group of M. The group of symplectic affine transformations $\widehat{\mathrm{Aut}}(M, \nabla, \omega)^0$ develops onto a subgroup $h(\widehat{\mathrm{Aut}}(M, \nabla, \omega)^0)$ of $\mathrm{Aff}(\omega_n)$ (compare Section 3.2). This group centralises Γ, and therefore also the Zariski closure $A(\Gamma) \leq \mathrm{Aff}(n)$. The Zariski closure $A(\Gamma)$ is a group of symplectic affine transformation which acts transitively on affine space $\mathbb{A}^{2n}$ (by Theorem 9.3). Thus, by Proposition 3.15, $h(\widehat{\mathrm{Aut}}(M, \nabla, \omega)^0)$ must be abelian. Hence, so is $\mathrm{Aut}(M, \nabla, \omega)^0$. □

It follows that a compact affine manifold which is homogeneous under the group of symplectic affine transformations must be diffeomorphic to a torus:

Corollary 3.19. *Let (M, ∇) be a compact flat affine manifold with parallel symplectic structure which is homogeneous for the group of symplectic affine transformations. Then M is diffeomorphic to a torus.*

Let $\mathfrak{s}(M, \nabla, \omega)$ denote the Lie algebra of symplectic affine vector fields on M. As shown above, if M is compact then $\mathfrak{s}(M, \nabla, \omega)$ is abelian. We further note:

Proposition 3.20. *Let (M, ∇) be a compact flat affine manifold with parallel symplectic structure ω. Then the following hold:*

(1) *The Lie algebra of symplectic affine vector fields $\mathfrak{s}(M, \nabla, \omega)$ is abelian, and $\dim \mathfrak{s}(M, \nabla, \omega) \leq \dim M$.*

(2) *For all $X, Y \in \mathfrak{s}(M, \nabla, \omega)$, $A_X A_X Y = 0$.*

Proof. The Lie algebra $\mathfrak{s}(M, \nabla, \omega)$ is abelian, by Theorem 3.18. Let $\bar{X}$ be the development image of X and Y. Let $Z \in \mathfrak{s}(\mathbb{A}^n, \omega_n)$ be a symplectic affine vector field on $\mathbb{A}^n$ centralising $\bar{X}$ and $\bar{Y}$. By Lemma 3.17, we obtain $\omega_n(A_{\bar{X}} A_{\bar{X}} \bar{Y}, Z) = 0$. Since $A(\Gamma)$ acts transitively, the vector fields Z commuting with $\bar{Y}$ and $\bar{X}$ span all tangent spaces. Therefore, $A_X A_X Y = 0$. □

If (M, ∇, ω) is homogeneous then $\mathfrak{s}(M, \nabla, \omega)$ forms a subalgebra of the algebra of affine vector fields:

Proposition 3.21. *If M is homogeneous under the group of symplectic affine transformations then the following hold:*

(1) *For* $X, Y \in \mathfrak{s}(M, \nabla, \omega)$, $A_X A_Y = 0$.

(2) $\mathfrak{s}(M, \nabla, \omega)$ *forms a subalgebra of the associative algebra* $(\mathfrak{aut}(M, \nabla, *_\nabla)$.

Proof. Since M is homogeneous, the vector fields $Y \in \mathfrak{s}(M, \nabla, \omega)$ span all tangent spaces. Therefore, $A_X A_X = 0$, for all $X \in \mathfrak{s}(M, \nabla, \omega)$. Since $\mathfrak{s}(M, \nabla, \omega)$ is abelian, also $A_X A_Y = 0$. □

Remark. The structure and classification of algebras which are appearing in Proposition 3.21 have been investigated in [10].

4 Homogeneous model spaces

Let G be a Lie group, and $H \leq G$ a closed subgroup. Then $X = G/H$ is a homogeneous space for G. If X admits a flat affine connection ∇, which is G-invariant, then (X, ∇) is called a *flat affine homogeneous space*. Let $\Gamma \leq G$ be a discrete subgroup, such that the quotient space

$$M = \Gamma \backslash G/H$$

is a manifold. Since ∇ is invariant by Γ, M inherits a flat connection ∇, such that the natural map $(X, \nabla) \to (M, \nabla)$ is an affine covering map. Then M is called a *Clifford–Klein form* for $X = G/H$, and the (simply connected) covering space (X, ∇) is called a global model for (M, ∇).

The Auslander and Markus conjectures suggest a strong relationship between the geometry of a flat affine homogeneous space (X, ∇), and the existence of compact Clifford–Klein forms for X.

The following result a consequence of Theorem 9.3:

Theorem 4.1. *Let* $(X = G/H, \nabla)$ *be a flat affine homogeneous space. If* X *has a compact Clifford–Klein form* (M, ∇) *with parallel volume then* (X, ∇) *is complete.*

In particular, if the universal covering (X, ∇) of a compact volume preserving flat affine manifold (M, ∇) is homogeneous under its group of affine transformations, then (M, ∇) is complete.

In general, it seems difficult to understand the precise conditions on an arbitrary flat affine homogeneous space (X, ∇), which ensure the completeness of (X, ∇). For flat affine Lie groups (G, ∇), a simple characterisation is known (see Section 5). An approach to the general problem will be introduced in Section 7 and Section 8. In particular, we will prove the following result:

Theorem 4.2. *Let* (M, ∇) *be a homogeneous flat affine manifold for a nilpotent Lie group* G. *Then* M *is complete if and only if* G *preserves a parallel volume form on* M.

The theorem is a direct consequence of a corresponding result for affine homogeneous domains of nilpotent groups (see Section 7.3).

In what follows, we will then mainly investigate *compact homogeneous* affine manifolds. Before doing so, we discuss the development map of a homogeneous space and indicate the proofs of Theorem 4.1 and Theorem 4.2 in Section 4.1. Next we provide a structure theorem for volume preserving compact homogeneous affine manifolds in Section 4.2.1. This result implies a direct proof (independent of Theorem 4.1) that every compact volume preserving homogeneous affine manifold is complete. As another consequence of the structure theorem, we derive a classification theorem for compact flat pseudo-Riemannian homogeneous manifolds. We also show that a compact symplectic homogeneous affine manifold is *diffeomorphic* to a torus. In Section 4.3 we further discuss the properties of the holonomy groups of homogeneous manifolds.

4.1 The development map of a homogeneous space

Recall that a group action on $\mathbb{A}^n$ is called *prehomogeneous* if it has a Zariski-dense open orbit. A subset of $\mathbb{A}^n$ is *semi-algebraic* if it is defined by polynomial equations and inequalities. The following is a basic observation:

Proposition 4.3. *Let (M, ∇) be a homogeneous flat affine manifold. Then the development map is a covering map, and its development image is a semi-algebraic subset in $\mathbb{A}^n$. Let G be a group which acts transitively on (M, ∇). Then its universal covering group acts prehomogeneously on $\mathbb{A}^n$ by the development homomorphism.*

Proof. Let G be a Lie group which acts transitively on (M, ∇), and $\widetilde{G} \leq \mathrm{Aff}(\widetilde{M})$ be a covering group, which lifts the action of G. Then the development $h(\widetilde{G}) \leq \mathrm{Aff}(n)$ acts transitively on the development image $U = D(\widetilde{M})$. In fact, it follows that $U \subseteq \mathbb{A}^n$ is an affinely homogeneous domain. By Proposition 6.8, every affinely homogeneous domain U is a semi-algebraic subset of $\mathbb{R}^n$. The development map $D\colon \widetilde{M} \to U$ is a covering map, because it identifies with a covering of homogeneous spaces $\widetilde{M} = \widetilde{G}/\widetilde{H} \to h(\widetilde{G})/L = U$, which is induced by the locally faithful homomorphism $h\colon \widetilde{G} \to \mathrm{Aff}(n)$. □

Proof of Theorem 4.1. Since X has a compact Clifford–Klein form with parallel volume, its development image is a homogeneous affine domain U, which is divisible by volume preserving affine transformations. Hence, by Corollary 9.4, $U = \mathbb{A}^n$. Since D is a covering, (X, ∇) must be complete. □

Proof of Theorem 4.2. Assume M has a transitive volume preserving action of some nilpotent Lie group. Then the universal covering X of M is a homogeneous space for a nilpotent Lie group N of affine transformations, which preserves a parallel volume form. The development of N preserves the parallel volume on $\mathbb{A}^n$, and acts prehomogeneously. By Corollary 7.8, the development of N acts transitively on $\mathbb{A}^n$,

and therefore $D(X) = \mathbb{A}^n$. Since D is a covering, it must be a diffeomorphism. Therefore, M is complete.

The converse is a direct consequence of Corollary 7.8. □

4.2 Compact homogeneous affine manifolds

Robert Hermann [52] observed that a compact pseudo-Riemannian manifold which admits a transitive group of isometries must be complete (see [75] for a complete proof).

Example 4.1. Consider $\mathrm{GL}(n,\mathbb{R})$ as an open subset in $\mathbb{R}^{n^2}$. Then the natural affine structure on $\mathrm{GL}(n,\mathbb{R})$ is invariant under left- and right-translations on the group. As a consequence of a result of Borel [16], there exist cocompact lattices $\Gamma \leq \mathrm{GL}(n,\mathbb{R})$. The compact manifold $M = \mathrm{GL}(n,\mathbb{R})/\Gamma$ inherits an affine flat structure, and $\mathrm{GL}(n,\mathbb{R})$ acts as a transitive group of affine transformations on (M,∇). Thus (M,∇) is a compact affinely homogeneous manifold, which is not complete.

Markus' conjecture asserts that an (orientable) compact flat affine manifold is complete if and only if it admits a parallel volume form. The conjecture is known to hold for compact affinely homogeneous affine manifolds. (This is proved in [42]. See Section 9.2 for further discussion.) We obtain an independent proof in Corollary 4.5 below.

4.2.1 Compact homogeneous manifolds with parallel volume. We have the following structure result for compact homogeneous flat affine manifold with parallel volume:

Theorem 4.4. *Let (M,∇) be a compact homogeneous flat affine manifold with volume preserving affine structure. Then the following hold:*

(1) *(M,∇) is affinely diffeomorphic to a quotient space of a simply connected nilpotent affine Lie group (N,∇) with biinvariant affine structure ∇ by a discrete subgroup $\Gamma \leq N$.*

(2) *(M,∇) is geodesically complete.*

Furthermore, every parallel tensor field on (M,∇), which is preserved by the action of N, pulls back to a biinvariant (and parallel) tensor field on N.

Proof. By Proposition 3.5, the group of affinities $\widehat{\mathrm{Aut}}(M,\nabla)^0$ develops to a connected unipotent Lie group. Moreover, M is a compact homogeneous space of the simply connected nilpotent Lie group $N = \widehat{\mathrm{Aut}}(M,\nabla)^0$. Since N acts almost effectively on M, if follows that $\dim N = \dim M$ (see [72]). Therefore, $M = N/\Gamma$ is a quotient of N by a uniform discrete subgroup $\Gamma \leq N$. Pull back ∇ to obtain a left-invariant affine

connection ∇ on N, which is right-invariant under the lattice Γ. Since Γ is Zariski-dense in the adjoint representation (see [87]), ∇ is a biinvariant connection on N. (In fact, ∇ is determined by the associative product structure $*_\nabla$ on the Lie algebra $\mathfrak{aut}(M,\nabla) = \mathfrak{n}$.) The volume preserving left-invariant connection ∇ on the nilpotent Lie group N is complete. (See Section 5.2.2.) Therefore, M is complete. □

Remark. By a result of Mostow [80] (see also, [87], [96]), a finite volume homogeneous space of a solvable Lie group is compact. Thus, the above proof also shows that a finite volume homogeneous affine manifold is compact and geodesically complete.

The Markus conjecture holds for compact homogeneous affine manifolds:

Corollary 4.5. *Let (M,∇) be a compact homogeneous flat affine manifold. Then (M,∇) is complete if and only if (M,∇) has parallel volume.*

Proof. If $(M,\nabla$ has parallel volume Theorem 4.4 applies.

Assume now that (M,∇) is complete. Then, as already remarked in the beginning of Section 3.3, $\mathrm{Aut}(M,\nabla)^0$ is a nilpotent Lie group, which develops to a unipotent subgroup of $\mathrm{Aff}(n)$. Therefore, as in the proof of Theorem 4.4, $M = N/\Gamma$, where N is a nilpotent Lie group with complete left invariant affine connection ∇, such that $(N,\nabla) \to (M,\nabla)$ is an affine covering. Since N is nilpotent and ∇ is complete, the left-multiplications on N are volume preserving affine transformations (see Section 5.2.2). Since N is nilpotent, also the right-multiplications preserve the left-invariant parallel volume on N. Since $\mathrm{hol}(\pi_1(M)) = \Gamma$ is contained in the right-multiplications on N, Γ preserves the parallel volume on N. This shows that (M,∇) has parallel volume. □

Note also that Theorem 4.4 strengthens the completeness result in [52], [75] considerably, since it is only required that the group of affinities $\mathrm{Aut}(M,\nabla_{\langle\,,\rangle})$ acts transitively on M, to ensure completeness.

4.2.2 Pseudo-Riemannian examples. Let $(M,\langle\,,\rangle)$ be a homogeneous flat pseudo-Riemannian manifold, which is compact. Then, by the above, $(M,\nabla_{\langle\,,\rangle})$ is complete. Moreover, the following structure theorem holds:

Theorem 4.6. *Let M be a compact (or finite volume) homogeneous flat pseudo-Riemannian manifold. Then M is isometric to a quotient of a flat pseudo-Riemannian Lie group N with biinvariant metric.*

Proof. In fact, by Theorem 4.4, $(M,\langle\,,\rangle)$ is isometric to a quotient of a nilpotent flat pseudo-Riemannian Lie group $(N,\langle\,,\rangle)$ with biinvariant metric $\langle\,,\rangle$, such that the natural (orbit-) map $N \to M = N/\Gamma$ is a pseudo-Riemannian covering. □

Theorem 4.6 implies the following strong geometric *rigidity property* for compact homogeneous flat pseudo-Riemannian manifolds:

Corollary 4.7. *Let M and M' be compact homogeneous flat pseudo-Riemannian manifolds with isomorphic fundamental groups. Then M and M' are affinely diffeomorphic.*

Proof. As above, we write $M = N/\Gamma$ and $M' = N'/\Gamma'$, where Γ, Γ' are lattices in the simply connected nilpotent Lie groups N and N', respectively. By Malcev-rigidity, cf. [72], every isomorphism $\phi\colon \Gamma \to \Gamma'$ extends to an isomorphism of Lie groups $\Phi\colon N \to N'$. We contend that Φ is an affine isomorphism of metric Lie groups $(N, \langle\,,\,\rangle) \to (N', \langle\,,\,\rangle')$. In fact, both $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ are biinvariant metrics, and therefore (see [83, Proposition 11.9]), the Levi-Civita connection ∇ (respectively ∇') is the canonical torsion-free connection on the Lie group N (respectively N'). That is, for left-invariant vector fields X and Y, $\nabla_X Y = \frac{1}{2}[X, Y]$. In particular, it follows that the isomorphism of Lie groups Φ is affine with respect to ∇ and ∇'. Therefore, the induced diffeomorphism $\bar{\Phi}\colon M \to M'$ is affine. □

In particular, the fundamental group $\pi_1(M)$ determines the Levi-Civita connection on a compact homogeneous flat pseudo-Riemannian manifold M up to affine equivalence.

Example 4.2. If $\pi_1(M)$ is abelian then $\pi_1(M) \cong \mathbb{Z}^n$, and $M = T^n$ is diffeomorphic to a torus. Moreover, by Corollary 4.7, every homogeneous flat manifold $(T^n, \langle\,,\,\rangle)$ is isometric to a quotient of $\mathbb{E}^s$ by a lattice of translations. The affine structure of $(T^n, \langle\,,\,\rangle)$ is the uniquely determined translational structure on T^n.

Remark. In fact, *every* flat metric on the torus T^n *is* homogeneous, and thus isometric to a quotient of $\mathbb{E}^s$ by a lattice of translations. To the contrary, the set of affine equivalence classes of volume preserving complete affine structures on T^n is rather large. See [9], [11] for further discussion.

A nilpotent Lie group N, which acts transitively by isometries on a compact flat pseudo-Riemannian manifold M is of rather restricted type:

Lemma 4.8. *Let $M = N/\Gamma$ be a compact homogeneous flat pseudo-Riemannian manifold. Then N is a two-step nilpotent Lie group.*

Proof. Remark (see [83, Proposition 11.9]), that the curvature of the Levi-Civita connection ∇ of the biinvariant flat metric $\langle\,,\,\rangle$ on N is computed as, $R_\nabla(X, Y)Z = \frac{1}{4}[X, [Y, Z]]$, for all left-invariant vector fields X, Y, Z. Since $\langle\,,\,\rangle$ is flat, this implies $[X, [Y, Z]] = 0$. It follows that N is two-step nilpotent. □

Note also that every quotient space $M = N/\Gamma$ of a flat pseudo-Riemannian Lie group N with biinvariant metric by a discrete subgroup Γ is a homogeneous flat pseudo-Riemannian manifold. In fact, left-multiplication in N induces a transitive isometric action of N on $M = N/\Gamma$.

In Section 5.3.1, we will discuss the structure of flat pseudo-Riemannian Lie groups with biinvariant metric in detail. In particular, see Corollary 5.14, we explicitly construct a family of such Lie groups, which gives rise to an interesting class of examples of compact (complete) homogeneous flat pseudo-Riemannian manifolds with non-abelian fundamental group. This is summarised in Corollary 4.10 below.

First, we state a few general consequences concerning the classification of *compact* homogeneous flat pseudo-Riemannian manifolds:

Corollary 4.9. *Let M be a compact homogeneous flat pseudo-Riemannian manifold. Then the following hold:*

(1) *If* $\dim M \leq 5$ *then M is isometric to a translation torus* (*that is, M is a quotient of $\mathbb{E}^s$, by a lattice of translations*).

(2) *The fundamental group Γ of M is two-step nilpotent.*

The first examples of compact homogeneous flat pseudo-Riemannian manifolds with *non-abelian* fundamental group arise in dimension six. More generally, there is the following interesting class of examples:

Corollary 4.10. *Let N/Γ be a compact two-step nilmanifold* (*where Γ is a lattice in the two-step nilpotent Lie group N, $n = \dim N$*). *Then there exists a flat* rank n *torus bundle*

$$T^n \to M \to N/\Gamma$$

such that M admits a homogeneous flat pseudo-Riemannian metric of signature $s = (n, n)$ with totally geodesic and isotropic fibers T^n.

Proof. Let $A\colon N \to \mathrm{GL}(\mathfrak{n}^*)$ denote the coadjoint representation of N on the dual $\mathfrak{n}^*$ of its Lie algebra $\mathfrak{n}$. By Corollary 5.3.1, the corresponding semidirect product $N \ltimes_A \mathfrak{n}^*$ admits a flat biinvariant pseudo-Riemannian metric of signature (n, n), $n = \dim M$. Now Γ is a lattice in N, and under the representation A, maps to a finitely generated subgroup of the group of unipotent upper triangular matrices with rational coefficients relative a basis of $\mathfrak{n}$. In particular, $A(\Gamma)$ preserves a lattice $\Lambda \cong \mathbb{Z}^n$ in $\mathfrak{n}^*$. The semidirect product $\Gamma' = \Gamma \ltimes_A \Lambda$ is a lattice in $N \ltimes_A \mathfrak{n}^*$, and the manifold $M = N \ltimes_A \mathfrak{n}^*/\Gamma'$ has the required properties. □

In particular, let Γ be a torsion-free finitely generated two-step nilpotent group. Then Γ arises as a quotient $\Gamma = \Gamma'/\Lambda$, where Γ' is the holonomy of a compact homogeneous flat pseudo-Riemannian manifold, and Λ is an abelian normal subgroup of Γ', with $\operatorname{rank}\Lambda = \operatorname{rank}\Gamma$.

Example 4.3. In dimension $n = 6$, there exist up to affine equivalence two classes of compact homogeneous flat pseudo-Riemannian manifolds. The six-dimensional translation torus T^6, and torus bundles $T^3 \to M_{6,\Gamma} \to H_3/\Gamma$, where Γ is a lattice in the three-dimensional Heisenberg group H_3.

4.2.3 Symplectic examples. As noted before, and contrasting the pseudo-Riemannian situation, only tori admit the structure of a homogeneous symplectic flat affine manifold. However, such a structure is, in general, not affinely equivalent to a translation torus:

Corollary 4.11. *Let M be a compact (or finite volume) homogeneous symplectic flat affine manifold. Then M is a compact abelian flat affine symplectic Lie group. In particular, M is diffeomorphic to a torus.*

Proof. By Theorem 4.4, (M, ∇) is a quotient of an affine symplectic Lie group N with biinvariant symplectic structure. By Proposition 3.16, N is abelian. (To show that M is a torus, we might have applied the classification results for connected groups of symmetries of compact symplectic manifolds given in [112], [47], as well.) □

Note that the space of affine equivalence classes of flat symplectic connections on the torus T^{2n} is very large. In fact, as is shown in [10], the set of simply connected abelian flat affine symplectic Lie groups forms an algebraic variety of (real) dimension n^3. For the precise classification result and further geometric properties of abelian flat affine symplectic Lie groups, see [10].

4.3 Holonomy of homogeneous affine manifolds

The fundamental group of a homogeneous flat Riemannian manifold is abelian, and its holonomy acts by translations. Starting in dimension five, there do exist complete flat homogeneous pseudo-Riemannian manifolds with abelian but non-translational holonomy group, see [102]. As Example 4.1 shows, the holonomy of a (compact) homogeneous affine manifold can be far away from being abelian or nilpotent. This example is however a non-complete manifold.

Theorem 4.12. *Let $\Gamma \leq \mathrm{Aff}(n)$ be the holonomy group of a complete affinely homogeneous affine manifold. Then Γ is a unipotent subgroup of $\mathrm{Aff}(n)$.*

Proof. Since M is homogeneous, the development G of $\mathrm{Aff}(M, \nabla)$ acts transitively on the development image of M, which is $\mathbb{A}^n$. Moreover, Γ centralises G. By Proposition 6.15, the centraliser of the transitive group G is a connected unipotent group. In particular, Γ is unipotent. □

In particular:

Corollary 4.13. *The fundamental group of a complete affinely homogeneous affine manifold is nilpotent.*

4.3.1 Holonomy of homogenous flat pseudo-Riemannian manifolds. The above result can be strengthened considerably for homogeneous flat pseudo-Riemannian manifolds even without the assumption of completeness.

Proposition 4.14. *Let* $\Gamma \leq \mathrm{E}(s)$ *be the holonomy group of a homogeneous flat pseudo-Riemannian manifold. Then* Γ *is a unipotent group, and* Γ *is nilpotent of nilpotency class at most two.*

Proof. Let $(M, \langle\,,\,\rangle)$ be a homogeneous flat pseudo-Riemannian manifold. The development image of M is a pseudo-Riemannian domain U in $\mathbb{E}^s$, which is homogeneous with respect to the development G of $\mathrm{Isom}(M, \langle\,,\,\rangle)$. The holonomy group Γ preserves the domain U and is contained in the centraliser of G in the full isometry group $\mathrm{E}(s)$. By Corollary 6.25, the centraliser is a connected unipotent group of nilpotency class at most two. In particular, Γ has this property. □

Remark. The proposition leads to a classification theory of flat homogeneous pseudo-Riemannian manifolds, which are complete. Wolf studied the classification of complete homogeneous pseudo-Riemannian manifolds with *abelian* holonomy Γ, discovering interesting phenomena, see [102], [107], [108]. In Section 4.2.2, we proved the existence of a six-dimensional compact (complete) homogeneous pseudo-Riemannian manifold with non-abelian holonomy Γ. Moreover, all examples constructed in Section 4.2.2 relate to certain two-step nilpotent Lie groups. *It remains to join these results to a full structure theory for complete flat homogeneous pseudo-Riemannian manifolds.*

The determination of all homogeneous flat pseudo-Riemannian manifolds (that is, including non-complete examples) seems not at hand, because of the difficulties associated with the classification of homogeneous pseudo-Riemannian domains. (See Section 6.6.)

We mention further that Proposition 4.14 admits an analogous version for symplectically homogeneous affine manifolds. This is a consequence of Proposition 6.28 below:

Proposition 4.15. *Let* $\Gamma \leq \mathrm{Aff}(\omega_n)$ *be the holonomy group of a symplectically homogeneous affine manifold with parallel symplectic form. Then* Γ *is an abelian group of unipotent symplectic affine transformations.*

5 Flat affine Lie groups

We consider a particular tractable class of homogeneous spaces, namely those which admit a *simply transitive* group of equivalences. Many phenomena can be illustrated, and in many cases examples constructed from flat affine Lie groups serve as the basic building blocks of the theory.

5.1 Left-invariant geometry on Lie groups

Let G be a Lie group. Recall that a geometric structure on G is called left-invariant if it is preserved by all left-multiplication maps $l_g : G \to G$, $g \in G$. A connection ∇ on G is called a *left-invariant connection* if, for all $g \in G$, l_g is a connection preserving diffeomorphism. A pseudo-Riemannian metric is called left-invariant if all left-multiplications are isometries. More generally, if G has the structure of an (X, A)-manifold then the structure on G is called left-invariant if all left multiplications are (X, A)-equivalences.

Definition 5.1. Let G be a Lie group and ∇ a connection on G. If ∇ is flat affine and left-invariant, then (G, ∇) is called a *flat affine Lie group*. Moreover, ∇ is called a flat left-invariant connection of type $\mathcal{A}$, where $\mathcal{A} \leq \mathrm{Aff}(n)$, if (G, ∇) has a compatible *left-invariant* locally homogeneous structure of type $(\mathbb{A}^n, \mathcal{A})$.

In the latter situation we shall call the pair (G, ∇) a *flat affine Lie group of type* $\mathcal{A}$. If $\langle\,,\,\rangle$ is a flat left-invariant metric, then $(G, \langle\,,\,\rangle)$ is called a *flat pseudo-Riemannian Lie group*.

Example 5.1. Let $\nabla_{\langle\,,\,\rangle}$ be the Levi-Civita connection of a flat pseudo-Riemannian Lie group. Then $(G, \nabla_{\langle\,,\,\rangle})$ is a flat affine Lie group of type $\mathrm{E}(s)$.

5.1.1 Left-symmetric algebras. There is a one-to-one correspondence of left-invariant flat affine connections ∇ on the simply connected group G, with left-symmetric algebra structures on the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. In fact, if X, Y are left-invariant then $\nabla_X Y = -Y *_\nabla X$ is a left-invariant vector field. Hence ∇ defines a left-symmetric product on $\mathfrak{g}$. (Compare Section 3.1.2.) The flat connection ∇ is a biinvariant connection if and only if the induced product is associative.

Example 5.2. Let N be a two-step nilpotent Lie group. The Lie algebra of left-invariant vector fields $\mathfrak{n}$ thus satisfies $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{0\}$. It follows that the canonical torsion-free connection on N, defined by

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad X, Y \in \mathfrak{n},$$

is flat. Therefore, N is naturally a flat affine Lie group with biinvariant flat connection ∇. The bilinear product $X \cdot Y := \nabla_X Y$ defines an associative algebra structure on $\mathfrak{n}$. (See [91], [92] for more constructions of left-invariant flat connections on nilpotent Lie groups.)

5.2 The development map of a flat affine Lie group

Let $\rho : G \to \mathrm{Aff}(n)$ be a homomorphism. Recall that ρ is called an *étale affine representation* (see Definition 6.3) if there exists $x \in \mathbb{R}^n$ such that the orbit map at x,

$$o_x : G \to \mathbb{A}^n, \quad g \mapsto \rho(g) \cdot x,$$

is a local diffeomorphism onto an open subset U of $\mathbb{A}^n$. The representation ρ is *simply transitive* if o_x is a diffeomorphism for one (and, hence, for all) $x \in \mathbb{A}^n$.

Example 5.3. Let $\rho\colon G \to \mathcal{A}$ be an étale affine representation, and $x \in \mathbb{A}^n$ such that $o_x\colon G \to \mathbb{A}^n$ is a local diffeomorphism. By pull back of the standard flat connection on $\mathbb{A}^n$ along o_x, we obtain a flat left invariant connection of type $\mathcal{A}$ on G.

Moreover, every flat left invariant connection of type $\mathcal{A}$ arises in this way:

Proposition 5.2. *Let G be a simply connected Lie group, and ∇ a torsion-free flat connection on G. Then the following are equivalent:*

- *∇ is a left-invariant connection of type $\mathcal{A}$.*
- *There exists an étale affine representation $\rho\colon G \to \mathcal{A}$, and $x \in \mathbb{R}^n$, such that ∇ and its compatible $(\mathbb{A}^n, \mathcal{A})$-structure are a pullback along the orbit map o_x.*

Moreover, the connection ∇ is complete if and only if ρ is simply transitive.

Proof. Let $D\colon G \to \mathbb{A}^n$ be any development map for the compatible left-invariant $(\mathbb{A}^n, \mathcal{A})$-structure. Let h denote the holonomy homomorphism with respect to D. Since the locally homogeneous structure is compatible with ∇, the map D is also an affine map $(G, \nabla) \to \mathbb{A}^n$. Since the maps $l_g\colon G \to G$ are equivalences for the locally homogeneous $(\mathbb{A}^n, \mathcal{A})$-structure on G, $\rho(g) = h(l_g)$ defines a representation $\rho\colon G \to \mathcal{A}$.

We put $x = D(1)$. Then, by (2.2),

$$D(g) = D(l_g 1) = \rho(g)D(1) = o_x(g).$$

Thus, the orbit map for ρ at $x = D(1)$ is the development map D. This shows that ρ is an étale affine representation, and the left-invariant $(\mathbb{A}^n, \mathcal{A})$-structure on G is obtained by pullback along o_x. □

5.2.1 The boundary of the development image. An étale affine representation $\rho\colon G \to \mathcal{A}$, which induces a flat left-invariant connection ∇ of type $\mathcal{A}$, as in Proposition 5.2, will be called a *compatible* étale affine representation for ∇.

Corollary 5.3. *The development map of a flat affine Lie group (G, ∇) is a covering map onto its image, and it is an orbit-map for any compatible étale affine representation of its universal covering group $(\widetilde{G}, \nabla)$.*

Proof. Since $D\colon \widetilde{G} \to \mathbb{A}^n$ is a local diffeomorphism and an orbit-map, it is a covering map. □

Let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation of G. Furthermore, if (G, ∇) is affine we put $\det_\nabla l\colon G \to \mathbb{R}^{\neq 0}$ for the character which is obtained by the left-action of G on parallel volume forms.

Theorem 5.4. *Let (G, ∇) be a flat affine Lie group. Then the following hold:*

(1) *Either (G, ∇) is complete or the development image of (G, ∇) is a connected component of the complement of a non-empty hypersurface in $\mathbb{R}^n$, $n = \dim G$. In particular, the development image is a semialgebraic subset of $\mathbb{R}^n$.*

(2) *Assume that G is simply connected. Then the connection ∇ is complete if and only if $\det \mathrm{Ad}(g)^{-1} \det_\nabla l_g = 1$, for all $g \in G$.*

Proof. In fact, the development image of G is the orbit $\rho(\widetilde{G})x$, for a compatible affine étale representation ρ for the induced flat structure on $\widetilde{G}$. By Section 5.4, $D(\widetilde{G}) = \rho(\widetilde{G})x$ is a connected component of the open semi-algebraic subset $U_\delta = \{x \in \mathbb{R}^n \mid \delta(x) \neq 0\} \subseteq \mathbb{R}^n$, where $\delta = \delta(\rho)$ is the relative invariant for ρ. This implies (1). Moreover, since D is a covering map, ∇ is complete if and only if D is surjective. If G is simply connected, the volume character of G satisfies $\det_\nabla l_g = \det \rho_\nabla(g)$, for any compatible étale affine representation $\rho_\nabla \colon \widetilde{G} \to \mathrm{Aff}(n)$. Thus (2) follows from Theorem 5.19 below. □

5.2.2 Completeness and unimodularity of flat affine Lie groups. Let (G, ∇) be a flat affine Lie group. If ∇ is complete then, by Proposition 5.2, the universal covering Lie group of G acts simply transitively on affine space. This, more or less, reduces the study of complete affine Lie groups to a study of simply transitive representations on affine space. (See [5], for a seminal paper on simply transitive groups of affine transformations.)

Corollary 5.5. *Let (G, ∇) be a flat affine Lie group. If ∇ is complete then G is a solvable Lie group.*

Proof. By Levi's theorem $\widetilde{G}$ splits as a semi-direct product of a semisimple group H and a solvable Lie group S. Now $\widetilde{G}$ can not have a reductive Levi-component H since every affine representation of H has a fixed point. Hence, $\widetilde{G} = S$ must be solvable. □

Recall that a Lie group G is called *unimodular* if $\det \mathrm{Ad}(g) = 1$, for all $g \in G$. If a flat Lie group (G, ∇) has a left-invariant parallel volume form, we call it *volume preserving*.

Corollary 5.6. *Let (G, ∇) be a volume-preserving flat affine Lie group. Then (G, ∇) is complete if and only if G is unimodular.*

Corollary 5.7. *Let (G, ∇) be a flat affine Lie group, where G is unimodular. Then (G, ∇) is complete if and only if (G, ∇) is a volume-preserving flat affine Lie group.*

Example 5.4 (Semi-simple Lie groups are not affine.). Let S be a semisimple Lie group. Then S does not admit a flat left-invariant connection.

For further generalisation, see Section 5.4.2.

5.3 Flat pseudo-Riemannian Lie groups

Corollary 5.8. *Let $(G, \langle\,,\,\rangle)$ be a flat pseudo-Riemannian Lie group. Then $(G, \nabla_{\langle\,,\,\rangle})$ is complete if and only if G is unimodular. In particular, a unimodular flat pseudo-Riemannian Lie group is solvable.*

Example 5.5. A flat pseudo-Riemannian Lie group $(G, \langle\,,\,\rangle)$, with biinvariant metric $\langle\,,\,\rangle$ is complete and G is a two-step nilpotent Lie group. (See Section 5.3.1 below.)

The above corollary already excludes a large class of Lie groups from carrying flat left-invariant metrics:

Example 5.6. Let $G = \mathrm{GL}(n, \mathbb{R})$. Since G is an open subset of $\mathbb{R}^{n^2}$, it has a natural structure of a flat affine Lie group. However, G does not admit a volume preserving flat left-invariant connection. In particular, G does not admit a flat pseudo-Riemannian metric.

The latter argument holds for all *reductive* Lie groups G. Here we call Lie group *reductive* if it is an almost semi-direct product of its center and a semisimple Lie group. Such a group is necessarily unimodular. Hence:

Corollary 5.9. *Let G be reductive. Then G does not admit a volume preserving flat left-invariant connection. In particular, G does not admit a left-invariant flat pseudo-Riemannian metric.*

Remark. Similar restrictions, as stated here for the pseudo-Riemannian case, also hold for symplectic affine Lie groups.

Concerning the existence of flat pseudo-Riemannian Lie groups $(G, \langle\,,\,\rangle)$, where G is not solvable, we observe:

Example 5.7. Every flat Lorentzian Lie group G is solvable. This is a consequence of the classification of homogeneous domains in $\mathbb{E}^{n-1,1}$. In fact, every such domain is diffeomorphic to $\mathbb{R}^n$. (See Section 6.6, respectively [30].) Therefore, the universal covering group $\widetilde{G}$ of G is diffeomorphic to $\mathbb{R}^n$. It can not have a reductive part, since $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ does not admit a faithful representation.

Not every flat pseudo-Riemannian Lie group is solvable, as the following example shows:

Example 5.8. Let $U = \mathrm{GL}(2, \mathbb{R}) \subset \mathbb{A}^4$. Then the dual tube domain $\check{T}(U) \leq \mathbb{E}^{4,4}$ is a homogeneous pseudo-Riemannian domain, which has a simply transitive isometric action of the group $\mathrm{GL}(4, \mathbb{R}) \ltimes (\mathbb{R}^4)^*$. (Compare Section 6.5.2.)

Question 4. Are flat pseudo-Riemannian Lie groups of signature $s = (n-2, 2)$ or $(n-3, 3)$ always solvable?

Milnor [78] gives the classification and structure of simply connected *flat Riemannian Lie groups*. These are automatically complete. In [45] (see also [3] for the case of nilpotent Lie groups) the classification of all simply connected *complete* flat Lorentzian Lie groups is described.

5.3.1 Flat biinvariant metrics on Lie groups. A left-invariant metric $\langle\,,\,\rangle$ on G is called *biinvariant* if the right-multiplications of G are isometries as well. If G is connected, $\langle\,,\,\rangle$ is biinvariant if and only if the induced scalar product $\langle\,,\,\rangle$ on the Lie algebra $\mathfrak{g}$ is skew with respect to the Lie bracket. That is, $\langle\,,\,\rangle$ is biinvariant iff, for all $X, Y, Z \in \mathfrak{g}$,

$$\langle [X, Y], Z\rangle = -\langle Y, [X, Z]\rangle. \tag{5.1}$$

The following fact was already used in the proof of Corollary 4.7:

Lemma 5.10. *Let $(G, \langle\,,\,\rangle)$ be a pseudo-Riemannian Lie group with biinvariant metric. Then the Levi-Civita connection $\nabla = \nabla_{\langle\,,\,\rangle}$ is the canonical torsion-free connection on G, that is, $\nabla_X Y = \frac{1}{2}[X, Y]$, for all $X, Y \in \mathfrak{g}$. Moreover, $(G, \langle\,,\,\rangle)$ is flat if and only if G is two-step nilpotent.*

Corollary 5.11. *Every flat pseudo-Riemannian Lie group with biinvariant metric is complete.*

5.3.2 Construction of flat biinvariant metric Lie algebras. We describe now a method, which allows to construct examples of flat biinvariant metric Lie algebras $(\mathfrak{g}, \langle\,,\,\rangle)$. This, of course, implies the construction of flat Lie groups with biinvariant metric.

Coadjoint extensions. Let $\mathfrak{n}$ denote a Lie algebra, and $\mathfrak{n}^*$ its dual vector space. By the coadjoint representation $\mathrm{ad}^*\colon \mathfrak{n} \to \mathfrak{gl}(\mathfrak{n}^*)$, $\mathfrak{n}^*$ is a module for $\mathfrak{n}$. Let $\omega \in Z^2(\mathfrak{n}, \mathfrak{n}^*)$ be a two-cocycle (cf. Appendix B). To ω there belongs a *Lie algebra extension*

$$0 \to \mathfrak{n}^* \to \mathfrak{t}_{\mathfrak{n},\omega} \to \mathfrak{n} \to 0,$$

where $\mathfrak{n}^*$ is an abelian ideal in $\mathfrak{t}_{\mathfrak{n},\omega}$. Put $W = \mathfrak{n}^* \oplus \mathfrak{n}$ for the direct sum of vector spaces. Explicitly, the Lie algebra $\mathfrak{t}_{\mathfrak{n},\omega}$ arises from the Lie product on W, which is declared by

$$[(\lambda, v), (\lambda', v')] = (\mathrm{ad}^*(v)\lambda' - \mathrm{ad}^*(v')\lambda + \omega(v, v'), [v, v']).$$

(See, for example, [71] for the general theory of Lie-algebra extensions.)

Next we define a natural scalar product $\langle\,,\,\rangle_o$ of signature $s = (n, n)$ on W, where $n = \dim \mathfrak{n}$. With respect to $\langle\,,\,\rangle_o$, $\mathfrak{n}$ and $\mathfrak{n}^*$ are dual totally isotropic subspaces under

$\langle\,,\,\rangle_o$. Moreover, if $v \in \mathfrak{n}$ and $\lambda \in \mathfrak{n}^*$, we put

$$\langle v, \lambda\rangle_o = \lambda(v).$$

We put $V = \mathfrak{n}$, and we define a three-form $F_\omega \in \bigwedge^2 V^* \otimes V^*$ by putting

$$F_\omega(u, v, w) = \langle \omega(u, v), w\rangle_o.$$

The following lemma is easily verified by direct calculation:

Lemma 5.12. *Let $\mathfrak{n}$ be two-step nilpotent, and $\omega \in Z^2(\mathfrak{n}, \mathfrak{n}^*)$. Then the following hold:*

(1) *The canonical scalar product $\langle\,,\,\rangle_o$ on $\mathfrak{t}_{\mathfrak{n},\omega}$ satisfies (5.1) if and only if F_ω is alternating (that is, if $F_\omega \in \bigwedge^3 V^*$).*

(2) *The Lie algebra $\mathfrak{t}_{\mathfrak{n},\omega}$ is two-step nilpotent, iff, for all $u, v, w, z \in \mathfrak{n}$,*

$$F_\omega(u, [v, w], z) = F_\omega(v, w, [u, z]). \tag{5.2}$$

Proof. Note, since $\mathfrak{n}$ is two-step nilpotent, triple commutators in $\mathfrak{t}_{\mathfrak{n},\omega}$ satisfy

$$[(\lambda, u), [(\mu, v), (\tau, w)]] = (ad_u^*\omega(v, w) + \omega(u, [v, w]), 0).$$

This proves (2). □

Thus we have, in particular:

Proposition 5.13. *Let $\mathfrak{n}$ be two-step nilpotent, and $\omega \in Z^2(\mathfrak{n}, \mathfrak{n}^*)$. If F_ω is alternating and satisfies (5.2), then the metric Lie algebra $(\mathfrak{t}_{\mathfrak{n},\omega}, \langle\,,\,\rangle_o)$ is flat and $\langle\,,\,\rangle_o$ is biinvariant.*

Proof. Since F_ω is alternating the metric $\langle\,,\,\rangle_o$ is biinvariant. Since , (5.2) is satisfied, $\mathfrak{t}_{\mathfrak{n},\omega}$ is two-step nilpotent, and therefore, the biinvariant metric $\langle\,,\,\rangle_o$ is flat. □

Split extensions. We may always choose $\omega = 0$. In this case, the Lie algebra $\mathfrak{t}_{\mathfrak{n},\omega}$ is the semi-direct product $\mathfrak{n}^* \oplus_{\mathrm{ad}^*} \mathfrak{n}$ of $\mathfrak{n}$ with its representation space $\mathfrak{n}^*$. Thus, a particular interesting and rich class of examples arises as follows:

Corollary 5.14. *Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra. Then the metric Lie algebra $(\mathfrak{n}^* \oplus_{\mathrm{ad}^*} \mathfrak{n}, \langle\,,\,\rangle_o)$ is flat with biinvariant metric.*

The corollary shows in particular that the class of Lie algebras with flat biinvariant metrics is as rich as the class of two-step nilpotent Lie algebras.

Twisted extensions with abelian base. If $\mathfrak{n} = \mathfrak{a}$ is abelian, examples may be constructed using alternating three-forms on $\mathfrak{a}$.

Example 5.9. Let $\mathfrak{n} = \mathfrak{a}_3$ be the three-dimensional abelian Lie algebra. Let $\det \in \bigwedge^3 \mathfrak{a}_3^*$ be a non-degenerate alternating three-form, and define

$$\omega_{\det}(X, Y) = \det(X, Y, \cdot).$$

Then $(\mathfrak{t}_{\mathfrak{a}_3, \omega_{\det}}, \langle\,,\,\rangle_o)$ is a non-abelian biinvariant flat metric Lie algebra.

The following result states that every flat metric Lie algebra $(\mathfrak{g}, \langle\,,\,\rangle)$ with biinvariant metric arises in this way:

Theorem 5.15. *Let $(\mathfrak{g}, \langle\,,\,\rangle)$ be a flat metric Lie algebra with biinvariant metric. Then there exists an abelian Lie algebra $\mathfrak{a}$ and an alternating three-form $F_\omega \in \bigwedge^3 \mathfrak{a}^*$, and an abelian Lie algebra $\mathfrak{z}$ such that $(\mathfrak{g}, \langle\,,\,\rangle)$ can be written as a direct product of metric Lie algebras*

$$(\mathfrak{g}, \langle\,,\,\rangle) = (\mathfrak{z}, \langle\,,\,\rangle) \oplus (\mathfrak{t}_{\mathfrak{a},\omega}, \langle\,,\,\rangle_o).$$

Proof. We show that it is possible to choose $\mathfrak{n}$ as an abelian Lie algebra, as in the previous example. Note that the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $(\mathfrak{g}, \langle\,,\,\rangle)$ is an isotropic ideal in $(\mathfrak{g}, \langle\,,\,\rangle)$, and its orthogonal complement with respect to $\langle\,,\,\rangle$ is the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$. This shows that we can choose an isotropic subspace $\mathfrak{a}$ of $(\mathfrak{g}, \langle\,,\,\rangle)$, such that there is a direct decomposition of subspaces $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{a} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z} \subset \mathfrak{z}(\mathfrak{g})$. □

Example 5.10. Let $(\mathfrak{g}, \langle\,,\,\rangle)$ be a flat biinvariant metric Lie algebra of dimension six. Then $\mathfrak{g}$ is abelian or $(\mathfrak{g}, \langle\,,\,\rangle) = (\mathfrak{t}_{\mathfrak{a}_3, \omega_{\det}}, \langle\,,\,\rangle_o)$. In particular, in the second case $\mathfrak{g}$, is a semidirect product of the three-dimensional Heisenberg algebra $\mathfrak{h}_3$ with its coadjoint representation.

Remark that a version of Theorem 5.15 already appeared in [104, Proposition 7.5]. A similar result is stated recently also in [26].

5.4 Étale affine representations

Let $\rho\colon G \to \mathrm{Aff}(n)$ be an étale affine representation, and let $\rho_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ be the corresponding Lie algebra representation.

5.4.1 The relative invariant for an étale affine representation. Choose a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$, and $x \in \mathbb{R}^n$. Let us consider the linear map $\tau_x(\rho)\colon \mathbb{R}^n \to \mathbb{R}^n$, which is defined by

$$\tau_x(\rho)\colon (\alpha_1, \ldots, \alpha_n) \longmapsto t_x\Big(\sum_{i=1}^{n} \alpha_i \rho_{\mathfrak{g}}(X_i)\Big).$$

Here, t_x is the derivative of the evaluation map at x, see (2.3). We easily see that $\tau_x(\rho)$ is an isomorphism if and only if the orbit map $o_x\colon G \to \mathbb{R}^n$ is non-singular, that is, if ρ is étale at x.

If $A \in \mathrm{Aff}(\mathbb{R}^n)$, we put $\rho^A(g) = A\,\rho(g)A^{-1}$. The following formulas are easily verified by direct calculation:

Lemma 5.16. *Let $A \in \mathrm{Aff}(\mathbb{R}^n)$, with $\ell(A) \in \mathrm{GL}(n,\mathbb{R})$ its linear part. Let $g \in G$, and $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$ its adjoint action on the Lie algebra $\mathfrak{g}$ of G. We put $B = \rho(g)$. Then the following hold:*

$$\det \tau_x(\rho^A) = (\det \ell(A)) \det \tau_{A^{-1}x}(\rho),$$
$$\det \tau_x(\rho^B) = (\det \mathrm{Ad}(g)) \det \tau_x(\rho).$$

We define the polynomial function

$$\delta = \delta(\rho)\colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \delta(x) = \det \tau_x(\rho)$$

which (by the above lemma) satisfies the transformation rule

$$\delta(\rho(g)x) = \det \mathrm{Ad}(g)^{-1} \det \ell(\rho(g))\delta(x). \tag{5.3}$$

Definition 5.17. We call $\delta = \delta(\rho)$ the *relative invariant* for the étale affine representation ρ. Its character is the function $\chi = \chi(\rho)\colon G \to \mathbb{R}^{>0}$, which is defined as

$$\chi(g) = \det \mathrm{Ad}(g)^{-1} \det \ell(\rho(g))$$

Note that an open orbit $\rho(G)x$ of an affine étale representation ρ is contained in (and it is actually a connected component of) the open semi-algebraic subset

$$U_\delta = \{x \in \mathbb{R}^n \mid \delta(x) \neq 0\} \subset \mathbb{R}^n.$$

By a theorem of Whitney [101], a real semi-algebraic set U_δ has only finitely many connected components in the standard Euclidean topology on $\mathbb{R}^n$. Thus we have proved the following result:

Corollary 5.18. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine étale representation. Then G has finitely many open orbits on $\mathbb{R}^n$. Moreover, either G is transitive or each orbit is a connected component of the complement of a hypersurface of degree n.*

Example 5.11. The list of étale affine representations of Lie groups G with $\dim G = 2$ is contained in Example 6.2.

The relative invariant δ may also be interpreted in terms of growth of the right-invariant volume on G, relative to a parallel volume on $\mathbb{A}^n$, see Example 9.1.

5.4.2 The relative invariant of a simply transitive representation is constant. The following theorem is actually a direct consequence of Proposition 7.7. For transparency, we provide a direct proof. (For a different proof see [42], [50], [62] for the case of nilpotent groups.)

Theorem 5.19. *Let G be a Lie group, and $\rho\colon G \to \mathrm{Aff}(n)$ be an affine étale representation. Then the following conditions are equivalent:*

(1) *ρ is simply transitive.*

(2) *The relative invariant $\delta(\rho)$ is a non-zero constant function.*

(3) $\chi(\rho) \equiv 1$.

Proof. Clearly, if $\delta(\rho)$ is non-vanishing, every orbit of G is open. Hence, there can be only one orbit, and $\rho(G)$ is a simply transitive group of affine transformations. Conversely, assume that $\rho(G)$ is simply transitive. Then the relative invariant $\delta(\rho)$ is a nowhere vanishing polynomial function. Let $\boldsymbol{G}$ denote the Zariski closure of $\rho(G)$ in $\mathrm{Aff}(\mathbb{R}^n)$. Then $\boldsymbol{G}$ is a transitive group of affine transformations on $\mathbb{R}^n$, and $\delta(\rho)$ remains a relative invariant, for some polynomial homomorphism $\chi_{\boldsymbol{G}}\colon \boldsymbol{G} \to \mathbb{R}^{>0}$, which extends χ. Note that also the unipotent radical $\boldsymbol{U}$ of $\boldsymbol{G}$ acts transitively on $\mathbb{R}^n$ (even simply transitively, see [5]), and the polynomial character $\chi_{\boldsymbol{G}}$ is trivial on $\boldsymbol{U}$. Therefore, by formula (5.3), evaluated on $\boldsymbol{U}$, $\delta(\rho)$ must be constant.

The latter fact clearly implies that $\chi(\rho)$ is constant. Conversely, if $\chi(\rho)$ is constant, by (5.3), $\delta(\rho)$ is constant and non-zero on an open subset of $\mathbb{R}^n$. Since, $\delta(\rho)$ is a polynomial it must be constant. □

Remark. In the case of an étale affine representation, the relative invariant $\delta(\rho)$ may be identified with the characteristic map $\Phi(\rho)\colon \mathbb{R}^n \to \bigwedge^n \mathfrak{g}^*$ (as defined in Section 6.3), using an isomorphism $\bigwedge^n \mathfrak{g}^* = \mathbb{R}$. See Section 8.4 for further discussion, and, in particular, Proposition 8.27.

We call ρ volume preserving if its linear part satisfies $\ell(\rho(G)) \leq \mathrm{SL}(n, \mathbb{R})$.

Corollary 5.20. *Let G be unimodular, and $\rho\colon G \to \mathrm{Aff}(n)$ an étale affine representation. Then G is simply transitive on $\mathbb{A}^n$ if and only if $\rho(G)$ is volume preserving.*

Here is an immediate application which shows that a large class of Lie groups does not admit étale affine representations. Consider the connected abelian group

$$H^1(G) = G/[G, G],$$

over which every homomorphism of G to an abelian group factorises.

Corollary 5.21. *If the group $H^1(G)$ is compact then G does not admit an étale affine representation.*

Proof. By our assumption, for every étale representation ρ of G, its character $\chi(\rho)$ must be trivial. Therefore, by the above theorem, ρ must be simply transitive, and G is simply connected solvable. This shows that, in fact, $H^1(G, \mathbb{R}) = \{1\}$, and since G is solvable, G must be trivial. A contradiction. □

Example 5.12. The criterion is satisfied, for example, if

$$H^1(\mathfrak{g}, \mathbb{R}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = \{0\}.$$

Hence, *no semisimple Lie group admits an étale affine representation*. Also *no compact group admits an étale affine representation*.

Corollary 5.22. *If the group G is compact then G does not admit an étale affine representation.*

Example 5.13. Let G be a reductive Lie group with one-dimensional center. Then every étale affine representation ρ of G is linear. Since G is unimodular, $\delta(\rho)$ is a non-zero homogeneous relative invariant polynomial for G of degree n, with non-trivial character $\det = \chi(\rho)$. Such representations exist, for example, for $G = \mathrm{GL}(n, \mathbb{R})$. There do exist other examples as well. See [8], [22], [63] for further reference on this problem.

5.4.3 The dual tube representation. The dual tube, as discussed in Section 6.5, is a useful tool in the construction of pseudo-Riemannian and symplectic affine étale representations. This is based on the following simple observation:

Lemma 5.23. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an étale affine representation, with open orbit U. Let $\ell(\rho)\colon G \to \mathrm{GL}(n)$ denote its linear part. Then the associated semi-direct product Lie group $G \ltimes_{\ell(\rho)^*} \mathbb{R}^n$ has an étale affine representation on $\mathbb{A}^{2n}$, which has the dual tube domain $\breve{T}(U)$ as open orbit. Moreover, $G \ltimes_{\ell(\rho)^*} \mathbb{R}^n$ preserves the natural flat pseudo-Riemannian metric of signature (n, n), and also the natural symplectic form on $\breve{T}(U)$.*

(Here, $\ell(\rho)^*$ denotes the dual representation for $\ell(\rho)$.)

The lemma is a particular case of Proposition 6.17 applied to étale affine representations.

6 Affinely homogeneous domains

Let $U \subseteq \mathbb{A}^n$ denote an open subset of $\mathbb{A}^n$. We put

$$\mathrm{Aff}(U) = \{A \in \mathrm{Aff}(n) \mid A(U) = U\}$$

for the *affine automorphism group* of U.

Definition 6.1. Let $U \subseteq \mathbb{A}^n$ be a connected open subset. Then U is called an *affinely homogeneous domain* if $\mathrm{Aff}(U)$ acts transitively on U.

A domain U in $\mathbb{E}^s$ is called homogeneous, or a *pseudo-Riemannian affine homogeneous domain* if its isometry group $\mathrm{Isom}(U) = \mathrm{Aff}(U) \cap \mathrm{E}(s)$ acts transitively on U. Similarly, a domain U in symplectic affine space $(\mathbb{A}^{2k}, \omega)$ is called a *symplectic affine homogeneous domain*, if the group of symplectic affine automorphisms of U, $\mathrm{Aff}(U, \omega) = \mathrm{Aff}(U) \cap \mathrm{Aff}(\omega)$ acts transitively.

An affine domain U is called *divisible* if there exists a *discrete* subgroup

$$\Gamma \leq \mathrm{Aff}(U)$$

such that U/Γ is compact.

6.1 Prehomogeneous affine representations

Let G be a Lie group and $\rho\colon G \to \mathrm{Aff}(n)$ a homomorphism. The homomorphism ρ is called an affine representation of G. Composition with the natural homomorphism $\mathrm{Aff}(n) \to \mathrm{GL}(n)$, defines a linear representation

$$\ell = \ell_\rho\colon G \to \mathrm{GL}(n)$$

which is called the linear part of ρ.

Definition 6.2. The affine representation ρ is called a *prehomogeneous affine representation*, if $\rho(G)$ acts transitively on an open set $U \subseteq \mathbb{A}^n$. If $G \leq \mathrm{Aff}(n)$ is a subgroup with an open orbit in $\mathbb{A}^n$ then G is called a *prehomogeneous subgroup*.

We let $U_\rho \subseteq \mathbb{A}^n$ denote the union of all open orbits for G.

Remark. If G admits a fixed point on $\mathbb{A}^n$, then the homomorphism ρ is conjugate by a translation to the linear representation ℓ_ρ on the vector space $V = \mathbb{R}^n$. Such a representation will be called *linear*. A vector space V with a linear G-action, which has an open orbit, is traditionally called a *prehomogeneous vector space* (see [63]).

Quite opposite to a linear representation is the situation that the affine representation ρ is *transitive* on $\mathbb{A}^n$. In this case, $U_\rho = \mathbb{A}^n$.

The following notion plays a special role.

Definition 6.3. A prehomogeneous representation ρ on $\mathbb{A}^n$ is called an *étale affine representation* if $\dim G = n$.

6.2 Some examples

There is a wealth of prehomogeneous affine representations ρ, and associated affinely homogeneous domains. We consider some simple examples.

Example 6.1. (1) The vector group $\mathbb{R}^n$ acts simply transitively by translations on $\mathbb{A}^n$.

(2) $G = \mathrm{GL}(n, \mathbb{R})$ acts by left- and also by right-multiplication on the vector space of matrices $\mathrm{Mat}(n \times n, \mathbb{R})$. Both actions have open orbit $U = \mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2} = \mathrm{Mat}(n \times n, \mathbb{R})$, and both are simply transitive on U.

(3) $\mathrm{SL}(n, \mathbb{R})$ acts transitively on $U = \mathbb{R}^n - \{0\}$. Also $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ has an open orbit $U \times U$ in $\mathbb{R}^{2n}$.

Note that all the homogeneous domains in the previous example are divisible by affine transformations. The first two examples admit transitive étale affine groups, the first one being the role model for a simply transitive affine representation. The second and third examples come from prehomogeneous linear representations of reductive groups, a topic which is studied extensively in the literature, see [63] and the references therein.

Two-dimensional affinely homogeneous domains. In dimension two there are up to affine equivalence six homogeneous affine domains. The following list also exhausts the set of all two-dimensional étale affine representations up to affine equivalence. (Compare [82], [33].) Note that each two-dimensional affinely homogeneous domain admits a simply transitive group of affine transformations.

Example 6.2 (Étale affine representations in dimension 2).

(1) ***The plane $U = \mathbb{A}^2$.***

$$\mathbb{R}^2 \quad \text{and} \quad \mathcal{U} = \left\{ \begin{pmatrix} 1 & v & u + \frac{1}{2}v^2 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{Aff}(2)$$

are simply transitive abelian groups of affine transformations.

The group

$$\mathcal{A}_\lambda = \left\{ \begin{pmatrix} \exp(t\lambda) & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{Aff}(2)$$

is a simply transitive solvable, non-abelian, group of affine transformations, for $\lambda \neq 0$.

(2) ***The halfspace $U = H_2$.*** Let H_2 be the halfspace $x > 0$. Then

$$\mathcal{B} = \left\{ \begin{pmatrix} \exp(t) & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{Aff}(H_2) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ z & \beta & v \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \alpha > 0 \right\}$$

is an abelian group of affine transformations. The half-spaces (x, y), $x > 0$ and $x < 0$ are open orbits.

The groups

$$\mathcal{C}_{\lambda,\tau} = \left\{ \begin{pmatrix} \exp(t\lambda) & 0 & s \\ 0 & \exp(-t\lambda) & (\exp(-t\lambda)-1)\tau \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{Aff}(2)$$

are solvable, non-abelian, groups of affine transformations, with two halfspaces as open orbits, $\lambda \neq 0$.

The group of upper triangular matrices

$$\mathcal{C} = \left\{ \begin{pmatrix} \exp(t\lambda) & b \\ 0 & \exp(-t\lambda) \end{pmatrix} \right\} \subseteq \mathrm{GL}(2)$$

is a solvable, non-abelian, *linear* group of transformations with two halfspaces as open orbits.

(3) ***The sector $U = Q_2$.***

$$\mathrm{Aff}(Q_2)^0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a > 0, b > 0 \right\} \subseteq \mathrm{GL}(2)$$

is an abelian, *linear* group of transformations, which has the four open sectors as open orbits.

(4) ***The parabolic domains $U = \mathcal{P}_2^+, \mathcal{P}_2^-$.*** Let $\mathcal{P}_2^+ = \{(y,x) \mid y > \frac{1}{2}x^2\}$ be the convex domain enclosed by a parabola.

$$\mathrm{Aff}(\mathcal{P}_2^{\pm})^0 = \left\{ \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v & \frac{1}{2}v^2 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \lambda > 0 \right\} \subseteq \mathrm{Aff}(2)$$

is solvable, non-abelian, with open orbits $\mathcal{P}_2^+$ and the exterior $\mathcal{P}_2^-$ of a parabola.

(5) ***The plane with a point removed, $U = \mathbb{A}^2 - 0$.***

$$\mathcal{E} = \left\{ \exp(t) \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \right\} \subseteq \mathrm{GL}(2) = \mathrm{Aut}(U)$$

is an abelian *linear* group, with open orbit $\mathbb{R}^2 - \{0\}$.

We remark:

Corollary 6.4. *Every two-dimensional affinely homogeneous domain $U \subseteq \mathbb{A}^2$ admits a simple transitive étale group of affine automorphisms in* $\mathrm{Aff}(U)$.

Corollary 6.5. *Every two-dimensional affinely homogeneous domain is divisible, except for the parabolic domains $\mathcal{P}^{\pm}$.*

Proof. If $U \subseteq \mathbb{A}^2$ admits a simply transitive abelian group G of affine transformations, then U is divisible by a lattice $\Gamma \leq G$, $\Gamma \cong \mathbb{Z}^2$. This is the case for all domains, but $\mathcal{P}^{\pm}$. The automorphism group $G = \mathrm{Aff}(\mathcal{P}^{\pm})$ is a solvable non-unimodular étale

group of affine transformations. Therefore, every Γ which divides $\mathcal{P}^{\pm}$ is a lattice in G. However, the existence of a lattice implies that G is unimodular (see Appendix C); a contradiction. □

Corollary 6.6 ([82]). *Let $U \subseteq \mathbb{A}^2$ be the development image of a compact affine two-torus. Then U is an affinely homogeneous domain which admits an abelian simply transitive group of affine transformations.*

This also implies:

Corollary 6.7. *Every two-dimensional divisible domain $U \subseteq \mathbb{A}^2$ is convex and homogeneous.*

There are attempts to classify étale affine representations in low dimensions, see [82], [38], [36], [12], for some results. In general, it is a difficult problem to decide which Lie groups G admit an étale affine representation. See Section 5.4, for further discussion.

6.3 Characteristic map for a prehomogeneous representation

Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine representation, and let $x \in \mathbb{A}^n$. We define the *orbit map* at x as

$$o_x\colon G \to \mathbb{R}^n, \quad g \mapsto \rho(g)x.$$

Let $\mathfrak{g}$ be the Lie algebra of G, and let

$$\tau_x = \tau_x(\rho)\colon \mathfrak{g} \to \mathbb{R}^n$$

denote the differential of o_x at the identity of G. We remark:

(1) The kernel of the linear map τ_x is the Lie algebra $\mathfrak{h} = \mathfrak{g}_x$ of the stabiliser $H = G_x \leq G$ of x under the action ρ.

(2) The orbit $\rho(G)x$ is open if and only if $o_x\colon G \to U = \rho(G)x$ is a submersion of G onto the open subset U of $\mathbb{A}^n$. Moreover, *G has an open orbit at x if and only if τ_x has maximal rank, that is, if τ_x is onto.*

Let us fix a parallel volume form ν on $\mathbb{A}^n$. We construct the *characteristic map for ρ*

$$\Phi = \Phi_\rho\colon \mathbb{A}^n \longrightarrow \bigwedge^n \mathfrak{g}^*, \quad \Phi(x) = \tau_x^*\nu,$$

by taking the pull back of ν along τ_x. Its image

$$\Phi(\mathbb{A}^n) \subseteq \bigwedge^n \mathfrak{g}^*$$

is called the *characteristic image* of ρ.

The map Φ and the characteristic image $\Phi(\mathbb{A}^n)$ carry fundamental information about ρ and the associated homogeneous domain U_ρ. This theme will be further developed in Section 7.2.

Note, in particular, that the map Φ is a *polynomial map* in the natural coordinates of $\mathbb{A}^n$. Moreover, by the above considerations, *G has an open orbit at x, if and only if the form $\Phi_x = \Phi(x)$ does not vanish.*

As a first application, we have two basic remarks:

Proposition 6.8. (1) *Every affinely homogenous domain U is an open semi-algebraic subset of $\mathbb{R}^n$.*

(2) *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine representation. Then G has only finitely many open orbits.*

Proof. Let G be a connected group which is transitive on U. The set $U_\rho = \{x \mid \Phi_x \neq 0\}$ is a Zariski open subset of $\mathbb{A}^n$. Its connected components are precisely the open orbits for G. By [101, Theorem 4], there are only finitely many components in the complement of a real algebraic variety. Therefore, U_ρ has only finitely many components. Each component is a semi-algebraic subset of $\mathbb{R}^n$. □

6.4 The automorphism group of an affine homogeneous domain

Let $U \subseteq \mathbb{A}^n$ be an affine homogeneous domain, and $\mathrm{Aff}(U)$ its affine automorphism group. The following explains the prominent role which algebraic groups play in the study of affinely homogeneous domains. (For a review on linear algebraic groups and basic definitions, consult Appendix A.)

Lemma 6.9. *Let $U \subseteq \mathbb{A}^n$ be an affine homogeneous domain. Then the automorphism group $\mathrm{Aff}(U)$ of U is a finite index subgroup of a real algebraic group. In particular, $\mathrm{Aff}(U)$ has finitely many connected components.*

Proof. Let $\widetilde{U} \subseteq \mathbb{A}^n$ be the domain, where the Lie group $\mathrm{Aff}(U)$ has open orbits. Then U is a connected component of $\widetilde{U}$. The complement $W = \mathbb{A}^n - \widetilde{U}$ is an algebraic subset of $\mathbb{A}^n$, by Proposition 6.8. Since $\mathrm{Aff}(\widetilde{U}) = \mathrm{Aff}(W)$, we conclude that $\mathrm{Aff}(\widetilde{U})$ is a real algebraic subgroup of $\mathrm{Aff}(n)$. Since $\widetilde{U}$ has only finitely many components (cf. Proposition 6.8), $\mathrm{Aff}(U)$ has finite index in $\mathrm{Aff}(\widetilde{U})$. □

Let $G \leq \mathrm{Aff}(n)$ be a prehomogeneous Lie subgroup. The real Zariski closure $\overline{G}^{\mathbb{R}} \leq \mathrm{Aff}(n)$ of G, is clearly also a prehomogeneous real linear algebraic group.

Corollary 6.10. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous representation. Then the Zariski closure $\overline{G}^{\mathbb{R}} \leq \mathrm{Aff}(n)$ preserves the maximal domain U_ρ.*

6.4.1 Automorphisms of complex affine domains. A connected domain $\mathcal{U} \subseteq \mathbb{C}^n$ is called a *complex affine homogeneous domain*, if its group of complex affine transformations

$$\mathrm{Aff}_{\mathbb{C}}(\mathcal{U}) = \{A \in \mathrm{Aff}(\mathbb{C}^n) \mid A\,\mathcal{U} = \mathcal{U}\}$$

acts transitively on $\mathcal{U}$. We mention:

Lemma 6.11. *Let $\mathcal{U} \subseteq \mathbb{C}^n$ be a complex affine homogeneous domain. Then $\mathcal{U}$ is a Zariski-open, hence irreducible and Zariski-dense connected subset of $\mathbb{C}^n$. The automorphism group $\mathrm{Aff}_{\mathbb{C}}(\mathcal{U}) \leq \mathrm{Aff}(\mathbb{C}^n)$ is a complex linear algebraic group.*

Proof. The proof follows along the lines of the proof of Lemma 6.9. Note that any open and connected orbit $\mathcal{U} = \boldsymbol{G}^0 x$, where $\boldsymbol{G}$ is an algebraic subgroup of $\mathrm{Aff}(\mathbb{C}^n)$, is Zariski-dense in $\mathbb{C}^n$. By Appendix A.3, $\boldsymbol{G}x$ is also dense in the Euclidean topology of $\mathbb{C}^n$. Therefore, there is only one such orbit. In particular, $\widetilde{\mathcal{U}} = \mathcal{U}$ is a connected domain, and $\mathrm{Aff}_{\mathbb{C}}(\mathcal{U}) = \mathrm{Aff}_{\mathbb{C}}(\widetilde{\mathcal{U}})$. □

6.4.2 Centralisers of prehomogeneous representations. Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous representation. We let

$$\mathrm{Z}_{\mathrm{Aff}(n)}(G) = \{A \in \mathrm{Aff}(n) \mid A\,\rho(g) = \rho(g)A \text{ for all } g \in G\} \leq \mathrm{Aff}(n)$$

denote the centraliser of G in $\mathrm{Aff}(n)$. Since the elements of $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ permute the open orbits of G, $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ preserves the maximal set U_ρ. In particular, it follows that

$$\mathrm{Z}_{\mathrm{Aff}(n)}(G) \leq \mathrm{Aff}(U_\rho).$$

Note that $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ is an algebraic subgroup of $\mathrm{Aff}(n)$. Moreover, we have the following fact:

Lemma 6.12. *The Lie algebra $\mathfrak{z}_\rho(G)$ of $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ forms an associative subalgebra of $\mathfrak{aff}(n)$.*

Proof. Let $\varphi \in \mathfrak{aff}(n)$. Clearly, $\varphi \in \mathfrak{z}_\rho(G)$ if and only if $A\varphi A^{-1} = \varphi$, for all $A \in \rho(G)$. Thus, if $\varphi, \psi \in \mathfrak{z}_\rho(G)$ then $A(\varphi\psi)A^{-1} = \varphi\psi \in \mathfrak{z}_\rho(G)$. □

The following lemma is useful in this context:

Lemma 6.13. *Let $\mathcal{A} \leq \mathrm{Mat}(n \times n)$ be an associative subalgebra of linear operators, such that* trace $\phi = 0$*, for all $\phi \in \mathcal{A}$. Then every element $\phi \in \mathcal{A}$ is nilpotent. Moreover, the Lie algebra which belongs to $\mathcal{A}$ (by taking commutators in $\mathcal{A}$) is nilpotent.*

Corollary 6.14. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous representation such that every element of the centraliser algebra $\mathfrak{z}_\rho(G)$ contains only elements of trace zero. Then the centraliser $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ contains only unipotent elements, and, in particular, it is a connected nilpotent Lie group.*

Proof. Let $A \in \mathrm{Z}_{\mathrm{Aff}(n)}(G)$. Viewing A as an element of $\mathrm{GL}(n+1, \mathbb{R})$, we obtain $A - E_{n+1} \in \mathfrak{z}_\rho(G)$. Thus, $A - E_{n+1}$ is nilpotent, by Lemma 6.13. This implies that the real algebraic group $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ has only unipotent elements. □

This situation occurs, for example, if ρ is transitive:

Proposition 6.15. *Let ρ be a transitive affine representation. Then the centraliser of $\rho(G)$ in $\mathrm{Aff}(n)$ is a connected unipotent group of dimension $\leq n$.*

Proof. If $G \leq \mathrm{Aff}(n)$ is a transitive subgroup, $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ acts without fixed points on $\mathbb{A}^n$. The centraliser $\mathrm{Z}_{\mathrm{Aff}(n)}(G)$ is an algebraic subgroup of $\mathrm{Aff}(n)$. Its reductive part T has a fixed point, by Lemma 2.2. Therefore, T must be trivial. □

We shall show below (see Proposition 6.26 and Proposition 6.28) that the centralisers of pseudo-Riemannian and symplectic prehomogeneous groups in the group of isometries of $\mathbb{E}^s$, respectively symplectic affine transformations of $(\mathbb{A}^n, \omega)$ are unipotent groups. Such results have immediate geometric implications. As the following example shows, the latter phenomena depend on the orthogonal and symplectic structure:

Example 6.3. $G = \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R})$ is a volume preserving prehomogeneous subgroup of $\mathrm{Aff}(\mathbb{R}^n \oplus \mathbb{R}^n)$, as in Example 6.1 (2). For $\lambda \in \mathbb{R}$, define $t_\lambda(u,v) = (\lambda u, \lambda^{-1} v)$. Then $T = \{t_\lambda \mid \lambda > 0\} \leq \mathrm{SL}(\mathbb{R}^n \oplus \mathbb{R}^n)$ is an abelian group of volume preserving linear maps, consisting of semisimple elements, and T is contained in the centraliser of G.

Note also that the volume preserving group T in the example leaves invariant the natural symplectic form ω on $\mathbb{R}^n \oplus \mathbb{R}^n$.

6.5 Tube like domains

Affinely homogeneous domains play an important role in the theory of Hermitian symmetric spaces, by the *tube construction*, which itself is an important source for the construction of complex homogeneous domains from affine homogeneous domains. An analogous dual construction plays a role for the theory of pseudo-Riemannian and symplectically affine homogeneous domains. We develop its basic properties and draw some consequences concerning the classification of pseudo-Riemannian and symplectically affine homogenous domains.

6.5.1 Complex tube domains. The classical construction of *tube domains* goes as follows. Let $U \subseteq \mathbb{R}^n$ be a connected open set. Then

$$T(U) = U + i\mathbb{R}^n \subseteq \mathbb{C}^n$$

is called the *complex tube domain* associated to U. Every $A \in \mathrm{Aff}(\mathbb{R}^n)$ extends to a complex affine transformation $A_{\mathbb{C}} \in \mathrm{Aff}(\mathbb{C}^n)$, where

$$A_{\mathbb{C}}(x + iy) = Ax + i\,\ell(A)y.$$

If $A \in \mathrm{Aff}(U)$ is an affine automorphism of U then $A_{\mathbb{C}} \in \mathrm{Aff}_{\mathbb{C}}(T(U))$. Also the group of purely imaginary translations $i\,\mathbb{R}^n \subset \mathbb{C}^n$ acts on $T(U)$, and this group is

normalised by the transformations $A_{\mathbb{C}}$. In particular, if U is a homogeneous affine domain then $T(U)$ admits a transitive group of complex affine transformations. That is, $T(U)$ is a *complex affine homogeneous domain*.

Example 6.4. Riemannian Hermitian symmetric spaces (cf. [49], Chapter VIII) are obtained by using the Bergman metric on a complex homogeneous domain $T(U)$, where $U \subset \mathbb{R}^n$ is a convex non-degenerate homogeneous selfdual cone.

Complex domains $T(U)$, where U is a convex non-degenerate cone are also called *Siegel-domains* of the first kind. Here a convex cone is called non-degenerate if it does not contain any straight line. See [69], [70] and Matsushimas expository paper [76] for results on homogeneous tube domains over convex non-degenerate cones, and their relation with *bounded domains* in $\mathbb{C}^n$. See [34] for a generalisation in the context of pseudo-Hermitian symmetric spaces.

6.5.2 Dual tube domains. A process which is dual to the construction of the tube domain $T(U) \subseteq \mathbb{C}^n$, $A \mapsto A_{\mathbb{C}}$, allows to construct pseudo-Riemannian and symplectically affine homogeneous domains from arbitrary homogeneous affine domains. Perhaps surprisingly, this construction suggests that *the determination of homogeneous pseudo-Riemannian and symplectic affine domains is at least as complicated as the classification of all affinely homogeneous domains.*

The dual tube domain $\check{T}(U)$. Let $V = \mathbb{R}^n$ denote a real vector space, and V^* the dual space of V. If φ is a linear map of V, we let φ^* denote the dual (transposed) map of V^*. If $U \subseteq V$ is a connected open subset we call

$$\check{T}(U) = U \times V^* \subseteq V \oplus V^*$$

the *dual tube domain over* U.

Let $A \in \mathrm{Aff}(\mathbb{R}^n)$. We extend A to an affine map $\check{A} \in \mathrm{Aff}(V \oplus V^*)$, by declaring

$$\check{A}(u + \lambda) = Au + (\ell(A)^{-1})^*\lambda.$$

If A is an affine automorphism of U then $\check{A}$ preserves $\check{T}(U)$. Note also that the group of translations $V^* \subseteq \mathrm{Aff}(\check{T}(U))$ is normalised by the transformation $\check{A}$. In particular, if U is a homogeneous affine domain then $\check{T}(U)$ is a homogeneous affine domain, as well.

Natural pseudo-Riemannnian metric on $\check{T}(U)$. The vector space $V \oplus V^*$ admits a naturally defined scalar product $\langle\!\langle \, , \, \rangle\!\rangle$ of signature (n, n). This is defined by

$$\overbrace{\langle u + \lambda, u' + \lambda' \rangle} = \lambda(u') + \lambda'(u).$$

The scalar product $\langle\!\langle \, , \, \rangle\!\rangle$ is evidently preserved by $\check{A}$, $A \in \mathrm{Aff}(n)$. Thus:

Proposition 6.16. *The map $A \mapsto \breve{A}$ defines a faithful homomorphism*

$$\mathrm{Aff}(U) \to \mathrm{Isom}(\breve{T}(U), \breve{\langle\,,\,\rangle}).$$

In particular, we remark:

Proposition 6.17. *Let $U \subseteq \mathbb{A}^n$ be an affine homogeneous domain. Then the tube domain $(\breve{T}(U), \breve{\langle\,,\,\rangle})$ is a pseudo-Riemannian homogeneous domain of signature (n,n).*

Proof. In fact let, $G \leq \mathrm{Aff}(n)$ be a subgroup which acts transitively on U. Then $V^* \rtimes \breve{G} \leq \mathrm{E}(n,n)$ acts transitively on $\breve{T}(U) = U \times V^*$. □

Note that V^* is a maximal isotropic subspace in $\mathbb{R}^{n,n} = V \oplus V^*$. The following geometric characterisation of pseudo-Riemannian domains of tube type is straightforward:

Lemma 6.18. *Let $D \subseteq \mathbb{E}^{n,n}$ be a homogeneous pseudo-Riemannian domain. If* $\mathrm{Isom}(\mathcal{D})$ *contains a maximal isotropic subgroup of translations then*

$$D = \breve{T}(U),$$

for some affinely homogeneous domain $U \subseteq \mathbb{A}^n$.

The following notion is related:

Definition 6.19. A domain $U \subseteq \mathbb{E}^s$ is called *translationally isotropic* if the group of translations $t(U)$ in $\mathrm{Isom}(U)$ satisfies $t(U)^\perp \subseteq t(U)$.

Every tube domain $(\breve{T}(U), \breve{\langle\,,\,\rangle}) \subseteq \mathbb{E}^{n,n}$ is *translationally isotropic*, since $V^* \subseteq t(U)$ is a maximal isotropic subspace, and thus $t(U)^\perp \subseteq V^*$.

Proposition 6.20 ([31][Theorem 5.3]). *Every translationally isotropic homogeneous pseudo-Riemannian domain $D \subseteq \mathbb{E}^{m,l}$ is of the form*

$$D = \breve{T}(U) \times \mathbb{E}^{m-k,l-k},$$

for some affinely homogeneous domain $U \subseteq \mathbb{A}^k$, $l, m \geq k$.

Question 5. By [30], the Lorentzian homogeneous domains are $\mathbb{E}^{n-1,1}$, and the translationally isotropic domains $\breve{T}(\mathbb{R}^{>0}) \times \mathbb{E}^{n-2}$. Is every homogeneous pseudo-Riemannian domain translationally isotropic?

The question suggests, in particular, that homogeneous pseudo-Riemannian domains admit large groups of translations.

Natural symplectic geometry on $\check{T}(U)$. On the vector space $V \oplus V^*$ there is also a naturally defined symplectic form $\check{\omega}$, which is given by

$$\check{\omega}(u + \lambda, u' + \lambda') = \lambda(u') - \lambda'(u).$$

The symplectic form $\check{\omega}$ is evidently preserved by $\check{A}$, $A \in \mathrm{Aff}(n)$. Most considerations for the pseudo-Riemannian case carry over analogously to the symplectic case. We just remark:

Proposition 6.21. *The map $A \mapsto \check{A}$ defines a faithful homomorphism*

$$\mathrm{Aff}(U) \to \mathrm{Aff}(\check{T}(U), \check{\omega}).$$

Thus:

Corollary 6.22. *Let U be an affine homogeneous domain. Then $(\check{T}(U), \check{\omega})$ is a symplectic affine homogeneous domain.*

Moreover, we mention:

Lemma 6.23. *Suppose that $D \subseteq \mathbb{A}^{2n}$ is a symplectic affine homogeneous domain. If $\mathrm{Aff}(\check{T}(U), \check{\omega})$ contains a Lagrangian subgroup of translations then*

$$D = \check{T}(U),$$

for some affinely homogeneous domain $U \subseteq \mathbb{A}^n$.

As in the pseudo-Riemannian case, one might wonder which role translationally isotropic symplectically affine homogeneous domains play in the classification of all symplectically affine homogeneous domains. However, here we have

Example 6.5. $(U = \mathbb{R}^2 - \{0\},\ \omega = dx \wedge dy)$ is a symplectically affine homogeneous domain, and $\mathrm{Aff}(U, \omega) = \mathrm{SL}(2, \mathbb{R})$. In particular, U is not translationally isotropic.

Para-Kähler geometry of $\check{T}(U)$. We remark that the natural geometry on $\check{T}(U)$, which is preserved by the transformations $\check{A}$, is actually determined by the symplectic form $\check{\omega}$ together with the metric $\langle \check{,} \rangle$. It may be expressed in a manner which is analogous to classical Kähler geometry as follows.

We define a linear operator $\check{J} \in \mathrm{GL}(V \oplus V^*)$, which is skew with respect to $\check{\omega}$ and satisfies $\check{J}^2 = \mathrm{id}$ by the relation

$$\check{g}(\cdot, \cdot) = \check{\omega}(\check{J}\cdot, \cdot).$$

It is computed by the formula

$$\check{J}(u + \lambda) = (-u + \lambda).$$

Note, in particular, that the transformations $\check{A}$ are $\check{J}$-linear, that is,

$$\ell(\check{A})\check{J} = \check{J}\ell(\check{A}).$$

A geometric structure $(\breve{\omega}, \breve{J})$ of this kind, is thus a sort of analogue of complex Kähler geometry. Such structures have attracted recent interest of mathematicians and also physicists under the names of *para-Kähler* or *bi-Lagrangian* geometry. See, for example, § 5.2 in [20] and, in particular, the work of Kaneyuki et al. on para-Kähler symmetric spaces [59], [60], [61].

6.6 Pseudo-Riemannian affine homogeneous domains

As follows from Proposition 6.17, there are as many pseudo-Riemannian affine homogeneous domains as there are homogeneous affine domains. In particular, the classification of pseudo-Riemannian affine homogeneous domains might be a quite untractable problem. However, for small index s, the possible types of homogeneous domains in $\mathbb{E}^s$ seem rather restricted.

Recall that a homogeneous Riemannian manifold is geodesically complete, since it is complete with respect to the Riemannian distance. Thus:

Example 6.6. There is only one Euclidean homogeneous domain $U \subseteq \mathbb{E}^n$, namely, Euclidean space $\mathbb{E}^n$ itself.

As follows from Example 6.2, we note for the case $n = 2$:

Example 6.7. The two-dimensional pseudo-Riemannian homogeneous domains are $\mathbb{E}^2$, $\mathbb{E}^{1,1}$, and the half-space $H_2 = T(\mathbb{R}^{>0}) \subset \mathbb{E}^{1,1}$.

In the Lorentzian case, we have the following result:

Proposition 6.24 ([30]). *Let $U \subseteq \mathbb{E}^{n-1,1}$ be a homogeneous Lorentzian domain. Then $U = \mathbb{E}^{n-1,1}$ or $U = H_2 \times \mathbb{E}^{n-2} \subset \mathbb{E}^{n-1,1}$.*

In particular, every Lorentzian affine homogeneous domain is translationally isotropic.

We call a pseudo-Riemannian domain *irreducible* if it does not admit a decomposition as in Proposition 6.20.

Example 6.8. The (translationally isotropic) irreducible domains in $\mathbb{E}^{2,2}$ are the dual tube domains $(\breve{T}(U), \langle \breve{,} \rangle)$, where $U = \mathbb{A}^2 - \{0\}$, H_2, Q_2, and $\mathcal{P}_2^{\pm}$. (Compare Example 6.2.)

It seems unknown, whether (up to products) the latter list exhausts all homogeneous domains in $\mathbb{E}^{n-2,2}$.

6.6.1 Centralisers of prehomogeneous groups of isometries. Let $\rho\colon G \to \mathrm{E}(s)$ be a prehomogeneous group of isometries. We consider the subgroup of isometries in $\mathrm{E}(s)$, which centralise $\rho(G)$.

Corollary 6.25. *Let $U \subseteq \mathbb{E}^s$ be a pseudo-Riemannian affine homogeneous domain. Then the centraliser of any prehomogeneous subgroup $G \leq \mathrm{Isom}(U)$ in the group $\mathrm{E}(s)$ is a connected unipotent group. Moreover, it is a nilpotent group of nilpotency class at most two.*

Proof. By Corollary 3.11, the Lie algebra of $\mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s)$ consists of nilpotent elements of $\mathfrak{aff}(n)$, and it is a nilpotent Lie algebra of nilpotency class at most two. Therefore, the identity component $(\mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s))^0$ is a unipotent group, and it is nilpotent of nilpotency class at most two. We remark next that $\mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s)$ is connected. (To prove that $\mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s)$ is connected, we can argue as in the proof of Proposition 6.28.) □

The following observation is due to Wolf [106] (for transitive groups), see also [31] for the general case:

Proposition 6.26. *The elements in $g \in \mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s)$ are unipotent, and satisfy $(g - E_{n+1})^2 = 0$.*

Proof. By the above, $\mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s)$ is a unipotent linear algebraic group. Using the exponential representation of $g \in \mathrm{Z}_{\mathrm{Aff}(n)}(G) \cap \mathrm{E}(s)$, Proposition 3.9 implies that $g - E_{n+1}$ represents a Killing vector field in $\mathfrak{aff}(n)$, and $(g - E_{n+1})^2 = 0$. □

The previous two results play a central role in the determination of all flat pseudo-Riemannian homogeneous manifolds, which are complete. (See Section 4.3.1 for further discussion.) Another application was observed in [31]:

Corollary 6.27. *Every nilpotent prehomogeneous group N of isometries of $\mathbb{E}^s$ is transitive on $\mathbb{E}^s$.*

Proof. The set of semisimple elements in a nilpotent algebraic group forms a central subgroup, see [17]. Thus, by Proposition 6.26, N must be a unipotent group. By Corollary 7.5, N acts transitively on $\mathbb{A}^n$. □

The latter result generalises to nilpotent groups of volume preserving affine transformations, see Corollary 7.8. For the proof, other methods are required.

6.7 Symplectic affine homogeneous domains

As for pseudo-Riemannian domains, the tube construction produces a symplectic affinely homogeneous domain $(\breve{T}(U), \breve{\omega}) \subseteq \mathbb{A}^{2k}$ from each affinely homogeneous domain $U \subseteq \mathbb{A}^k$. However, we already noted in Example 6.5 that this construction does not exhaust the set of symplectic affinely homogeneous domains:

Example 6.9. The two-dimensional symplectic affine homogeneous domains of tube type are $\mathbb{A}^2$ and the halfspace $H_2 = \breve{T}(\mathbb{R}^{>0})$.

On the other hand, from Example 6.2, we note:

Example 6.10. The two-dimensional symplectic affinely homogeneous domains are $\mathbb{A}^2 - \{0\}$, and the tube type domains $\mathbb{A}^2$, H_2.

Here is another similarity to the pseudo-Riemannian situation (compare Corollary 6.25):

Proposition 6.28. *Let $U \subseteq \mathbb{A}^{2k}$ be a symplectic affine homogeneous domain. Then the centraliser of any prehomogeneous subgroup $G \leq \mathrm{Aff}(U, \omega_k)$ in the group of symplectic affine transformations of $\mathbb{A}^{2k}$ is a connected unipotent group (and it is also abelian).*

(The connected component of the symplectic centraliser of a prehomogeneous group of symplectic transformations is abelian, by Proposition 3.15.)

Hence, we get:

Corollary 6.29. *Every nilpotent prehomogeneous subgroup of $\mathrm{Aff}(\omega_k)$ is transitive on $\mathbb{A}^{2k}$.*

For the proof of Proposition 6.28, let us first formulate a lemma:

Lemma 6.30. *Let T be an abelian subgroup of semisimple elements in $\mathrm{Sp}(V, \omega)$, and $G \leq \mathrm{Sp}(V, \omega)$ a subgroup which centralises T. Let V^T denote the subspace of invariants for T. Then there exists a decomposition $V = V^T \oplus (W_1 \oplus W_2)$, which is invariant by G and T, such that W_1 and W_2 are isotropic subspaces, and the restriction of ω to the subspaces V^T and $W_1 \oplus W_2$ is non-degenerate. Moreover, V^T and $W_1 \oplus W_2$ are orthogonal with respect to ω.*

The lemma is a direct consequence of the decomposition of V in eigenspaces with respect to T.

Proof. Proof of Proposition 6.28. Let $G \leq \mathrm{Aff}(U, \omega_k)$ be a prehomogeneous subgroup with orbit U. For $g \in \mathrm{Aff}(n)$, $t(g) \in V = \mathbb{R}^n$ denotes the translation part of g. Let T denote a subgroup of semisimple elements in $\mathrm{Z}_{\mathrm{Aff}(\omega_k)}(G)$. After a change of origin, we may assume that T is linear, and, in particular, $T \leq \mathrm{Sp}(\omega_k)$. Since T centralises G, $t(g) \in V^T$, for all $g \in G$, and the subspace V^T is invariant by $\ell(G)$. Let $x = x_0 + x_1 \in U$, where $x_0 \in V^T$ and $x_1 \in W_1 \oplus W_2$, as in Lemma 6.30. Let $\pi_2 \colon V \to W_1 \oplus W_2$ be the corresponding projection operator. We conclude that $\pi_2(U) = \pi_2(Gx) = \pi_2(Gx_1) = \pi_2(\ell(G)x_1) = \ell(G)x_1$. Since $\ell(G)$ also preserves a Lagrange decomposition of $W_1 \oplus W_2$, it cannot have an open orbit in $W_1 \oplus W_2$. (In fact, the dual pairing of W_1 and W_2 with respect to ω produces a non-trivial $\ell(G)$-invariant polynomial.) Unless $T = \{1\}$, this is a contradiction, since $\pi_2(U)$ is open in $W_1 \oplus W_2$. □

7 A criterion for transitivity of prehomogeneous representations

A distinguished class of prehomogeneous affine representations is formed by those representations, which are transitive on affine space $\mathbb{A}^n$. Our main concern in this section will be to formulate a criterion, which ensures that a given prehomogeneous affine representation $\rho\colon G \to \mathrm{Aff}(n)$ is transitive on $\mathbb{A}^n$.

In Section 7.1, we recall how transitivity depends on the unipotent radical of the Zariski closure. A closer analysis of the characteristic map $\Phi(\rho)$, for a prehomogeneous representation ρ, is given in Section 7.2. This allows to interpret transitivity for ρ in terms of the characteristic image

$$\Phi(\rho)(\mathbb{A}^n) \subseteq \bigwedge^n \mathfrak{g}^*.$$

An immediate application (see Corollary 7.8) is the following result:

Theorem 7.1. *Let U be an affine homogeneous domain, which admits a nilpotent transitive subgroup of volume preserving affine transformations. Then $U = \mathbb{A}^n$.*

In particular, for nilpotent prehomogeneous groups, transitivity is equivalent to volume preservation. Various special cases of this result have been treated in the literature before. For the case of étale affine representations see [62], [37], and for pseudo-Riemannian prehomogeneous affine representations of nilpotent groups see [31, Corollary 6.27].

7.1 Transitivity for prehomogeneous groups

As a first remark, we note that transitivity may be recognised by looking at the real Zariski closure:

Corollary 7.2. *Let $G \leq \mathrm{Aff}(n)$ be a prehomogeneous subgroup. Then G is transitive on $\mathbb{A}^n$ if and only if its Zariski closure $\bar{G}^{\mathbb{R}} \leq \mathrm{Aff}(n)$ is transitive.*

The corollary is a direct consequence of Corollary 6.10.

Transitivity for prehomogeneous algebraic groups is determined by the unipotent radical, as follows:

Proposition 7.3. *Let $G \leq \mathrm{Aff}(n)$ be a prehomogeneous real algebraic group. Then the following are equivalent:*

(1) *G is transitive on $\mathbb{A}^n$.*

(2) *The unipotent radical $U(G)$ is transitive on $\mathbb{A}^n$.*

Proof. By Section A, $G = U(G)H$, where H is reductive. Moreover, H has a fixed point on $\mathbb{A}^n$, by Lemma 2.2. Thus, if G is transitive, $U(G)$ must be transitive as well. □

Let $G \leq \mathrm{Aff}(n)$ be a prehomogeneous subgroup. Let $\boldsymbol{G} = \bar{G}^Z \leq \mathrm{Aff}(\mathbb{C}^n)$ denote the complex Zariski closure of G. Then $\boldsymbol{G}$ acts prehomogeneously on complex affine space $\mathbb{C}^n$.

Corollary 7.4. *Let $G \leq \mathrm{Aff}(n)$ be prehomogeneous. Then G is transitive on $\mathbb{A}^n$ if and only if its complex Zariski closure $\boldsymbol{G}$ is transitive on $\mathbb{C}^n$.*

Proof. If G is transitive, so is the unipotent radical U of its real closure $\bar{G}^{\mathbb{R}}$. The Zariski closure of U, $\boldsymbol{U} \leq \boldsymbol{G}$ is prehomogeneous on $\mathbb{C}^n$ and unipotent. By the closed orbit property, $\boldsymbol{U}$ must be transitive. Therefore, $\boldsymbol{G}$ is transitive.

Conversely, assume $\boldsymbol{G}$ is transitive. Note that Proposition 7.3 holds analogously for complex algebraic actions on $\mathbb{C}^n$. We deduce that $\boldsymbol{U}$ acts transitively on $\mathbb{C}^n$. Note that $U = \boldsymbol{U}_{\mathbb{R}}$ is Zariski-dense in $\boldsymbol{U}$ (see [17]). In particular, $\dim_{\mathbb{R}} U = \dim \boldsymbol{U}$. Thus U, and, in particular, also $\bar{G}^{\mathbb{R}}$ acts transitively on $\mathbb{A}^n$. By Corollary 7.2, G is transitive, as well. □

Transitivity of nilpotent groups. We remark that unipotent prehomogeneous groups are always transitive:

Corollary 7.5. *Let $N \leq \mathrm{Aff}(n)$ be a prehomogeneous group which is nilpotent. Then N is transitive on $\mathbb{A}^n$ if and only if N is unipotent.*

Proof. In fact, if N is unipotent every orbit on $\mathbb{A}^n$ is closed (cf. Appendix A, Proposition A.2). The converse is a consequence of Proposition 6.15. □

7.2 The fundamental diagram

We let $N = N_{\mathrm{Aff}(n)}(\rho(G)) \leq \mathrm{Aff}(n)$ denote the normaliser of G in $\mathrm{Aff}(n)$. Then N acts by conjugation on the image $\rho(G)$. Assuming that ρ is faithful, this gives also rise to an action $C : N \to \mathrm{Aut}(G)$, where, for $A \in N$,

$$C(A) \colon G \to G$$

is defined by the relation

$$\rho\,(C(A)(g)) = A\rho(g)A^{-1}.$$

Let further $c : N \to \mathrm{Aut}(\mathfrak{g})$ denote the induced representation on the Lie algebra $\mathfrak{g}$.

We pick up the notation used in Section 6.3. Recall, in particular, that $\tau_x : \mathfrak{g} \to \mathbb{R}^n$ denotes the differential of the orbit map $o_x : G \to \mathbb{A}^n$ at $x \in \mathbb{A}^n$. For all $x \in \mathbb{A}^n$, $A \in N$, the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\tau_x} & \mathbb{R}^n \\ {\scriptstyle c(A)}\downarrow & & \downarrow{\scriptstyle \ell(A)} \\ \mathfrak{g} & \xrightarrow[\tau_{Ax}]{} & \mathbb{R}^n. \end{array} \tag{7.1}$$

Now consider the characteristic map

$$\Phi = \Phi(\rho)\colon \mathbb{A}^n \to \bigwedge^n \mathfrak{g}^*, \quad \Phi(x) = \tau_x^* \nu$$

for ρ. The commutative diagram (7.1) implies the following relation, which holds for all $A \in \mathrm{N}_{\mathrm{Aff}(n)}(\rho(G))$:

$$\Phi_{Ax} = \det \ell(A)\, \big((c(A)^{-1})^* \Phi_x\big). \tag{7.2}$$

By Section 6.3, the following is evident:

Proposition 7.6. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous affine representation. Then the following conditions are equivalent:*

i) *G acts transitively on $\mathbb{A}^n$.*

ii) *0 is not contained in the characteristic image $\Phi(\mathbb{A}^n)$.*

For an algebraic representation, we note the following refinement:

Proposition 7.7. *Let $\rho\colon G \to \mathrm{Aff}(G)$ be a prehomogeneous algebraic representation with maximal domain U_ρ. Then the following are equivalent:*

(i) *G acts transitively on $\mathbb{A}^n$.*

(ii) *0 is not contained in the closure of the characteristic image $\Phi(\mathbb{A}^n)$.*

(iii) *The image $\Phi(U_\rho) \subseteq \bigwedge^n \mathfrak{g}^*$ is Zariski-closed.*

(iv) *The image $\Phi(U_\rho) \subseteq \bigwedge^n \mathfrak{g}^*$ is closed in the Euclidean topology.*

Proof. We show first, if G acts transitively on $\mathbb{A}^n$ then $\Phi(\mathbb{A}^n)$ is Zariski closed in $\bigwedge^n \mathfrak{g}^*$. In fact, if G acts transitively then its unipotent radical $U(G)$ acts transitively as well. Thus, using relation (7.2), we deduce that

$$\Phi(\mathbb{A}^n) = \Phi(U(G)x) = c(U(G))\Phi_x$$

is the orbit of a unipotent linear algebraic group. Therefore, $\Phi(\mathbb{A}^n)$ is Zariski closed. Hence, (i) implies (ii) and also (iii). In particular, by Proposition 7.6, (i) is equivalent to (ii).

Now, assume condition (iii). Then $\Phi(U_\rho)$ is Zariski-closed. Note that the open set U_ρ is Zariski-dense in $\mathbb{A}^n$. Since Φ is a polynomial map (a morphism of varieties), it preserves the Zariski closure. Thus,

$$\Phi(\mathbb{A}^n) = \Phi(\overline{U_\rho}^{\mathbb{R}}) \subseteq \overline{\Phi(U_\rho)}^{\mathbb{R}} = \Phi(U_\rho).$$

Since Φ is non-zero on U_ρ, 0 is not contained in $\Phi(\mathbb{A}^n)$. This shows that $U_\rho = \mathbb{A}^n$, and, hence, G acts transitively. We proved that (i) is equivalent to (iii).

Since $\Phi(U_\rho)$ is a finite union of orbits, $\Phi(U_\rho)$ is Zariski closed if and only if it is closed in the Euclidean topology. (See the remarks in Appendix A.) Hence, (iii) is equivalent to (iv). □

Remark. It might well happen that the characteristic image $\Phi(\mathbb{A}^n)$ is Zariski-closed in $\bigwedge^n \mathfrak{g}^*$ and also contains 0.

For étale affine representations the above criterion implies:

Example 7.1. Let $n = \dim G$. Then G acts simply transitively on $\mathbb{A}^n$ if and only if Φ is constant, and non-zero. In particular, $\Phi(\mathbb{A}^n) = \{\Phi_0\}$, where $\Phi_0 \neq 0$.

7.3 Transitivity of nilpotent prehomogeneous groups

As a direct application of Proposition 7.7, we can deduce that, for nilpotent groups, transitivity is equivalent with volume preservation:

Corollary 7.8. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous affine representation, where G is a connected nilpotent Lie group. Then the following are equivalent:*

(i) *ρ is transitive.*

(ii) *$\rho(G)$ is unipotent.*

(iii) *ρ is volume preserving (that is, $\ell(\rho(G)) \subset \mathrm{SL}(n)$).*

Proof. Suppose G is acting transitively on $\mathbb{A}^n$. Without loss of generality, we may assume that G is an algebraic subgroup of $\mathrm{Aff}(n)$. A connected nilpotent linear algebraic group is a direct product $G = U(G)T$, where $U(G)$ is unipotent and T is a connected abelian group of diagonalisable matrices, which is contained in the center of G, cf. [17]. Since, $U(G)$ acts transitively, its centraliser is unipotent. Hence, $T = \{1\}$ and $G = U(G)$ is unipotent. Thus, i) implies ii).

Since every algebraic character of a unipotent group is trivial, ii) implies iii).

Suppose now that $\rho(G)$ is volume preserving. Let $x_0 \in U_\rho$. Then the characteristic orbit $\Phi(G\,x_0) = c(G)^*\Phi_{x_0}$ is (Zariski-) closed, since the adjoint action of G on $\mathfrak{g}$ is unipotent. Then $\Phi(\mathbb{A}^n) = \Phi(\overline{G\,x_0}^Z) \subseteq \overline{\Phi(G\,x_0)}^Z = c(G)^*\Phi_{x_0}$ does not contain 0. Hence, G acts transitively. Thus, iii) implies i). □

8 Characteristic cohomology classes associated with affine representations

Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine representation. The characteristic map Φ_ρ gives rise to certain characteristic cohomology classes, which carry information about Φ_ρ, and, in particular, about the transitivity properties of ρ.

8.1 Construction of the characteristic classes

Let $\bar{\rho}\colon \mathfrak{g} \to \mathfrak{aff}(n)$ denote the differential of ρ, and $\bar{\ell}\colon \mathfrak{g} \to \mathfrak{gl}(n)$ the corresponding linear part. The representation $\bar{\ell}$ turns $\mathbb{R}^n$ into a $\mathfrak{g}$-module, which we denote by $\mathbb{R}^n_{\bar{\ell}}$. Furthermore, we let $\mathbb{R}_{\bar{\ell}}$ denote the one-dimensional $\mathfrak{g}$-module, which is induced by the trace of $\bar{\ell}$. We shall also consider the Lie algebra cohomology groups $H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$, $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$, where $n = \dim \mathfrak{g}/\mathfrak{h}$. (See Appendix B for definitions and notation on Lie algebra cohomology.)

Lemma 8.1. *Let $x \in \mathbb{A}^n$ and $\mathfrak{h} = \mathfrak{g}_x$ the stabiliser of x under the action $\bar{\rho}$. Let $\Phi\colon \mathbb{A}^n \to \bigwedge^n \mathfrak{g}^*$ be the characteristic map for ρ. Then Φ_x is an element of $C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})^{\mathfrak{h}} \subseteq C^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$. In particular,*

$$d_{\bar{\ell}}\, \Phi_x = 0.$$

Moreover, if $x \in U_\rho$ then $\Phi_x \neq 0$, and $\mathfrak{h}$ acts trivially on $C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})$.

Proof. Clearly, Φ_x is in $C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})$. By the relation (7.2), the form Φ_x is stabilised by the twisted adjoint action of $H = G_x$. Taking derivatives shows that $\Phi_x \in C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})^{\mathfrak{h}}$ is an invariant for $\mathfrak{h}$. Since $C^*(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})^{\mathfrak{h}}$ is a subcomplex for $d_{\bar{\ell}}$ and $C^k(\mathfrak{g}/\mathfrak{h}, M) = \{0\}$, $k > n$, (for any module M), we deduce that $d_{\bar{\ell}}\Phi_x = 0$. Since $C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ is one-dimensional and spanned by Φ_x, $\mathfrak{h}$ acts trivially. □

The absolute class. As a consequence of relation (7.2), we deduce:

Proposition 8.2. *The forms Φ_x, $x \in \mathbb{A}^n$, are elements of the cocycle vector space $Z^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$, and represent a unique cohomology class $\bar{c} = [\Phi_x] \in H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$, which does not depend on x.*

Proof. By Lemma 8.1, Φ_x is a cocycle. We show now that the associated cohomology classes $[\Phi_x] \in H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$ do not depend on x. We argue as follows. Recall that the representation ℓ defines a twisted adjoint action of G on the cohomology groups $H^k(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$, whose derivative is induced by the operators L_X, $X \in \mathfrak{g}$, cf. Appendix B. Assuming G is connected, this inner action of G on the cohomology is trivial, cf. [85]. If ρ is not prehomogeneous then $\Phi_x = 0$, for all $x \in \mathbb{A}^n$. Hence, $[\Phi_x] = 0$, and the claim is proved. Otherwise, let $x_0 \in U_\rho$. The transformation formula (7.2) shows that the forms Φ_x, $x \in G\, x_0$, form a twisted adjoint orbit in $Z^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$. By the above remark, they represent the same cohomology class $c \in H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$. Therefore, the polynomial map $\mathbb{A}^n \to H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$, $x \mapsto [\Phi_x]$ is constant on the Zariski dense subset $G\, x_0$. Hence, it is constant. □

Definition 8.3. We call $\bar{c} = \bar{c}(\rho) \in H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$ the (absolute) characteristic class of the affine representation ρ.

The geometric importance of the absolute class $\bar{c}(\rho)$ stems from:

Proposition 8.4. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine representation. If the absolute characteristic class $\bar{c}(\rho)$ is different from 0 then ρ is transitive.*

Proof. Assume that ρ is not transitive. Then, by Proposition 7.6, $\Phi(\mathbb{A}^n)$ contains 0. By Proposition 8.2, $\bar{c}(\rho) = 0$. □

However, the class $\bar{c}$ may vanish for a transitive affine representation. The following example of a nilpotent transitive group is taken from [42]:

Example 8.1. Consider

$$\mathfrak{g} = \left\{ X(s,t,u,v) = \begin{pmatrix} 0 & t & v & s \\ 0 & 0 & u & t \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; s,t,u,v \in \mathbb{R} \right\} \subset \mathfrak{aff}(3)$$

Then $\mathfrak{g}$ is the tangent algebra to a transitive action of a four-dimensional unipotent group $G \leq \mathrm{Aff}(n)$. The tangent algebra of the stabiliser at $0 \in \mathbb{A}^3$ is $\mathfrak{h} = \{X(0,0,0,v) \mid v \in \mathbb{R}\}$. We put $\sigma, \tau, \omega, \nu$, for the left invariant one-forms dual to $S = X(1,0,0,0), T = X(0,1,0,0), U = X(0,0,1,0), V = X(0,0,0,1) \in \mathfrak{g}$. Then $d\tau = d\omega = 0$, and $d\nu = -\tau \wedge \omega$, $d\sigma = \omega \wedge \nu$. Thus $\Phi_0 = \sigma \wedge \tau \wedge \omega = -d(\sigma \wedge \nu)$. But note that $H^3(\mathfrak{g}, \mathfrak{h}, \mathbb{R}) = \langle [\Phi_0] \rangle \neq \{0\}$.

The example motivates the following construction of the relative classes.

The relative classes. Let $x \in \mathbb{A}^n$. We put $\mathfrak{h} = \mathfrak{g}_x$ for the stabiliser of x under the action $\bar{\rho}$. By Lemma 8.1, the form Φ_x represents a cohomology class $\bar{c}(\rho, x) \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$.

Definition 8.5. We call $\bar{c}(\rho, x) = [\Phi_x] \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ the *relative* characteristic class of the affine representation ρ, at $x \in \mathbb{A}^n$.

Note that the forms $\Phi(\rho, x)$ vanish precisely outside the union of all open orbits of G. Hence:

Lemma 8.6. *If $\bar{c}(\rho, x) \neq 0$, for some $x \in \mathbb{A}^n$, then ρ is prehomogeneous.*

In fact, we also have:

Lemma 8.7. *Let $x \in U_\rho$. Then $\bar{c}(\rho, x) \neq 0$ if and only if $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq 0$.*

As example 8.1 shows, there exist transitive representations ρ, where G is nilpotent, and the absolute class $\bar{c}(\rho)$ is zero, while all relative versions $\bar{c}(\rho, x)$ are non-zero. In fact, in Section 8.2 below we show that $\bar{c}(\rho, x)$ is non-zero, whenever ρ is transitive. We will also show that, *for nilpotent groups G*, the non-vanishing of the class $\bar{c}(\rho, x)$, for some $x \in \mathbb{A}^n$, implies transitivity of ρ.

Naturality properties

(1) By construction, the relative classes $\bar{c}(\rho, x)$ map to the absolute class $\bar{c}(\rho)$ under the natural homomorphism

$$H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \to H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}}).$$

(2) Let $L \leq G$ be a closed subgroup, and let $\rho_L : L \to \mathrm{Aff}(n)$ denote the restriction of ρ to L. Then in the following commutative diagram (vertical arrows denoting restriction homomorphisms)

$$\begin{array}{ccc} H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) & \longrightarrow & H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}}) \\ \downarrow & & \downarrow \\ H^n(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) & \longrightarrow & H^n(\mathfrak{l}, \mathbb{R}_{\bar{\ell}}) \end{array}$$

the classes $\bar{c}(\rho, x)$, $c(\rho)$, map to $\bar{c}(\rho_L, x)$, $c(\rho_L)$ respectively.

(3) Let $A \in \mathrm{N}_{\mathrm{Aff}(n)}(\rho(G))$, and $c(A) : \mathfrak{g} \to \mathfrak{g}$ the induced automorphism of $\mathfrak{g}$. If $\mathfrak{h} = \mathfrak{g}_x$ then $c(A)\mathfrak{h} = \mathfrak{g}_{Ax}$. Consider the map of cohomology groups

$$c(A)^* : H^n(\mathfrak{g}, \mathfrak{g}_{Ax}, \mathbb{R}_{\bar{\ell}}) \to H^n(\mathfrak{g}, \mathfrak{g}_x, \mathbb{R}_{\bar{\ell}}).$$

Then, by (7.2),

$$c(A)^* \bar{c}(\rho, Ax) = \det \ell(A) \bar{c}(\rho, x).$$

8.1.1 Alternative construction of the characteristic classes. Let Γ be a group. To any affine representation $\rho : \Gamma \to \mathrm{Aff}(n)$ the translation part $t : \Gamma \to \mathbb{R}^n$ defines a cohomology class $[t] \in H^1(\Gamma, \mathbb{R}^n_\ell)$, *which vanishes if and only if* Γ *has a fixed point on* $\mathbb{A}^n$. (See [53, 41] for some applications.)

Analogously, for a Lie algebra representation $\bar{\rho} : \mathfrak{g} \to \mathfrak{aff}(n)$, the translation part

$$t : \mathfrak{g} \to \mathbb{R}^n$$

defines a class $u \in H^1(\mathfrak{g}, \mathbb{R}^n_{\bar{\ell}})$, which is called the *radiance obstruction*. Goldman and Hirsch [41], [42] considered the exterior powers

$$u^i = \Lambda^i u \in H^i(\mathfrak{g}, \textstyle\bigwedge^i \mathbb{R}^n_{\bar{\ell}}).$$

By [42], if $u^i \neq 0$ then every orbit of G has at least dimension i. Note that the fixed n-form ν defines an isomorphism of $\mathfrak{g}$-modules

$$\textstyle\bigwedge^n \mathbb{R}^n_{\bar{\ell}} \cong \mathbb{R}_{\bar{\ell}}.$$

Under this isomorphism, the n-th exterior power of u identifies with $\bar{c}(\rho)$:

$$\bar{c}(\rho) = \Lambda^n u.$$

The representation of $\bar{c}(\rho)$ as an exterior product of a class in degree one, leads to an important application for compact affine manifolds, see Section 9.2.

8.2 Significance of the relative classes

Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous representation. We show below that the cohomology groups $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ are computed by a characteristic character $\chi_{G/H}(\rho)$, which is associated to ρ and G. We then explain how the group $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ is linked to the geometry of the semi-invariant measure, which is induced by ρ on the homogeneous space $X = G/H$. As a first application, we deduce that, *for a transitive affine representation* ρ, all relative cohomology classes $\bar{c}(\rho, x)$ are non-zero (contrary to what may happen for the absolute class $\bar{c}(\rho)$).

8.2.1 The character $\chi_{G/H}(\rho)$. Let $x \in U_\rho$. We put $N_G(H)$ for the normaliser of $H = G_x$ in G. The adjoint representation $\mathrm{Ad}\colon N(G) = N_{\mathrm{Aff}(n)}(G) \to \mathrm{GL}(\mathfrak{g})$ induces a quotient representation

$$\mathrm{Ad}_{G/H}\colon N_G(H) \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h}).$$

We then define the *unimodular character for* G/H *and* ρ as

$$\chi_{G/H}(\rho) = \det \ell(\rho) \det \mathrm{Ad}_{G/H}^{-1}\colon N_G(H) \to \mathbb{R}^{>0}.$$

The group $N(G)$ acts on $C^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$ by the twisted adjoint action as in equation (7.2). Since $N_G(H)$ normalises H and $\mathfrak{h}$, this action preserves the subspace

$$C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \subseteq C^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}}).$$

Then $N_G(H)$ acts on the one-dimensional space

$$C^n(\mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \cong \mathrm{Hom}(\Lambda^n \mathfrak{g}/\mathfrak{h}, \mathbb{R}_{\bar{\ell}})$$

with the character $\chi_{G/H}(\rho)$. For all $A \in N_G(H)$, equation (7.2) turns to

$$\Phi_{Ax} = \chi_{G/H}(\rho)(A)\Phi_x. \tag{8.1}$$

In particular, this implies that $\chi_{G/H}(\rho)$ factorises over H, that is,

$$\chi_{G/H}(\rho)_{|H} \equiv 1.$$

Associated Lie algebra cohomology group. By Proposition C.3, the character $\chi_{G/H}(\rho)$ computes the cohomology group $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$:

Proposition 8.8. *The following conditions are equivalent:*

(1) $\chi_{G/H}(\rho)\colon (N_G(H)/H)^0 \to \mathbb{R}^{>0} \equiv 1$.

(2) $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq \{0\}$.

Example 8.2. If $N_G(H)^0 = H^0$ then $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq \{0\}$. For example, the automorphism group $G = \mathrm{Aut}(H_2)$ of a halfspace has $H^2(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) = \mathbb{R}$.

Induced semi-invariant measure on $X = G/H$. We put $X = G/H$ for the associated homogeneous space, where $H = G_x$. The pull back of the parallel volume form ν on $\mathbb{A}^n$ defines a semi-invariant measure on X. Its properties strongly interact with the geometry of the representation ρ.

The following geometric interpretation of the characteristic character $\chi_{G/H}(\rho)$ is a consequence of Lemma C.2:

Proposition 8.9. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous representation, $x \in U_\rho$ and $X = G/H$ the associated homogeneous space. Then the following conditions are equivalent:*

(1) $|\chi_{G/H}(\rho)|\colon N_G(H) \to \mathbb{R}^{>0} \equiv 1$.

(2) *The semi-invariant measure on X, which is induced from a parallel volume form ν on $\mathbb{A}^n$, is invariant by the right-action of $N_G(H)$ on X.*

Homogeneous domains with non-vanishing relative class. We say that the affinely homogeneous domain U has a *non-vanishing relative class*, if the relative cohomology groups for the group $G = \mathrm{Aff}(U)$ are non-vanishing.

Corollary 8.10. *Let U be a homogeneous affine domain, $G = \mathrm{Aff}(U)$, and $H = G_x$, for $x \in U_\rho$. If $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\tilde{\ell}}) \neq \{0\}$ then the center $\mathrm{Z}(G)$ of $\mathrm{Aff}(U)$ is a connected unipotent group.*

Proof. Clearly, $Z(\mathrm{Aff}(n), G)^0 \leq N_G(H)$. Thus, by our assumption, the character $\chi_{G/H}(\rho)$ is trivial on $Z(\mathrm{Aff}(n), G)^0$. Since $Z(\mathrm{Aff}(n), G)^0$ is in the kernel of the representation $\mathrm{Ad}_{G/H}$, this implies that the elements of $Z(\mathrm{Aff}(n), G)^0$ are volume preserving on $\mathbb{A}^n$. By Corollary 6.14, $Z(\mathrm{Aff}(n), G)$ is a connected unipotent group. Thus, $\mathrm{Z}(G) = Z(\mathrm{Aff}(n), G)$. □

Example 8.3. Among the two-dimensional homogeneous domains, precisely $U = \mathbb{R}^2 - \{0\}$, and the sector $U = Q_2$ have vanishing relative class.

8.2.2 Transitive representations. As remarked before, $\chi_{G/H}(\rho)$ is trivial on H, for any prehomogeneous affine representation ρ. If ρ is transitive then $\chi_{G/H}(\rho)$ is trivial on all of $N_G(H)$:

Corollary 8.11. *If ρ is a transitive affine representation then $|\chi_{G/H}(\rho)| \equiv 1$ on $N_G(H)$.*

Proof. By Proposition 7.7, the orbit Φ_{Ax}, $A \in N_G(H)$ must be bounded away from 0. By formula (8.1), this can only be if $\chi_{G/H}(A) \in \{1, -1\}$. □

We have the following consequence:

Theorem 8.12. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a transitive affine representation. For $x \in \mathbb{A}^n$, put $\mathfrak{h} = \mathfrak{g}_x$. Then, for all $x \in \mathbb{A}^n$, the cohomology class $\bar{c}(\rho, x) \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ is non-zero.*

Proof. Since G is transitive, the character $\chi_{G/H}$ is trivial, by Corollary 8.11. This implies, $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \cong \mathbb{R}$, by Proposition 8.8. More specifically, the natural map induces an isomorphism of the group of relative n-cocycles $C^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ with $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \cong \mathbb{R}$. Since $\Phi_x \neq 0$, $[\Phi_x] = \bar{c}(\rho, x) \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ is a non-zero generator of $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$. □

Remark that, conversely, if, for all $x \in \mathbb{A}^n$, $\bar{c}(\rho, x) \neq 0$ then G acts transitively.

Corollary 8.13. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a transitive affine representation. Then $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq \{0\}$.*

We thus note (compare Proposition 8.9):

Corollary 8.14. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a transitive affine representation. Then the semi-invariant measure on $X = G/H$ with character $\det \ell(\rho)$ is right-invariant by $N_G(H)$.*

For a transitive *algebraic* representation, we add the following observation:

Proposition 8.15. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a transitive algebraic affine representation. Then the following are equivalent:*

(1) *ρ is volume preserving.*

(2) *$\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} \equiv 1$ on H.*

(3) *$\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} \equiv 1$ on $N_G(H)^0$.*

Proof. Put $\lambda = \det \ell(\rho)$. If ρ is volume preserving then $H \leq \ker \lambda$. Moreover, $\chi_{G/H}(\rho) \equiv 1$ on H, as for any prehomogeneous representation. Thus, $\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} \equiv 1$ on H. Conversely, assume $\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} \equiv 1$ on H. Then $H \leq \ker \lambda$. Since the unipotent radical of $\rho(G)$ acts transitively, $\ker \lambda$ acts transitively. Thus $\rho(G) = \rho((\ker \lambda)\, H) = \rho(\ker \lambda)$. Therefore, $\lambda \equiv 1$. It follows the equivalence of (1) and (2).

By Corollary 8.11, the analogous argument may be used to show the equivalence of (1) and (3). □

Corollary 8.16. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a transitive affine representation. Then ρ is volume preserving if and only if $X = G/H$ admits a G-invariant measure.*

8.3 Affine representations of nilpotent groups

In the case of nilpotent groups, we can summarise as follows:

Theorem 8.17. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine prehomogeneous representation, where G is a connected nilpotent Lie group. Let $x \in U_\rho$, and put $H = G_x$. Then the following are equivalent:*

(1) *ρ is transitive on $\mathbb{A}^n$.*

(2) *$\chi_{G/H}(\rho) \equiv 1$ on $N_G(H)^0$.*

(3) *$H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq 0$.*

(4) *The semi-invariant measure induced by a parallel volume on $X = G/H$ is invariant by the right-action of $N_G(H)^0$.*

(5) *The induced semi-invariant measure on X is a G-invariant measure.*

(6) *$\rho(G)$ is volume preserving.*

(7) *$\rho(G)$ is unipotent.*

Proof. If ρ is transitive then $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq 0$, by Theorem 8.12. By Proposition 8.8 and Proposition 8.9, this implies that $\chi_{G/H}(\rho) \equiv 1$ on $N_G(H)^0$, and also that the induced semi-invariant measure is invariant by $N_G(H)^0$. Since G is nilpotent this implies (cf. Example C.1) that the character $\det \ell(\rho)$ is trivial on $N_G(H)^0$. Moreover $\rho(G) \leq \overline{\rho(G)}^{\mathbb{R}} = TU$, where T is a central subgroup of diagonalisable elements, and U is unipotent. By functoriality we may assume now that $\rho(G) = \overline{\rho(G)}^{\mathbb{R}}$. It follows that $\det \ell(\rho) = 1$ on the central subgroup T. Since $\det = 1$ on U, $\rho(G)$ is volume preserving, and, in particular, the induced measure is an invariant measure. By Corollary 7.8, $\rho(G)$ is volume preserving implies that ρ is transitive. □

Moreover, transitivity of a nilpotent affine representation is determined by the relative class $\bar{c}(\rho_L, x)$ at an arbitrary point $x \in \mathbb{A}^n$:

Corollary 8.18. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine representation, where G is a connected nilpotent Lie group, and let $x \in \mathbb{A}^n$. Then ρ is a transitive representation if and only if*

$$0 \neq \bar{c}(\rho, x) \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}).$$

Proof. Assume that $\bar{c}(\rho, x) \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$ is different from zero. In particular, this implies that $0 \neq \Phi_x \in C^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}})$. Thus, the representation ρ is prehomogeneous at x. By Theorem 8.17, ρ is transitive. □

Note, that in the situation of Theorem 8.17, $\mathbb{R}_{\bar{\ell}} = \mathbb{R}$ is the trivial $\mathfrak{g}$-module.

Theorem 8.17, together with Corollary 8.18 characterise transitivity for nilpotent affine actions. In particular, the results summarise various special cases, which have been obtained in the literature before: see [62] for étale affine representations of nilpotent groups, [42] for relation with the absolute class $\bar{c}(\rho)$, and [31] for nilpotent representations with invariant scalar product (compare also Sections 6.6, 6.7).

8.3.1 Minimal classes for algebraic group actions. Let ρ be an algebraic representation.

Definition 8.19. A relative class $\bar{c}(x,\rho)$ will be called *minimal* if $N_G(H)$ contains a maximal $\mathbb{R}$-split torus in G, where $H = G_x$.

For example, if G is nilpotent then $\bar{c}(x,\rho)$ is minimal, for all $x \in \mathbb{A}^n$. Every reductive subgroup of G fixes a point $x \in \mathbb{A}^n$. Therefore, every ρ has minimal relative classes $\bar{c}(x,\rho)$.

The following result generalises Corollary 8.18:

Theorem 8.20. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an affine representation, where G is algebraic. Let $x \in \mathbb{A}^n$ such that $\bar{c}(\rho,x)$ is minimal, and put $\mathfrak{h} = \mathfrak{g}_x$. Then ρ is transitive if and only if $\bar{c}(\rho,x) \in H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_{\bar{\ell}})$ is different from zero.*

Proof. Assume $\bar{c}(\rho,x) \neq 0$. Then, in particular, ρ is prehomogeneous at x. Since $H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_{\bar{\ell}})$ is different from zero, $\chi_{G/H}(\rho) \equiv 1$, on $N_G(H)^0$. In particular, by (8.1), $N_G(H)^0$ is contained in the stability group of Φ_x. By Proposition A.3, the orbit $c(G)^*\Phi_x$ is closed in $C^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_{\bar{\ell}})$. Hence, Proposition 7.7 implies that ρ is transitive. □

8.4 Prehomogeneous representations with reductive stabiliser

A homogeneous space G/H is called *reductive* if the adjoint action of H on $\mathfrak{g}$ is reductive. In particular, the Lie algebra $\mathfrak{h}$ acts reductively on $\mathfrak{g}$, and there is a $\mathfrak{h}$-invariant direct decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a vector subspace isomorphic to $\mathfrak{g}/\mathfrak{h}$, and $\mathfrak{h}$ acts reductively on $\mathfrak{g}/\mathfrak{h}$. Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous affine representation. We call ρ *reductive* if, for some $x \in U_\rho$, the isotropy algebra $\mathfrak{h}$ for the homogeneous space G/H, $H = G_x$ is reductive in $\mathfrak{g}$.[1] In particular H^0 is a reductive subgroup in $\mathrm{Aff}(n)$. Also ρ is reductive if $\mathfrak{h} = \{0\}$. This covers the important special case of étale affine representations, as well.

The character χ. Let $\rho\colon G \to \mathrm{Aff}(n)$ be a prehomogeneous representation, where G is connected. We define the *characteristic character*

$$\chi = \chi(\rho)\colon G \to \mathbb{R}^{>0}$$

by putting

$$\chi(\rho)(g) = \det \ell(g) \det \mathrm{Ad}(g)^{-1}.$$

Let $\mathrm{Ad}_{\mathfrak{h}}$ denote the adjoint representation of $N_G(H)$ restricted to $\mathfrak{h}$. On $N_G(H)$ we have $\chi(\rho) = (\det \mathrm{Ad}_{\mathfrak{h}})^{-1}\, \chi_{G/H}(\rho)$.

[1] A particular important class of examples appears in the work of Sato and Kimura [90] on regular reductive prehomogeneous vector spaces, where both G and H are assumed reductive. See also [63], and Example 8.4.

Lemma 8.21. *Let ρ be a reductive representation, where G is connected. Then $\chi(\rho) = \chi_{G/H}(\rho)$ on $N_G(H)^0$. Moreover, $\chi(\rho) \equiv 1$ if and only if $\chi_{G/H}(\rho) \equiv 1$ on $N_G(H)^0$.*

Proof. Since $\mathfrak{h}$ is a reductive Lie Algebra $\det \mathrm{Ad}_{\mathfrak{h}}(h) = 1$, for all $h \in H^0$. Let $\mathfrak{p}^{\mathfrak{h}}$ be the elements of $\mathfrak{p}$, centralised by $\mathfrak{h}$. Since $\mathfrak{h}$ acts reductively,

$$\mathfrak{n}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{h} + \mathfrak{p}^{\mathfrak{h}} \tag{8.2}$$

and

$$\mathfrak{g} = \mathfrak{n}(\mathfrak{g}, \mathfrak{h}) + [\mathfrak{p}, \mathfrak{h}]. \tag{8.3}$$

It follows from (8.2) that $\det \mathrm{Ad}_{\mathfrak{h}}(n) = 1$, for all $n \in N_G(H)^0$. Therefore, $\chi(\rho) = \chi_{G/H}(\rho)$ on $N_G(H)^0$. In particular, $\chi(\rho) \equiv 1$, implies $\chi_{G/H}(\rho) \equiv 1$ on $N_G(H)^0$.

For the converse, assume that $\chi_{G/H}(\rho) \equiv 1$ on $N_G(H)^0$. Then the corresponding infinitesimal character $\bar{\chi}_{G/H} \colon \mathfrak{n}(\mathfrak{g}, \mathfrak{h}) \to \mathbb{R}$ vanishes. Since $\bar{\chi}_{G/H}$ is the restriction of the infinitesimal character $\bar{\chi} \colon \mathfrak{g} \to \mathbb{R}$, which belongs to $\chi(\rho)$, the decomposition (8.3) implies that $\bar{\chi} \equiv 0$. Hence, the character $\chi(\rho)$ is constant. □

Note, in particular, that $\chi(\rho) \equiv 1$ on H.

Corollary 8.22. *Let $\rho \colon G \to \mathrm{Aff}(n)$ be a reductive prehomogeneous representation, which is transitive. Then $|\chi(\rho)| \equiv 1$.*

Recall (cf. [68], [55]) that $\mathfrak{h}$ is called *not homologous to zero in* $\mathfrak{g}$ if the restriction homomorphism $H^q(\mathfrak{g}, \mathbb{R}) \to H^q(\mathfrak{h}, \mathbb{R})$ is surjective, for all q. For example, if $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{h}$, where $\mathfrak{j}$ is an ideal in $\mathfrak{g}$ then $\mathfrak{h}$ is not homologous to zero. If $\mathfrak{h}$ is reductive in $\mathfrak{g}$, then surjectivity of $H^q(\mathfrak{g}, \mathbb{R}) \to H^q(\mathfrak{h}, \mathbb{R})$, $q = \dim \mathfrak{h}$, is sufficient for $\mathfrak{h}$ being not homologous to zero. If $\mathfrak{h}$ is not homologous to zero then (cf. [55, Theorem 12]) it follows, in particular, that the natural map on cohomology $H^\bullet(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_\lambda) \to H^\bullet(\mathfrak{g}, \mathbb{R}_\lambda)$ is injective, for any one-dimensional module $\mathbb{R}_\lambda$.

We summarise:

Corollary 8.23. *Let $\rho \colon G \to \mathrm{Aff}(n)$ be a reductive prehomogeneous representation. If $\mathfrak{h} = \mathfrak{g}_x$ is not homologous to zero in $\mathfrak{g}$, then the following are equivalent:*

(1) *ρ is transitive.*

(2) $|\chi(\rho)| \equiv 1$.

(3) $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq \{0\}$.

(4) $H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}}) \ni \bar{c}(\rho) \neq 0$.

Remark. If $\mathfrak{h}$ is reductive in $\mathfrak{g}$, we could (in lieu of Lemma 8.21) have applied Poincaré duality for relative cohomology (see [64]). Thereby,

$$H^k(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_\lambda) \cong H^{n-k}(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{ad} \otimes \mathbb{R}_\lambda^*)^*,$$

where $n = \dim \mathfrak{g}/\mathfrak{h}$. The module $\mathbb{R}_{ad} \otimes \mathbb{R}^*_{\bar{\ell}}$ is determined by the character $-\bar{\chi}$. Moreover,

$$H^0(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{ad} \otimes \mathbb{R}^*_{\bar{\ell}}) = H^0(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\chi}}) = \mathbb{R}^{\mathfrak{g}}_{\bar{\chi}}$$

is non-zero if and only if $\bar{\chi} = 0$.

8.4.1 Prehomogeneous domains of reductive algebraic representations

Corollary 8.24. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be a reductive prehomogeneous representation, which is algebraic. Then ρ is transitive if and only if the absolute class $\bar{c}(\rho) \in H^n(\mathfrak{g}, \mathbb{R}_\ell)$ is non-vanishing.*

Proof. If ρ is a transitive algebraic representation then $\rho(G) = UT$, where T is maximal reductive. In particular, $T = G_x$ fixes a point $x \in \mathbb{A}^n$. By Theorem 8.12, $0 \neq \bar{c}(\rho, x) \in H^n(\mathfrak{g}, \mathfrak{t}, \mathbb{R}_{\bar{\ell}})$. Since $\mathfrak{t}$ is not homologous to zero in $\mathfrak{g}$, its image $\bar{c}(\rho) \in H^n(\mathfrak{g}, \mathbb{R}_\ell)$ is non-zero. □

Example 8.4 (Regular prehomogeneous vector spaces). Let ρ be a linear prehomogeneous algebraic representation. If H is reductive then the set U_ρ is the complement of a hypersurface, and it is also the set of real points of an affine algebraic variety. (See, for example, [63, p. 41ff], for reference.) In particular, any open orbit $\rho(G)x$ is (a connected component) of the set of real points of an affine algebraic variety. If furthermore $\rho(G)$ is reductive then the representation space $(\rho(G), \mathbb{R}^n)$ is called a *regular* prehomogeneous vector space. By [63, Proposition 2.26] there exists a *non-degenerate* relative invariant for $\rho(G)$ which has character $(\det \rho)^2$. In particular, $\chi_\rho = \det \rho \neq 1$. *It follows, in particular, that $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_\ell) = \{0\}$, for a regular prehomogeneous vector space.*

Using arguments as in [63, loc. cit.], we note:

Proposition 8.25. *Let ρ be a prehomogeneous algebraic representation. If H is reductive then the affine homogeneous domain U_ρ is the complement of a hypersurface, and it is also the set of real points of an affine algebraic variety.*

In particular, in the situation of the proposition, there exists a relative polynomial invariant

$$\delta\colon \mathbb{A}^n \to \mathbb{R},$$

satisfying $\delta\ \rho(g) = \tau(g)\,\delta$, for some character $\tau\colon G \to \mathbb{R}^{\neq 0}$, and

$$U_\rho = U_\delta = \{x \in \mathbb{A}^n \mid \delta(x) \neq 0\}.$$

One can show that there is a one-to-one correspondence between algebraic characters of G which factor over H, and the irreducible components of the complement of U_ρ. Hence, the group of characters, which factor over H is of rank one if $U_\rho \neq \mathbb{A}^n$. We conclude that $\chi(\rho)$ must be a power of τ.

Corollary 8.26. *Let $\rho\colon G \to \mathrm{Aff}(n)$ be an algebraic reductive prehomogeneous representation. Then the following are equivalent:*

(1) *ρ is transitive.*

(2) $|\chi(\rho)| \equiv 1$.

(3) $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\ell}}) \neq \{0\}$.

(4) $H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}}) \ni \bar{c}(\rho) \neq 0$.

Proof. By the previous corollary, we have the equivalence of (1), (3) and (4) Assume that (1) holds. By the above remarks, this implies that the character $|\chi(\rho)|$ is constant. Conversely, if $|\chi(\rho)|$ is constant, then $\chi_{G/H}(\rho)$ is constant on $N_G(H)^0$. By Proposition 8.8, this implies (3). □

By Corollary 7.8, the equivalences (1)–(3) of Corollary 8.23 and Corollary 8.26 also hold if G is nilpotent. However, in general, $\chi(\rho) \equiv 1$ does not imply transitivity:

Example 8.5. Consider the 2-dimensional linear prehomogeneous representation of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$. It has open orbit $\mathbb{R}^2 - \{0\}$. The stabiliser $H = G_x$, $x \neq 0$, is a unipotent subgroup. Moreover, $\chi_{\mathrm{SL}(2,\mathbb{R})} \equiv 1$, but $\chi_{\mathrm{SL}(2,\mathbb{R})/H} \neq 1$. In particular, $H^2(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{h}, \mathbb{R}) = \{0\}$.

Also, the character χ may not be trivial for a transitive representation, as the following example shows:

Example 8.6. Let $G = \mathbb{R}^2 \rtimes H \leq \mathrm{Aff}(2)$, where $H \leq \mathrm{SL}(2,\mathbb{R})$ denotes the 2-dim. solvable group of upper triangular matrices, and $\mathbb{R}^2$ is the group of translations. Then G is transitive and $\chi_G \neq 1$.

8.4.2 The case of étale affine representations. If the affine representation ρ is étale then $\mathfrak{h} = \{0\}$, and there are no relative versions of the characteristic class $c(\rho)$. As a special case of Corollary 8.23, we obtain:

Proposition 8.27. *Let G be a Lie group, and $\rho\colon G \to \mathrm{Aff}(n)$ be an affine étale representation. Then the following are equivalent*

(1) *ρ is transitive.*

(2) $\chi(\rho) \equiv 1$.

(3) $H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}}) \neq \{0\}$.

This result (in particular, the equivalence of (1) and (2)) appears in the work of Helmstetter [50], Kim [62], whereas the relation with the cohomology group $H^n(\mathfrak{g}, \mathbb{R}_{\bar{\ell}})$ is introduced in [42]. It also plays a role in the theory of prehomogeneous vector spaces, see [63]. As shown in Section 5.4, if ρ is an étale affine representation, the character χ relates to a relative invariant δ of ρ. This may also be interpreted as follows:

Example 8.7. Any right-invariant volume form on G induces a volume form η on U_ρ. By comparison with the parallel volume ν on $\mathbb{A}^n$, we can write

$$\eta = f\nu,$$

for some function f on U_ρ. It follows that $\delta = f^{-1}$ is a relative invariant for G with character χ, satisfying $\delta\,\eta = \nu$.

Choose linear independent affine vector fields $\bar{X}_1, \dots \bar{X}_n$, whose flows generate $\rho(G)$, and such that $\eta(\bar{X}_1, \dots, \bar{X}_n) = 1$. Then

$$\delta(x) = \nu(\bar{X}_1(x), \dots, \bar{X}_n(x)).$$

It follows that the function δ is a polynomial on $\mathbb{R}^n$, and the zeros of δ describe the boundary of U_ρ.

9 Compact affine manifolds and prehomogeneous algebraic groups

Here, we develop a theme, which is contained in a series of papers [53], [41], [42] by Goldman and Hirsch. Their ideas show a strong link of the parallel volume conjecture with the theory of prehomogeneous affine representations, as presented in the previous two sections.

9.1 Holonomy of compact complete affine manifolds

A basic remark concerning compact complete affine manifolds is:

Theorem 9.1. *Let M be a compact complete affine manifold, and $\Gamma = h(\pi_1(M))$ its affine holonomy group. Then the Zariski closure $A(\Gamma)$ of Γ acts transitively on $\mathbb{A}^n$.*

Proof. Let U be the unipotent radical of $A(\Gamma)$. Every orbit Ux is a contractible closed submanifold of $\mathbb{A}^n$ which is preserved by Γ. Since Γ acts properly discontinuously and freely on Ux, the quotient $\Gamma\backslash Ux$ is a compact manifold. Considering the cohomological dimension $cd(\Gamma)$ of the group Γ over $\mathbb{Z}$ (see [19]), we have $n = \dim \mathbb{A}^n = \dim M = cd(\Gamma) = \dim \Gamma\backslash Ux$. Thus $Ux = \mathbb{A}^n$. □

Polynomial volume forms. A volume form on M is called a *polynomial volume form* if it is expressed by a polynomial function in the affine coordinates of M. The following observation is thus an immediate consequence of Theorem 9.1 and Corollary 8.16:

Proposition 9.2. *Let M be a compact complete affine manifold. Put $G = A(\Gamma)$, and $H = G_x$, for some $x \in \mathbb{A}^n$. Then the following conditions are equivalent:*

(1) *M has a parallel volume form.*

(2) *M has a polynomial volume form.*

(3) *The homogeneous space G/H admits a G-invariant volume.*

In particular, *every polynomial volume form on a compact complete affine manifold is parallel*.

It seems natural to conjecture that on a compact affine manifold M a polynomial volume form must be parallel. In [37], this is proved under the assumption that the holonomy group Γ of M is nilpotent. Not much seems to be known in the general case.

Question 6. Let M be a compact affine manifold. Is every polynomial volume form on M parallel?

Example 9.1. Let $\rho\colon G \to \mathrm{Aff}(n)$ be an étale affine representation. Assume that G admits a uniform lattice Γ. Every polynomial volume form on

$$M = \Gamma\backslash G$$

lifts to a Γ-left-invariant volume form μ on G, which is polynomial. We may integrate μ with respect to a finite G-invariant volume λ on $\Gamma\backslash G$, to obtain a polynomial volume form η, which is left-invariant on G. Explicitly, if X_i are vector fields on G,

$$\eta(X_1, \ldots, X_n) = \int_{\Gamma\backslash G} l_g^* \mu\,(X_1, \ldots, X_n)\, d\lambda.$$

Since Γ is uniform, G is unimodular and, therefore, η is also right-invariant on G. As in Example 8.7, we can write

$$\eta = f\nu,$$

where $\delta = f^{-1}$ is a polynomial. Since η is polynomial, also f is a polynomial. It follows that f must be constant. Hence, the parallel volume on G is left-invariant. This implies that G and M are complete (see Corollary 5.7). By the above, since M is complete, the polynomial volume form μ is parallel.

9.2 Holonomy of compact volume preserving affine manifolds

Let M be a compact affine manifold, and $\Gamma = \mathrm{hol}(\pi_1(M))$ its affine holonomy group. We put $A(\Gamma)$ for the Zariski closure of Γ in $\mathrm{Aff}(n)$. The following striking observation is the main result of [42].

Theorem 9.3 (Goldman-Hirsch, [42]). *Let M be a compact affine manifold with parallel volume. Then $A(\Gamma)$ acts transitively on $\mathbb{A}^n$.*

We put $G = A(\Gamma)$, and $H = G_x$, for some $x \in \mathbb{A}^n$. The proof of Theorem 9.3, which we will explain below, then shows:

Corollary 9.4. *Let M be a compact affine manifold with parallel volume. Then the absolute characteristic class $\bar{c}(G) \in H^n(\mathfrak{g}, \mathbb{R})$ is non-zero.*

We also obtain the following strong restriction on the homogeneous spaces G/H, which may appear as Zariski closures of the holonomy groups of compact volume preserving affine manifolds.

Corollary 9.5. *Let M be a compact affine manifold with parallel volume. Then*

$$\dim H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}) = 1,$$

and the natural homomorphism

$$H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}) \to H^n(\mathfrak{g}, \mathbb{R})$$

is injective.

Proof. By Theorem 9.3, $A(\Gamma)$ acts transitively, and by Corollary 9.4, the absolute class $\bar{c}(G) \in H^n(\mathfrak{g}, \mathbb{R})$ is non-zero. Since, as described in Section 8.1, $\bar{c}(G)$ is the image of a relative class $\bar{c}(G, x) \in H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R})$, the claim follows. □

Here is an important consequence of Theorem 9.3 concerning the *divisibility of homogeneous domains*.

Corollary 9.6. *Let U be a homogenous domain which is divisible by a properly discontinuous group Γ of affine transformations. Then either $U = \mathbb{A}^n$ or there exists an element $\gamma \in \Gamma$ with* $\det \gamma > 1$.

Proof. Follows immediately, since $A(\Gamma)^0 \leq \mathrm{Aff}(U)$ preserves U. (See Section 6.4.) □

More generally, Theorem 9.3 implies that the development image of a compact volume preserving affine manifold M is not contained in a proper semi-algebraic subset of $\mathbb{A}^n$. (See [42]).

One can also use Theorem 9.3 to deduce:

Corollary 9.7 ([42]). *A compact homogeneous affine manifold M with parallel volume is complete.*

Proof. In fact, the universal cover X of a homogeneous affine manifold M develops onto an affine homogeneous domain U, and the development map is a covering map. If M is volume preserving U is divisible by a volume preserving properly discontinuous group Γ of affine transformations. Hence, $U = \mathbb{A}^n$. It follows that the development map of M is a covering map onto $\mathbb{A}^n$. Therefore, it is a diffeomorphism. Thus, M is complete. □

See Section 4.2, for an independent proof of the latter fact.

For an application of Theorem 9.3 concerning the structure of the groups of isometries of compact flat pseudo-Riemannian manifolds, see Section 3.3.

9.2.1 Proof of Theorem 9.3. We outline the proof of Theorem 9.3 using the main ideas of [42], which relate nicely to the concepts discussed in Section 8.1. The basic new tools which we require stem from the cohomology theory of discrete groups (as documented for example in [19]), and the cohomology of algebraic linear groups G, as developed in [54].

Cohomology of algebraic linear groups. Let $G = UT$ be an algebraic linear group, where U is the unipotent radical and T is a maximal reductive subgroup. We let $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{t}$ denote their corresponding Lie algebras, which are subalgebras of $\mathfrak{aff}(n)$. Let V be a rational G-module. The algebraic cohomology groups $H^i_{\text{alg}}(G, V)$ are defined in [54]. For any subgroup Γ of G, there exists then a natural restriction homomorphism

$$r_\Gamma \colon H^i_{\text{alg}}(G, V) \to H^i(\Gamma, V)$$

into the ordinary cohomology groups of Γ. Moreover, the groups $H^i_{\text{alg}}(G, V)$ may be computed by the Hochschild isomorphism

$$\text{hs} \colon H^i_{\text{alg}}(G, V) \to H^i(\mathfrak{u}, V)^T,$$

which identifies $H^i_{\text{alg}}(G, V)$ with the T-invariants of the Lie algebra cohomology of the unipotent radical. (See [54] for reference.)

Discrete cohomology of Γ, and cohomology of M. Let M be a manifold, $\Gamma = \pi_1(M)$. There exists a natural homomorphism

$$q \colon H^i(\Gamma, V) \to H^i(M, V)$$

where V is the local coefficient system on M induced by V. If M has contractible universal cover X then q is an isomorphism. Moreover q is an isomorphism on H^1. (See for example [19, VII, §7] or [71, IV, §11].)

The proof of Theorem 9.3 then builds on the following two remarks:

Functoriality of the characteristic class u^n. Let $\Gamma = \text{hol}(\pi_1(M))$ be the holonomy of a compact volume preserving affine manifold, and $G = A(\Gamma) = UT$ its Zariski closure in $\text{Aff}(n)$. As remarked in Section 8.1.1, the translation part t, defines a characteristic class u^n, within the n-th cohomology groups with coefficients in $\mathbb{R} = \bigwedge^n \mathbb{R}^n$. The characteristic class u (and therefore u^n) is naturally defined in the discrete group-, algebraic group- and Lie algebra cohomology theory, and, as observed in [42], it is compatible with the restriction homomorphisms and the Hochschild isomorphism hs.

Representation of u^n in de Rham cohomology. It is proved in [41] that $q(u^n) \in H^n(M, \mathbb{R})$ is represented by the parallel volume form in the de Rham cohomology group $H^n_{dR}(M, \mathbb{R})$ of M.

Proof of Theorem 9.3. Since M is compact, the de Rham cohomology class $q(u^n)$ of the parallel volume form on M does not vanish. Therefore, in particular $u^n \neq 0$.

By the Hochschild isomorphism this also holds for $u^n \in H^n(\mathfrak{u}, \mathbb{R})$. As remarked in Section 8.1.1, u^n corresponds to the class $\bar{c}(\mathfrak{u})$, which therefore is non-vanishing. Hence, U acts transitively on $\mathbb{A}^n$. □

Appendices

A Linear algebraic groups

We provide some background material and basic facts from the theory of linear algebraic groups and algebraic group actions.

A.1 Definition of linear algebraic groups

A subgroup $\boldsymbol{G} \leq \mathrm{GL}(n, \mathbb{C})$ is a *linear algebraic group* if it is the zero locus of polynomial equations in its matrix entries. A subgroup $G \leq \mathrm{GL}(n, \mathbb{R})$ is said to be a *real linear algebraic group* if it is closed with respect to the Zariski topology on $\mathrm{GL}(n, \mathbb{R})$. A real linear algebraic group G is thus defined as the real zero locus of a set of real polynomials in its matrix entries. In particular, G is of the form $G = \boldsymbol{G}(\mathbb{R})$, for some linear algebraic group $\boldsymbol{G}$, which is defined over the reals. A homomorphism $\rho\colon \boldsymbol{G} \to \boldsymbol{L}$ between linear algebraic groups is called a *morphism of algebraic groups* or algebraic homomorphism if it may be expressed by polynomials in the matrix entries of $\boldsymbol{G}$.

If $G \leq \mathrm{GL}(n, \mathbb{R})$ is any subgroup, we let $\bar{G}^{\mathbb{R}} \leq \mathrm{GL}(n, \mathbb{R})$ denote its real Zariski closure, which is the smallest real algebraic subgroup containing G. Correspondingly, let $G \leq \mathrm{GL}(n, \mathbb{C})$. Then its Zariski closure $\bar{G}^{Z} \leq \mathrm{GL}(n, \mathbb{C})$ is the smallest linear algebraic group containing G.

A.2 Structure theory

Unipotent groups. A linear algebraic group is called *unipotent* if it is conjugate to a subgroup of the group of all upper triangular matrices which have only 1 as eigenvalue. All its elements are unipotent matrices. Every connected subgroup of unipotent matrices in $\mathrm{GL}(n, \mathbb{R})$ is a unipotent real linear algebraic group. Also every unipotent linear algebraic group is connected. (See [17] or [84] for more details.)

Reductive groups. A subgroup G of $\mathrm{GL}(n, \mathbb{R})$ is called *reductive* if every G-invariant subspace in $\mathbb{R}^n$ has a G-invariant complement in $\mathbb{R}^n$. Main examples are compact subgroups, semisimple groups and (complex-) diagonalisable groups. A linear algebraic group $\boldsymbol{G}$ is reductive if and only if its unipotent radical $\boldsymbol{U}$ is trivial. A connected abelian group of semisimple elements is called an algebraic torus.

Levi-splitting. Every linear algebraic group $\boldsymbol{G} \leq \mathrm{GL}(n,\mathbb{C})$ splits as a semidirect product of linear algebraic groups $\boldsymbol{G} = \boldsymbol{H}\boldsymbol{U}$, where $\boldsymbol{H}$ is reductive and $\boldsymbol{U}$ is the maximal unipotent normal subgroup of $\boldsymbol{G}$. The group $\boldsymbol{U}$ is called the *unipotent radical* of $\boldsymbol{G}$. The splitting induces a corresponding splitting of the groups of real points. In particular, every real linear algebraic group $G \leq \mathrm{GL}(n,\mathbb{R})$ splits as a semidirect product of linear algebraic groups $G = HU(G)$, where H is reductive and $U(G)$ is the unipotent radical of G. See [17], [84].

A.3 Orbit closure

Every orbit $\boldsymbol{G}x$ of a linear algebraic group $\boldsymbol{G}$ on a vector space (or algebraic variety) contains a closed orbit in its Zariski closure $\overline{\boldsymbol{G}x}^{Z}$, and the orbit $\boldsymbol{G}x$ is open in its closure. This is called the *closed orbit lemma*, see [17, 1.8]. By the latter fact and [81, I.10], *orbit closure in the Zariski-topology and in the Euclidean* (*Hausdorff*) *topology coincide*. Concerning the topology of real algebraic group actions we have:

Proposition A.1. *If $\boldsymbol{G}$ is a linear algebraic group defined over $\mathbb{R}$, then the real points $\boldsymbol{X}_{\mathbb{R}}$ of the orbit $\boldsymbol{X} = \boldsymbol{G}x$ form a finite union of orbits of $\boldsymbol{G}_{\mathbb{R}}$. Moreover, if $\boldsymbol{X}$ is closed, the orbits of $\boldsymbol{G}_{\mathbb{R}}$ in the real algebraic variety $\boldsymbol{X}_{\mathbb{R}}$ are closed in the Euclidean topology.*

For this result, see [18, Proposition 2.3]. If $\boldsymbol{G}$ is reductive also the converse holds, cf. [15].

Proposition A.2 (See [15], [17], [89]). *Let $\boldsymbol{U}$ be a unipotent linear algebraic group, which acts on a vector space* (*or on an affine algebraic variety*)*. Then every orbit of $\boldsymbol{U}$ is closed.*

The analogous result holds for actions of unipotent real linear algebraic groups. That is, every orbit of a unipotent real algebraic group is closed in the real Zariski-topology.

For algebraic actions, which are defined over $\mathbb{R}$, the preceding result generalises as follows. Let $\boldsymbol{G} \leq \mathrm{GL}(n,\mathbb{C})$ be a linear algebraic group which is defined over $\mathbb{R}$. Recall that a torus $\boldsymbol{T} \leq \boldsymbol{G}$ is called $\mathbb{R}$-split if it can be diagonalised with respect to a real basis.

Proposition A.3 (See [15]). *If $\boldsymbol{G}$ is defined over $\mathbb{R}$, and the stabiliser $\boldsymbol{G}_x$, of $x \in \mathbb{R}^n$, contains a maximal $\mathbb{R}$-split torus of $\boldsymbol{G}$ then the orbit $\boldsymbol{G}x$ is Zariski closed.*

A corresponding result holds for real algebraic group actions. Namely, if G_x contains the maximal connected diagonalisable subgroup A of G, then Gx is closed in the Euclidean topology. (This can be proved directly as a consequence of Proposition A.2 and the Iwasawa decomposition (see [84]) $G = KAN = KNA$, where K is compact, N is unipotent.)

B Lie algebra cohomology

Here we introduce Lie algebra cohomology, and relative Lie algebra cohomology. We compute the top cohomology group in relative Lie algebra cohomology with one-dimensional coefficients.

B.1 Definition of Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra over the reals. (We shall use the real numbers as ground field, for convenience.) Let $V = V_\lambda$ be the $\mathfrak{g}$-module, corresponding to an action $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Let

$$C^k(\mathfrak{g}, V) = \left(\bigwedge^k \mathfrak{g}\right)^* \otimes V$$

be the module of k-forms on $\mathfrak{g}$ with values in V. The Lie algebra $\mathfrak{g}$ acts on $C^k(\mathfrak{g}, V)$ by the adjoint action on $\bigwedge^k \mathfrak{g}^*$ twisted with λ. For $X \in \mathfrak{g}$, the corresponding operator $L_X\colon C^k(\mathfrak{g}, V) \to C^k(\mathfrak{g}, V)$ is called the Lie derivative. For $\omega \in C^k(\mathfrak{g}, V)$, we put $\iota_X\omega \in C^{k-1}(\mathfrak{g}, V)$ to denote the contraction with X. Then the following commutation formula holds:

$$\iota_Y\, L_X = L_X\, \iota_Y - \iota_{[X,Y]}. \tag{B.1}$$

The boundary operator $d_V\colon C^k(\mathfrak{g}, V) \to C^{k+1}(\mathfrak{g}, V)$ is a differential of degree one, which commutes with the operators L_X. It is defined inductively by the relation

$$\iota_X\, d_V + d_V\, \iota_X = L_X. \tag{B.2}$$

Recall, that d_V may be explicitly computed as

$$\begin{aligned} &d_V\omega\,(Y_1 \wedge \cdots \wedge Y_k) \\ &\quad= \sum_{l=1}^{k} (-1)^{l+1} \lambda(Y_l)\left(\omega(Y_1 \wedge \cdots \wedge \widehat{Y}_l \wedge \cdots \wedge Y_k)\right) \\ &\qquad + \sum_{r<s} (-1)^{r+s} \omega([Y_r, Y_s] \wedge Y_1 \wedge \cdots \wedge \widehat{Y}_r \wedge \cdots \wedge \widehat{Y}_s \wedge \cdots \wedge Y_k). \end{aligned}$$

As usual, $Z^k(\mathfrak{g}, V) = \{\omega \in C^k(\mathfrak{g}, V) \mid d_V\omega = 0\}$ denotes the group of k-cocycles, and $B^k(\mathfrak{g}, V) = \{d_V\eta \mid \eta \in C^{k-1}(\mathfrak{g}, V)\} \subset Z^k(\mathfrak{g}, V)$ the group of coboundaries. The complex $(C^\bullet(\mathfrak{g}, V), d_V)$ is called the *Koszul-complex*. Its cohomology vector spaces $H^k(\mathfrak{g}, V_\lambda) = Z^k(\mathfrak{g}, V)/B^k(\mathfrak{g}, V)$ are called the cohomology groups of $\mathfrak{g}$ with coefficients in V.

B.2 Relative Lie algebra cohomology

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. We shall also consider the relative cohomology groups $H^k(\mathfrak{g}, \mathfrak{h}, V)$. Let $C^k(\mathfrak{g}/\mathfrak{h}, V)$ denote the subspace of cochains in $C^k(\mathfrak{g}, V)$ which

vanish if one argument is contained in $\mathfrak{h}$. Note that $C^k(\mathfrak{g},\mathfrak{h},V) \cong \Lambda^k(\mathfrak{g}/\mathfrak{h}^*,V)$. As follows from (B.2), the $\mathfrak{h}$-invariants

$$C^\bullet(\mathfrak{g}/\mathfrak{h},V)^{\mathfrak{h}} \cong \mathrm{Hom}_{\mathfrak{h}}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h},V),$$

are preserved by d_V, and thus form a subcomplex $C^\bullet(\mathfrak{g},\mathfrak{h},V)$ of the Koszul complex $(C^\bullet(\mathfrak{g},V),d_V)$. This subcomplex is called the complex of relative cochains. Its cohomology groups are the *relative cohomology groups* $H^k(\mathfrak{g},\mathfrak{h},V)$. Note, in particular, that the inclusion of cochain complexes, induces natural homomorphisms

$$H^k(\mathfrak{g},\mathfrak{h},V) \to H^k(\mathfrak{g},V).$$

See [55], [64], [68] for detailed reference on Lie algebra (co-) homology.

B.2.1 Top cohomology group with one-dimensional coefficients. Observe that $H^k(\mathfrak{g},\mathfrak{h},V) = \{0\}$, $k > n = \dim \mathfrak{g}/\mathfrak{h}$. Let $\lambda\colon \mathfrak{g} \to \mathbb{R}$ be a one dimensional representation. We shall compute the top cohomology group $H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_\lambda)$.

Let $\mathfrak{n}(\mathfrak{g},\mathfrak{h})$ denote the normaliser of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote the adjoint representation, $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}\colon \mathfrak{n}(\mathfrak{g},\mathfrak{h}) \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ the quotient representation. Note, in particular, that (induced by the adjoint action) $\mathfrak{n}(\mathfrak{g},\mathfrak{h})$ acts on the one dimensional module $C^n(\mathfrak{g}/\mathfrak{h},\mathbb{R})$. Since $C^n(\mathfrak{g}/\mathfrak{h},\mathbb{R}) \cong \Lambda^n(\mathfrak{g}/\mathfrak{h})^*$, $\mathfrak{n}(\mathfrak{g},\mathfrak{h})$ acts with the character $-\operatorname{trace}\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$. The Lie derivatives L_X, for $X \in \mathfrak{n}(\mathfrak{g},\mathfrak{h})$, preserve $C^n(\mathfrak{g}/\mathfrak{h},\mathbb{R}_\lambda) \subseteq C^n(\mathfrak{g},\mathbb{R}_\lambda)$, and, thereby, $\mathfrak{n}(\mathfrak{g},\mathfrak{h})$ acts with the character

$$\bar{\chi}_\lambda = \lambda - \operatorname{trace}\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}} \tag{B.3}$$

on $C^n(\mathfrak{g}/\mathfrak{h},\mathbb{R}_\lambda)$, that is, for all $\tau \in C^n(\mathfrak{g}/\mathfrak{h},\mathbb{R}_\lambda)$,

$$L_X\tau = \bar{\chi}_\lambda(X)\,\tau. \tag{B.4}$$

Now the following holds:

Proposition B.1. *Let $n = \dim \mathfrak{g}/\mathfrak{h}$. Then the group $H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_\lambda)$ is non-zero if and only if $\bar{\chi}_\lambda \equiv 0$.*

Proof. Assume that $H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_\lambda) \neq \{0\}$. Then, in particular, there exists a non-zero generator τ of the module $C^n(\mathfrak{g},\mathfrak{h},\lambda) = (\Lambda^n(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$. We compute the boundary operator $d_\lambda\colon C^{n-1}(\mathfrak{g},\mathfrak{h},\lambda) \to C^n(\mathfrak{g},\mathfrak{h},\lambda)$: Let $\omega \in \left(\Lambda^{n-1}(\mathfrak{g}/\mathfrak{h})^*\right)^{\mathfrak{h}}$ be a relative $n-1$ cochain. By duality in $C^\bullet(\mathfrak{g}/\mathfrak{h},\mathbb{R})$, there exists $X \in \mathfrak{g}$, such that $\iota_X\tau = \omega$. Moreover, by (B.1), $0 = L_H\omega = L_H\iota_X\tau = \iota_{[H,X]}\tau$, for all $H \in \mathfrak{h}$. In fact, this implies that $X \in \mathfrak{n}(\mathfrak{g},\mathfrak{h})$. Using (B.2), we compute

$$d_\lambda\omega = d_\lambda\iota_X\tau = L_X\tau = \bar{\chi}_\lambda(X)\tau. \tag{B.5}$$

Therefore, if $\bar{\chi}_\lambda \neq 0$, $H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_\lambda) = \{0\}$.

For the converse, assume $H^n(\mathfrak{g},\mathfrak{h},\mathbb{R}_\lambda) = \{0\}$. Then either $(\Lambda^n\mathfrak{g}/\mathfrak{h}^*)^{\mathfrak{h}} = \{0\}$, or $d_\lambda \neq 0$. In the first case, we must have $\bar{\chi}_\lambda(H) \neq 0$, for some $H \in \mathfrak{h}$. In the second case, we may use the computation of the boundary (B.5), to conclude $\bar{\chi}_\lambda \neq 0$. □

The special case, $H^n(\mathfrak{g}) \neq 0$, $n = \dim \mathfrak{g}$, if and only if $\mathfrak{g}$ is unimodular, is due to Koszul [68].

Example B.1. If $\mathfrak{g}$ is nilpotent then the group $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_\lambda)$ is non-zero if and only if $\lambda \equiv 0$ on $\mathfrak{n}(\mathfrak{g}, \mathfrak{h})$.

For an interpretation of $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_\lambda)$ concerning invariant measures on homogeneous spaces, see Proposition C.3.

C Invariant measures on homogeneous spaces

We briefly give some standard background material on the existence of invariant measures on homogeneous spaces. One may consult [87], Chapter I, or [100] for more detailed reference. We then proceed to show that the top relative Lie algebra cohomology group with one-dimensional coefficients, which is associated to a semi-invariant measure on a homogeneous space, carries information about the measure preserving automorphisms of the space. This extends a well known non-vanishing result of Koszul [68] on the top cohomology of unimodular Lie groups.

C.1 Semi-invariant and invariant measures

Let G be a Lie group, $H \leq G$ is a closed subgroup. We put $X = G/H$ for the associated G-homogeneous space, $n = \dim X$. Let $L_g : X \to X$ denote left-multiplication with $g \in G$. A Borel measure μ on X is called *semi-invariant* with character λ if there exists a continuous homomorphism $\lambda : G \to \mathbb{R}^{>0}$ such that $L_g^* \mu = \lambda(g)\mu$, for all $g \in G$. The measure μ is called *invariant* if $\lambda \equiv 1$.

Haar measure and unimodular character. Every locally compact group G has a (up to scalar multiple) unique (left)-invariant measure $\mu = \mu_G$, which is called the *Haar measure* of G.

Let $R_g : G \to G$ denote right-multiplication with $g \in G$. Then $R_g^* \mu = \Delta_G(g)\mu$ is another Haar measure for G. The homomorphism

$$\Delta = \Delta_G : G \to \mathbb{R}^{>0}$$

is called the *unimodular character* of G. If $\Delta_G \equiv 1$ the Haar-measure is also right-invariant. Therefore, G has a *biinvariant measure* if and only if $\Delta_G \equiv 1$. In this case, G is called *unimodular*.

Since G is a Lie group, the Haar measure can be computed by integration relative to a left-invariant n-form $\omega \neq 0$ on G, $n = \dim G$. We have

$$\Delta_G(g) = |\det \mathrm{Ad}(g)|,$$

where $\det \mathrm{Ad}(g)$ is the determinant of the adjoint representation. In particular, G is unimodular if and only if the determinant of the adjoint representation has absolute value one.

Existence of semi-invariant measures. Not every homogeneous space admits an invariant measure. A precise criterion is as follows:

For $h \in H$, define $\Delta_{G/H}(h) = \Delta_G(h)\Delta_H(h)^{-1}$.

Proposition C.1 ([87], Lemma 1.4). *The homogeneous space $X = G/H$ admits a (unique up to scalar) semi-invariant measure with character λ if and only if, for all $h \in H$,*

$$\lambda(h) = \Delta_{G/H}(h).$$

In particular, X admits an invariant measure if and only if $\Delta_{G/H} \equiv 1$ on H.

Let $\mathrm{Ad}_{G/H}$ denote the adjoint representation of H on $\mathfrak{g}/\mathfrak{h}$. Then, for all $h \in H$, $\Delta_{G/H}(h) = |\det \mathrm{Ad}_{G/H}(h)|$.

C.2 The unimodular character of $X = G/H$

Let $N_G(H)$ denote the normaliser of H in G. The adjoint representation $G \to \mathrm{GL}(\mathfrak{g})$ induces a quotient representation

$$\mathrm{Ad}_{G/H} \colon N_G(H) \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h}).$$

Note that the restriction of $\mathrm{Ad}_{G/H}$ to H corresponds to the isotropy representation of H on the tangent space of X at H.

For every $g \in N_G(H)$, define

$$C(g) = L_g R_g^{-1} \colon X \to X,$$

where R_g denotes right-multiplication on X. Let μ be a semi-invariant measure on X with character λ. The relation

$$C(g)^*\mu = \Delta_{G/H}(g)\mu,$$

defines a unimodular character,

$$\Delta_{G/H} \colon N_G(H) \to \mathbb{R}^{>0},$$

independently of λ. In fact, $\Delta_{G/H} = |\det \mathrm{Ad}_{G/H}|$. Obviously, we have:

Lemma C.2. *The semi-invariant measure μ with character λ is right-invariant by $g \in N_G(H)$ if and only if $\lambda(g)\,\Delta_{G/H}^{-1}(g) = 1$.*

In addition, we obtain the following:

Proposition C.3. *Let $X = G/H$ be a homogeneous space and μ a semi-invariant measure with (smooth) character λ. Let $\bar{\lambda}\colon \mathfrak{g} \to \mathbb{R}$ denote the derivative of λ. Then the following conditions are equivalent:*

(1) *μ is invariant by the right action of $N_G(H)^0$ on X.*

(2) $\chi_\lambda = \lambda \Delta_{G/H}^{-1} \colon N_G(H)^0 \to \mathbb{R}^{>0} \equiv 1$.

(3) $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\lambda}}) \neq \{0\}$.

Proof. This a direct consequence of the above lemma and Proposition B.1. □

Example C.1. Let G be nilpotent. Then $\Delta_{G/H} \equiv 1$. Therefore, $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}_{\bar{\lambda}}) \neq \{0\}$ is equivalent to $\lambda \cong 1$ on $N_G(H)^0$.

Corollary C.4. *Let $X = G/H$ be a homogeneous space which admits an invariant measure μ. Then μ is right-invariant by the action of $N_G(H)^0$ if and only if $H^n(\mathfrak{g}, \mathfrak{h}, \mathbb{R}) \neq \{0\}$.*

As a special case of the corollary, we have the well known theorem of Koszul [68], which states that $H^n(\mathfrak{g}, \mathbb{R}) \neq 0$ if and only if G is unimodular.

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Part G

Conformal geometry

Chapter 23

The conformal analog of Calabi–Yau manifolds

Helga Baum

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1 Introduction

In this survey we intend to introduce the reader to Cartan connections, their holonomy theory and applications to conformal supersymmetries. The concept of nowadays called 'Cartan connections' was introduced and successfully applied by E. Cartan during the first decades of 20th century. Cartan studied geometric structures associated with solution spaces of classical ordinary differential equations using moving frames[1]. Thereby, Cartan found a uniform concept that generalizes F. Klein's approach to geometry by studying invariants of group actions as well as B. Riemann's concept of curved Riemannian manifolds to a broad class of geometries, such as conformal, projective, Einstein–Weyl, 3-dimensional CR and so on. The renewed interest in conformal and quaternionic structures as well as relations to complex analysis via CR structures brought Cartan connections back to general interest during the last decade. In particular, the systematic development of parabolic differential geometry in modern language of principle fibre bundles by the group in Vienna, Brno and Prag (A. Cap, J. Slovak, V. Soucek and others) over the last years turned out to be an

[1] Cf. [Nu05] for a reformulation and extension of some of Cartan's work.

extremely useful tool in studying geometric problems and structures in an unified and systematic way. Let us mention here for example the classification of different conformal and CR invariants, the classification of conformally invariant operators, the classification of higher symmetries of differential operators or the progress in understanding T. Branson's Q-curvature (cf. for example [FG85], [CSS01], [CD01], [E05], [J06]).

The common feature of many different kinds of geometries is the existence of a distinguished (normal) Cartan connection which characterizes the geometry in question, similarly as the Levi-Civita connection in Riemannian geometry. An example is conformal geometry, which we will focus on in this chapter. Many symmetries and super-symmetries in conformal geometry (conformal vector fields, conformal Killing forms, conformal Killing tensors, conformal Killing spinors) have an interpretation as invariant objects for the holonomy group of the normal conformal Cartan connection and can be classified in this way. We will explain this here for the case of conformal Killing spinors.

It is well known that the existence of parallel spinors on (pseudo)-Riemannian spin manifolds (M^n, g), i.e., spinors φ with

$$\nabla_X \varphi = 0 \quad \text{for all vector fields } X, \tag{1}$$

is characterized by the holonomy group of (M, g) – the group of all parallel displacements with respect to the Levi-Civita connection along curves which are closed in a fixed point. By the holonomy principle, any of these holonomy groups correspond to a special geometric structure. Using the known classification of Riemannian holonomy groups, McK. Wang ([Wan89]) showed that the holonomy group of simply connected indecomposable *Riemannian* manifolds with parallel spinors is $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)$, G_2 or $\mathrm{Spin}(7)$. The corresponding special geometric structures are Ricci-flat Kähler metrics, hyper-Kähler metrics, a special parallel 3-form ($n = 7$) or a special parallel 4-form ($n = 8$), respectively. In 2003, T. Leistner ([Lei03]) classified the holonomy groups of indecomposable simply connected *Lorentzian* manifolds. As an application he showed that this holonomy group in presence of a parallel spinor is $G \ltimes \mathbb{R}^{n-2}$, where G is a product of $\{1\}$, $\mathrm{SU}(k)$, $\mathrm{Sp}(l)$, G_2 and $\mathrm{Spin}(7)$.

Another already 'classical' kind of spinor fields occurred in the 80th in studying eigenvalues of the Dirac operator (cf. [Fr81]) as well as in supergravity theories (cf. [DNP86]). Let D be the Dirac operator on a compact Riemannian spin manifold (M^n, g) of positive scalar curvature scal^g. All eigenvalues μ of D are bounded from below by

$$\mu^2 \geq \frac{n}{4(n-1)} \min \mathrm{scal}^g. \tag{2}$$

The eigenspinors φ in the limiting case (equality in (2)) are geometric Killing spinors, that is, they satisfy the stronger equation

$$\nabla_X \varphi + \alpha X \cdot \varphi = 0 \quad \text{for all vector fields } X, \tag{3}$$

where α is a real number related to the eigenvalue μ. This initiated an intensive search for geometric structures with geometric Killing spinors (cf. [BFGK91], [Kat99]). In 1993 Ch. Bär observed that there is a Cartan connection of type $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ behind (3) and gave a reinterpretation of geometric Killing spinors as invariants of the holonomy group of this Cartan connection. In particular, this Cartan connection is the same as the Levi-Civita connection of the Riemannian cone over (M, g), hence Killing spinors on (M, g) correspond to parallel spinors on the cone. This observation together with Wang's result led him to the complete classification of all geometric structures with Killing spinors of type (3) (cf. [Bär93]).

There are a lot of different spinor field equations in mathematics as well as in physics (string theory), generalizing (1) and (3). In the present article we deal with *conformal Killing spinors*, spinors satisfying

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \quad \text{for all vector fields } X. \tag{4}$$

These spinors occur in conformal geometry as the solutions of the 'twistor equation', one of the two conformally invariant spinor field equations on 1/2 spinors. They were first studied by R. Penrose and his collaborators in General Relativity (cf. [PR86]), later on in mathematics (cf. [BFGK91]). In [ACDS98], D. Alekseevski, V. Cortés et all gave an interpretation of conformal Killing spinors as infinitesimal symmetries of supermanifolds. J. Figueroa O'Farrill and H. Rajaniemi used it to give geometric constructions of superalgebras (cf. [R06]). It is natural to ask for a classification of conformal structures which admit conformal Killing spinors. The aim of the present survey is to explain the holonomy theory behind (4), which turns out to be the holonomy theory of the normal conformal Cartan connection of the conformal manifold $(M, [g])$. We call its holonomy group, the 'conformal holonomy group'. The conformal holonomy group of a manifold (M, g) with a metric of signature (p, q) is contained in $\mathrm{SO}(p+1, q+1)$. In Riemannian and algebraic geometry as well as in string theory Calabi–Yau manifolds play a special role. A Calabi–Yau manifold is a compact $2m$-dimensional Riemannian manifold with holonomy group $\mathrm{SU}(m) \subset \mathrm{SO}(2m)$. Such a manifold is Ricci-flat, Kähler and admits a 2-parameter family of parallel spinors. In view of that we will focus attention in particular to Lorentzian manifolds with conformal holonomy group in $\mathrm{SU}(1, m) \subset \mathrm{SO}(2, 2m)$, which can be viewed as conformal analog of Calabi–Yau manifolds. Such Lorentzian manifolds have an interesting geometric structure. They are Fefferman spaces, which occur as S^1-bundles over strictly pseudoconvex CR spin manifolds, thus relating conformal Lorentzian geometry to CR geometry. Furthermore, they admit a 2-parameter family of conformal Killing spinors.

The chapter is organized as follows. In Section 2 we will introduce the notion of a Cartan connection and of its holonomy group. In particular, we relate this holonomy group to the holonomy group of covariant derivatives on associated tractor bundles. Section 3 contains an outline of the construction of the normal conformal Cartan connection in its invariant form as well as its description in terms of the metrics in the

conformal class (3.1.–3.3.). This leads to the 'metric' formula for the associated normal derivative on tractor bundles, usually used in conformal geometry (Section 3.4). In Section 3.5. we give a review on recent classification results for conformal holonomy groups. In particular we explain the relation to the existence of Einstein metrics in the conformal class. In 3.6. we introduce conformal Killing spinors and give the reinterpretation as parallel sections in the spin tractor bundle with respect to the covariant derivative induced by the normal Cartan connection. We close this section with the recent classification result of F. Leitner ([L05a], who used the methods of Cartan geometry to give a local conformal classification of all Lorentzian geometries admitting generic conformal Killing spinors. In Section 4 we deal with Lorentzian manifolds with conformal holonomy in $\mathrm{SU}(1,m)$, the 'conformal analog of Calabi–Yau-manifolds'. For this aim we explain in 4.1. the construction of Fefferman spaces, which are S^1-bundles over strictly pseudoconvex CR-manifolds. Finally, using our former results about conformal Killing spinors on Fefferman spaces, we explain in 4.2. why the conformal holonomy group of Fefferman spin spaces is contained in $\mathrm{SU}(1,m)$.

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2 Cartan connections and holonomy groups

In this section we give a short introduction to Cartan connections and their holonomy groups. In particular, we explain the relation to holonomy groups of principal fibre bundle connections and to holonomy groups of covariant derivatives in associated vector bundles. Details can be found in [KN63] and [S97].

Let M be a smooth manifold and let $\mathcal{L}(M,x)$ denote the set of all piecewise smooth loops in a point $x \in M$. Usually, holonomy groups are defined by parallel displacements along loops in x with respect to a connection form on a principal fibre bundle or with respect to a covariant derivative in a vector bundle over M given by the geometric structure in question. Let us shortly recall this in order to fix some notation.

First, we consider a vector bundle E over M with covariant derivative ∇^E. For any piecewise smooth curve $\gamma\colon [\alpha,\beta] \subset \mathbb{R} \to M$ the covariant derivative gives rise to a parallel displacement

$$\mathcal{P}_\gamma^{\nabla^E}\colon e \in E_{\gamma(\alpha)} \longmapsto \varphi_e(\beta) \in E_{\gamma(\beta)}, \tag{5}$$

where $\varphi_e\colon [\alpha,\beta] \to E$ is the unique parallel section in E along γ defined by

$$\frac{\nabla^E \varphi_e}{dt} = 0 \quad \text{and} \quad \varphi_e(\alpha) = e.$$

The *holonomy group of* (E, ∇^E) *with respect to the base point* x is the Lie group of all parallel displacements along loops in x:

$$\mathrm{Hol}_x(E, \nabla^E) := \{\mathcal{P}_\gamma^{\nabla^E} \mid \gamma \in \mathcal{L}(M, x)\} \subset \mathrm{GL}(E_x).$$

Next, let $(P, \pi, M; B)$ be a principal fibre bundle over M with structure group B. We consider a connection form A on P, i.e., a 1-form $A \in \Omega^1(P, \mathfrak{b})$ with values in the Lie algebra $\mathfrak{b}$ of B, which satisfies the following invariance conditions.

(1) $R_b^* A = \mathrm{Ad}(b^{-1}) \circ A$ for all $b \in B$, and

(2) $A(\widetilde{X}) = X$ for all $X \in \mathfrak{b}$,

where R_b denotes the right action of B on the total space P of the principal fibre bundle and $\widetilde{X}$ is the fundamental vector field of the B-action on P defined by the element $X \in \mathfrak{b}$:

$$\widetilde{X}(u) = \frac{d}{dt}\big(u \cdot \exp(tX)\big)_{|t=0}.$$

The connection form A defines a right invariant horizontal distribution $Th^A P$ on P by

$$Th^A \colon u \in P \longrightarrow Th_u^A P := \mathrm{Ker}\, A_u \subset T_u P.$$

Any piecewise smooth curve $\gamma \colon [\alpha, \beta] \to M$ admits a unique horizontal lift $\gamma_u^* \colon [\alpha, \beta] \to P$ with fixed initial point $u \in P_x := \pi^{-1}(x)$, defined by the conditions

$$\dot{\gamma}^*(t) \in Th^A_{\gamma^*(t)} P, \quad \pi(\gamma^*(t)) = \gamma(t), \quad \gamma^*(\alpha) = u.$$

Thus, we obtain a parallel displacement along γ by

$$\mathcal{P}_\gamma^A \colon u \in P_x \longmapsto \gamma_u^*(\beta) \in P_{\gamma(\beta)}. \tag{6}$$

If γ is a loop in x, then the points u and $\mathcal{P}_\gamma^A(u)$ lie in the same fibre P_x. Hence there is a unique element $\mathrm{hol}_u^A(\gamma) \in B$ with

$$u = \mathcal{P}_\gamma^A(u) \cdot \mathrm{hol}_u^A(\gamma),$$

called the holonomy of the loop γ with respect to A and u. The *holonomy group of* (P, A) *with respect to the base point* u is the Lie group of all of these holonomies,

$$\mathrm{Hol}_u(P, A) := \{\mathrm{hol}_u^A(\gamma) \mid \gamma \in \mathcal{L}(M, x)\} \subset B.$$

Now, let $\rho \colon B \to \mathrm{GL}(V)$ be a representation of the Lie group B over a vector space V. There is a standard way to associate a vector bundle E over M to the principal fibre bundle P by means of ρ. The total space E is defined to be the orbit space of the right action of B on $P \times V$ given by

$$(u, v) \cdot b := (u \cdot b,\ \rho(b^{-1})v), \quad (u, v) \in P \times V,\ b \in B.$$

We denote this orbit space by $E := P \times_B V := (P \times V)/B$ and the elements of E by $[u, v] := \{(u \cdot b, \rho(b^{-1})v) \mid b \in B\}$. Obviously, E is a vector bundle over M

with fibre type V and projection $\pi_E([u,v]) := \pi(u)$. Any point u in the fibre P_x of P over $x \in M$ gives rise to a linear isomorphism

$$[u]\colon v \in V \longrightarrow [u,v] \in E_x := \pi_E^{-1}(x) \tag{7}$$

between the fibre type V and the fibre E_x of E. A smooth section ϕ in the vector bundle E can locally be represented in the form $\phi_{|U} = [s,v]$, where $s\colon U \subset M \to P$ is a smooth local section in the principal fibre bundle P and $v \in C^\infty(U,V)$ is a smooth function on U with values in the vector space V. Any connection form $A \in \Omega^1(P,\mathfrak{b})$ on P induces a covariant derivative $\nabla^A\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ by

$$\nabla^A_X \phi_{\,|U} := [s, X(v) + \rho_*(A(ds(X)))v], \quad \text{where } \phi_{|U} = [s,v]. \tag{8}$$

The formulas (5), (6) and (8) show that the holonomy groups of (E,∇^A) and (P,A) are related by

$$\mathrm{Hol}_x(E,\nabla^A) = [u] \circ \rho(\mathrm{Hol}_u(P,A)) \circ [u]^{-1}. \tag{9}$$

Hence, if ρ is faithful, the holonomy groups $\mathrm{Hol}_x(E,\nabla^A)$ and $\mathrm{Hol}_u(P,A)$ are isomorphic.

Now, let us come to the object of our interest, to Cartan connections and their holonomy groups. For that we consider in addition a Lie group G that contains the structure group B of P as a closed subgroup. A *Cartan connection of type* G/B on the B-principal fibre bundle P is a 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ on P with values in the Lie algebra $\mathfrak{g}$ (!!) such that

(1) $R_b^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in B \subset G$,

(2) $\omega(\widetilde{X}) = X$ for all $X \in \mathfrak{b} \subset \mathfrak{g}$, and

(3) ω is a global parallelism, i.e., $\omega_p\colon T_pP \to \mathfrak{g}$ is an isomorphism for all $p \in P$.

Note that the third condition implies that the base manifold M has the same dimension as the homogeneous space G/B. Furthermore, any vector $Z \in \mathfrak{g}$ induces a global nowhere vanishing vector field Z_ω defined by

$$Z_\omega(p) := (\omega_p)^{-1}(Z).$$

Hence, the manifold P is parallelizable, i.e., it has a global basis field. A Cartan connection is called *complete*, if all these vector fields Z_ω are complete (meaning that the integral curves of Z_ω are defined for all parameters $t \in \mathbb{R}$).

Let us consider two examples of Cartan connections.

Example 1. Let G be a Lie group and ω_G its Maurer–Cartan form:

$$\omega_G\colon X \in T_gG \longmapsto dL_{g^{-1}}(X) \in T_eG = \mathfrak{g}.$$

If $B \subset G$ is a closed subgroup, then the projection $\pi\colon G \to G/B$ gives rise to a principal fibre bundle with structure group B over the homogeneous space $M = G/B$. The Maurer–Cartan form ω_G is a complete Cartan connection of type G/B on this homogeneous bundle.

Example 2. Let M^n be a smooth manifold and P be the frame bundle of M. Then the tangent bundle of M is given by $TM = P \times_{\mathrm{GL}(n)} \mathbb{R}^n$. Let $A \in \Omega^1(P, \mathfrak{gl}(n))$ be an arbitrary connection form on P and let $\theta \in \Omega^1(P, \mathbb{R}^n)$ be the displacement form of P:

$$\theta_u(X) := [u]^{-1} d\pi_u(X), \quad u \in P, X \in T_u P. \tag{10}$$

Then the 1-form given by $\omega := A + \theta \in \Omega^1(P, \mathfrak{a}(n))$ is a Cartan connection of type $A(n)/\mathrm{GL}(n)$, where $A(n) := \mathrm{GL}(n) \ltimes \mathbb{R}^n$ is the affine group and $\mathfrak{a}(n)$ its Lie algebra.

Although, contrary to usual connections, Cartan connections do not allow to distinguish a right invariant horizontal distribution on P, one can define the notion of a holonomy group also for Cartan connections using the development of the connection form along curves. By the development of a 1-form along a curve we understand the following: Let N be a manifold and $\sigma \in \Omega^1(N; \mathfrak{g})$ a 1-form with values in the Lie algebra $\mathfrak{g}$ of a Lie group G. For any piecewise smooth curve $\lambda\colon [\alpha, \beta] \to N$ there is an unique curve $\lambda_{[\sigma]}\colon [\alpha, \beta] \to G$ starting in the unit element $e \in G$ such that

$$\lambda'_{[\sigma]}(t) = dL_{\lambda_{[\sigma]}(t)} \omega(\lambda'(t)), \tag{11}$$

where L_g denotes the left action on G by the element $g \in G$. We call $\lambda_{[\sigma]}\colon [\alpha, \beta] \to G$ the *development of* σ *along* λ and its endpoint

$$\mathrm{hol}^\sigma(\lambda) := \lambda_{[\sigma]}(\beta) \tag{12}$$

the *holonomy of* λ *with respect to* σ.

Example 3. Let $G = (\mathbb{R}^n, +)$ be the additive group. Then the development of a 1-form σ on N along a curve λ in N is the usual curve integral of σ along the curve λ,

$$\lambda_{[\sigma]}(t) := \int_\alpha^t \sigma(\lambda'(s)) ds \quad \text{and} \quad \mathrm{hol}^\sigma(\lambda) = \int_\lambda \sigma.$$

Example 4. Let ω_G be the Maurer–Cartan form of a Lie group G and λ a curve in G from g_0 to g_1. Then the development of ω_G along λ is the curve $\lambda_{[\omega_G]}$ given by

$$\lambda_{[\omega_G]}(t) = g_0^{-1} \cdot \lambda(t).$$

The holonomy is $\mathrm{hol}^{\omega_G}(\lambda) = g_0^{-1} \cdot g_1$. In particular, the holonomy of any closed curve λ with respect to ω_G is trivial.

Now, let us come back to the B-principal fibre bundle P with a given Cartan connection $\omega \in \Omega^1(P, \mathfrak{g})$. We fix a point u in the fibre P_x over $x \in M$ and denote by $\pi(\mathcal{L}(P, u))$ the set of all loops in $\mathcal{L}(M, x)$ which admit a closed lift in $\mathcal{L}(P, u)$. One observes that $\mathrm{hol}^\omega(\delta_1) = \mathrm{hol}^\omega(\delta_2)$ for loops $\delta_1, \delta_2 \in \mathcal{L}(P, u)$ with the same projection $\pi \circ \delta_1 = \pi \circ \delta_2 \in \mathcal{L}(M, x)$. Thus we can define the *holonomy of a loop* $\gamma \in \pi(\mathcal{L}(P, u)) \subset \mathcal{L}(M, x)$ *with respect to* ω *and* u by

$$\mathrm{hol}^\omega_u(\gamma) := \mathrm{hol}^\omega(\tilde{\gamma}) \in G, \quad \text{where } \tilde{\gamma} \in \mathcal{L}(P, u) \text{ and } \pi \circ \tilde{\gamma} = \gamma. \tag{13}$$

The *holonomy group of the Cartan connection ω with respect to the base point u* is the group

$$\mathrm{Hol}_u(P,\omega) := \{\mathrm{hol}_u^\omega(\gamma) \mid \gamma \in \pi(\mathcal{L}(P,u)) \subset \mathcal{L}(M,x)\} \subset G.$$

$\mathrm{Hol}_u(P,\omega)$ is an immersed Lie subgroup of G with the *reduced holonomy group*

$$\mathrm{Hol}_u^0(P,\omega) := \{\mathrm{hol}_u^\omega(\gamma) \mid \gamma \in \mathcal{L}(M,x) \text{ null homotop}\}$$

as connected component of the unit element. Holonomy groups for different reference points u are conjugated in G. Therefore, we often omit the base point and understand the holonomy group as a class of conjugated subgroups in the group G in question.

Example 5. Consider the Maurer–Cartan form ω_G as Cartan connection on the homogeneous principal fibre bundle $(G,\pi,G/B;B)$. Then, by Example 4, the holonomy group is trivial: $\mathrm{Hol}_u(G,\omega_G) = \{e\}$.

Now, let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of the larger group G on a vector space V. Of course, by restriction, this gives a representation of B on V and an associated vector bundle

$$E := P \times_B V.$$

Bundles of this form (defined using a representation of the larger group G) are called *tractor bundles* and play a crucial role in Cartan geometry – since, contrary to the case of arbitrary associated vector bundles, on a tractor bundle there is a covariant derivative associated to the Cartan connection ω.

To see this, we extend the B-principal fibre bundle P to a G-principal fibre bundle $\bar{P} := P \times_B G$. Let $\mathrm{pr}_B\colon P \times G \mapsto \bar{P}$ be the natural projection. We use the Cartan connection $\omega \in \Omega^1(P,\mathfrak{g})$ and the Maurer–Cartan form of G to define a 1-form $\hat{\omega} \in \Omega^1(P\times G,\mathfrak{g})$ by

$$\hat{\omega}_{(u,g)} := \mathrm{Ad}(g^{-1}) \circ (\pi_P^*\omega)_{(u,g)} + (\pi_G^*\omega_G)_{(u,g)}, \tag{14}$$

where π_P and π_G are the projections from $P\times G$ onto P and G, respectively. $\hat{\omega}$ is invariant under the B-action on $P\times G$ and projects to a 1-form $\bar{\omega} \in \Omega^1(\bar{P},\mathfrak{g})$, $\mathrm{pr}_B^*\bar{\omega} = \hat{\omega}$, which is a usual connection form on the G-principal fibre bundle $\bar{P}$. For the canonical embedding $\iota\colon u \in P \mapsto [u,e] \in \bar{P}$ obviously holds $\iota^*\bar{\omega} = \omega$. Since $\rho\colon G \to \mathrm{GL}(V)$ is a G-representation,

$$E = P \times_B V \simeq \bar{P} \times_G V.$$

Hence, any Cartan connection $\omega \in \Omega^1(P,\mathfrak{g})$ defines a covariant derivative ∇^ω on E via its extension to the principal fibre bundle connection $\bar{\omega}$ on $\bar{P}$, $\nabla^\omega := \nabla^{\bar{\omega}}$. According to formula (8), ∇^ω is given by

$$(\nabla_X^\omega \phi)_{|U} = [s, X(v) + \rho_*(\omega(ds(X)))v], \tag{15}$$

where $\phi_{|U} = [s,v] \in \Gamma(E_{|U})$ for a local section $s\colon U \to P$ and a smooth function $v\colon U \to V$. By definition, ∇^ω is a metric covariant derivative with respect to any

metric $\langle\cdot,\cdot\rangle$ on the bundle E, which is induced by a G-invariant inner product $\langle\cdot,\cdot\rangle_V$ on V, i.e.,

$$X(\langle\varphi,\psi\rangle) = \langle\nabla^{\omega}_X\varphi,\psi\rangle + \langle\varphi,\nabla^{\omega}_X\psi\rangle.$$

Example 6. We consider the situation of Example 2. Let P be the frame bundle over M^n with Cartan connection $\omega = A + \theta$, where A is a connection form and θ the displacement form on P. We realize the affine group $A(n)$ as a subgroup of the linear group $\mathrm{GL}(n+1)$ by

$$(A,x) \in A(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n \mapsto \begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1)$$

and consider its matrix representation ρ on $\mathbb{R}^{n+1}$. The restriction of this representation to $\mathrm{GL}(n)$ decomposes into the sum $\mathbb{R}^n \oplus \mathbb{R}$, where the first factor is the matrix representation of $\mathrm{GL}(n)$ on $\mathbb{R}^n$ and the second the trivial representation. Hence, the tractor bundle $\mathcal{T}$ associated to P by ρ is

$$\mathcal{T} = P \times_{\mathrm{GL}(n)} \mathbb{R}^{n+1} \approx TM \oplus \underline{\mathbb{R}},$$

where $\underline{\mathbb{R}}$ denotes the trivial line bundle on M. For the tractor connection ∇^{ω} on $\mathcal{T}$ induced by ω we obtain from (8), (10) and (15)

$$\nabla^{\omega}_X \begin{pmatrix} Z \\ f \end{pmatrix} = \begin{pmatrix} \nabla^A_X Z + fX \\ X(f) \end{pmatrix},$$

where f is a function and Z a vector field on M and ∇^A the covariant derivative on the tangent bundle induced by A.

Now, let us discuss the relation between the holonomy groups of ω, $\bar{\omega}$ and ∇^{ω} which we defined above.

Proposition 1. (1) *The connected components of the holonomy groups coincide*

$$\mathrm{Hol}^0_u(P,\omega) = \mathrm{Hol}^0_{[u,e]}(\bar{P},\bar{\omega}) \tag{16}$$

(2) *If the structure group B of P is connected or if the base space M of P is simply connected, then*

$$\mathrm{Hol}_u(P,\omega) = \mathrm{Hol}_{[u,e]}(\bar{P},\bar{\omega}) \subset G, \tag{17}$$

$$\rho(\mathrm{Hol}_u(P,\omega)) = [u]^{-1} \circ \mathrm{Hol}_x(E,\nabla^{\omega}) \circ [u] \subset \mathrm{GL}(V). \tag{18}$$

Proof. Let $\gamma \in \mathcal{L}(M,x)$ be a loop in x which has a closed lift $\tilde{\gamma} \in \mathcal{L}(P,u)$. First we will show that the holonomy of such curves γ with respect to the Cartan connection $\omega \in \Omega^1(P,\mathfrak{g})$ and the point $u \in P$ coincide with the holonomy of γ with respect to the connection form $\bar{\omega} \in \Omega^1(\bar{P},\mathfrak{g})$ and the point $\bar{u} := [u,e] \in \bar{P}$.

Let $\gamma^*_{\bar{u}}\colon [\alpha,\beta] \to \bar{P} := P \times_B G$ be the $\bar{\omega}$-horizontal lift of γ in $\bar{P}$ with initial point $\bar{u}$. By definition of $\bar{P}$ we can represent the curve $\gamma^*_{\bar{u}}$ in the form $\gamma^*_{\bar{u}} = [\tilde{\gamma},g]$,

where $g\colon [\alpha,\beta] \to G$ and $g(\alpha) = e$. Then

$$\gamma^*_{\bar{u}}(\beta) = [\tilde{\gamma}(\beta), g(\beta)] = [u, g(\beta)] = [u,e]\cdot g(\beta) = \bar{u}\cdot g(\beta).$$

Hence, by definition, $\mathrm{hol}^{\bar{\omega}}_{\bar{u}}(\gamma) = g(\beta)^{-1}$. Since $\gamma^*_{\bar{u}}$ is $\bar{\omega}$-horizontal, we obtain from the definition of $\bar{\omega}$ and formula (14) that

$$\begin{aligned} 0 &= \bar{\omega}_{\gamma^*_{\bar{u}}(t)}(\dot{\gamma}^*_{\bar{u}}(t)) \\ &= \hat{\omega}_{(\tilde{\gamma}(t),g(t))}(\dot{\tilde{\gamma}}(t), \dot{g}(t)) \\ &= \mathrm{Ad}(g^{-1}(t))\omega(\dot{\tilde{\gamma}}(t)) + \omega_G(\dot{g}(t)). \end{aligned}$$

Therefore,

$$\omega(\dot{\tilde{\gamma}}(t)) = -\mathrm{Ad}(g(t))\,\omega_G(\dot{g}(t)) = -dR_{g^{-1}(t)}\dot{g}(t) = dL_{a(t)^{-1}}\dot{a}(t),$$

where $a(t) := g(t)^{-1}$. Thus, $a := g^{-1}\colon [\alpha,\beta] \to G$ is the development of ω along $\tilde{\gamma}$ and by definition of the holonomy of γ with respect to ω (cf. (11), (12) and (13)) follows that $\mathrm{hol}^{\omega}_{u}(\gamma) = g(\beta)^{-1}$. Thus, $\mathrm{hol}^{\omega}_{u}(\gamma) = \mathrm{hol}^{\bar{\omega}}_{\bar{u}}(\gamma)$ for all loops $\gamma \in \mathcal{L}(M,x)$ which admit a closed lift $\tilde{\gamma} \in \mathcal{L}(P,u)$.

It remains to show that $\mathcal{L}(M,x) = \pi(\mathcal{L}(P,u))$ if B is connected or if M is simply connected. From the homotopy lifting lemma for fibrations follows that

$$\pi(\mathcal{L}(P,u)) = \{\gamma \in \mathcal{L}(M,x) \mid [\gamma] \in \pi_\sharp(\pi_1(P,u))\}.$$

The exact homotopy sequence of the fibration $(P,\pi,M;B)$ shows that the induced map $\pi_\sharp$ on the fundamental groups is surjective if the structure group B is connected. Therefore, $\mathcal{L}(M,x) = \pi(\mathcal{L}(P,u))$ if the B is connected or if M is simply connected. The statements (16) and (17) follow then from the definition of the holonomy groups in question. Statement (18) is a consequence of (9). □

This coincidence between the holonomy groups allows us to translate the generally known properties of holonomy groups of connection forms on principal fibre bundles (resp. covariant derivatives) into properties of holonomy groups of Cartan connections. In particular, the holonomy algebra is generated by the curvature.

In general, the *curvature of a* 1*-form* $\sigma \in \Omega^1(N,\mathfrak{g})$ on a manifold N with values in a Lie algebra $\mathfrak{g}$ is the 2-form $\Omega^\sigma \in \Omega^2(N,\mathfrak{g})$ defined by

$$\Omega^\sigma := d\sigma + \frac{1}{2}[\sigma,\sigma].$$

Hence, the *curvature* Ω^ω *of a Cartan connection* $\omega \in \Omega^1(P,\mathfrak{g})$ is a 2-form on P with values in the Lie algebra $\mathfrak{g}$. The invariance properties of a Cartan connection yield that its curvature is horizontal and of type Ad, i.e.,

$$\begin{aligned} TvP \lrcorner \Omega^\omega &= 0, \\ R_b^*\Omega^\omega &= \mathrm{Ad}(b^{-1}) \circ \Omega^\omega \quad \text{for all } b \in B, \end{aligned}$$

where $TvP \subset TP$ is the vertical tangent bundle of P. The *torsion of a Cartan connection* is the 2-form Tor^ω, which arises from the curvature form by projection

onto the factor space $\mathfrak{g}/\mathfrak{b}$:

$$\mathrm{Tor}^{\omega} := \mathrm{proj}_{\mathfrak{g}/\mathfrak{b}} \circ \Omega^{\omega} \in \Omega^2(P, \mathfrak{g}/\mathfrak{b}).$$

We call a Cartan connection ω *torsion free*, if its torsion is zero, that is, if the curvature Ω^{ω} takes its values in the subalgebra $\mathfrak{b} \subset \mathfrak{g}$. We call ω *flat*, if its curvature is zero. For example, the Maurer–Cartan form ω_G of a Lie group G considered as Cartan connection on the homogeneous bundle $(G, \pi, G/B; B)$ over the homogeneous space G/B is flat. The Uniformization Theorem in Cartan geometry says that up to discrete groups this is the only situation, where flat connections appear. More precisely:

Proposition 2 (cf. [S97], Chapter 5.5). *Let $(P, \pi, M; B)$ be a principal fibre bundle with connected structure group B over a connected manifold M which admits a complete flat Cartan connection $\omega \in \Omega^1(P, \mathfrak{g})$. Then, there exists a connected Lie group G with Lie algebra $\mathfrak{g}$, containing B as closed subgroup, such that M is diffeomorphic to the locally homogeneous space $\Gamma \setminus G/B$, where $\Gamma \subset G$ is a discrete subgroup. Moreover, the B-bundle $(\Gamma \setminus G, \mathrm{pr}, \Gamma \setminus G/B; B)$ with the Cartan connection $\omega_{\Gamma\setminus G}$ given by the Maurer–Cartan form of G, is isomorphic to (P, ω).*

The next proposition shows, how the Lie algebra of the holonomy group of a Cartan connection is determined by its curvature.

Proposition 3. *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a Cartan connection on the B-principal fibre bundle P over M and $u \in P_x$. Then the Lie algebra of the holonomy group $\mathrm{Hol}_u(P, \omega)$ is given by*

$$\mathfrak{hol}_u(P, \omega) = \mathrm{span}\big\{\mathrm{Ad}(\mathrm{hol}^{\omega}(\lambda)) \circ \Omega^{\omega}_p(X, Y) \mid p \in P,\ X, Y \in T_p P,\ \lambda \text{ a path in } P \text{ from } u \text{ to } p\big\} \subset \mathfrak{g}.$$

Furthermore, if $\rho\colon G \to \mathrm{GL}(V)$ is a G-representation, $E := P \times_B V$ the associated tractor bundle, ∇^{ω} the covariant derivative on E induced by the Cartan connection ω and $R^{\nabla^{\omega}}(X, Y) := [\nabla^{\omega}_X, \nabla^{\omega}_Y] - \nabla^{\omega}_{[X,Y]}$ its curvature endomorphism. Then

$$\mathfrak{hol}_x(E, \nabla^{\omega}) = \mathrm{span}\big\{\mathcal{P}^{\nabla^{\omega}}_{\gamma^{-1}} \circ R^{\nabla^{\omega}}_y(X, Y) \circ \mathcal{P}^{\nabla^{\omega}}_{\gamma} \mid X, Y \in T_y M,\ \gamma \text{ a path in } M \text{ from } x \text{ to } y\big\}.$$

Proof. Let $\bar{\omega}$ be the connection form on the extended G-bundle $\bar{P}$, induced by the Cartan connection ω (cf. formula (14)). Furthermore, let $\bar{u} := [u, e] \in \bar{P}$. According to (16) the Lie algebras $\mathfrak{hol}_u(P, \omega)$ and $\mathfrak{hol}_{\bar{u}}(\bar{P}, \bar{\omega})$ coincide. By the Ambrose–Singer Theorem (cf. [KN63], Chapter II, Theorem 8.1) the holonomy algebra of the connection form $\bar{\omega}$ is given by

$$\mathrm{hol}_{\bar{u}}(\bar{P}, \bar{\omega}) = \mathrm{span}\big\{\Omega^{\bar{\omega}}_{[p,g]}(\bar{X}, \bar{Y}) \mid \text{there exists a } \bar{\omega}\text{-horizontal path in } \bar{P} \text{ from } \bar{u} \text{ to } [p, g]\bar{X}, \bar{Y} \in T_{[p,g]}\bar{P}\big\}.$$

Let $\delta(t) = [\lambda(t), g(t)]$ be a path in $\bar{P}$ starting in $\bar{u} = [u, e]$. As in the proof of Proposition 1 follows that $\delta(t) = [\lambda(t), g(t)]$ is $\bar{\omega}$-horizontal if and only if λ is an arbitrary path in P starting from u and $a(t) := g(t)^{-1}$ is the development of ω along λ. Thus

$$\mathfrak{hol}_{\bar{u}}(\bar{P}, \bar{\omega}) = \operatorname{span}\big\{\Omega^{\bar{\omega}}_{[p,a^{-1}]}(\bar{X}, \bar{Y}) \mid p \in P,\ a = \operatorname{hol}^{\omega}(\lambda) \text{ for a path } \lambda \text{ in } P \text{ from } u \text{ to } p,\ \bar{X}, \bar{Y} \in T_{[p,a^{-1}]}\bar{P}\big\}.$$

Using formula (14), we obtain for the curvatures of the Cartan connection ω on P and the induced connection form $\bar{\omega}$ in $\bar{P}$ that

$$\Omega^{\bar{\omega}}_{[p,a^{-1}]}(\bar{X}, \bar{Y}) = \operatorname{Ad}(a)\, \Omega^{\omega}_{p}(X, Y)$$

for appropriate $X, Y \in T_p P$. Hence,

$$\begin{aligned}\mathfrak{hol}_u(P, \omega) &= \mathfrak{hol}_{\bar{u}}(\bar{P}, \bar{\omega}) \\ &= \operatorname{span}\big\{\operatorname{Ad}(\operatorname{hol}^{\omega}(\lambda)) \circ \Omega^{\omega}_{p}(X, Y) \mid p \in P,\ X, Y \in T_p P, \\ &\qquad\qquad \lambda \text{ a path in } P \text{ from } u \text{ to } p\big\}.\end{aligned}$$

The second formula in the proposition comes from the relation between the curvature form $\Omega^{\bar{\omega}}$ on the principal bundle $\bar{P}$ and the curvature endomorphism $R^{\nabla^{\omega}}$ on E:

$$\rho_*(\Omega^{\bar{\omega}}_{q}(\bar{X}, \bar{Y})) = [q]^{-1} \circ R^{\nabla^{\omega}}_{y}(X, Y) \circ [q],$$

where $q \in \bar{P}_y$ and $\pi(\bar{X}) = X, \pi(\bar{Y}) = Y \in T_y M$. For the holonomy algebra we have to take into account all points q that arise by parallel displacement from u along curves γ connecting x with y in M, i.e., all points q with $[q] = \mathcal{P}^{\nabla^{\omega}}_{\gamma} \circ [u]$. This gives the second statement. □

In special situations there is a more appropriate formula for the holonomy algebra of a Cartan connection. Let us denote by ∇^{End} the covariant derivative in the endomorphism bundle $\operatorname{End}(E)$ induced by ∇^{ω}:

$$\nabla^{\mathrm{End}}_X F := [\nabla^{\omega}_X, F] = \nabla^{\omega}_X \circ F - F \circ \nabla^{\omega}_X \quad \text{for } F \in \Gamma(\operatorname{End}(E)).$$

Furthermore, we fix an arbitrary covariant derivative D on TM. For a 2-form σ on M with values in $\operatorname{End}(E)$ we define by induction

$$(\tilde{\nabla}_{V_1}\sigma)(X, Y) := \nabla^{\mathrm{End}}_{V_1}(\sigma(X, Y)) - \sigma(D_{V_1}X, Y) - \sigma(X, D_{V_1}Y)$$

and

$$\begin{aligned}(\tilde{\nabla}^{k}_{V_1 \dots V_k}\sigma)(X, Y) := {} & \nabla^{\mathrm{End}}_{V_k}((\tilde{\nabla}^{k-1}_{V_1 \dots V_{k-1}}\sigma)(X, Y)) \\ & - (\tilde{\nabla}^{k-1}_{V_1 \dots V_{k-1}}\sigma)(D_{V_k}X, Y) - (\tilde{\nabla}^{k-1}_{V_1 \dots V_{k-1}}\sigma)(X, D_{V_k}Y) \\ & - \sum_{i=1}^{k-1} (\tilde{\nabla}^{k-1}_{V_1 \dots D_{V_k} V_i \dots V_{k-1}}\sigma)(X, Y).\end{aligned}$$

Proposition 4. *Let ∇^ω be the covariant derivative on the tractor bundle E defined by the Cartan connection ω and let $x \in M$. Consider the following subspace in* $\mathrm{GL}(E_x)$*:*

$$\mathfrak{hol}'_x(E,\nabla^\omega) := \operatorname{span}\left\{(\widetilde{\nabla}^k_{V_1,\dots,V_k} R^{\nabla^\omega})_x(X,Y) \mid X,Y,V_1,\dots,V_k \in T_xM,\ 0 \le k \le \infty\right\}.$$

If the dimension of $\mathfrak{hol}'_x(E,\nabla^E)$ is constant in x, then this vector space coincides with the holonomy algebra $\mathfrak{hol}_x(E,\nabla^\omega)$. In particular this is the case if all data are analytic.

Proof. This is a translation of the corresponding statements for the infinitesimal holonomy group of affine connections in [KN63], vol. 1. □

Finally, let us state the basic relation between the holonomy group of a Cartan connection ω and ∇^ω-parallel tractors.

Proposition 5. *Let ω be a Cartan connection on the B-principal fibre bundle P over M, $E = P \times_B V$ a tractor bundle associated to P, ∇^ω the induced covariant derivative on E and $u \in P_x$ a fixed point. Let B be connected or M be simply connected. Then there is a* 1-1 *correspondence between the space of parallel sections*

$$\{\varphi \in \Gamma(E) \mid \nabla^\omega\varphi = 0\}$$

and the space of holonomy invariant vectors

$$\{v \in V \mid \rho(\mathrm{Hol}_u(P,\omega))\, v = v\} \overset{[u]}{\simeq} \{e \in E_x \mid \mathrm{Hol}_x(E,\nabla^\omega)\, e = e\}.$$

Proof. For $v \in V$ we define a section $\varphi_v \in \Gamma(E)$ by the parallel transport of the element $e = [u,v] \in E_x$ with respect to ∇^ω:

$$\varphi_v(y) := \mathcal{P}^{\nabla^\omega}_\gamma(e) \in E_y,$$

where γ is a curve in M from x to y. The section φ_v does not depend on the choice of γ, is parallel and the map $v \mapsto \varphi_v$ is bijective between the space of holonomy invariant vectors and the space of parallel sections. □

This proposition shows on one hand, how useful the knowledge of the holonomy group is if one wants to determine the spaces of parallel sections. The task to solve the differential equation $\nabla^\omega\varphi = 0$ reduces to a problem of linear algebra. On the other hand, if one has a special kind of geometry (for example conformal geometry) which is characterized by the Cartan connection ω, then the holonomy group – by its fixed vectors on a representation space – defines distinguished geometric invariants (∇^ω-parallel object) of the manifold.

3 Holonomy groups of conformal structures

In this section we consider Cartan connections which are associated to conformal manifolds. A conformal manifold of signature (p,q) is a pair (M,c), where M is a smooth manifold of dimension $n = p+q$ and c is an equivalence class of metrics of signature (p,q), where two metrics g and $\tilde{g}$ are called equivalent if $\tilde{g} = fg$ for a positive smooth function f on M. We always assume that $n = p+q \geq 3$.

For a metric g there is a distinguished covariant derivative on the tangent bundle and a corresponding distinguished connection form on the frame bundle of M, the Levi-Civita connection. Contrary to that, there cannot be a distinguished covariant derivative on the tangent bundle or a connection form on the frame bundle of M depending only on the conformal structure c itself, since one needs derivatives up to second order to determine conformal diffeomorphisms uniquely.

In order to establish a conformally invariant differential calculus for conformal manifolds one uses techniques of Cartan geometry. In this approach the bundle of frames on M is replaced by a larger bundle $\mathcal{P}^1$, which is obtained from the conformal frame bundle by a process called 1. *prolongation*. This first prolongation comes with a distinguished Cartan connection, the *normal conformal Cartan connection* ω^{nor}. The normal conformal Cartan connection plays in conformal geometry the same role the Levi-Civita connection is playing in (pseudo)Riemannian geometry. It is used to define conformally invariant differentiations on all tractor bundles defined on conformal manifolds.

We define *the holonomy group of the conformal manifold* (M,c) to be the holonomy group of this normal conformal Cartan connection

$$\mathrm{Hol}(M,c) := \mathrm{Hol}(\mathcal{P}^1, \omega^{\mathrm{nor}}).$$

For the convenience of the reader and to fix the notation we recall shortly some basic facts on conformal Cartan geometry, in particular the definition of the normal conformal Cartan connection (for details cf. for example [K72], [CSSI], [CSSII], [F05]).

3.1 Cartan connections in conformal geometry

First, we want to describe the groups B and G (compare Section 2) which are used in conformal geometry. For this purpose let us consider the isotropic cone

$$C^{p,q} := \{x \in \mathbb{R}^{p+1,q+1} \setminus \{0\} \mid \langle x,x\rangle_{p+1,q+1} = 0\}$$

in the pseudo-Euclidean space $(\mathbb{R}^{p+1,q+1}, \langle\cdot,\cdot\rangle_{p+1,q+1})$. The projectivization $Q^{p,q} := \mathbb{P}C^{p,q}$ can be equipped with a conformal structure c of signature (p,q):

$$c := [\mu^* \langle\cdot,\cdot\rangle_{p+1,q+1}],$$

where $\mu\colon Q^{p,q} \to C_+^{p,q}$ is an arbitrary section from the projectivization into a fixed component $C_+^{p,q}$ of the cone. The conformal manifold $(Q^{p,q}, c)$ is called *Möbius sphere of signature* (p, q).

For Riemannian conformal structures (signature $(0, n)$) the Möbius sphere is just the sphere S^n equipped with the conformal class of the round metric. It is the conformal boundary of the hyperbolic space H^{n+1} as well as the conformal compactification of the Euclidean space. For Lorentzian conformal structures (signature $(1, n-1)$) the Möbius sphere appears in physics literature under the name 'Einstein universe Ein_n'. It is the conformal boundary of the Anti de Sitter space AdS^{n+1} as well as the conformal compactification of the Minkowski space.

The group G we are working with in conformal geometry is $G := \mathrm{SO}(p+1, q+1)$. It acts transitively and conformally on the Möbius sphere. Moreover, G (if n is odd) resp. $G/\pm I$ (if n is even) is isomorphic to the group of orientation preserving conformal diffeomorphisms of the Möbius sphere $(Q^{p,q}, c)$. For the group $B \subset G$ we take the stabilizer of a fixed isotropic line $p_\infty \in Q^{p,q}$. Hence, the Möbius sphere can be identified with the n-dimensional homogeneous space G/B. This explains the choice of these groups. Let $\mathfrak{b} \subset \mathfrak{g} := \mathfrak{so}(p+1, q+1)$ be the Lie algebra of B and let us denote by $\mathfrak{b}_1 \subset \mathfrak{b}$ the n-dimensional abelian ideal in $\mathfrak{b}$ obtained as the orthogonal complement of $\mathfrak{b} \subset \mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Then $B_1 := \exp \mathfrak{b}_1 \subset B$ is a normal abelian subgroup in B and the factor group B/B_1 is isomorphic to the conformal group

$$\begin{aligned}\mathrm{CO}(p, q) &:= \{A \in \mathrm{GL}(n) \mid \text{there exists a } \theta > 0 \text{ such that} \\ &\qquad \langle Ax, Ay\rangle_{p,q} = \theta\langle x, y\rangle_{p,q} \text{ for all } x, y \in \mathbb{R}^n\} \\ &\simeq \mathbb{R}^+ \times \mathrm{O}(p, q) \simeq \mathbb{R}^* \cdot \mathrm{SO}(p, q).\end{aligned}$$

For short, we will denote $B_0 := \mathrm{CO}(p, q)$. The projection from B to the linear conformal group B_0 corresponds geometrically to the mapping $f \in \mathrm{Conf}_+(Q^{p,q}) \mapsto df_{p_\infty} \in \mathrm{Conf}(T_{p_\infty} Q^{p,q})$.

The sequence $\mathfrak{b}_1 \subset \mathfrak{b} \subset \mathfrak{g}$ is an $\mathrm{Ad}(B)$-invariant filtration of $\mathfrak{g}$. The Killing form of $\mathfrak{g}$ gives an $\mathrm{Ad}(B)$-equivariant identification between $\mathfrak{b}_1$ and the dual Lie algebra $(\mathfrak{g}/\mathfrak{b})^*$.

In the following we will consider $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$ as $|1|$-graded Lie algebra, i.e., we decompose $\mathfrak{g}$ into a sum of subspaces

$$\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \quad \text{with } [\mathfrak{b}_i, \mathfrak{b}_j]_{\mathfrak{g}} \subset \mathfrak{b}_{i+j},$$

where furthermore, $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$. Let us describe such a splitting in term of matrices.

We fix a basis $(f_0, e_1, \ldots, e_n, f_{n+1})$ in $\mathbb{R}^{p+1,q+1} = \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}$ such that the Gram matrix of $\langle \cdot, \cdot \rangle_{p+1,q+1}$ has the form $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & \mathbb{J} & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ where $\mathbb{J} = \left(\begin{smallmatrix} -\mathbb{I}_p & 0 \\ 0 & \mathbb{I}_q \end{smallmatrix}\right)$ and $\mathbb{I}_k$ is the identity matrix with k rows. For a row vector $z \in (\mathbb{R}^n)^*$ let $z^\sharp := \mathbb{J} z^\top$. For a column vector $x \in \mathbb{R}^n$ let $x^\flat := x^\top \mathbb{J}$. With respect to the basis $(f_0, e_1, \ldots, e_n, f_{n+1})$

the Lie algebra $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$ is given by the matrices

$$\mathfrak{so}(p+1,q+1) = \left\{ \mathbb{M}(x,(A,a),z) := \begin{pmatrix} -a & z & 0 \\ x & A & -z^\sharp \\ 0 & -x^\flat & a \end{pmatrix} \;\middle|\; \begin{array}{l} x \in \mathbb{R}^n,\ z \in (\mathbb{R}^n)^* \\ a \in \mathbb{R},\ A \in \mathfrak{so}(p,q) \end{array} \right\}.$$

We set

$$\begin{aligned} \mathfrak{b}_{-1} &:= \{\mathbb{M}(x,(0,0),0) \mid x \in \mathbb{R}^n\} \simeq \mathbb{R}^n, \\ \mathfrak{b}_1 &:= \{\mathbb{M}(0,(0,0),z) \mid z \in (\mathbb{R}^n)^*\} \simeq \mathbb{R}^{n*}, \\ \mathfrak{b}_0 &:= \{\mathbb{M}(0,(A,a),0) \mid A \in \mathfrak{so}(p,q),\ a \in \mathbb{R}\} \simeq \mathfrak{co}(p,q) = \mathfrak{so}(p,q) \oplus \mathbb{R} \end{aligned} \tag{19}$$

For elements $x \in \mathfrak{b}_{-1}$, $(A,a) \in \mathfrak{b}_0$, $z \in \mathfrak{b}_1$ we have

$$\begin{aligned} [(A,a),x] &= ax + Ax \in \mathfrak{b}_{-1}, \\ [(A,a),z] &= -az - zA \in \mathfrak{b}_1, \\ [x,z] &= (xz - z^\sharp x^\flat, zx) \in \mathfrak{b}_0. \end{aligned} \tag{20}$$

Clearly, $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$. We identify $\mathfrak{b}_1$ and $\mathfrak{b}_{-1}^*$ using the $\mathrm{Ad}(B_0)$-equivariant isomorphism ϕ given by the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$

$$\phi(z)x := \frac{1}{2n} B_\mathfrak{g}(x,z), \quad x \in \mathfrak{b}_{-1},\ z \in \mathfrak{b}_1.$$

This corresponds to the standard dual pairing between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. The adjoint representation of B_0 on $\mathfrak{b}_{-1}$ yields an isomorphism between B_0 and the group of linear conformal isomorphisms of $(\mathfrak{b}_{-1} = \mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q})$.

Now, let (M^n, c) be a conformal manifold. We call a frame $(x; s_1, \ldots, s_n)$ on M *conformal frame*, if the vectors $(s_1, \ldots, s_n)$ form a pseudo-orthonormal basis in T_xM with respect to a metric g in the conformal class c. We denote by $\pi^0 \colon \mathcal{P}^0 \to M$ the bundle of all conformal frames of (M, c). This bundle is a reduction of the frame bundle of M to the linear conformal group $B_0 = \mathrm{CO}(p,q) \subset \mathrm{GL}(n)$.

The basic B-principal fibre bundle over M, which is used in conformal geometry, is obtained from the conformal frame bundle $\mathcal{P}^0$ by 1. prolongation. There are several equivalent models for the first prolongation of $\mathcal{P}^0$. We will describe the first prolongation using horizontal subspaces in the tangent space of $\mathcal{P}^0$.

For this purpose, let $u \in \mathcal{P}^0$ be a conformal frame in $x \in M$. The vertical tangent space of $\mathcal{P}^0$ in u is the tangent space of the fibre $\mathcal{P}^0_x$ in u:

$$Tv_u\mathcal{P}^0 := T_u(\mathcal{P}^0_x).$$

By a *horizontal subspace* $H \subset T_u\mathcal{P}^0$ we mean a subspace which is complementary to the vertical tangent space $Tv_u\mathcal{P}^0 \subset T_u\mathcal{P}^0$. To any horizontal subspace $H \subset T_u\mathcal{P}^0$ we can associate a 2-form $t(H) \in \Lambda^2(\mathfrak{b}_{-1}^*) \otimes \mathfrak{b}_{-1}$ called the *torsion of* H, which we want to define now: First note that the identification between the matrix representation of B_0 on $\mathbb{R}^n$ and the adjoint representation of B_0 on $\mathfrak{b}_{-1}$ allows us to represent the tangent bundle of M in the form

$$TM = \mathcal{P}^0 \times_{(B_0,\mathrm{Ad})} \mathfrak{b}_{-1}.$$

In this representation the displacement form θ (cf. Example 2) is a 1-form on $\mathcal{P}^0$ with values in $\mathfrak{b}_{-1}$:

$$\theta_u(X) := [u]^{-1} d\pi_u^0(X) \in \mathfrak{b}_{-1} \quad \text{for } u \in \mathcal{P}^0, X \in T_u\mathcal{P}^0.$$

In particular, $\theta_{|H}\colon H \to \mathfrak{b}_{-1}$ is an isomorphism for any horizontal space $H \subset T_u\mathcal{P}^0$. Thus we can define the 2-form $t(H) \in \Lambda^2(\mathfrak{b}_{-1}^*) \otimes \mathfrak{b}_{-1}$ as

$$t(H)(v,w) := d\theta_u\big((\theta_{|H})^{-1}(v), (\theta_{|H})^{-1}(w)\big) \in \mathfrak{b}_{-1}, \quad v, w, \in \mathfrak{b}_{-1}.$$

A horizontal space H is called *torsion-free*, if $t(H) = 0$. With this preparations we can define the *first prolongation* $\mathcal{P}^1$ by

$$\mathcal{P}^1 := \{H \subset T_u\mathcal{P}^0 \mid u \in \mathcal{P}^0 \text{ and } H \text{ horizontal and torsion-free}\}.$$

There are natural projections

$$\pi\colon \mathcal{P}^1 \xrightarrow{\pi^1} \mathcal{P}^0 \xrightarrow{\pi^0} M, \quad H \subset T_u\mathcal{P}^0 \longmapsto u \in \mathcal{P}^0_x \longmapsto x;$$

$\pi\colon \mathcal{P}^1 \to M$ is a principal fibre bundle with structure group B, whereas $\pi^1\colon \mathcal{P}^1 \to \mathcal{P}^0$ is a principal fibre bundle with structure group B_1. Let us explain the action of B on $\mathcal{P}^1$. Any element $b \in B = B_0 \ltimes B_1$ can be represented by $b = b_0 \cdot \exp(Z)$, where $b_0 \in B_0$ and $Z \in \mathfrak{b}_1$. An element $b_0 \in B_0$ 'transports' the horizontal space $H \subset T_u\mathcal{P}^0$ into the point ub_0 using the right action of B_0 on $\mathcal{P}^0$:

$$H \cdot b_0 := dR_{b_0}(H) \subset T_{ub_0}\mathcal{P}^0.$$

An element $\exp Z \in B_1$ acts on $\mathcal{P}^1$ by 'rotating' H inside $T_u\mathcal{P}^0$:

$$H \cdot \exp Z := \{X + [Z, \widetilde{\theta}(X)]_{\mathfrak{g}}(u) \mid X \in H\}.$$

The bundle $(\mathcal{P}^1, \pi, M; B)$ is called the *first prolongation of the conformal frame bundle* $\mathcal{P}^0$. It is the basic bundle that is used in conformal differential geometry. Here the Cartan connections live, in particular the distinguished normal Cartan connection of conformal geometry.

3.2 The normal conformal Cartan connection in invariant form

In order to distinguish a special Cartan connection, we first describe a certain affine subspace of the space of all Cartan connections of type G/B on $\mathcal{P}^1$, the space of *admissible* Cartan connections.

Let $\omega \in \Omega^1(\mathcal{P}^1, \mathfrak{g})$ be a Cartan connection on $\mathcal{P}^1$. With respect to the grading $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ the 1-form ω splits into the components

$$\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1.$$

The invariance properties of a Cartan connection show that the 1-forms ω_{-1} and ω_0 are horizontal with respect to the B_1-bundle $\pi^1\colon \mathcal{P}^1 \to \mathcal{P}^0$. Moreover, the right

translation of ω_{-1} and ω_0 with respect to $b_1 = \exp Z \in B_1$ is given by

$$R^*_{b_1}\omega_{-1} = 0,$$
$$R^*_{b_1}\omega_0 = \omega_0 + [Z, \omega_{-1}]_{\mathfrak{g}}\ .$$

We call the Cartan connection ω *admissible* if its $\mathfrak{b}_{-1}$-part and its $\mathfrak{b}_0$-part are given by the following conditions:

(1) $\omega_{-1} = (\pi^1)^*\theta$

(2) $d\pi^1_H(\xi) - \widetilde{(\omega_0)_H(\xi)}(u) \in H \subset T_u\mathcal{P}^0 = H \oplus Tv_u\mathcal{P}^0$ for all $\xi \in T_H\mathcal{P}^1$.

Clearly, two admissible Cartan connections differ only by the $\mathfrak{b}_1$-component. Let us note that any admissible Cartan connection ω is torsion-free, i.e., the curvature Ω^ω takes its values in the subalgebra $\mathfrak{b} \subset \mathfrak{g}$. To see this, remember that the 2-form Ω^ω is horizontal with respect to the B-bundle $\pi\colon \mathcal{P}^1 \to M$ and check for the $\mathfrak{b}_{-1}$-part of the curvature the formula

$$(\Omega^\omega_{-1})_H(\xi, \eta) = t(H)(v, w), \quad \text{where } \omega_H(\xi) = v,\ \omega_H(\eta) = w \in \mathfrak{b}_{-1}.$$

The *normal* conformal Cartan connection is an admissible Cartan connection which is determined by an additional condition on its curvature.

The curvature of a Cartan connection ω can obviously be identified with the curvature function $\kappa^\omega \in C^\infty(\mathcal{P}^1, \Lambda^2(\mathfrak{b}^*_{-1}) \otimes \mathfrak{g})$,

$$\kappa^\omega(H)(v, w) := \Omega^\omega_H(\omega_H^{-1}(v), \omega_H^{-1}(w)), \quad v, w \in \mathfrak{b}_{-1}.$$

Let

$$\partial^*\colon \Lambda^2(\mathfrak{b}^*_{-1}) \otimes \mathfrak{g} \longrightarrow \mathfrak{b}^*_{-1} \otimes \mathfrak{g}$$

be defined by

$$\partial^*\eta(X) := \sum_{i=1}^{n} [v_i^*, \eta(v_i, X)]_{\mathfrak{g}}, \quad X \in \mathfrak{b}_{-1},$$

where $(v_1, \dots, v_n)$ is a basis in $\mathfrak{b}_{-1}$ and $(v_1^*, \dots, v_n^*)$ is the dual basis in $\mathfrak{b}_1 \simeq \mathfrak{b}^*_{-1}$.

The *normal conformal Cartan connection* ω^{nor} on $\mathcal{P}^1$ is characterized in the following way:

Proposition 6. *There is a unique admissible Cartan connection* $\omega^{\mathrm{nor}} \in \Omega^1(\mathcal{P}^1, \mathfrak{g})$ *such that*

$$\partial^*\kappa^{\omega^{\mathrm{nor}}} = 0.$$

Proof. Admissible Cartan connections ω satisfy $\kappa^\omega_{-1} = 0$. Hence, for admissible Cartan connections the condition $\partial^*\kappa^\omega = 0$ is equivalent to $\partial^*\kappa^\omega_0 = 0$, where κ^ω_i denotes the projection of κ^ω onto $\mathfrak{b}_i$. Then, using the Lie algebra structure of $\mathfrak{g}$, one shows that the condition $\partial^*\kappa^\omega_0 = 0$ determines ω_1 uniquely. For details cf. [K72] Theorem IV.4.2, [CSSII], or [F05], Chapter 6. □

We defined the normal conformal Cartan connection starting with a conformal manifold (M, c). In Cartan geometry often the opposite viewpoint is used. One works with an *arbitrary* B-principal fibre bundle $(P, \pi, M; B)$ over a manifold M and a Cartan connection $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying $\partial^* \kappa^\omega = 0$, where $G = \mathrm{SO}(p+1, q+1)$ and $B = \mathrm{SO}(p+1, q+1)_{p_\infty}$ as above. In this case, $P^0 := P/B_1$ is a B_0-principal fibre bundle over M. The $\mathfrak{b}_{-1}$-part ω_{-1} of the Cartan connection projects down to a 1-form $\theta \in \Omega^1(P^0, \mathfrak{b}_{-1})$ of type Ad with the vertical tangent bundle TvP^0 as kernel. Such a pair (P^0, θ) defines a reduction of the frame bundle of M to the linear conformal group $\mathrm{CO}(p, q) = B_0$ and therefore determines a conformal structure c on M. Let $\mathcal{P}^1$ be the 1. prolongation of the conformal frame bundle of (M, c) for this conformal structure c. Then there exists a principal fibre bundle isomorphism $\phi\colon P \to \mathcal{P}^1$ such that $\phi^* \omega^{\mathrm{nor}} = \omega$.

3.3 The normal conformal Cartan connection in its metric form

For many applications it is useful to describe the invariantly defined normal conformal Cartan connection ω^{nor} and the induced covariant derivative on tractor bundles in terms of the metrics g in the conformal class c.

In order to state these formulas we need various curvature tensors of the pseudo-Riemannian manifold (M, g). In the following we will denote by R^g the *curvature endomorphism* given by the Levi-Civita connection ∇^g of g,

$$R^g(X, Y) := \nabla^g_X \nabla^g_Y - \nabla^g_Y \nabla^g_X - \nabla^g_{[X,Y]},$$

by Ric^g the *Ricci tensor* and by scal^g the *scalar curvature*,

$$\begin{aligned} \mathrm{Ric}^g(X, Y) &:= \mathrm{trace}(Z \mapsto R^g(Z, X)Y), \\ \mathrm{scal}^g &:= \mathrm{trace}_g \mathrm{Ric}^g. \end{aligned}$$

Further, K^g denotes the *Schouten tensor*:

$$K^g := \frac{1}{n-2}\left(\frac{1}{2(n-1)}\mathrm{scal}^g \cdot g - \mathrm{Ric}^g\right).$$

We often consider the Schouten tensor as endomorphism $K^g\colon TM \to T^*M$, using the identification $K^g(X)(Y) := K^g(X, Y)$. The skew-symmetric derivative of the endomorphism K^g gives the *Cotton–York tensor* C^g:

$$C^g(X, Y) := \nabla^g_X(K^g)(Y) - \nabla^g_Y(K^g)(X).$$

Finally, W^g is the *Weyl tensor* considered as 2-form with values in the g-skewsymmetric endomorphism on TM:

$$W^g(X, Y) := R^g(X, Y) - X^\flat \otimes K^g(Y)^\sharp - K^g(X) \otimes Y + K^g(Y) \otimes X + Y^\flat \otimes K^g(X)^\sharp,$$

where $X^\flat$ for a vector field X denotes the dual 1-form $X^\flat(Z) := g(X, Z)$ and $\mu^\sharp$ for a 1-form μ denotes the dual vector field $\mu(Z) = g(\mu^\sharp, Z)$.

Before we describe the normal conformal Cartan connection in its metric form, we show how torsion-free connection forms on the conformal frame bundle $\mathcal{P}^0$ over (M, c) induce admissible Cartan connections on the first prolongation $\mathcal{P}^1$. Let $A \in \Omega^1(\mathcal{P}^0, \mathfrak{b}_0)$ be a connection form on the conformal frame bundle $\mathcal{P}^0$ and $\theta \in \Omega^1(\mathcal{P}^0, \mathfrak{b}_{-1})$ the displacement form as above. The torsion of A is the 2-form $T^A \in \Omega^2(\mathcal{P}^0, \mathfrak{b}_{-1})$ defined by

$$T^A := d\theta + [A, \theta].$$

The connection form A defines a right invariant distribution of horizontal spaces $H_u := \operatorname{Ker} A_u \subset T_u\mathcal{P}^0$ with torsion

$$t(H_u)(\theta(X), \theta(Y)) = T^A_u(X, Y), \quad X, Y \in T_u\mathcal{P}^0.$$

Therefore, any torsion-free connection form A on $\mathcal{P}^0$ defines a B_0-equivariant smooth section

$$\sigma^A \colon \mathcal{P}^0 \longrightarrow \mathcal{P}^1, \quad u \longmapsto \operatorname{Ker} A_u,$$

in the B_1-bundle $\pi^1 \colon \mathcal{P}^1 \to \mathcal{P}^0$ (and vice versa).

Proposition 7. *For any torsion-free connection form $A \in \Omega^1(\mathcal{P}^0, \mathfrak{b}_0)$ there exists a unique admissible Cartan connection $\omega^A \in \Omega^1(\mathcal{P}^1, \mathfrak{g})$ such that*

$$(\sigma^A)^*\omega^A = \theta + A. \tag{21}$$

Proof. Let $\sigma^A \colon \mathcal{P}^0 \to \mathcal{P}^1$ be the B_0-equivariant section defined by A. We consider the map $\tau \colon \mathcal{P}^1 \to \mathfrak{b}_1$ given by

$$H = \sigma(\pi^1(H)) \cdot \exp(\tau(H)) \quad \text{for all } H \in \mathcal{P}^1$$

and define $\omega^A \in \Omega^1(\mathcal{P}^1, \mathfrak{g})$ by

$$\begin{aligned} \omega^A_{\sigma(u)} &:= ((\pi^1)^*\theta)_{\sigma(u)} + ((\pi^1)^*A)_{\sigma(u)} + d\tau_{\sigma(u)}, && u \in \mathcal{P}^0, \\ \omega^A_H &:= \operatorname{Ad}(b_1^{-1}) \circ \omega^A_{\sigma(u)} \circ dR_{b_1^{-1}}, && H = \sigma(u) \cdot b_1 \in \mathcal{P}^1. \end{aligned}$$

A direct calculation shows that ω^A is an admissible Cartan connection. Obviously, $(\sigma^A)^*\omega^A = \theta + A$. □

Now, let us consider a metric g in the conformal class c. We denote by $\mathcal{P}^g \subset \mathcal{P}^0$ the subbundle of all g-orthonormal frames and by A^g the Levi-Civita connection of g on $\mathcal{P}^g$ (which extends to a torsion-free connection on $\mathcal{P}^0$). The choice of $g \in c$ gives us two data:

(1) A reduction of the B-bundle $\mathcal{P}^1$ to the bundle of g-orthonormal frames $\mathcal{P}^g$

$$\sigma^g \colon \mathcal{P}^g \xhookrightarrow{i} \mathcal{P}^0 \xrightarrow{\sigma^{A^g}} \mathcal{P}^1.$$

(2) An admissible Cartan connection ω^{A^g} on $\mathcal{P}^1$ with $(\sigma^g)^*\omega^{A^g} = \theta + A^g$.

Now, we compare the admissible Cartan connections ω^{nor} and ω^{A^g}.

For this purpose we fix an orthonormal basis $(e_1, \dots, e_n)$ in $(\mathfrak{b}_{-1} = \mathbb{R}^n, \langle \cdot, \cdot \rangle_{p,q})$ and denote by $(e_1^*, \dots, e_n^*)$ the dual basis in $\mathfrak{b}_1 \simeq \mathfrak{b}_{-1}^*$. Let $u = (s_1, \dots, s_n)$ be a conformal frame in a point $x \in M$. Recall that we identified the tangent bundle TM with the associated bundle $\mathcal{P}^0 \times_{B_0} \mathfrak{b}_{-1}$ in such a way that $s_j = [u, e_j]$. If $u \in \mathcal{P}^g \subset \mathcal{P}^0$, then $(s_1, \dots, s_n)$ is g-orthonormal.

Proposition 8. *Let g be a metric in the conformal class c. Then the normal conformal Cartan connection is given by*

$$\omega_H^{\mathrm{nor}}(\xi) = \omega_H^{A^g}(\xi) + \sum_{j=1}^{n} K_x^g(d\pi_H(\xi), s_j)\, e_j^* \quad \text{for } \xi \in T_H\mathcal{P}^1, \tag{22}$$

where $x = \pi(H)$ and $(s_1, \dots, s_n) = \pi^1(H)$.

In particular, if $\sigma^g \colon \mathcal{P}^g \to \mathcal{P}^1$ is the $\mathrm{O}(p,q)$-reduction of $\mathcal{P}^1$ given by the metric g, then

$$\begin{aligned}((\sigma^g)^*\omega^{\mathrm{nor}})_u(X) &= \theta_u(X) + A_u^g(X) \\ &\quad + \sum_{j=1}^{n} K_x^g\big(d\pi_u^0(X), s_j\big)\, e_j^* \quad \text{for } X \in T_u\mathcal{P}^g,\end{aligned} \tag{23}$$

where $u = (s_1, \dots, s_n) \in \mathcal{P}^g$ and $x = \pi^0(u)$.

Proof. The calculation of the difference $\omega^{\mathrm{nor}} - \omega^{A^g}$, which is a 1-form on $\mathcal{P}^1$ with values in $\mathfrak{b}_1$, is straight forward. Details can be found in [CSSII] or [F05], Satz 6.8., Formula (23) follows from (21) in Proposition 7. □

3.4 The tractor connection and its curvature

As we explained in Section 2, any Cartan connection on $\mathcal{P}^1$ defines a covariant derivative on an associated tractor bundle. In this section we want to express the covariant derivative induced by the normal conformal Cartan connection and its curvature in terms of a metric g in the conformal class c.

Let $\rho \colon G \to \mathrm{GL}(V)$ be a representation of the group $G = \mathrm{SO}(p+1, q+1)$ on a vector space V, $E = \mathcal{P}^1 \times_B V$ the associated tractor bundle over the conformal manifold (M, c) and $\nabla^{\mathrm{nor}} \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ the covariant derivative on E induced by the normal conformal Cartan connection ω^{nor} according to (15).

Let us fix again a metric g in the conformal class c. As we saw in Section 3.3, the choice of g allows us to reduce the bundles $\mathcal{P}^0$ and $\mathcal{P}^1$ to the bundle of g-orthonormal frames $\mathcal{P}^g$. Hence, the vector bundles E, TM, T^*M and the bundle $\mathfrak{so}(TM, g)$ of skew-symmetric endomorphisms on (TM, g) can be expressed as bundles associated

to the $\mathrm{O}(p,q)$-bundle $\mathcal{P}^g$:

$$\begin{aligned} E &= \mathcal{P}^g \times_{(\mathrm{O}(p.q),\rho)} V, \\ TM &= \mathcal{P}^g \times_{(\mathrm{O}(p,q),\mathrm{Ad})} \mathfrak{b}_{-1}, \\ T^*M &= \mathcal{P}^g \times_{(\mathrm{O}(p,q),\mathrm{Ad})} \mathfrak{b}_1, \\ \mathfrak{so}(TM,g) &= \mathcal{P}^g \times_{(\mathrm{O}(p,q),\mathrm{Ad})} \mathfrak{so}(p,q). \end{aligned}$$

The Levi-Civita connection of g defines a covariant derivative $\nabla^g : \Gamma(E) \to \Gamma(T^*M \otimes E)$ (cf. (8)). Moreover, the representation ρ induces g-dependent maps

$$\begin{aligned} \rho^g : TM &\longrightarrow \mathrm{End}(E,E), \\ \rho^g : T^*M &\longrightarrow \mathrm{End}(E,E), \\ \rho^g : \mathfrak{so}(TM,g) &\longrightarrow \mathrm{End}(E,E). \end{aligned}$$

To define these map, choose a frame $u \in \mathcal{P}^g_x$ and set

$$\rho^g(\Upsilon)\varphi := \big[u, \rho_*\big([u]^{-1}\Upsilon\big)\,[u]^{-1}\varphi\big] \in E_x, \tag{24}$$

where $\varphi \in E_x$ and Υ is a tangent vector, a covector or a skew-symmetric map in x. It is easy to check that the right hand side does not depend on the choice of $u \in \mathcal{P}^g_x$.

We obtain the following 'metric' formula for the normal tractor derivative ∇^{nor} and its curvature endomorphism $R^{\mathrm{nor}}(X,Y) := [\nabla^{\mathrm{nor}}_X, \nabla^{\mathrm{nor}}_Y] - \nabla^{\mathrm{nor}}_{[X,Y]}$.

Proposition 9. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of $G := \mathrm{SO}(p+1,q+1)$ and $E := \mathcal{P}^1 \times_B V$ the associated tractor bundle. Then, for any metric $g \in c$ the normal tractor derivative on E and its curvature are given by*

$$\nabla^{\mathrm{nor}}_X = \nabla^g_X + \rho^g(X) + \rho^g(K^g(X)), \tag{25}$$

$$R^{\mathrm{nor}}(X,Y) = \rho^g(W^g(X,Y)) + \rho^g(C^g(X,Y)). \tag{26}$$

Proof. Let $\phi \in \Gamma(E)$ be locally represented by $\phi_{|U} = [u,v] = [\sigma^g \circ u, v]$, where $u\colon U \to \mathcal{P}^g$ is a local smooth section in $\mathcal{P}^g$ and $v\colon U \to V$ is a smooth function. Then, from the formulas (8),(15), (23) and (24) we obtain

$$\begin{aligned} \nabla^{\mathrm{nor}}_X\phi &\overset{(15)}{=} \big[u, X(v) + \rho_*\big(\omega^{\mathrm{nor}}(d(\sigma^g \circ u)(X))\big)v\big] \\ &\overset{(23)}{=} \Big[u, X(v) + \rho_*\Big(\theta(du(X)) + A^g(du(X)) + \sum_{j=1}^{n} K^g(X,s_j)e_j^*\Big)\,v\Big] \\ &\overset{(8)}{=} \nabla^g_X\phi + \big[u, \rho_*\big([u]^{-1}X\big)\,v\big] + \big[u, \rho_*\big([u]^{-1}(K^g(X)\big)\,v\big] \\ &\overset{(24)}{=} \nabla^g_X\phi + \rho^g(X)\phi + \rho^g(K^g(X))\phi. \end{aligned}$$

Inserting (25) into the formula for the curvature endomorphism R^{nor} gives

$$\begin{aligned} R^{\mathrm{nor}}(X,Y) = {} & R^{\nabla^g}(X,Y) + [\rho^g(X),\rho^g(Y)] + [\rho^g(K^g(X)),\rho^g(K^g(Y))] \\ & + [\rho^g(X),\rho^g(K^g(Y))] + [\rho^g(K^g(X)),\rho^g(Y)] \\ & + [\nabla^g_X,\rho^g(Y)] + [\rho^g(X),\nabla^g_Y] - \rho^g([X,Y]) \\ & + [\nabla^g_X,\rho^g(K^g(Y))] + [\rho^g(K^g(X)),\nabla^g_Y] - \rho^g(K^g([X,Y])). \end{aligned}$$

Using (24) for the endomorphism ρ^g, formula (20) for the Lie bracket $[\mathfrak{b}_{-1},\mathfrak{b}_1]_{\mathfrak{g}}$ and taking into account that the subalgebras $\mathfrak{b}_{-1}$ and $\mathfrak{b}_1$ of $\mathfrak{g}$ are abelian, we obtain

$$\begin{aligned} R^{\mathrm{nor}}(X,Y) &= R^{\nabla^g}(X,Y) + [\rho^g(X),\rho^g(K^g(Y))] \\ &\quad - [\rho^g(Y),\rho^g(K^g(X))] + \rho^g(C^g(X,Y)) \\ &= \rho^g(W^g(X,Y)) + \rho^g(C^g(X,Y)). \end{aligned}$$

□

As a next step, we apply these formulas to the standard representation $\rho\colon G \to \mathrm{GL}(\mathbb{R}^{p+1,q+1})$ given by the matrix action. We denote the associated tractor bundle by

$$\mathcal{T}(M) = \mathcal{P}^1 \times_B \mathbb{R}^{p+1,q+1}$$

and call it the *standard tractor bundle*. The scalar product $\langle\cdot,\cdot\rangle_{p+1,q+1}$ on $\mathbb{R}^{p+1,q+1}$ induces a bundle metric $\langle\cdot,\cdot\rangle$ on $\mathcal{T}(M)$. The normal covariant derivative ∇^{nor} on $\mathcal{T}(M)$ is metric

$$X(\langle\phi,\psi\rangle) = \langle\nabla^{\mathrm{nor}}_X\phi,\psi\rangle + \langle\phi,\nabla^{\mathrm{nor}}_X\psi\rangle.$$

We fix again a metric g in the conformal class c and reduce $\mathcal{P}^1$ to the frame bundle $\mathcal{P}^g$. If we restrict the representation ρ to the subgroup $\mathrm{O}(p,q)$ we obtain the splitting

$$\mathbb{R}^{p+1,q+1} \simeq \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}, \quad \alpha f_0 + y + \beta f_{n+1} \longmapsto (\alpha, y, \beta)$$

into three $\mathrm{O}(p,q)$-representations, where $\mathrm{O}(p,q)$ acts trivial on the both 1-dimensional summands and by matrix action on $\mathbb{R}^{p,q}$. Hence, any metric g in the conformal class c defines the following splitting of the standard tractor bundle

$$\mathcal{T}(M) \overset{g}{\simeq} \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}} \tag{27}$$

where $\underline{\mathbb{R}}$ denotes the trivial line bundle over M. In this identification the bundle metric is given by

$$\langle(\alpha_1,Y_1,\beta_1)\,,\,(\alpha_2,Y_2,\beta_2)\rangle = \alpha_1\beta_2 + \alpha_2\beta_1 + g(Y_1,Y_2).$$

Let X be a vector field and μ a 1-form on M and τ a g-skew-symmetric endomorphism on TM. Using (19) and (24) we obtain for the endomorphisms $\rho^g(X)$, $\rho^g(\mu)$ and $\rho^g(\tau)$ the following formulas on sections of $\mathcal{T}(M)$, represented as a triple

(α, Y, β) of two functions α, β and a vector field Y on M:

$$\rho^g(X)\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ \alpha\, X\\ -g(X,Y)\end{pmatrix}, \quad \rho^g(\mu)\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix} = \begin{pmatrix}\mu(Y)\\ -\beta\,\mu^\sharp\\ 0\end{pmatrix},$$

$$\rho^g(\tau)\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ \tau(Y)\\ 0\end{pmatrix}.$$

Using these formulas in Proposition 9 we obtain for the normal derivative ∇^{nor} on the standard tractor bundle the following

Proposition 10. *Suppose that g is a metric in the conformal class c and let $\mathcal{T}(M) \overset{g}{\simeq} \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$ be the corresponding splitting of the standard tractor bundle. The smooth sections of $\mathcal{T}(M)$ are identified with triples (α, Y, β), where α and β are smooth functions and Y is a smooth vector field on M. Then the tractor derivative ∇^{nor} induced on $\mathcal{T}(M)$ by the normal Cartan connection ω^{nor} is given by*

$$\nabla^{\mathrm{nor}}_X\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix} = \begin{pmatrix}X(\alpha) + K^g(Y,X)\\ \nabla^g_X Y + \alpha X - \beta K^g(X)^\sharp\\ X(\beta) - g(X,Y)\end{pmatrix}. \tag{28}$$

The curvature endomorphism of ∇^{nor} satisfies

$$R^{\mathrm{nor}}(X_1,X_2)\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix} = \begin{pmatrix}C^g(X_1,X_2)Y\\ W^g(X_1,X_2)Y - \beta\, C^g(X_1,X_2)^\sharp\\ 0\end{pmatrix}. \tag{29}$$

□

3.5 Conformal holonomy and Einstein metrics

The conformal holonomy group is an invariant of the conformal structure and it is natural to study which information about the conformal structure it contains. In this section we shortly review the relation between the conformal holonomy group of (M, c) and the existence of Einstein metrics in the conformal class c. For different proofs we refer to the papers of S. Armstrong [A05] (Riemannian case), F. Leitner [L05a] (arbitrary signature) and T. Leistner [Lei05] (Lorentzian case). To begin with let us state the following direct application of the 'metric' formula for the tractor connection ∇^{nor} on $\mathcal{T}(M)$ (cf. Proposition 10).

Proposition 11. *Let (M, c) be a conformal manifold. If there is an Einstein metric g in the conformal class c, then there exists a nontrivial parallel section $\phi \in \mathcal{T}(M)$: $\nabla^{\mathrm{nor}}\phi = 0$. On the other hand, if we suppose that there is a nontrivial ∇^{nor}-parallel section $\phi \in \mathcal{T}(M)$, then there is an open dense subset $\widetilde{M} \subset M$ and an Einstein*

metric g in the conformal class $c_{|\widetilde{M}}$. In both cases, for the scalar curvature scal^g *of g holds:*

- $\mathrm{scal}^g > 0 \iff \phi$ *is timelike.*
- $\mathrm{scal}^g < 0 \iff \phi$ *is spacelike.*
- $\mathrm{scal}^g = 0 \iff \phi$ *is lightlike.*

Proof. Let $g \in c$ be an Einstein metric. Then the Schouten tensor K^g is a multiple of the metric g,

$$K^g = -\frac{1}{2n(n-1)}\,\mathrm{scal}^g \cdot g.$$

Hence, decomposing $\mathcal{T}(M)$ with respect to g as it was described in the latter section, we obtain

$$\nabla^{\mathrm{nor}} \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} d\alpha - \frac{\mathrm{scal}^g}{2n(n-1)} Y^\flat \\ \nabla^g Y + (\alpha + \beta \frac{\mathrm{scal}^g}{2n(n-1)})\,\mathrm{Id} \\ d\beta - Y^\flat \end{pmatrix}.$$

If $\mathrm{scal}^g = 0$, then $\phi = (0,0,1)^\perp$ is a lightlike parallel section in $(\mathcal{T}(M), \nabla^{\mathrm{nor}})$. If $\mathrm{scal}^g \neq 0$, then $\phi = (-\frac{\mathrm{scal}^g}{2n(n-1)}, 0, 1)^\perp$ is parallel and timelike if $\mathrm{scal}^g > 0$ resp. spacelike if $\mathrm{scal}^g < 0$. Now, let us suppose that there is a nontrivial parallel section $\phi \in \Gamma(\mathcal{T}(M))$. We fix an arbitrary metric $g \in c$. Then with respect to this metric g we represent ϕ as $\phi = (\alpha, Y, \beta)^\perp$. The parallelity condition $\nabla^{\mathrm{nor}}\phi = 0$ implies

$$\begin{aligned} Y &= \mathrm{grad}^g \beta, \\ d\alpha &= -K^g(\mathrm{grad}^g \beta), \\ \alpha g &= \beta K^g - \mathrm{Hess}^g(\beta). \end{aligned} \tag{30}$$

Therefore, $\widetilde{M} := \{x \in M \mid \beta(x) \neq 0\}$ is an open dense subset in M, otherwise ϕ would vanish on an open set, hence on M, since ϕ is parallel. Now, consider the metric $\tilde{g} := \beta^{-2} g$ on $\widetilde{M}$. With the general transformation formulas (cf. [Be87], p. 58) and (30) we obtain for the Schouten tensor of $\tilde{g}$ on $\widetilde{M}$

$$\begin{aligned} K^{\tilde{g}} &= K^g + \frac{1}{2}\beta^{-2}|d\beta|^2 g - \beta^{-1}\mathrm{Hess}^g \beta \\ &= \left(\alpha\beta + \frac{1}{2}|d\beta|^2\right)\tilde{g}. \end{aligned}$$

Therefore, $\tilde{g}$ is an Einstein metric and the scalar curvature is given by

$$-\frac{1}{2n(n-1)}\mathrm{scal}^{\tilde{g}} = \alpha\beta + \frac{1}{2}|d\beta|^2 = \alpha\beta + \frac{1}{2}g(Y,Y) = \frac{1}{2}\langle\phi,\phi\rangle.$$

This concludes the proof. □

This result can be generalized to the following statements: Let $(M^{p,q}, c)$ be a simply connected conformal manifold of dimension $n = p + q$ and let us denote by $H = \mathrm{Hol}(M, c)$ the conformal holonomy group of (M, c).

(1) *Holonomy characterization of conformal Einstein spaces.* If there is a 1-dimensional H-invariant subspace $V^1 \subset \mathbb{R}^{p+1,q+1}$, then there exists an Einstein metric g in the conformal class c over an open dense subset $\widetilde{M} \subset M$. The scalar curvature of g is positive, negative or zero, if V is timelike, spacelike or lightlike, respectively.

(2) *Local splitting theorem.* If there is a non-degenerate H-invariant subspace $V^k \subset \mathbb{R}^{p+1,q+1}$ of dimension $k \geq 2$. Then, locally outside a singular set, (M, c) is a product $(M_1^{k-1} \times M_2^{n-k+1}, [g_1 \times g_2])$, where (M_i, g_i) are Einstein manifolds with scalar curvature

$$\mathrm{scal}^{g_1} = -\frac{k(k-1)}{(n-k)(n-k-1)}\mathrm{scal}^{g_2}.$$

The conformal holonomy group of (M, c) is the product of the conformal holomomy groups of $(M_1, [g_1])$ and $(M_2, [g_2])$.

(3) *Holonomy of conformal Einstein spaces.* Let $g \in c$ be an Einstein metric in the conformal class c. Then the conformal holonomy can be expressed by the metric holonomy of a certain ambient space. In case of non-zero scalar curvature scal^g we consider the Ricci-flat pseudo-Riemannian manifold $(\widetilde{M}, \tilde{g})$ and the cone (CM, g_C) over (M, g):

$$\widetilde{M} := \mathbb{R} \times M \times \mathbb{R}^+, \qquad \tilde{g} := \frac{n(n-1)}{\mathrm{scal}^g}(dt^2 - ds^2) + t^2 g,$$
$$CM := \mathbb{R}^+ \times M, \qquad g_C := \frac{n(n-1)}{\mathrm{scal}^g} d\tau^2 + \tau^2 g.$$

We obtain

$$\mathrm{Hol}_x(M, c) = \mathrm{Hol}_{(1,x,1)}(\widetilde{M}, \tilde{g}) = \mathrm{Hol}_{(1,x)}(CM, g_C).$$

In case $\mathrm{scal}^g = 0$, we consider the Ricci-flat pseudo-Riemannian manifold

$$\widetilde{M} := \mathbb{R} \times M \times \mathbb{R}^+, \quad \tilde{g} := 2dsdt + t^2 g$$

for which

$$\mathrm{Hol}_x(M, c) = \mathrm{Hol}_{(1,x,1)}(\widetilde{M}, \tilde{g}).$$

These statements can be used to give a complete classification of the conformal holonomy groups of simply connected *Riemannian* conformal structures (cf. [A05]). To understand this, remember that for a Riemannian conformal structure the holonomy group $\mathrm{Hol}(M^n, c)$ is a subgroup of $\mathrm{SO}(1, n+1)$. It is known that any connected irreducible acting subgroup of $\mathrm{SO}(1, n+1)$ equals $\mathrm{SO}_0(1, n+1)$ (cf. [DO01]). Hence, if the conformal holonomy group of a simply connected Riemannian conformal manifold (M^n, c) is not the full group $\mathrm{SO}_0(1, n+1)$, it admits a holonomy invariant subspace. But then, either c contains an Einstein metric on an open dense set or (locally) a product of Einstein metrics. Therefore, one has only to classify the conformal holonomy groups of indecomposable Einstein spaces. The identification of the conformal holonomy group of an Einstein spaces with the metric holonomy group of a

Ricci-flat ambient space allows then to apply known classification results for metric holonomy groups.

Proposition 12 ([A05]). *Let* $(M^n, [g])$ *be a simply connected indecomposable Riemannian conformal Einstein space of dimension* $n \geq 4$ *and let* $\mathrm{Hol}(M, [g]) \subset \mathrm{SO}_0(1, n+1)$ *be its conformal holonomy group.*

(1) *If* $\mathrm{scal}^g < 0$, *then* $\mathrm{Hol}(M, [g]) = \mathrm{SO}_0(1, n)$.

(2) *If* $\mathrm{scal}^g > 0$, *then* $\mathrm{Hol}(M, [g])$ *is one of the groups:* $\mathrm{SO}(n+1)$, $\mathrm{SU}(\frac{n+1}{2})$, $\mathrm{Sp}(\frac{n+1}{4})$, G_2 *if* $n = 6$ *or* $\mathrm{Spin}(7)$ *if* $n = 7$.

(3) *If* $\mathrm{scal}^g = 0$, *then* $\mathrm{Hol}(M^n, [g])$ *is one of the groups:* $\mathrm{SO}(n) \ltimes \mathbb{R}^n$, $\mathrm{SU}(\frac{n}{2}) \ltimes \mathbb{R}^n$, $\mathrm{Sp}(\frac{n}{4}) \ltimes \mathbb{R}^n$, $G_2 \ltimes \mathbb{R}^7$ *if* $n = 7$ *or* $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ *if* $n = 8$.

For indefinite metrics and indefinite conformal structures the classification of holonomy groups is open and much more difficult than in the Riemannian case.

Let us consider the case of a *Lorentzian* conformal manifold $(M^{1,n-1}, c)$. Then the conformal holonomy group $\mathrm{Hol}(M, c)$ is contained in $\mathrm{SO}(2, n)$. If it acts non-irreducible, than – locally – c contains an Einstein metric (1-dimensional invariant subspace), a product of 2 Einstein metrics with related scalar curvatures (k-dimensional non-degenerate invariant subspace, $k \geq 2$) or a metric with totally isotropic Ricci tensor and recurrent lightlike vector field (2-dimensional totally isotropic invariant subspace) (see [Lei05]). The classification of Lorentzian conformal structures with irreducible holonomy group is not completed yet. In the Section 4 we will describe even-dimensional Lorentzian conformal manifolds with conformal holonomy group in the irreducible subgroup $\mathrm{SU}(1, \frac{n}{2}) \subset \mathrm{SO}(2, n)$.

3.6 Conformal holonomy and conformal Killing spinors

In this section we want to explain the relation between conformal holonomy groups and solutions of the conformally invariant twistor equation on spinors (conformal Killing spinors). To begin with, we recall the definition of conformal Killing spinors. Afterwards we show that conformal Killing spinors correspond to parallel sections in the spin tractor bundle for the normal tractor derivative. Therefore, they are directly related to conformal holonomy groups. For an detailed introduction to spin geometry of Riemannian and pseudo-Riemannian manifolds we refer to [B81] or [Fr00], for basic results about conformal Killing spinors see [PR86], [BFGK91], [Lew91], [KR96], [KR96], [B99], [L01], [BL04], [L05a] and the references therein.

Let $(M^{p,q}, g)$ be a space- and time-oriented semi-Riemannian spin manifold of dimension $n = p + q \geq 3$. We denote by S^g the spinor bundle with respect to the metric g and by $\mu^g : T^*M \otimes S^g \to S^g$ the Clifford multiplication. The spinor bundle S^g is equipped with a natural inner product $\langle \cdot, \cdot \rangle$, which is non-degenerate (but indefinite, if g is indefinite). The bundle of 1-forms with values in the spinor

bundle decomposes into two subbundles

$$T^*M \otimes S^g = V^g \oplus \operatorname{Ker} \mu^g,$$

where V^g, being the orthogonal complement to the subbundle $\operatorname{Ker} \mu^g$, is isomorphic to S^g. Obviously, we obtain two differential operators of first order by composing the spinor derivative ∇^{S^g} induced by the Levi-Civita connection of g with the orthogonal projections onto each of these subbundles, the *Dirac operator* $D(g)$,

$$D(g)\colon \Gamma(S^g) \xrightarrow{\nabla^{S^g}} \Gamma(T^*M \otimes S^g) = \Gamma(S^g \oplus \operatorname{Ker} \mu^g) \xrightarrow{\mathrm{pr}_{S^g}} \Gamma(S^g),$$

and the *twistor operator* $P(g)$,

$$P(g)\colon \Gamma(S^g) \xrightarrow{\nabla^{S^g}} \Gamma(T^*M \otimes S^g) = \Gamma(S^g \oplus \operatorname{Ker} \mu^g) \xrightarrow{\mathrm{pr}_{\operatorname{Ker} \mu^g}} \Gamma(\operatorname{Ker} \mu^g).$$

Both operators are conformally invariant. More precisely, if $\tilde{g} = e^{2\sigma} g$ is a conformal change of the metric, the Dirac and the twistor operator change by

$$D(\tilde{g}) = e^{-\frac{n+1}{2}\sigma} D(g) e^{\frac{n-1}{2}\sigma},$$
$$P(\tilde{g}) = e^{-\frac{\sigma}{2}} P(g) e^{-\frac{\sigma}{2}}.$$

This relates the study of solutions of the *Dirac equation* $D(g)\varphi = 0$ and of the *twistor equation* $P(g)\varphi = 0$ to conformal geometry: it is enough to study these equations for an appropriate "simple" metric in the conformal class of g. Studying these two equations more closely, it turns out that they are rather different in nature. The latter one, the twistor equation, is of so-called twistor type, that means it is equivalent to parallel sections of a certain covariant derivative on a certain vector bundle. To see this, let us first remark that a spinor field $\varphi \in \Gamma(S)$ is a solution of the twistor equation $P(g)\varphi = 0$ if and only if it satisfies the *conformal Killing spinor equation*

$$\nabla^{S^g}_X \varphi + \frac{1}{n} X \cdot D(g)\, \varphi = 0 \quad \text{for all vector fields } X, \tag{31}$$

where the $\cdot$ denotes the Clifford multiplication. It is obvious that parallel spinors ($\nabla^{S^g} \varphi = 0$) and Killing spinors ($\nabla^{S^g}_X \varphi + \lambda X \cdot \varphi = 0$ for some $\lambda \in \mathbb{C} \setminus \{0\}$) are special solutions of (31). The name *conformal Killing spinor* for solutions of (31) reflects the fact that the Dirac current of φ, i.e., the vector field V_φ on M associated to the spinor φ by

$$g(V_\varphi, X) = i^{\,p+1} \langle X \cdot \varphi, \varphi \rangle \quad \text{for all vector fields } X, \tag{32}$$

is a conformal Killing vector field. Using (31) one obtains for any conformal Killing spinor φ the integrability condition

$$\nabla^{S^g}_X D(g)\varphi = \frac{n}{2} K^g(X) \cdot \varphi, \tag{33}$$

where K^g is the Schouten tensor of g (cf. [BFGK91], Theorem 1.3). Then, a direct calculation using (31) and (33) yields the following result.

Proposition 13 ([BFGK91], Theorem 1.4). *Let E^g be the vector bundle $E^g = S^g \oplus S^g$ and ∇^{E^g} the covariant derivative on E^g defined by*

$$\nabla^{E^g}_X := \begin{pmatrix} \nabla^{S^g}_X & X\cdot \\ -\frac{1}{2}K^g(X)\cdot & \nabla^{S^g}_X \end{pmatrix}. \tag{34}$$

Then, for any conformal Killing spinor φ holds $\nabla^{E^g}\begin{pmatrix}\varphi\\ \frac{1}{n}D(g)\varphi\end{pmatrix} = 0$. Conversely, if $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ is ∇^{E^g}-parallel, then φ is a conformal Killing spinor and $\psi = \frac{1}{n}D(g)\varphi$.

The latter proposition shows that conformal Killing spinors, i.e., the solutions of the twistor equation $P(g)\varphi = 0$, are in 1-1-correspondence with the ∇^{E^g}-parallel sections in the double spinor bundle E^g. Hence, by the general philosophy which we explained in Section 1, we can use the holonomy group of the covariant derivative ∇^{E^g} to study the existence of solutions of the twistor equation on spinors. We now want to give a reinterpretation of the situation in the framework of conformal Cartan geometry. We will show that the bundle E^g is isomorphic to the spin tractor bundle over the conformal manifold $(M, c = [g])$ and that the covariant derivative ∇^{E^g} is the normal covariant derivative on this tractor bundle. Hence, the holonomy group of ∇^{E^g} is just given by the holonomy group of the conformal structure $c = [g]$ of the metric g in question. Therefore, the conformal holonomy group $\mathrm{Hol}(M, c)$ contains the full information about the existence of conformal Killing spinors.

To explain this in more detail, we first extend the basics of conformal Cartan geometry which we described in Sections 3.1–3.4. to the spin case in the obvious way. Let (M, c) be a time- and space-oriented conformal manifold of signature (p, q). In this section we denote by $\mathcal{P}^0$ the bundle of all conformal frames which respect the given time- and space-orientation. The structure group is the connected component $\mathrm{CO}_0(p, q) := \mathbb{R}^+ \times \mathrm{SO}_0(p, q)$ of the linear conformal group. Let $\mathrm{CSpin}_0(p, q)$ be the conformal spin group

$$\mathrm{CSpin}_0(p, q) := \mathbb{R}^+ \times \mathrm{Spin}_0(p, q)$$

and $\lambda : \mathrm{CSpin}_0(p, q) \to \mathrm{CO}_0(p, q)$ the 2-fold covering of the linear conformal group given by the identity in the first component and the usual 2-fold covering of the orthogonal group by the spin group in the second component, which we will denote for simplicity also by λ. A *conformal spin structure* of (M, c) is a $\mathrm{CSpin}_0(p, q)$-principal fibre bundle $\mathcal{Q}^0$ over M with a smooth map $f^0 : \mathcal{Q}^0 \to \mathcal{P}^0$ which respects the bundle projections and the group actions

$$\begin{aligned} f^0(q \cdot A) &= f^0(q) \cdot \lambda(A) \quad \text{for all } q \in \mathcal{Q}^0,\ A \in \mathrm{CSpin}_0(p, q), \\ \pi^0 \circ f^0 &= \pi^0. \end{aligned}$$

We call the conformal manifold (M, c) a *spin manifold*, if it admits a conformal spin structure. This is equivalent to the existence of spin structures for any metric $g \in c$. In fact, any conformal spin structure $(\mathcal{Q}^0, f^0)$ of (M, c) induces a metric spin structure

$(\mathcal{Q}^g, f^g)$ for the metric $g \in c$ by reduction:

$$\mathcal{Q}^g := (f^0)^{-1}(\mathcal{P}^g), \quad f^g := f^0|_{\mathcal{Q}^g},$$

where $\mathcal{P}^g \subset \mathcal{P}^0$ denotes the subbundle of time- and space-oriented g-orthonormal frames. On the other hand, since $\mathcal{P}^0 = \mathcal{P}^g \times_{\mathrm{SO}_0(p,q)} \mathrm{CO}_0(p,q)$, any spin structure $(\mathcal{Q}^g, f^g)$ of a metric $g \in c$ gives rise to a conformal spin structure for the conformal class c by enlargement:

$$\mathcal{Q}^0 := \mathcal{Q}^g \times_{\mathrm{Spin}_0(p,q)} \mathrm{CSpin}_0(p,q), \quad f^0 := f^g \times \lambda.$$

Hence, due to the orientability assumption, (M, c) is spin if and only if the second Stiefel–Whitney class of M vanishes: $w_2(M) = 0$.

Now, let $G = \mathrm{SO}_0(p+1, q+1)$ be the connected component of the pseudo-orthogonal group and $B \subset G$ the stabilizer of the isotropic line p_∞ of the Möbius sphere $Q^{p,q}$. Then the spin group $\widetilde{G} := \mathrm{Spin}_0(p+1, q+1)$ acts transitively on $Q^{p,q}$ with p_∞-stabilizer $\widetilde{B}$. If $\widetilde{B}_1 := \exp(\lambda_*^{-1}(\mathfrak{b}_1)) \subset \widetilde{B} \subset \widetilde{G}$ then $\widetilde{B}/\widetilde{B}_1$ is isomorphic to the conformal spin group $\widetilde{B}_0 := \mathrm{CSpin}_0(p,q)$. Let $(\mathcal{Q}^0, f^0)$ be a conformal spin structure of (M, c). We define the first prolongation of the $(\mathcal{Q}^0, f^0)$ as a reduction of $\mathcal{P}^1$ with respect to $\lambda\colon \widetilde{B} \to B$. For that, consider the set

$$\mathcal{Q}^1 := \{H \subset T_q\mathcal{Q}^0 \mid q \in \mathcal{Q}^0 \text{ and } df_q^0(H) \subset T_{f^0(q)}\mathcal{P}^0 \text{ horizontal and torsion-free}\}$$

with the $\widetilde{B}$-action

$$H \cdot \tilde{b} := (df_{q\cdot b_0}^0)^{-1}(df_q^0(H) \cdot \lambda(\tilde{b})), \quad H \subset \mathcal{Q}^1 \cap T_q\mathcal{Q}^0,\ \tilde{b} = \tilde{b}_0 \cdot \tilde{b}_1 \in \widetilde{B} = \widetilde{B}_o \cdot \widetilde{B}_1$$

and the natural projections $\pi^1(H) := q$ and $\pi(H) := \pi^0(q)$. This gives us a $\widetilde{B}$-principal bundle $(\mathcal{Q}^1, \pi, M; \widetilde{B})$ over M and a 2-fold covering

$$f^1\colon H \in \mathcal{Q}^1 \longmapsto df^0(H) \in \mathcal{P}^1$$

which commutes with the group actions and the projections π and π^1. We call the pair $(\mathcal{Q}^1, f^1)$ the *first prolongation of the conformal spin structure* $(\mathcal{Q}^0, f^0)$.

Now, let $\omega \in \Omega^1(\mathcal{P}^1, \mathfrak{g})$ be a Cartan connection on $\mathcal{P}^1$. Since λ_* and df_H^1 are isomorphisms, the 1-form

$$\tilde{\omega} := \lambda_*^{-1} \circ \omega \circ df^1 \in \Omega^1(\mathcal{Q}^1, \tilde{\mathfrak{g}})$$

is a Cartan connection on $\mathcal{Q}^1$ with values in $\tilde{\mathfrak{g}} := \mathfrak{spin}(p+1, q+1)$. In particular, the normal conformal Cartan connection ω^{nor} on $\mathcal{P}^1$ defines a normal conformal Cartan connection $\tilde{\omega}^{\mathrm{nor}}$ on $\mathcal{Q}^1$. Any representation

$$\tilde{\rho}\colon \mathrm{Spin}(p+1, q+1) \longrightarrow \mathrm{GL}(V)$$

of the spin group defines a tractor bundle $E := \mathcal{Q}^1 \times_{\widetilde{B}} V$ with covariant derivative ∇^{nor} induced by the normal Cartan connection $\tilde{\omega}^{\mathrm{nor}}$ in the same way as we explained in Section 3.4.

Now, we fix again a metric g in the conformal class c and reduce $\mathcal{Q}^1$ to the spin bundle $\mathcal{Q}^g$. Then $E \simeq \mathcal{Q}^g \times_{\mathrm{Spin}_0(p,q)} V$. Let F be one of the bundles TM, T^*M or

$\mathfrak{so}(TM, g)$. Analogously to (24) the choice of g allows us to define the maps

$$\tilde{\rho}^g : F \longrightarrow \mathrm{End}(E, E),$$

$$\tilde{\rho}^g(\Upsilon)\varphi := \left[\tilde{u}, \tilde{\rho}_*\left(\lambda_*^{-1}([u]^{-1}\Upsilon)\right)[\tilde{u}]^{-1}\varphi\right] \in E_x \tag{35}$$

where $\Upsilon \in F_x$, $\varphi \in E_x$ and $\tilde{u} \in \mathcal{Q}^g_x$.

By ∇^g we denote the covariant derivative on E, which is defined by the Levi-Civita connection of g. In the same way as in Proposition 9 we obtain

Proposition 14. *Let $\tilde{\rho}\colon \widetilde{G} \to \mathrm{GL}(V)$ be a representation of the spin group $\widetilde{G} = \mathrm{Spin}_0(p+1, q+1)$ and $E = \mathcal{Q}^1 \times_{\widetilde{B}} V$ the associated tractor bundle. For any metric g in the conformal class c the normal tractor connection on E is given by*

$$\nabla^{\mathrm{nor}}_X = \nabla^g_X + \tilde{\rho}^g(X) + \tilde{\rho}^g(K^g(X)). \tag{36}$$

The curvature endomorphism of ∇^{nor} is

$$R^{\mathrm{nor}}(X, Y) = \tilde{\rho}^g(W^g(X, Y)) + \tilde{\rho}^g(C^g(X, Y)). \tag{37}$$

Let us apply this to the standard spinor representation

$$\kappa\colon \mathrm{Spin}_0(p+1, q+1) \longrightarrow \Delta_{p+1,q+1}.$$

We denote the corresponding tractor bundle on a conformal spin manifold (M, c) by

$$\mathcal{S}(M) := \mathcal{Q}^1 \times_{\widetilde{B}} \Delta_{p+1,q+1}$$

and call it *spin tractor bundle* of (M, c). In the same way as for metric spinor bundles we define a Clifford multiplication

$$\mu\colon \mathcal{T}(M) \times \mathcal{S}(M) \longrightarrow \mathcal{S}(M)$$

and a bundle metric $\langle \cdot, \cdot \rangle$ on $\mathcal{S}(M)$ using the corresponding Spin_0-invariant data on the fibers.

Now, we will prove that any metric $g \in c$ defines a canonical isomorphism between the spin tractor bundle $\mathcal{S}(M)$ and the double metric spin bundle $E^g = S^g \oplus S^g$ in such a way that the normal tractor connection ∇^{nor} corresponds to the covariant derivative ∇^{E^g} defined in (34).

For this purpose, let us denote by $W_\pm$ the following $\mathrm{Spin}(p, q)$-invariant subspaces of the spinor module $\Delta_{p+1,q+1}$:

$$W_- := \{v \in \Delta_{p+1,q+1} \mid f_0 \cdot v = 0\},$$
$$W_+ := \{v \in \Delta_{p+1,q+1} \mid f_{n+1} \cdot v = 0\}.$$

As $\mathrm{Spin}(p, q)$-representation, W_+ is isomorphic to $\Delta_{p,q}$. If we restrict the spin representation κ to $\mathrm{Spin}(p, q)$ we obtain the decomposition of $\Delta_{p+1,q+1}$ into the sum of two $\mathrm{Spin}(p, q)$-representations $W_+ \simeq \Delta_{p.q}$:

$$\Delta_{p+1,q+1} \simeq W_+ \oplus W_+, \quad w_1 + f_0 \cdot w_2 \longmapsto (w_1, w_2), \quad w_1, w_2 \in W_+.$$

Therefore,

$$\mathcal{S}(M) = \mathcal{Q}^g \times_{\mathrm{Spin}_0(p,q)} \Delta_{p+1,q+1} \simeq \mathcal{Q}^g \times_{\mathrm{Spin}_0(p,q)} (W_+ \oplus W_+) \simeq S^g \oplus S^g = E^g.$$

The inverse of $\lambda_* \colon \mathfrak{spin}(p+1,q+1) \to \mathfrak{so}(p+1,q+1)$ applied to $x \in \mathfrak{b}_{-1} \simeq \mathbb{R}^{p,q}$, $z \in \mathfrak{b}_1 \simeq (\mathbb{R}^{p,q})^*$ and $(A,a) \in \mathfrak{b}_0 = \mathfrak{co}(p,q)$ is

$$\begin{aligned}
\lambda_*^{-1}(x) &= -\frac{1}{2} x \cdot f_{n+1}, \\
\lambda_*^{-1}(z) &= +\frac{1}{2} z \cdot f_0, \\
\lambda_*^{-1}((A,a)) &= \lambda_*^{-1}(A) + \frac{a}{4}(f_0 \cdot f_{n+1} - f_{n+1} \cdot f_0).
\end{aligned}$$

Furthermore, if we represent $v \in \Delta_{p+1.q+1}$ in the form $v = w_1 + f_0 \cdot w_2$ with $w_1, w_2 \in W_+$, then

$$\begin{aligned}
f_{n+1} \cdot v &= f_{n+1} \cdot f_0 \cdot w_2 = -f_0 \cdot f_{n+1} \cdot w_2 - 2\langle f_0, f_{n+1}\rangle w_2 = -2w_2, \\
f_0 \cdot v &= f_0 \cdot w_1 + f_0 \cdot f_0 \cdot w_2 = f_0 \cdot w_1.
\end{aligned}$$

With these formulas we obtain for the action $\kappa^g(\Upsilon)$ on $\mathcal{S}(M) = S^g \oplus S^g$ defined in (35) in our special case

$$\begin{aligned}
\kappa^g(X) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} &= \begin{pmatrix} (X \cdot \psi \\ 0 \end{pmatrix}, \\
\kappa^g(\mu) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} &= \begin{pmatrix} 0 \\ -\frac{1}{2}\mu \cdot \varphi \end{pmatrix}, \\
\kappa^g(\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} \tau \cdot \varphi \\ \tau \cdot \phi \end{pmatrix},
\end{aligned}$$

where $\varphi, \psi \in \Gamma(S^g)$, X is a vector field, μ a 1-form on M and τ a skew-symmetric endomorphism $\tau \in \Gamma(\mathfrak{so}(TM,g)) = \Gamma(\Lambda^2(T^*M))$.

The application of Proposition 14 to this special case gives

Proposition 15. *Let g be a metric in the conformal class c. Then the spin tractor bundle $\mathcal{S}(M)$ can be identified with $S^g \oplus S^g$, and in this identification the normal tractor derivative on $\mathcal{S}(M)$ is given by*

$$\nabla^{\mathrm{nor}}_X \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \nabla^{S^g}_X \varphi + X \cdot \psi \\ \nabla^{S^g}_X \psi - \frac{1}{2} K^g(X) \cdot \varphi \end{pmatrix} = \begin{pmatrix} \nabla^{S^g}_X & X \cdot \\ -\frac{1}{2} K^g(X) \cdot & \nabla^{S^g}_X \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{38}$$

The curvature of ∇^{nor} is

$$R^{\mathrm{nor}}(X,Y) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} W^g(X,Y) \cdot \varphi \\ W^g(X,Y) \cdot \psi - C^g(X,Y) \cdot \varphi \end{pmatrix} \tag{39}$$

Recently, studying invariants of the conformal holonomy group, F. Leitner (cf. [L05a]) was able to prove a local classification result for generic conformal Killing

spinors on Lorentzian manifolds. To state this result, let us recall that any spinor field $\varphi \in \Gamma(S^g)$ on a Lorentzian manifold (M, g) induces a causal (i.e., time- or lightlike) vector field V_φ on (M, g) by

$$g(V_\varphi, X) = -\langle X \cdot \varphi, \varphi \rangle \quad \text{for all } X \in \mathfrak{X}(M).$$

If φ is a conformal Killing spinor, then V_φ is conformal as well. We call the spinor φ *generic* if it has no zeros, V_φ does not change the causal type and the dual 1-form $V_\varphi^\flat$ has constant rank, where the rank of a 1-form σ is the maximum of all k with $\sigma \wedge (d\sigma)^k \neq 0$.

Proposition 16 ([L05a]). *Let (M, g) be a Lorentzian spin manifold with a generic conformal Killing spinor. Then (M, g) is locally conformal equivalent to one of the following spaces:*

(1) $(\mathbb{R}, -dt^2) \times (N_1, h_1) \times \cdots \times (N_r, h_r)$, *where (N_j, h_j) are Ricci-flat Kähler, hyper-Kähler, G_2- or* Spin(7)*-manifolds;*

(2) *Lorentzian Einstein–Sasaki manifold;*

(3) $(N_1, h_1) \times (N_2, h_2)$, *where (N_1, h_1) is a Lorentzian Einstein–Sasaki manifold and (N_2, h_2) a Riemannian Einstein–Sasaki manifold, a* 3*-Sasaki-manifold, a nearly Kähler manifold or a Riemannian sphere;*

(4) *Fefferman space;*

(5) *Brinkmann space with parallel spinor.*

All Lorentzian manifolds which appear in the list, admit global solutions of the conformal Killing spinor equation. In the final section we will describe Fefferman spin manifolds and use the global existence of conformal Killing spinors on it to prove that the conformal holonomy group of Fefferman spin manifolds is contained in $\mathrm{SU}(1, \frac{n}{2})$.

4 Fefferman spin manifolds – conformal structures with holonomy group in SU(1, *m*)

Fefferman metrics are Lorentzian metrics on S^1-bundles over strictly pseudoconvex CR manifolds. Such metric was first discovered by Ch. Fefferman in [F76]. Fefferman studied the boundary behaviour of the Bergman kernel of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and in connection with this the solution u of the Dirichlet problem for the complex Monge–Ampère equation on Ω:

$$(-1)^n \det \begin{pmatrix} u & \partial u/\partial \bar{z}_k \\ \partial u/\partial z_j & \partial^2 u/\partial z_j \partial \bar{z}_k \end{pmatrix} = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{40}$$

For existence, uniqueness and regularity of the solution cf. [CY77]. Let u be a solution of (40) and $H : \bar{\Omega} \times \mathbb{C}^* \to \mathbb{R}$ the function

$$H(z_1, \ldots, z_n, z_0) = |z_0|^{2/(n+1)} u(z_1, \ldots, z_n).$$

Then the tensor field

$$G = \frac{\partial^2 H}{\partial z_\alpha \, \partial \bar{z}_\beta} dz_\alpha \odot d\bar{z}_\beta$$

on $\bar{\Omega} \times \mathbb{C}^*$ is non-degenerate and the pull back $g = j^*G$ for $j : \partial\Omega \times S^1 \hookrightarrow \bar{\Omega} \times \mathbb{C}^*$ is a Lorentzian metric on $\partial\Omega \times S^1$. If $\gamma = \arg(z_0)$ is the coordinate on S^1, the metric g is given by

$$g = -\frac{\mathrm{i}}{n+1} j^*(\partial u - \bar{\partial} u) \odot d\gamma + j^*(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \odot d\bar{z}_k). \tag{41}$$

Any biholomorphism of $\Omega \subset \mathbb{C}^n$ lifts to a smooth conformal diffeomorphism of $(\partial\Omega \times S^1, g)$. This indicated a close and interesting link between CR geometry and Lorentzian geometry. Of course, a solution u of (40) and hence the metric (41) is not explicitly given. Fefferman used a formal approximation procedure to describe the ambient metric G on a neighborhood of $\bar{\Omega}$. Some years later, Farris ([F86]) and Lee ([Lee86]) gave an explicit intrinsic description of the Fefferman metric g as Lorentzian metric on the canonical S^1-bundle over the strictly pseudoconvex CR manifold $\partial\Omega$ and extended the construction to abstract CR manifolds. Graham ([Gra87]) characterized the Fefferman metric locally by curvature properties. Other approaches studied the Fefferman construction and its relation to conformal geometry in the frame work of Cartan geometry (see [BDS77], [Ku97], [C02] and [C05]).

Inspired by the close link and the analogies the Fefferman construction provides between CR geometry and conformal Lorentzian geometry there is now a growing number of geometric and analytic investigations of Fefferman spaces. We refer for details to [DT06].

In [Lew91], J. Lewandowski studied the twistor equation for spinors on 4-dimensional space-times and found *local* solutions on Fefferman spaces. In the following section we will explain the intrinsic geometric construction of Fefferman spaces in CR geometry. In particular, we will describe Fefferman spin manifolds, which admit *global* solutions of the conformal Killing spinor equation. Details of the proofs can be found in [B99]. As a consequence, we will conclude that the conformal holonomy group of any Fefferman spin manifold is contained in $\mathrm{SU}(1, m)$.

4.1 CR geometry and Fefferman spaces

Let us first explain the necessary notations from CR geometry. Let N^{2m+1} be a smooth oriented manifold of odd dimension $2m+1$. A *CR structure on* N is a pair (H, J) where

1. $H \subset TN$ is a real $2m$-dimensional subbundle.
2. $J : H \to H$ is an almost complex structure on H, i.e., $J^2 = -\mathrm{Id}$.
3. If $X, Y \in \Gamma(H)$, then $[JX, Y] + [X, JY] \in \Gamma(H)$ and $J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \equiv 0$ (integrability condition).

Example 7 (Real hypersurfaces of complex manifolds). Any real hypersurface $N \subset \widetilde{N}$ of a complex manifold $(\widetilde{N}, \tilde{J})$ is a CR-manifold. Set $H := TN \cap \tilde{J}(TN)$ and $J = \tilde{J}_{|H}$.

Example 8 (The complex Möbius space). Let $L := \{z \in \mathbb{C}^{m+1} \setminus \{0\} \mid \langle z, z\rangle_{1,m} = 0\}$ be the isotropic cone in the complex vector space $\mathbb{C}^{1,m}$ with hermitian form of signature $(1, m)$. The projectivization $\mathbb{P}L$ is diffeomorphic to the sphere $S^{2m-1} \subset \mathbb{C}^m \simeq \{1\} \times \mathbb{C}^m \subset \mathbb{C}^{1,m}$ and has therefore a canonical CR-structure. $\mathbb{P}L$ is the boundary of the complex hyperbolic space $H^m_{\mathbb{C}} = \{z \in \mathbb{C}^{m+1} \mid \langle z, z\rangle_{1,m} = -1\}$.

Example 9 (Heisenberg manifolds). Let $\mathrm{He}(m)$ be the Heisenberg group

$$\mathrm{He}(m) := \left\{ \begin{pmatrix} 1 & x^\perp & z \\ 0 & \mathbb{I}_m & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}^m,\ z \in \mathbb{R} \right\}.$$

and Γ a discrete subgroup in $\mathrm{He}(m)$. Than the manifold $\mathrm{He}(m)/\Gamma$ is a CR manifold with the canonical CR structure induced by the Lie algebra structure of $\mathrm{He}(m)$.

Example 10 (Sasaki manifolds). Let (N, g) be an odd-dimensional Riemannian manifold with a Killing vector field ξ and let $\varphi := -\nabla^g \xi$. (N, g, ξ) is called *Sasaki-*manifold if

$$\begin{aligned} g(\xi, \xi) &= 1, \\ \varphi^2(X) &= -X + g(X, \xi)\xi, \\ (\nabla^g_X \varphi)(Y) &= g(X, Y)\xi - g(Y, \xi)X. \end{aligned}$$

A Riemannian manifold (N, g) admits a Sasaki structure if and only if the cone $(CN = \mathbb{R}^+ \times N, g_C = dt^2 + t^2 g)$ is a Kähler manifold. Obviously, $H := \xi^\perp$ and $J := \varphi_{|H}$ is a CR structure on the Sasaki manifold (N, g, ξ).

Now, let (N, H, J) be an oriented CR manifold. In order to define Fefferman spaces we fix a contact form $\theta \in \Omega^1(N)$ on N such that $\theta|_H = 0$. Let us denote by T the Reeb vector field of θ. In the following we suppose that the Levi form $L_\theta \colon H \times H \to \mathbb{R}$

$$L_\theta(X, Y) := d\theta(X, JY)$$

is positive definite. In this case, (N, H, J, θ) is called a *strictly pseudoconvex manifold*. The tensor field $g_\theta := L_\theta + \theta \odot \theta$ defines a Riemannian metric on N. There is a special metric covariant derivative on a strictly pseudoconvex manifold, the *Tanaka–Webster connection* $\nabla^W \colon \Gamma(TN) \to \Gamma(TN^* \otimes TN)$. Since ∇^W is metric ($\nabla^W g_\theta = 0$) it is uniquely defined by its torsion tensor Tor^W which is given by

$$\begin{aligned} \mathrm{Tor}^W(X, Y) &= L_\theta(JX, Y) \cdot T, \\ \mathrm{Tor}^W(T, X) &= -\frac{1}{2}([T, X] + J[T, JX]) \end{aligned}$$

for $X, Y \in \Gamma(H)$. Let us denote by $T_{10} \subset TN^{\mathbb{C}}$ the eigenspace of the complex extension of J on $H^{\mathbb{C}}$ to the eigenvalue i. Then L_θ extends to a Hermitian form on T_{10} by

$$L_\theta(U, V) := -\mathrm{i}\, d\theta(U, \bar{V}), \quad U, V \in T_{10}.$$

For a complex 2-form $\omega \in \Lambda^2 N^{\mathbb{C}}$ we denote by $\mathrm{trace}_\theta \omega$ the θ-trace of ω:

$$\mathrm{trace}_\theta \omega := \sum_{\alpha=1}^{m} \omega(Z_\alpha, \bar{Z}_\alpha),$$

where $(Z_1, \ldots, Z_m)$ is a unitary basis of (T_{10}, L_θ). Let R^W be the $(4,0)$-curvature tensor of the Tanaka–Webster connection ∇^W on the complexified tangent bundle of N

$$R^W(X, Y, Z, V) := g_\theta(([\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]})Z, \bar{V}).$$

and let us denote by

$$\mathrm{Ric}^W := \mathrm{trace}^{(3,4)}_\theta := \sum_{\alpha=1}^{m} R^W(\cdot, \cdot, Z_\alpha, \bar{Z}_\alpha)$$

the *Tanaka–Webster Ricci curvature* and by $\mathrm{scal}^W := \mathrm{trace}_\theta \mathrm{Ric}^W$ the *Tanaka–Webster scalar curvature*.

We now suppose in addition that N has a spin structure. This spin structure defines a square root $\sqrt{\Lambda^{m+1,0} N}$ of the canonical line bundle

$$\Lambda^{m+1,0} N := \{\omega \in \Lambda^{m+1} N^{\mathbb{C}} \mid V \lrcorner\, \omega = 0 \text{ for all } V \in \bar{T}_{10}\}.$$

We denote by $(F, \pi, N; S^1)$ the S^1-principal bundle associated to $\sqrt{\Lambda^{m+1,0} N}$. Let A^W denote the connection form on F defined by the Tanaka–Webster connection ∇^W and A_θ the modified connection form

$$A_\theta := A^W - \frac{\mathrm{i}}{4(m+1)} \mathrm{scal}^W \cdot \theta,$$

which we call *Fefferman connection*. Then

$$h_\theta := \pi^* L_\theta - \mathrm{i}\, \frac{8}{m+2} \pi^* \theta \odot A_\theta$$

is a Lorentzian metric on F. Note that the Fefferman connection and the constants in the Fefferman metric are chosen in this way, in order to ensure that the conformal class $[h_\theta]$ does not depend on the choice of θ. Hence, the conformal structure $[h_\theta]$ on F is an invariant of the CR structure (N, H, J) itself. We call (F^{2m+2}, h_θ) with its canonically induced spin structure *Fefferman spin space of the strictly pseudoconvex spin manifold* (N, H, J, θ).

Let us describe some further geometric properties of the Lorentzian manifold (F, h_θ). By construction, the metric h_θ is S^1-invariant, the fibres of the S^1-bundle are lightlike. We denote by T^* the horizontal lift of the Reeb vector field T with

respect to the Fefferman connection A_θ and by ξ the vertical fundamental vector field normalized by $h_\theta(\xi, T^*) = 1$. Then ξ is a lightlike Killing field and T^* is lightlike as well. Furthermore, for the Schouten tensor K^{h_θ}, the Cotton–York tensor C^{h_θ} and the Weyl tensor W^{h_θ} the following hold:

$$\begin{aligned} 4K^{h_\theta}(\xi, Y) &= -h_\theta(T^*, Y) \quad \text{for all } Y \in \mathcal{X}(F), \\ 4K^{h_\theta}(\xi, \xi) &= -1, \\ \xi \lrcorner W^{h_\theta} &= 0, \\ \xi \lrcorner C^{h_\theta} &= 0. \end{aligned}$$

Moreover, if H^* denotes the horizontal lift of H with respect to A_θ and $J^*X^* := (JX)^*$ for $X \in H$, then

$$J^*X^* = 2\nabla^{h_\theta}_{X^*}\xi.$$

In [B99] we studied the twistor equation on Fefferman spin manifolds. We gave explicit formulas for the solutions and studied their properties. Taking into account results of [BL04] we obtain

Proposition 17 ([B99]). *Let (N, H, J, θ) be a strictly pseudoconvex spin manifold with the Fefferman spin space (F, h_θ). Then there exists a 2-parameter family of conformal Killing spinors φ on (F, h_θ) which satisfy*

a) *V_φ is a regular lightlike Killing field, and*

b) *$\nabla^{S^{h_\theta}}_{V_\varphi}\varphi = \mathrm{i}c\,\varphi$, where $c \in \mathbb{R}\backslash\{0\}$.*

Conversely, if (M, g) is an even dimensional Lorentzian spin manifold with a conformal Killing spinor satisfying a) *and* b)*, then there exists a strictly pseudoconvex spin manifold (N, H, J, θ) such that its Fefferman space is locally isometric to (M, g).*

4.2 The conformal holonomy group of Fefferman spin spaces

In this section we will show that the conformal holonomy group $\mathrm{Hol}(F, [h_\theta])$ of an n-dimensional Fefferman spin manifolds is contained in $\mathrm{SU}(1, \frac{n}{2}) \subset \mathrm{SO}(2, n)$. In the first step we show that $\mathrm{Hol}(F, [h_\theta]) \subset \mathrm{U}(1, \frac{n}{2})$. For this purpose, we will define an isometric almost complex structure on the standard tractor bundle $\mathcal{T}(F)$, which is parallel with respect to the normal tractor connection ∇^{nor}. The further reduction to $\mathrm{SU}(1, \frac{n}{2})$ is then a consequence of the existence of two global conformal Killing spinors.

Let $\mathcal{T}(F)$ be the standard tractor bundle of the Fefferman spin manifold $(F, c = [h_\theta])$. As we saw in Section 3.4, the choice of a metric $h_\theta \in c$ gives rise to a splitting of $\mathcal{T}(F)$ into

$$\mathcal{T}(F) \overset{h_\theta}{\simeq} \underline{\mathbb{R}} \oplus TF \oplus \underline{\mathbb{R}}.$$

Using this splitting, we define an almost complex structure $J^{\mathcal{T}}$ on $\mathcal{T}(F)$ by

$$J^{\mathcal{T}}\begin{pmatrix}\alpha\\ \gamma T^*+\delta\xi+X^*\\ \beta\end{pmatrix}:=\begin{pmatrix}-\frac{\delta}{2}\\ 2\alpha\xi+\frac{\beta}{2}T^*+2\nabla^{h_\theta}_{X^*}\xi\\ -2\gamma\end{pmatrix}, \tag{42}$$

where $\alpha, \beta, \gamma, \delta$ are functions on F and T^*, ξ, X^* are the vector fields on F defined in the previous section. Using the geometric properties of h_θ described above, one obtains (by a direct but lengthy calculation) the following proposition. For a more conceptional proof see [L05b].

Proposition 18. *The endomorphism $J^{\mathcal{T}}: \mathcal{T}(F) \to \mathcal{T}(F)$ satisfies*

(1) $(J^{\mathcal{T}})^2 = -\mathrm{Id}_{\mathcal{T}(F)}$;

(2) $\langle J^{\mathcal{T}}\phi_1, J^{\mathcal{T}}\phi_2\rangle_{\mathcal{T}} = \langle\phi_1,\phi_2\rangle_{\mathcal{T}}$ *for all* $\phi_1, \phi_2 \in \mathcal{T}(F)$;

(3) $\nabla^{\mathrm{nor}}(J^{\mathcal{T}}) = 0$.

According to Proposition 5, the parallel almost complex structure $J^{\mathcal{T}}$ defines a hermitian almost complex structure J_0 on the fibre $\mathbb{R}^{2,n}$ of $\mathcal{T}(F)$, which is invariant under the action of the conformal holonomy group $\mathrm{Hol}(F, h_\theta)$. This means that

$$\mathrm{Hol}(F, h_\theta) \subset \mathrm{U}(1, \tfrac{n}{2}) \subset \mathrm{SO}(2, n).$$

Now, in order to reduce the situation further to $\mathrm{SU}(1, \frac{n}{2})$, we use the following Lemma, which can be proved considering the different Jordan normal forms of a matrix in $\mathrm{U}(1, m)$.

Lemma 1. *Let $A \in \mathrm{U}(1, m) \subset \mathrm{SO}(2, 2m)$ and let $\tilde{A} \in \lambda^{-1}(A) \subset \mathrm{Spin}(2, 2m)$ be one of the two elements of the spin group that cover A. If there are two linearly independent spinors in the spin-module $\Delta_{2,2m}$, that are fixed by the action of $\tilde{A}$, then $A \in \mathrm{SU}(1, m)$.*

Now, according to Proposition 17 there exists at least 2 linearly independent conformal Killing spinors on the Fefferman spin manifold (F, h_θ). Hence, by Proposition 15 there are 2 linearly independent ∇^{nor}-parallel sections in the spin tractor bundle $\mathcal{S}(F)$. Again Proposition 5 shows us that the conformal holonomy group $\mathrm{Hol}(F, [h_\theta])$ has 2 linearly independent fixed spinors in $\Delta_{2,n}$. Since we already know that any element of this holonomy group is unitary, Lemma 1 yields

Proposition 19. *The conformal holonomy group of a $2m$-dimensional Fefferman spin manifold is special unitary:* $\mathrm{Hol}(F, [h_\theta]) \subset \mathrm{SU}(1, m)$.

Using the local curvature characterization of Fefferman spaces given by R. Graham ([Gra87]) and further, more detailed investigations in conformal Cartan geometry one proves the converse of Proposition 19 (cf. for example [L06]):

Proposition 20. *Let (M, g) be a $2m$-dimensional Lorentzian manifold and let the connected component of the conformal holonomy group be special unitary:*

$$\mathrm{Hol}_0(M, [g]) \subset \mathrm{SU}(1, m).$$

Then (M, g) is locally conformal equivalent to a Fefferman space.

Finally, let us close with a recent result of F. Leitner ([L06]) which is a bit surprising. Using curvature arguments in Cartan geometry, he proved that for any conformal manifold with holonomy group $\mathrm{Hol}(M, c) \subset \mathrm{U}(1, m)$ the *connected component* $\mathrm{Hol}_0(M, c)$ is already contained in $\mathrm{SU}(1, m)$.

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Chapter 24

Nondegenerate conformal structures, CR structures and quaternionic CR structures on manifolds

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Introduction

The Weyl curvature tensor is a conformal invariant of Riemannian manifolds and the Chern–Moser curvature tensor is a CR invariant of strictly pseudo-convex CR-manifolds. The geometric significance of the vanishing of these curvature tensors is the appearance of the finite dimensional Lie group $\mathcal{G}$ with homogeneous space X, where $(\mathcal{G}, X) = (\mathrm{PO}(n+1,1), S^n)$ or $(\mathrm{PU}(n+1,1), S^{2n+1})$, respectively. On the other hand, the complete simply connected $(n+1)$-dimensional quaternionic hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{H}}$ with the group of isometries $\mathrm{PSp}(n+1,1)$ has a natural compactification inside the quaternionic projective space $\mathbb{HP}^{n+1}$. The subgroup $\mathrm{PSp}(n+1,1)$ of the group of quaternionic projective transformations $\mathrm{PGL}(n+2,\mathbb{H})$ acts smoothly on the boundary sphere S^{4n+3} of $\mathbb{H}^{n+1}_{\mathbb{H}} \subset \mathbb{HP}^{n+1}$. The action of $\mathrm{PSp}(n+1,1)$ is transitive on S^{4n+3}, and we obtain a flat (spherical) quaternionic CR geometry $(\mathrm{PSp}(n+1,1), S^{4n+3})$. Combined with conformally flat and spherical CR geometries, this exhibits *parabolic geometry* on the boundary of the compactification of a rank-one symmetric space of noncompact type over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We note that these model spaces are hyperspheres in the $\mathbb{K}$-projective spaces $\mathbb{KP}^{n+1}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$. This observation naturally raises the following problems.

(1) Construction of (positive definite) conformal, CR and quaternionic CR structures on hyperquadrics in $\mathbb{KP}^{n+1}$. More precisely, a generalization to (possibly indefinite) nondegenerate conformal, CR, quaternionic CR structure on manifolds. Notion of curvature forms, construction of flat model spaces, uniformization.

(2) Construction of a nondegenerate integrable geometric structure on a $(4n+3)$-dimensional manifold M and of the curvature form of a geometric invariant whose vanishing implies that M is locally equivalent to the flat quaternionic CR geometry $(\mathrm{PSp}(p+1,q+1), S^{4p+3,4q})$.

For (1), we will review and generalize the known results from the conformal viewpoint. For (2), we have introduced the notion of a *pseudo-conformal quaternionic CR* (*p-c qCR*) *structure* $(\mathcal{D}, \{\omega_\alpha\}_{\alpha=1,2,3})$ on a $(4n+3)$-dimensional manifold M ([3]). This geometric structure is a special case of an *almost pseudo-conformal quaternionic CR structure* (*almost p-c qCR structure*) $(\mathcal{D}, Q)$, which is discussed in [4].

(3) Which geometry relates to the above (indefinite) geometric structures?

(4) Which compact manifolds admit a nondegenerate conformal structure, nondegenerate CR structure, pseudo-conformal quaternionic CR structure, respectively?

For (3), we have introduced pseudo-Sasakian structures (also pseudo-Sasakian 3-structures). From the fibration associated to a nondegenerate spherical CR structure, we obtain Bochner flat pseudo-Kähler structures and also locally conformal Kähler structures. The question (4) is not well understood so far in the indefinite case, but we will give a few compact examples. In this chapter of the Handbook, we shall discuss these problems and explain the current results. In Section 2 we will introduce model spaces. In order to study flatness, we construct principal bundles as open dense subsets

of the model space in Section 3. Section 4 is concerned with conformal structures on pseudo-Riemannian manifolds. In Section 5 we will discuss nondegenerate CR geometry. In Section 6 we will study Bochner flat pseudo-Kähler geometry as a related geometry. We will explain the notion of pseudo-conformal quaternionic CR geometry in Section 7. It is introduced in [3]. For the viewpoint of G-structures of parabolic geometry of semisimple type, we refer to [5], [13] and references therein for the Weyl structures and Cartan connections. In the final Section 8, we discuss some problems related to these geometries.

2 Geometric structures on hypersurfaces

2.1 Model geometry

Let $\mathbb{K}$ be the field of real numbers, complex numbers or quaternions, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and denote by $\mathbb{K}^{n+2} = \mathbb{K}^{p+1,q+1}$ the n-dimensional K-vector space ($p+q=n$). Let $\mathrm{GL}(n+2,\mathbb{K})$ be the group of all invertible $(n+2)\times(n+2)$-matrices with entries in $\mathbb{K}$. Put

$$\mathrm{GL}(n+2,\mathbb{K})\cdot\mathbb{K}^* = \begin{cases} \mathrm{GL}(n+2,\mathbb{R}) & (\mathbb{K}=\mathbb{R}),\\ \mathrm{GL}(n+2,\mathbb{C}) & (\mathbb{K}=\mathbb{C}),\\ \mathrm{GL}(n+2,\mathbb{H})\cdot\mathbb{H}^* & (\mathbb{K}=\mathbb{H}). \end{cases}$$

In the quaternion case $\mathbb{K}=\mathbb{H}$, the group of all invertible matrices $\mathrm{GL}(n+1,\mathbb{H})$ acts from the left and $\mathbb{H}^* = \mathrm{GL}(1,\mathbb{H})$ acts as scalar multiplication from the right on $\mathbb{H}^{n+1}$, which forms the group $\mathrm{GL}(n+1,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H}) = \mathrm{GL}(n+1,\mathbb{H})\times_{\mathbb{R}^*}\mathrm{GL}(1,\mathbb{H})$. For the first two cases, the center of $\mathrm{GL}(n+2,\mathbb{K})$ coincides with $\mathbb{K}^*$. Consider the equivariant principal bundle of $\mathbb{K}$-projective space:

$$\mathbb{K}^* \to \big(\mathrm{GL}(n+2,\mathbb{K})\cdot\mathbb{K}^*, \mathbb{K}^{n+2}-\{0\}\big) \xrightarrow{P} \big(\mathrm{PGL}(n+2,\mathbb{K}), \mathbb{K}\mathbb{P}^{n+1}\big).$$

The Hermitian form on $\mathbb{K}^{n+2}$ is defined by

$$\mathcal{B}(x,y) = \bar{x}_1 y_1 + \cdots + \bar{x}_{p+1}y_{p+1} - \bar{x}_{p+2}y_{p+2} - \cdots - \bar{x}_{n+2}y_{n+2} \tag{2.1}$$

($n=p+q$). Denote by $\mathrm{O}(p+1,q+1;\mathbb{K})$ the subgroup of $\mathrm{GL}(n+2,\mathbb{K})$ given by

$$\{A\in\mathrm{GL}(n+2,\mathbb{K}) \mid \mathcal{B}(Ax,Ay)=\mathcal{B}(x,y),\ x,y\in\mathbb{K}^{n+2}\}.$$

Consider the following subset in $\mathbb{K}^{n+2}-\{0\}$:

$$V_0^{c(n+2)-1} = \{x\in\mathbb{K}^{n+2}-\{0\} \mid \mathcal{B}(x,x)=0\},$$

where $c = 1, 2, 4$, respectively. Let $z = (z_1, \ldots, z_{p+1}), w = (w_1, \ldots, w_{q+1})$, respectively, and put

$$\begin{aligned} S^{c(p+1)-1,cq} &= \{[z,w] \in \mathbb{K}\mathbb{P}^{n+1} \mid |z_1|^2 + \cdots + |z_{p+1}|^2 \\ &\qquad - |w_1|^2 - \cdots - |w_{q+1}|^2 = 0\} \qquad (2.2) \\ &= S^{c(p+1)-1} \times S^{c(q+1)-1}/S^{c-1}. \end{aligned}$$

The latter correspondence is given by

$$[z,w] \mapsto \left[\frac{z}{\|z\|}, \frac{w}{\|w\|}\right].$$

Let $\mathrm{PO}(p+1,q+1;\mathbb{K})$ denote the image of $\mathrm{O}(p+1,q+1;\mathbb{K})$ in $\mathrm{PGL}(n+2,\mathbb{K})$. Since $S^{c(p+1)-1,cq} = P(V_0^{c(n+2)-1})$ and $V_0^{c(n+2)-1}$ is invariant under $\mathrm{O}(p+1,q+1;\mathbb{K})$, we obtain the model geometry

$$(\mathrm{PO}(p+1,q+1;\mathbb{K}), S^{c(p+1)-1,cq}) \qquad (2.3)$$

inside $(\mathrm{PGL}(n+1,\mathbb{K}), \mathbb{K}\mathbb{P}^{n+1})$. It is customary to write *nondegenerate*,

conformally flat geometry: $(\mathrm{PO}(p+1,q+1), S^{p,q})$;
spherical CR geometry: $(\mathrm{PU}(p+1,q+1), S^{2p+1,2q})$;
flat pseudo-conformal qCR geometry: $(\mathrm{PSp}(p+1,q+1), S^{4p+3,4q})$.

Since $S^{p,q} = S^p \times S^q/\mathbb{Z}_2$ is not necessarily simply connected for $q \neq 0$ $(c = 1)$, instead of it we usually take the universal covering as $(\mathrm{O}(p+1,q+1), S^p \times S^q)$ for $p \geq q \geq 2$, $(\mathrm{O}(p+1,2)^\sim, S^p \times \mathbb{R})$ for $p > q = 1$, where $\mathrm{O}(p+1,2)^\sim$ is a lift of $\mathrm{PO}(p+1,2)$ associated to the universal covering $S^p \times \mathbb{R}$. For $p = q = 1$, $(\widetilde{\mathrm{O}(2,2)}, \mathbb{R} \times \mathbb{R})$, where $\mathbb{Z} \times \mathbb{Z} \to \widetilde{\mathrm{O}(2,2)} \to \mathrm{O}(2,2)$ is a central group extension. Note that $\mathrm{O}(2,2)^0 \cong \mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$. See [6] for related results.

2.2 Pseudo $\mathbb{K}$-hyperbolic geometry

Put $V_-^{c(n+2)} = \{z \in \mathbb{K}^{n+2} \mid \mathcal{B}(z,z) < 0\}$. The pseudo $\mathbb{K}$- hyperbolic space $\mathbb{H}_{\mathbb{K}}^{p+1,q}$ is defined to be $P(V_-^{c(n+2)})$, where the action $\mathrm{O}(p+1,q+1,\mathbb{K})$ on $V_-^{c(n+2)}$ induces an action on $\mathbb{H}_{\mathbb{K}}^{p+1,q}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, and S^1 if $\mathbb{K} = \mathbb{C}$ whose quotient is the pseudo $\mathbb{K}$-hyperbolic group $\mathrm{PO}(p+1, q+1, \mathbb{K})$. When $q = 0$, $p = n$, remark that $P(V_-^{c(n+2)})$ is the real hyperbolic space $\mathbb{H}_{\mathbb{R}}^{n+1}$, complex Kähler hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$ or quaternionic Kähler hyperbolic space $\mathbb{H}_{\mathbb{H}}^{n+1}$ with the group of isometries $\mathrm{PO}(n+1,1)$, $\mathrm{PU}(n+1,1)$, $\mathrm{PSp}(n+1,1)$, respectively. The $\mathbb{K}$-projective compactification of $\mathbb{H}_{\mathbb{K}}^{p+1,q}$ is obtained by taking the closure $\bar{\mathbb{H}}_{\mathbb{K}}^{p+1,q}$ in $\mathbb{K}\mathbb{P}^{n+1}$. Then it is easy to check that $\bar{\mathbb{H}}_{\mathbb{K}}^{p+1,q} = \mathbb{H}_{\mathbb{K}}^{p+1,q} \cup S^{c(p+1)-1,cq}$. From this viewpoint, the pseudo hyperbolic action of $\mathrm{PO}(p+1,$

$q+1, \mathbb{K})$ on $\mathbb{H}^{p+1,q}_{\mathbb{K}}$ (for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively) extends to a *conformal* action on $S^{p,q}$, CR-action on $S^{2p+1,2q}$ and a smooth action on $S^{4p+3,4q} = P(V_0^{4n+7})$ as projective transformations.

3 Model flat space forms of type $(c(p+1)-1, cq)$

Suppose that $p+q=n$ and $c=1,2,4$. Let $\mathbb{K}^{n+1}$ be the $\mathbb{K}$-number space with the nondegenerate quaternionic Hermitian form

$$\langle x, y\rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_{p+1} y_{p+1} - \bar{x}_{p+2} y_{p+2} - \cdots - \bar{x}_{n+1} y_{n+1}. \tag{3.1}$$

If $\mathrm{Re}\langle x, y\rangle$ denotes the real part of $\langle x, y\rangle$, observe that $\mathrm{Re}\langle\,,\rangle$ is a nondegenerate symmetric bilinear form on $\mathbb{R}^{c(n+1)} = \mathbb{K}^{n+1}$. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ denote by $O(p+1, q; \mathbb{K})$ the subgroup of $\mathrm{GL}(n+1, \mathbb{K})$ preserving the nondegenerate form $\mathrm{Re}\langle\,,\rangle$. Then $O(p+1, q; \mathbb{K}) = \mathrm{O}(p+1, q), \mathrm{U}(p+1, q)$, respectively. The group $\mathrm{Sp}(p+1, q)\cdot\mathrm{Sp}(1)$ is a subgroup of $\mathrm{GL}(n+1, \mathbb{H})\cdot\mathrm{GL}(1, \mathbb{H})$ preserving the nondegenerate bilinear form $\mathrm{Re}\langle\,,\rangle$. Denote by $\Sigma^{c(p+1)-1,cq}_{\mathbb{K}}$ the $(c(n+1)-1)$-dimensional quadric space:

$$\big\{(z_1, \ldots, z_{p+1}, w_1, \ldots, w_q) \in \mathbb{K}^{n+1} \mid \textstyle\sum_{k=1}^{p+1} |z_k|^2 - \sum_{k=1}^{q} |w_k|^2 = 1\big\}.$$

In particular, the group $O(p+1, q; \mathbb{K})$ leaves $\Sigma^{c(p+1)-1,cq}_{\mathbb{K}}$ invariant $(c=1,2)$ and $\mathrm{Sp}(p+1, q)\cdot\mathrm{Sp}(1)$ leaves $\Sigma^{3+4p,4q}_{\mathbb{H}}$ invariant. Let $\langle\,,\rangle_x$ be the nondegenerate quaternionic Hermitian inner product on the tangent space $T_x\mathbb{K}^{n+1}$ obtained from the parallel translation of $\langle\,,\rangle$ to the point $x \in \mathbb{K}^{n+1}$. Recall that J_0 is the standard complex structure on $\mathbb{C}^{n+1}$ acting as $J_0 z = z\boldsymbol{i}$, and $\{I, J, K\}$ is the standard hypercomplex structure on $\mathbb{H}^{n+1}$ which operates as $Iz = z\boldsymbol{i}$, $Jz = z\boldsymbol{j}$, or $Kz = z\boldsymbol{k}$. As the action of J_{0x} on $T_x\mathbb{C}^{n+1}$, $\{I_x, J_x, K_x\}$ also acts on $T_x\mathbb{H}^{n+1}$ at each point x. Then it is easy to see that, for all $X, Y \in T_x\mathbb{K}^{n+1}$, $g^{\mathbb{K}}_x(X, Y) = \mathrm{Re}\langle X, Y\rangle_x$ is the standard pseudo-euclidean metric of signature $(p+1, q)$ on $\mathbb{K}^{n+1}$ which is invariant under J_0 and $\{I, J, K\}$, respectively. ($g^{\mathbb{R}}_x(X, Y)$ is the usual pseudo-euclidean metric of type $(p+1, q)$ on $\mathbb{R}^{n+1}$.) Restricting $g^{\mathbb{K}}$ to the quadric $\Sigma^{c(p+1)-1,cq}_{\mathbb{K}}$ in $\mathbb{K}^{n+1}$, we obtain a nondegenerate pseudo-Riemannian metric of signature $(c(p+1)-1, cq)$ which we also denote by $g^{\mathbb{K}}$.

Definition 3.1. The quadric $(\Sigma^{c(p+1)-1,cq}_{\mathbb{K}}, g^{\mathbb{K}})$ is referred to as the pseudo $\mathbb{K}$-Riemannian space form of signature $(c(p+1)-1, cq)$ of constant curvature 1 endowed with transitive group of isometries $\mathrm{O}(p+1, q)$, $\mathrm{U}(p+1, q)$, $\mathrm{Sp}(p+1, q)\cdot\mathrm{Sp}(1)$ for which $\Sigma^{p,q}_{\mathbb{R}} = \mathrm{O}(p+1, q)/\mathrm{O}(p, q)$, $\Sigma^{2p+1,2q}_{\mathbb{C}} = \mathrm{U}(p+1, q)/\mathrm{U}(p, q)$, $\Sigma^{4p+3,4q}_{\mathbb{H}} = \mathrm{Sp}(p+1, q)\cdot\mathrm{Sp}(1)/\mathrm{Sp}(p, q)\cdot\mathrm{Sp}(1)$. Here $\mathrm{O}(p, q)$, $\mathrm{U}(p, q)$ and $\mathrm{Sp}(p, q)\cdot\mathrm{Sp}(1)$ are, respectively, the stabilizers at $(1, 0, \ldots, 0)$.

Compare [35], [26] for this definition. When $(\Sigma^{3+4p,4q}_{\mathbb{H}}, g^{\mathbb{H}})$ is viewed as a real pseudo-Riemannian space form, then the full group of isometries is $\mathrm{O}(4p+4, 4q)$.

Notice that the intersection of $\mathrm{O}(4p+4, 4q)$ with $\mathrm{GL}(n+1, \mathbb{H}) \cdot \mathrm{GL}(1, \mathbb{H})$ is $\mathrm{Sp}(p+1, q) \cdot \mathrm{Sp}(1)$. (Similarly, $\mathrm{O}(2p+2, 2q) \cap \mathrm{GL}(n+1, \mathbb{C}) = \mathrm{U}(p+1, q)$.) Now, $\mathrm{O}(1, \mathbb{K}) = \mathrm{O}(1), \mathrm{U}(1), \mathrm{Sp}(1)$ acts freely on $\Sigma_{\mathbb{K}}^{c(p+1)-1, cq}$ as right translations ($\lambda \in \mathrm{O}(1, \mathbb{K})$):

$$(\lambda, (z_1, \ldots, z_{p+1}, w_1, \ldots, w_q)) = (z_1 \cdot \bar{\lambda}, \ldots, z_{p+1} \cdot \bar{\lambda}, w_1 \cdot \bar{\lambda}, \ldots, w_q \cdot \bar{\lambda}).$$

Definition 3.2. The orbit space $\Sigma_{\mathbb{K}}^{c(p+1)-1, cq}/\mathrm{O}(1, \mathbb{K})$ is said to be the pseudo-$\mathbb{K}$ projective space $\mathbb{KP}^{p,q}$ of type (cp, cq).

Note that $\mathbb{CP}^{p,q}$ is a pseudo-Kähler manifold provided that $2n \geq 4$, and $\mathbb{HP}^{p,q}$ is a pseudo-quaternionic Kähler manifold provided that $4n \geq 8$. When $p = n$, $q = 0$, then $\mathbb{KP}^{n,0}$ is the standard $\mathbb{K}$-projective space $\mathbb{KP}^n$. When $p = 0, q = n$, then $\mathbb{KP}^{0,n}$ is the quaternionic hyperbolic space $\mathbb{H}_{\mathbb{K}}^n$. It is easy to see that $\mathbb{KP}^{p,q}$ is *homotopic* to the canonical $\mathbb{K}^q$-bundle over $\mathbb{KP}^p$. There are the equivariant principal bundles

$$\begin{aligned}
&\mathrm{O}(1) \to \big(\mathrm{O}(p+1, q), \Sigma_{\mathbb{R}}^{p,q}\big) \xrightarrow{\pi} \big(\mathrm{PO}(p+1, q), \mathbb{RP}^{p,q}\big),\\
&\mathrm{U}(1) \to \big(\mathrm{U}(p+1, q), \Sigma_{\mathbb{C}}^{2p+1,2q}\big) \xrightarrow{\pi} \big(\mathrm{PU}(p+1, q), \mathbb{CP}^{p,q}\big),\\
&\mathrm{Sp}(1) \to \big(\mathrm{Sp}(p+1, q) \cdot \mathrm{Sp}(1), \Sigma_{\mathbb{H}}^{3+4p,4q}\big) \xrightarrow{\pi} \big(\mathrm{PSp}(p+1, q), \mathbb{HP}^{p,q}\big).
\end{aligned} \tag{3.2}$$

3.1 Invariant subbundle of $T\Sigma_{\mathbb{K}}^{c(p+1)-1, cq}$

If N_x is the normal vector at $x \in \Sigma_{\mathbb{K}}^{c(p+1)-1, cq} \subset \mathbb{K}^{n+1}$, it then follows that

$$T_x \Sigma_{\mathbb{K}}^{c(p+1)-1, cq} = N_x^{\perp} \quad \text{with respect to } g^{\mathbb{K}}. \tag{3.3}$$

If N is a normal vector field on $\Sigma_{\mathbb{C}}^{2p+1,2q}$, then $J_0 N \in T\Sigma_{\mathbb{C}}^{2p+1,2q}$, where

$$T\Sigma_{\mathbb{C}}^{2p+1,2q} = \{J_0 N\} \oplus \{J_0 N\}^{\perp}. \tag{3.4}$$

If N is a normal vector field on $\Sigma_{\mathbb{H}}^{3+4p,4q}$, then $IN, JN, KN \in T\Sigma_{\mathbb{H}}^{3+4p,4q}$ such that we have the decomposition

$$T\Sigma_{\mathbb{H}}^{3+4p,4q} = \{IN, JN, KN\} \oplus \{IN, JN, KN\}^{\perp}. \tag{3.5}$$

We denote

$$\mathcal{D}_{\mathbb{C}} = \{J_0 N\}^{\perp} \quad \text{and} \quad \mathcal{D}_{\mathbb{H}} = \{IN, JN, KN\}^{\perp} \tag{3.6}$$

the $2n$-dimensional subbundle on $\Sigma_{\mathbb{C}}^{2p+1,2q}$ and the $4n$-dimensional subbundle on $\Sigma_{\mathbb{H}}^{4p+3,4q}$, respectively.

Lemma 3.3. *$\mathcal{D}_{\mathbb{C}}$ is J_0-invariant. Similarly, $\mathcal{D}_{\mathbb{H}}$ is $\{I, J, K\}$-invariant.*

Proof. Note that $g^{\mathbb{C}}|\mathcal{D}_{\mathbb{C}}$ is J_0-invariant and $g^{\mathbb{H}}$ is an $\{I, J, K\}$-invariant metric on

$\mathcal{D}_{\mathbb{H}}$. As $T_x\Sigma_{\mathbb{C}}^{2p+1,2q} = N_x^{\perp} = \{J_0N_x\} \oplus \{J_0N_x\}^{\perp}$,

$$\mathcal{D}_{\mathbb{C}} = \bigcup_{x\in\Sigma_{\mathbb{C}}^{2p+1,2q}} \{X_x \in T_x\mathbb{C}^{n+1} \mid g^{\mathbb{C}}(X_x, N_x) = g^{\mathbb{C}}(X_x, J_0N_x) = 0\}.$$

It is obvious that $\mathcal{D}_{\mathbb{C}}$ is J_0-invariant. Similarly, it follows that

$$\mathcal{D}_{\mathbb{H}} = \bigcup_{x\in\Sigma_{\mathbb{H}}^{3+4q,4q}} \big\{X_x \in T_x\mathbb{H}^{n+1} \mid g^{\mathbb{H}}(X_x, N_x) = g^{\mathbb{H}}(X_x, IN_x) = g^{\mathbb{H}}(X_x, JN_x) = g^{\mathbb{H}}(X_x, KN_x) = 0\big\}. \quad \square$$

We shall see later that $(\mathcal{D}, J_0)$ is a nondegenerate spherical CR structure and $(\mathcal{D}_{\mathbb{H}}, \{I, J, K\})$ is a flat pseudo-conformal quaternionic CR structure.

4 Conformal structures

The notion of conformal structure was introduced as the conformal equivalence class of Riemannian metrics on smooth manifolds. The Weyl conformal curvature tensor W is an invariant of conformal equivalence on pseudo-Riemannian metrics and its vanishing implies that the class locally contains an indefinite euclidean metric (flat pseudo-Riemannian metric). A nondegenerate conformally flat manifold is a pseudo-Riemannian manifold which admits a pseudo-Riemannian flat metric in its conformal class. As a G-structure, the conformal structure on an n-manifold M is equivalent to the reduction of the structure group of the principal frame bundle over M to a subgroup of $\mathrm{O}(p,q)\times\mathbb{R}^+$ $(p+q=n)$. An integrable nondegenerate $\mathrm{O}(p,q)\times\mathbb{R}^+$-structure is said to be a conformally flat structure on M, if each frame contains a basis consisting of coordinate frames on a neighborhood of M. We shall explain this equivalence between flatness of pseudo-Riemannian metrics and integrable $\mathrm{O}(p,q)\times\mathbb{R}^+$-structures.

Recall that $g_0 = ds^2 = dx_1^2+\cdots+dx_p^2-dy_1^2-\cdots-dy_q^2$ is the pseudo-euclidean metric on $\mathbb{R}^n$ of signature (p,q) with $p+q=n$.

Proposition 4.1. *There exists the standard pseudo-Riemannian metric g_1 on $S^{p,q}$ which is locally conformal to the euclidean metric g_0 on $\mathbb{R}^n$ $(p+q=n\geq 2)$.*

Proof. First of all, we embed $\mathbb{R}^n$ into $S^{p,q}$ in the following way:

$$\iota\colon (x,y) \mapsto \left[\frac{|x|^2-|y|^2}{2} - 1, \sqrt{2}x, \sqrt{2}y, \frac{|x|^2-|y|^2}{2}+1\right], \tag{4.1}$$

$x=(x_1,\ldots,x_p), y=(y_1,\ldots,y_q), |x| = \sqrt{x_1^2+\cdots+x_p^2}, |y| = \sqrt{y_1^2+\cdots+y_q^2}$. Note that the embedding $\iota\colon \mathbb{R}^n \to S^{p,q}$ is equivariant with respect to the subgroups $\mathbb{R}^n \times (\mathrm{O}(p,q)\times\mathbb{R}^+) \subset \mathrm{PO}(p+1,q+1)$. Put $U_\alpha = f_\alpha^{-1}(\iota(\mathbb{R}^n))$ for each

$f_\alpha \in \mathrm{PO}(p+1,q+1)$ and define the indefinite metric $g_\alpha = f_\alpha^* g_0$ on U_α. Since the open sets U_α cover $S^{p,q}$, suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Recall that the elements of $\mathrm{PO}(p+1,q+1)$ preserve the integrable $\mathrm{O}(p,q)\times\mathbb{R}^+$-structure by definition (cf. Section 4.1). As $g_{\beta\alpha} = f_\beta \circ f_\alpha^{-1}\colon f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$ is a local conformal change of $\iota(\mathbb{R}^n)$ $(\subset S^{p,q})$, it extends uniquely to $g_{\beta\alpha}\colon \iota(\mathbb{R}^n) \to \iota(\mathbb{R}^n)$ as an element of $\mathrm{PO}(p+1,q+1)$. Note that, by definition, $g_{\beta\alpha}^* g_0 = \lambda_{\beta\alpha}\cdot g_0$ for some positive function $\lambda_{\beta\alpha}$ on $\iota(\mathbb{R}^n)$.

Since $g_{\gamma\alpha} = g_{\gamma\beta}\cdot g_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ by the definition,

$$\lambda_{\gamma\alpha}\cdot g_0 = g_{\gamma\alpha}^* g_0 = g_{\beta\alpha}^* g_{\gamma\beta}^* g_0 = g_{\beta\alpha}^*(\lambda_{\gamma\beta}\cdot g_0) = g_{\beta\alpha}^*\lambda_{\gamma\beta}\cdot\lambda_{\beta\alpha}\cdot g_0$$

on $\iota(\mathbb{R}^n)$, we obtain that

$$\lambda_{\gamma\alpha} = g_{\beta\alpha}^*\lambda_{\gamma\beta}\cdot\lambda_{\beta\alpha}.$$

Define a local function $\{\mu\}_{\alpha,\beta\in\Lambda}$ to be

$$\mu_{\beta\alpha} = \lambda_{\beta\alpha}\circ f_\alpha\colon U_\alpha\cap U_\beta \to \mathbb{R}^+ \quad (\alpha,\beta\in\Lambda).$$

Then, $\{\mu\}_{\alpha,\beta\in\Lambda}$ satisfy on $U_\alpha\cap U_\beta\cap U_\gamma$ that

$$\begin{aligned}\mu_{\gamma\alpha} &= \lambda_{\gamma\alpha}\circ f_\alpha = g_{\beta\alpha}^*\lambda_{\gamma\beta}\cdot\lambda_{\beta\alpha}\circ f_\alpha\\ &= (\lambda_{\gamma\beta}(g_{\beta\alpha})\cdot\lambda_{\beta\alpha})\circ f_\alpha = \lambda_{\gamma\beta}(g_{\beta\alpha}\circ f_\alpha)\cdot\lambda_{\beta\alpha}(f_\alpha)\\ &= \lambda_{\gamma\beta}(f_\beta)\cdot\lambda_{\beta\alpha}(f_\alpha) = \mu_{\gamma\beta}\cdot\mu_{\beta\alpha}.\end{aligned}$$

This implies that $\{\mu\}_{\alpha,\beta\in\Lambda}$ defines a 1-cocycle on $S^{p,q}$. Since $\mathbb{R}^+$ is a fine sheaf viewed as the germ of local continuous functions, recall that the first cohomology $H^1(\mathcal{U},\mathbb{R}_R^+) = 0$. (Here $\mathcal{U}$ is a chain complex of covers running over all open covers of $S^{p,q}$.) Therefore there exists a local function $\{h\}_{\alpha\in\Lambda}$ such that $\delta h(\beta,\alpha) = \mu_{\beta\alpha}$, i.e., $h_\alpha\cdot h_\beta^{-1} = \mu_{\beta\alpha}$ on $U_\alpha\cap U_\beta$. We obtain that

$$\begin{aligned}f_\beta^* g_0 &= f_\alpha^* g_{\beta\alpha}^* g_0 = f_\alpha^*(\lambda_{\beta\alpha}\cdot g_0)\\ &= \lambda_{\beta\alpha}\circ f_\alpha\cdot f_\alpha^* g_0 = \mu_{\beta\alpha}\cdot f_\alpha^* g_0\\ &= h_\alpha\cdot h_\beta^{-1}\cdot f_\alpha^* g_0.\end{aligned}$$

In particular, $h_\beta\cdot f_\beta^* g_0 = h_\alpha\cdot f_\alpha^* g_0$ on $U_\alpha\cap U_\beta$. As $S^{p,q} = \bigcup_{\alpha\in\Lambda} U_\alpha$, setting

$$g_1|U_\alpha = h_\alpha\cdot f_\alpha^* g_0,$$

this defines a pseudo-Riemannian metric g_1 on $S^{p,q}$ which is locally conformal to g_0. □

Theorem 4.1 (Coincidence with the classical definition). *An n-dimensional $(n\geq 3)$ smooth manifold M is uniformizable over $(\mathrm{PO}(p+1,q+1), S^{p,q})$ if and only if M admits a pseudo-Riemannian metric locally conformal to the euclidean metric g_0.*

Proof. $\Longrightarrow$: Given a conformally flat structure $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\Lambda}$ on M, there are elements $g_{\beta\alpha}, g_{\gamma\beta}, g_{\gamma\alpha}\in\mathrm{PO}(p+1,q+1)$ defined on the intersection $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$

such that

$$U_\alpha \cap U_\beta \begin{array}{c} \overset{\varphi_\alpha}{\nearrow} \\ \circlearrowright \\ \underset{\varphi_\beta}{\searrow} \end{array} \begin{array}{l} S^{p,q} \\ \downarrow g_{\beta\alpha} \\ S^{p,q}, \end{array} \qquad U_\beta \cap U_\gamma \begin{array}{c} \overset{\varphi_\beta}{\nearrow} \\ \circlearrowright \\ \underset{\varphi_\gamma}{\searrow} \end{array} \begin{array}{l} S^{p,q} \\ \downarrow g_{\gamma\beta} \\ S^{p,q}, \end{array} \qquad U_\alpha \cap U_\gamma \begin{array}{c} \overset{\varphi_\alpha}{\nearrow} \\ \circlearrowright \\ \underset{\varphi_\gamma}{\searrow} \end{array} \begin{array}{l} S^{p,q} \\ \downarrow g_{\gamma\alpha} \\ S^{p,q}. \end{array}$$

Let g_1 be the standard pseudo-Riemannian metric on $S^{p,q}$ which is locally conformal to g_0 on $\iota(\mathbb{R}^n) \subset S^{p,q}$ by Proposition 4.1. As $g_{\beta\alpha}$ is a conformal transformation of $S^{p,q}$, there is a function $\lambda_{\beta\alpha} > 0$ such that $g_{\beta\alpha}^* g_1 = \lambda_{\beta\alpha} \cdot g_1$ on $S^{p,q}$. Putting $\mu_{\beta\alpha} = \lambda_{\beta\alpha} \circ \varphi_\alpha \colon U_\alpha \cap U_\beta \to \mathbb{R}^+$, the same argument as in the proof of Proposition 4.1 applies to show that $f_\beta \cdot \varphi_\beta^* g_1 = f_\alpha \cdot \varphi_\alpha^* g_1$ on $U_\alpha \cap U_\beta$ for some local functions $\{f\}_{\alpha\in\Lambda}$. Setting $g|U_\alpha = f_\alpha \cdot \varphi_\alpha^* g_1$, this defines a pseudo-Riemannian metric g on M which is locally conformal to g_1. By our choice of g_1, $(M, [g])$ is a nondegenerate conformally flat manifold. (Note that this proof works for $n \geq 2$.)

$\Longleftarrow$: There is a coordinate chart (U, φ) at each point of M such that U is an open subset and $\varphi \colon U \to \mathbb{R}^n$ is a homeomorphism onto its image. Suppose that $g|U = \lambda \cdot \varphi^* g_0$ and $\lambda \colon U \to \mathbb{R}^+$ is a function. For an open subset V, suppose that $U \cap V \neq \emptyset$ and $\psi \colon V \to \mathbb{R}^n$ is a coordinate function such that $g|V = \mu \cdot \psi^* g_0$. Then the local change of $\mathbb{R}^n$, $\psi \cdot \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ satisfies that $(\psi \cdot \varphi^{-1})^* g_0 = v \cdot g_0$ where $v = \mu^{-1}(\varphi^{-1}) \cdot \lambda(\varphi^{-1})$. As $\iota(\mathbb{R}^n) \subset S^{p,q}$, φ is viewed as an immersion $\varphi \colon U \to S^{p,q}$ such that $\psi \cdot \varphi^{-1}$ is a locally conformal transformation of $S^{p,q}$. By Liouville's theorem ($n \geq 3$), a local conformal change can be extended to a conformal transformation of $S^{p,q}$, i.e., there exists an element $h \in \mathrm{PO}(p+1, q+1)$ such that

$$\psi \cdot \varphi^{-1} = h|\varphi(U \cap V). \tag{4.2}$$

Hence a collection of such coordinate neighborhoods (U, φ) satisfying (4.2) gives a nondegenerate conformally flat structure on M. □

4.1 Weyl tensor

A nondegenerate conformal ($\mathrm{O}(p,q) \times \mathbb{R}^+$-) structure is a conformal class of nondegenerate pseudo-Riemannian metric g on a manifold M. Since we know that $\mathrm{O}(p,q) \times \mathbb{R}^+$ is of finite type (depth 1), an *integrable* $\mathrm{O}(p,q) \times \mathbb{R}^+$-structure is equivalent to the existence of a conformally flat pseudo-Riemannian metric, i.e., g is locally conformally equivalent to the standard euclidean metric g_0. As a matter of fact, the existence of an integrable $\mathrm{O}(p,q) \times \mathbb{R}^+$-structure is equivalent with vanishing Weyl curvature tensor of M provided that $\dim M \geq 4$. The vanishing of Weyl curvature tensor on a Riemannian metric on M gives a uniformization over the standard pseudo-sphere $S^{p,q}$ with respect to the group of conformal transformations $\mathrm{PO}(p+1, q+1)$. The pair $(\mathrm{PO}(p+1, q+1), S^{p,q})$ is said to be the *nondegenerate* (*indefinite*) *conformally flat geometry*.

A conformally flat manifold is a smooth n-manifold locally modeled on $S^{p,q}$ with coordinate changes belonging to $\mathrm{PO}(p+1, q+1)$. By the monodromy argument,

there is a developing map dev from the universal covering space $\widetilde{M}$ into the universal covering space of $S^{p,q}$ which is equivariant under the holonomy homomorphism ρ, i.e.,

$$(\rho, \mathrm{dev})\colon (\pi_1(M), \widetilde{M}) \to (\widetilde{\mathrm{PO}}(p+1, q+1), \widetilde{S}^{p,q}),$$
$$\mathrm{dev}(\gamma \cdot x) = \rho(\gamma)\mathrm{dev}(x) \quad \text{for all } x \in M,\ \gamma \in \pi_1(M),$$

where $\pi_1(M)$ is the fundamental group of M.

5 Nondegenerate CR geometry

A nondegenerate CR structure on a $(2n+1)$-manifold M is a pair $(\mathcal{D}, J)$ consisting of a codimension 1 subbundle of TM and a complex structure on it. D is a contact bundle such that $[\mathcal{D}, \mathcal{D}] \cup \mathcal{D} = TM$ at each point of M. When J is an almost complex structure on $\mathcal{D}$ and $D \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ is the eigenspace decomposition, J is a complex structure if and only if $[T^{1,0}, T^{1,0}] \subset T^{1,0}$. If ω is a 1-form whose kernel represents $\mathcal{D}$, it satisfies that

(1) $d\omega(JX, JY) = d\omega(X, Y)$ for all $X, Y \in D$,

(2) $g^{\mathcal{D}}(X, Y) = d\omega(JX, Y)$ is a nondegenerate symmetric bilinear form on D with type $(2p, 2q)$.

If $q = 0$, i.e., $g^{\mathcal{D}}$ is positive definite, then the CR structure $(\mathcal{D}, J)$ is said to be strictly pseudoconvex.

5.1 Existence of CR structure on $\Sigma_{\mathbb{C}}^{2p+1,2q}$

As before, let $\Sigma_{\mathbb{C}}^{2p+1,2q}$ be the complex pseudo-Riemannian space form of signature $(2p+1, 2q)$. Put

$$\omega_0 = -(\bar{z}_1 dz_1 + \cdots + \bar{z}_{p+1} dz_{p+1} - \bar{w}_1 dw_1 - \cdots - \bar{w}_q dw_q). \tag{5.1}$$

Then it is easy to check that ω_0 is an $\mathrm{Im}\,\mathbb{C}$-valued 1-form on $\Sigma_{\mathbb{C}}^{2p+1,2q}$. Let ξ_1 be the vector field on $\Sigma_{\mathbb{C}}^{2p+1,2q}$ induced by the one-parameter subgroup $\{e^{i\theta}\}_{\theta \in \mathbb{R}}$. Note that if J_0 is the standard complex structure on $\mathbb{C}^{n+1}$ (cf. Section 3), then $\xi_1 = J_0 N$. A calculation shows that $\omega_0(\xi_1) = i$. Let S^1 be the center of $\mathrm{U}(p+1, q)$. As $\mathrm{U}(p+1, q)$ leaves ω_0 invariant, it follows that $\mathrm{R}_a^* \omega_0 = \omega_0$. Therefore, ω_0 is a connection form of the principal bundle $S^1 \to \Sigma_{\mathbb{C}}^{2p+1,2q} \xrightarrow{\pi} \mathbb{CP}^{p,q}$ of (3.2). Put $\omega_0 = \omega_1 i$. Then ω_1 is a nondegenerate contact form on $\Sigma_{\mathbb{C}}^{2p+1,2q}$ because $\omega_1 \wedge \underbrace{(d\omega_1 \wedge \cdots \wedge d\omega_1)}_{n\text{-times}} \neq 0$ at any point of $\Sigma_{\mathbb{C}}^{2p+1,2q}$ from (5.1). (Compare [19], [29], for example). Noting that $\omega_1(\xi_1) = 1$, we have that $\mathcal{L}_{\xi_1} \omega_1 = \iota_{\xi_1} d\omega_1 = 0$ and so ξ_1 is the characteristic vector

field for ω_1. From Section 3.1 it follows that

$$\mathcal{D}_{\mathbb{C}} = \operatorname{Null}\omega_1, \tag{5.2}$$

where $\operatorname{Null}\omega_1 = \{X \in T\Sigma^{2p+1,2q}_{\mathbb{C}} \mid \omega_1(X) = 0\}$. It is easily checked that $d\omega_1(X,Y) = g^{\mathbb{C}}(X, J_0Y)$ $(X, Y \in \mathcal{D}_{\mathbb{C}})$. (Compare Lemma 7.8.) In particular, if θ^α is a coframe field of $T^{1,0} = \{e_1, \dots, e_n\}$ and $g_{\alpha\bar\beta} = g^{\mathbb{C}}(e_\alpha, e_{\bar\beta})$, then this gives the following structure equation of the bundle (3.2):

$$d\omega_1 = -\boldsymbol{i}\, g_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}. \tag{5.3}$$

It is shown that $(\Sigma^{2p+1,2q}_{\mathbb{C}}, J_0, g^{\mathbb{C}})$ is a pseudo-Sasakian space form of constant positive curvature with signature $(2p+1, 2q)$. (See [2], [20].) We note that

$$g^{\mathbb{C}}(X,Y) = \omega_1(X)\cdot\omega_1(Y) + d\omega_1(J_0X, Y)$$

which is called the pseudo-Sasakian metric of $\Sigma^{2p+1,2q}_{\mathbb{C}}$. In particular, $g^{\mathbb{C}}$ induces a pseudo-Kähler metric $\hat g_1$ of constant positive holomorphic curvature on $\mathbb{CP}^{p,q}$, see [31], [32] for details.

Corollary 5.1. *$(\Sigma^{2p+1,2q}_{\mathbb{C}}, \omega_1, J_0, g^{\mathbb{C}})$ is a $(2n+1)$-dimensional homogeneous CR-manifold of signature $(2p+1, 2q)$ equipped with the nondegenerate CR structure $(\mathcal{D}_{\mathbb{C}}, J_0)$ $(p+q = n \geq 1)$. Moreover, there exists the equivariant principal bundle of the pseudo-Riemannian submersion over the (homogeneous) pseudo-complex projective space $\mathbb{CP}^{p,q}$ of signature $(2p, 2q)$:*

$$S^1 \to (\mathrm{U}(p+1,q), \Sigma^{2p+1,2q}_{\mathbb{C}}, g^{\mathbb{C}}) \xrightarrow{\pi} (\mathrm{PU}(p+1,q), \mathbb{CP}^{p,q}, \hat g).$$

Remark 5.2. It is generally difficult to find compact nondegenerate CR-manifolds. There are discrete cocompact subgroups of $\mathrm{U}(1,n)$ that act properly and freely on $\Sigma^{1,2n}_{\mathbb{C}} = V^{2n+1}_{-1}$. We obtain compact nondegenerate CR-manifolds. (1) The pseudo-Riemannian standard space form V^{2n+1}_{-1}/Γ of signature $(1, 2n)$ with constant sectional curvature -1 which is an S^1-bundle over the Kähler hyperbolic orbifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma^*$ of constant negative sectional curvature. ($\Gamma^* \subset \mathrm{PU}(1,n)$ is a discrete subgroup.) (2) The indefinite Heisenberg nilmanifold of signature $(1+2p, 2q)$ which is a compact nondegenerate *CR*-manifold whose quotient is the complex euclidean orbifold (i.e., pseudo-Kähler orbifold of zero Ricci curvature), see Section 6.1.

5.2 Existence of flat CR structure on $S^{2p+1,2q}$

Recall that if N_x is the normal vector at $x \in \Sigma^{2p+1,2q}_{\mathbb{C}} \subset \mathbb{C}^{n+1}$, then $(\operatorname{Null}\omega_1)_x = \mathcal{D}_x = \{J_0N_x\}^\perp \subset T_x\Sigma^{2p+1,2q}_{\mathbb{C}} = N_x^\perp$ with respect to $g^{\mathbb{C}}$. If we note that $\{N_x, J_0N_x\} = x\mathbb{C}$ and $g^{\mathbb{C}} = \mathrm{Re}\langle\ ,\ \rangle$, it follows that $\mathcal{D}_x = x\mathbb{C}^\perp$ with respect to $\langle\ ,\ \rangle$. In particular, this shows that $\mathcal{D}_x$ is J_0-invariant. Let $\iota\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ be

the embedding defined by $\iota(z) = (z, 1)$ so that

$$P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q}) \subset P(V_0^{2n+3}) = S^{2p+1,2q}, \tag{5.4}$$

where

$$P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q}) = \{[(z_1, \ldots, z_{p+1}, w_1, \ldots, w_q, 1)]\}.$$

As $\Sigma_{\mathbb{C}}^{2p+1,2q}$ has the transitive isometry group $\mathrm{U}(p+1,q)$, this embedding implies that $\mathrm{U}(p+1,q)$ is identified with the subgroup $P(\mathrm{U}(p+1,q) \times \mathrm{U}(1))$ of $\mathrm{PU}(p+1,q+1)$. We define a 1-form η_1 on $P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q})$ by

$$(P \circ \iota)^* \eta_1 = \omega_1. \tag{5.5}$$

Fix $x \in \Sigma_{\mathbb{C}}^{2p+1,2q}$. As the inner product satisfies

$$\langle (\iota_*((\mathrm{Null}\,\omega_1)_x), \iota(x)\rangle = \langle ((\mathrm{Null}\,\omega_1)_x, 0), (x, 1))\rangle = \langle (\mathrm{Null}\,\omega_1)_x, x\rangle - \langle 0, 1\rangle = 0,$$

we obtain that

$$\iota_*((\mathrm{Null}\,\omega_1)_x) = \iota(x)^\perp = (\iota(x)\mathbb{C})^\perp.$$

Since $z^\perp$ is obviously J_0-invariant for any $z \in V_0^{2n+3}$, J_0 induces a complex structure J on $P_*(z^\perp)$ for all $P(z) \in S^{2p+1,2q}$. (5.5) implies that

$$(\mathrm{Null}\,\eta_1)_{[\iota(x)]} = P_*\iota_*((\mathrm{Null}\,\omega_1)_x) = P_*((\iota(x)\mathbb{C})^\perp). \tag{5.6}$$

If $f \in \mathrm{PU}(p+1,q+1)$, then

$$f_*((\mathrm{Null}\,\eta_1)_{[\iota(x)]}) = f_*P_*(\iota(x)^\perp) = P_*(f(\iota(x)^\perp)) = P_*((f\iota(x))^\perp). \tag{5.7}$$

As

$$f_*((\mathrm{Null}\,\eta_1)_{[\iota(x)]}) = (\mathrm{Null}\,f^{-1*}\eta_1)_{[f(\iota(x))]}$$

holds on $f(P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q}))$, $(\mathrm{Null}\,f^{-1*}\eta_1)_{[f(\iota(x))]}$ is J_0-invariant. Furthermore, if $h([\iota(x)]) = [\iota(x)]$, then $h(\iota(x)) = \iota(x) \cdot \lambda$ for $\lambda \in \mathbb{C}^*$. It follows that

$$h_*((\mathrm{Null}\,\eta_1)_{[\iota(x)]}) = h_*(P_*((\iota(x)\mathbb{C})^\perp)) = P_*((\iota(x)\lambda\mathbb{C})^\perp) = (\mathrm{Null}\,\eta_1)_{[\iota(x)]}. \tag{5.8}$$

Since $P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q})$ is an open dense subset in $S^{2p+1,2q}$, obviously $\{f(P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q}))\}_{f \in \mathrm{PU}(p+1,q+1)}$ covers $S^{2p+1,2q}$. For $a \in S^{2p+1,2q}$, choose $f \in \mathrm{PU}(p+1,q+1)$ such that $f([\iota(x)]) = a$. In view of (5.8), we obtain the well-defined subspace $\mathcal{H}_a = (\mathrm{Null}\,f^{-1*}\eta_1)_{[f(\iota(x))]}$. Hence the distribution

$$\mathcal{H} = \{(\mathrm{Null}\,f^{-1*}\eta_1)_{[f(\iota(x))]}\}_{f \in \mathrm{PU}(p+1,q+1)}$$

endowed with J is a nondegenerate CR structure on $S^{2p+1,2q}$.

Theorem 5.3 (cf. Definition 5.5). *There exists a pseudo-Hermitian pair (ω, J) on $S^{2p+1,2q}$ such that*

$$\mathcal{H} = \mathrm{Null}\,\omega.$$

Moreover, the group of CR transformations $\mathrm{Aut}_{\mathrm{CR}}(S^{2p+1,2q})$ *is* $\mathrm{PU}(p+1,q+1)$.

Proof. Put $U_\alpha = h_\alpha(P \circ \iota(\Sigma^{2p+1,2q}_{\mathbb{C}}))$ for each $h_\alpha \in \mathrm{PU}(p+1, q+1)$ and the form $\eta_\alpha = h_\alpha^{-1*}\eta_1$ on U_α. Since the open sets U_α covers $S^{2p+1,2q}$, suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Consider the local change $h_{\beta\alpha} = h_\beta^{-1} \circ h_\alpha \colon h_\alpha^{-1}(U_\alpha \cap U_\beta) \to h_\beta^{-1}(U_\alpha \cap U_\beta)$ which extends to $P \circ \iota(\Sigma^{2p+1,2q}_{\mathbb{C}})$. Then since $h_{\beta\alpha}$ preserves the CR structure on $P \circ \iota(\Sigma^{2p+1,2q}_{\mathbb{C}})$, $(h_{\beta\alpha})_* \operatorname{Null}\eta_1 = \operatorname{Null}\eta_1$, which implies that $(h_{\beta\alpha})^{-1*}\eta_1 = u_{\beta\alpha}\cdot\eta_1$ for some $u_{\beta\alpha} > 0$ on $U_\alpha \cap U_\beta$. As in the proof of Proposition 4.1, there is the set of local functions $\{u\}_{\alpha\in\Lambda}$ such that $u_\beta \cdot \eta_\beta = u_\alpha \cdot \eta_\alpha$ on $U_\alpha \cap U_\beta$. Setting

$$\omega | U_\alpha = u_\alpha \cdot \eta_\alpha,$$

this defines a nondegenerate 1-form ω on $S^{2p+1,2q}$ for which $\mathcal{H} = \operatorname{Null}\omega$. □

Remark 5.4. From (2.2), the flat nondegenerate CR space form

$$S^{2p+1,2q} = S^{2p+1} \times S^{2q+1}/S^1$$

has the principal circle bundle

$$T^1 = S^1 \times S^1/S^1 \to S^{2p+1} \times S^{2q+1}/S^1 \xrightarrow{\pi} \mathbb{CP}^p \times \mathbb{CP}^q.$$

There is the connection ω_2 of this bundle such that $\pi^*(\Omega_p \times -\Omega_q) = d\omega_2$ where $[\Omega_p]\times[-\Omega_q]$ is the Kähler class of the product $\mathbb{CP}^p \times \mathbb{CP}^q$. In particular, $(\operatorname{Null}\omega_2, J)$ gives a CR structure on $S^{2p+1,2q}$ of signature $(2p, 2q)$. Here J is a complex structure which is the pullback of the complex structure under the isomorphism $\pi_* \colon \operatorname{Null}\omega_2 \to T(\mathbb{CP}^p \times \mathbb{CP}^q)$ at each point. The restriction of this structure provides a nondegenerate CR structure to the embedding $P \circ \iota(\Sigma^{2P+1,2q}_{\mathbb{C}})$ of (5.4). However, this CR structure is different from (η_1, J). For this, the above T^1-action restricted to $P \circ \iota(\Sigma^{2P+1,2q}_{\mathbb{C}})$ is described by

$$t[(z_1, \ldots, z_{p+1}, w_1, \ldots, w_q, 1)] = [(t \cdot z_1, \ldots, t z_{p+1}, w_1, \ldots, w_q, 1])$$

which is different from the principal action of S^1 on $P \circ \iota(\Sigma^{2P+1,2q}_{\mathbb{C}})$ because these circle actions are linear on $\mathbb{C}^{n+2}$ and are non-equivalent. Hence, this structure $(\operatorname{Null}\omega_2, J)$ is not flat.

5.3 Conformal viewpoint of pseudo-Hermitian geometry and uniformization

There is no canonical choice of contact forms ω which represents $\mathcal{D}$ leaving J fixed. If θ is another form, then there exists a positive function u on M such that

$$\theta = u \cdot \omega. \tag{5.9}$$

Definition 5.5. A pair (ω, J) is said to be a nondegenerate pseudo-Hermitian structure on M.

Another nondegenerate pseudo-Hermitian structure (θ, J) gives a nondegenerate CR structure $(\mathcal{D}, J) = (\mathrm{Null}\,\theta, J)$ on M. So to say, a CR structure is the *conformal class of the pseudo Hermitian forms with J fixed.* Chern and Moser have defined a curvature tensor $S(\omega, J)$ for the pseudo-Hermitian pair (ω, J). From this conformal viewpoint it is a CR invariant, as proved by the same authors and Webster (cf. [15], [34], [33]).

$$S = S(\omega, J) = S(\theta, J). \tag{5.10}$$

The vanishing of Chern–Moser curvature tensor S provides a uniformization of a nondegenerate CR-manifold $(M, \{\mathcal{D}, J\})$ with respect to *flat CR geometry* $(\mathrm{PU}(p+1, q+1), S^{2p+1,2q})$. When $q = 0$, i.e., in the strictly pseudokonvex case, $(\mathrm{PU}(n+1, 1), S^{2n+1})$ is usually called *spherical CR geometry*.

Theorem 5.6. *If $S = 0$ identically on a $(2n+1)$-dimensional nondegenerate CR-manifold $(n \geq 2)$, then M admits a developing pair (ρ, dev) up to composition with elements of* $\mathrm{PU}(p+1, q+1)$:

$$(\rho, \mathrm{dev})\colon (\mathrm{Aut}_{\mathrm{CR}}(\tilde{M}), \tilde{M}) \to (\mathrm{PU}(p+1, q+1), S^{2p+1,2q}),$$
$$\mathrm{dev}(h \cdot x) = \rho(h)\,\mathrm{dev}(x) \quad (h \in \mathrm{Aut}_{\mathrm{CR}}(\tilde{M})),$$

where $\mathrm{Aut}_{\mathrm{CR}}(\tilde{M})$ is the group of CR-automorphisms of the universal covering space $\tilde{M}$.

5.4 CR structures from the viewpoint of Cartan geometry

Given a nondegenerate CR structure (H, J) on a $(2n+1)$-dimensional manifold M, there is an $\mathfrak{su}(p+1, q+1)$-valued 1-form κ called a Cartan connection whose associated curvature form Π vanishes if and only if M is locally isomorphic to $\mathrm{PU}(p+1, q+1)/\mathrm{P}^+(\mathbb{C})$ where $\mathrm{P}^+(\mathbb{C})$ denotes the maximal parabolic subgroup $(p + q = n)$. (See [3].) The fourth-order Chern–Moser CR curvature tensor $S = (S_{\alpha}{}^{\beta}{}_{\rho\sigma})$ is the coefficient of the component Φ_{α}^{β} of the curvature matrix Π. Observed by Webster (cf. [34], [33]), the other components are obtained from S by further covariant differentiation for $n > 1$. In the CR case, the Chern–Moser curvature tensor S vanishes on M if and only if so does the $\mathfrak{su}(p+1, q+1)$-valued Cartan curvature form Π

6 Bochner flat pseudo-Kähler geometry

Webster [33] has proved that the formula of the Chern–Moser CR curvature tensor coincides with the Bochner curvature tensor of a Kähler manifold. In fact, suppose that a nondegenerate CR-manifold M admits a Seifert fibration

$$T \to M \xrightarrow{\nu} Y,$$

where T is either S^1 or $\mathbb{R}$ acting properly on M as CR-transformations. (In general, this works locally and Y is an orbifold.) Let ω be a contact form representing the CR structure $\mathcal{D}$ (keeping J fixed) and ξ the CR-vector field which generates T. If $\omega(\xi) \neq 0$ on a domain X of M, then we can define a contact form as follows:

$$\eta = \frac{1}{\omega(\xi)} \cdot \omega \quad \text{on } X. \tag{6.1}$$

Then ξ becomes a characteristic CR-vector field for η. In other words, ξ generates T which is a 1-parameter subgroup of pseudo-Hermitian transformations of η, i.e.,

$$\begin{aligned} &\eta(\xi) = 1, \\ &\mathcal{L}_\xi \eta = \iota_\xi d\eta = 0. \end{aligned} \tag{6.2}$$

When we put $W = X/T$, the complex structure J on $\mathcal{D}$ induces a complex structure $\hat{J}$ on W. Letting

$$\nu^* \Omega = d\eta, \tag{6.3}$$

Ω gives a $\hat{J}$-invariant closed 2-form on W. As η is a contact form, Ω becomes a Kähler form on W for which $\hat{g}(\cdot, \cdot) = \Omega(\hat{J}\cdot, \cdot)$ becomes a pseudo-Kähler metric (indefinite metric) on $(W, \hat{J})$. Note that the Bochner curvature tensor $B(\hat{g}, \hat{J})$ of the pseudo-Kähler metric $\hat{g}$ on W coincides with $S(\omega, J)$ of X by (6.3), (5.10). As a consequence, if $(M, \{\mathcal{D}, J\})$ is a flat nondegenerate CR-manifold, then $(W, \{\hat{g}, \hat{J}\})$ is a Bochner flat pseudo-Kähler manifold.

6.1 Bochner flat pseudo-Kähler structures on $\mathbb{C}^{p,q}$ and CR structures on the Heisenberg space $\mathcal{N}$

Let $\mathrm{Aut}_{\mathrm{CR}}(S^{2p+1,2q}) = \mathrm{PU}(p+1, q+1)$ be the group of all automorphisms preserving the flat CR structure of $S^{2p+1,2q}$. We consider the stabilizer of the point at infinity $\{\infty\} = [1, 0, \ldots, 0, 1] \in P \circ \iota(\Sigma^{2p+1,2q}_{\mathbb{C}}) \subset S^{2p+1,2q}$. Recall the notion of the (indefinite) Heisenberg nilpotent Lie group $\mathcal{N} = \mathcal{N}(p, q)$ from [19]. It is the product $\mathbb{R} \times \mathbb{C}^n$ with group law

$$(a, y) \cdot (b, z) = (a + b - \mathrm{Im}\langle y, z\rangle, y + z).$$

Here $\langle\, , \rangle$ is the Hermitian inner product of signature (p, q) on $\mathbb{C}^n$ as in (2.1) and $\mathrm{Im}\langle\, , \rangle$ is the imaginary part ($p + q = n$). It is nilpotent because the commutator subgroup $[\mathcal{N}, \mathcal{N}] = \mathbb{R}$ coincides with the center of $\mathcal{N}$ consisting of elements of the form $(a, 0)$. In particular, there is the central group extension

$$1 \to \mathbb{R} \to \mathcal{N} \longrightarrow \mathbb{C}^n \to 1. \tag{6.4}$$

Denote by $\mathrm{Sim}(\mathcal{N})$ the semidirect product $\mathcal{N} \rtimes (\mathrm{U}(p, q) \times \mathbb{R}^+)$ where the action $(A, t) \in \mathrm{U}(p, q) \times \mathbb{R}^+$ on $(a, y) \in \mathcal{N}$ is given by

$$(A, t) \circ (a, y) = (t^2 \cdot a,\ t \cdot Ay). \tag{6.5}$$

Denote the origin by $O = [1, 0, \ldots, 0, -1] \in P \circ \iota(\Sigma_{\mathbb{C}}^{2p+1,2q}) - \{\infty\}$. The stabilizer $\mathrm{Aut}_{\mathrm{CR}}(S^{2p+1,2q})_\infty$ is isomorphic to $\mathrm{Sim}(\mathcal{N})$ (cf. [21]). The orbit $\mathcal{N} \cdot O$ is an open dense subset of $S^{2p+1,2q}$. The embedding μ is defined by

$$(a, (z_+, z_-)) \in \mathcal{N} \xrightarrow{\mu} \begin{bmatrix} \frac{\|z_+\|^2 - \|z_-\|^2}{2} - 1 + \boldsymbol{i}\, a \\ \sqrt{2} z_+ \\ \sqrt{2} z_- \\ \frac{\|z_+\|^2 - \|z_-\|^2}{2} + 1 + \boldsymbol{i}\, a \end{bmatrix}. \tag{6.6}$$

Then the pair $(\mathrm{Sim}(\mathcal{N}), \mathcal{N})$ is said to be a *flat Heisenberg CR geometry* which is a subgeometry of the flat CR geometry $(\mathrm{Aut}_{\mathrm{CR}}(S^{2p+1,2q}), S^{2p+1,2q})$. The rest of this section is devoted to the construction of *Bochner flat pseudo-Kähler structures* on $\mathbb{C}^n$ from $\mathcal{N}$. The contact form $\omega_{\mathcal{N}}$ on $\mathcal{N}$ is described as follows. If $(t, (z_1, \ldots, z_n))$ is the coordinate of $\mathcal{N} = \mathbb{R} \times \mathbb{C}^n$, then

$$\begin{aligned} \omega_{\mathcal{N}} &= dt + \sum_{j=1}^{p} (x_j dy_j - y_j dx_j) - \sum_{j=p+1}^{n} (x_j dy_j - y_j dx_j) \\ &= dt + \mathrm{Im}\langle z, dz \rangle. \end{aligned} \tag{6.7}$$

The subgroup $\mathrm{Psh}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(p, q)$ leaves the form $\omega_{\mathcal{N}}$ invariant. In fact, if $\gamma = ((a, w), A) \in \mathcal{N} \rtimes \mathrm{U}(p, q)$, then

$$((a, w), A) \cdot (t, z) = (a + t - \mathrm{Im}\langle w, Az \rangle, w + Az),$$

and so

$$\gamma^* \omega_{\mathcal{N}} = dt - d\,\mathrm{Im}\langle w, Az \rangle + \mathrm{Im}\langle w + Az, d(w + Az) \rangle.$$

Since $d\,\mathrm{Im}\langle w, Az \rangle = \mathrm{Im}\langle w, dAz \rangle$, it is easy to see that

$$\gamma^* \omega_{\mathcal{N}} = dt + \mathrm{Im}\langle z, dz \rangle = \omega_{\mathcal{N}}.$$

Let $(\mathrm{Null}\,\omega, J)$ be the flat CR structure on $S^{2p+1,2q}$ (cf. Theorem 5.5). Restricted to $\mathcal{N} \subset S^{2p+1,2q} - \{0\}$, we have a flat CR structure $(\mathrm{Null}\,\omega_{\mathcal{N}}, J)$ on $\mathcal{N}$. In general, for $h \in \mathrm{Aut}_{\mathrm{CR}}(\mathcal{N})$, there exists a positive function u on $\mathcal{N}$ such that

$$h^* \omega_{\mathcal{N}} = u \cdot \omega_{\mathcal{N}}.$$

Moreover, h is holomorphic (Cauchy–Riemann) on $\mathrm{Null}\,\omega_{\mathcal{N}}$, so every element h of $\mathrm{Aut}_{\mathrm{CR}}(\mathcal{N})$ preserves the CR structure $(\mathrm{Null}\,\omega_{\mathcal{N}}, J)$.

On the other hand, we have the canonical principal fibration

$$\mathbb{R} \to (\mathcal{N}, \omega_{\mathcal{N}}) \xrightarrow{P} (\mathbb{C}^n, \Omega_0), \tag{6.8}$$

where $d\omega_{\mathcal{N}} = P^* \Omega_0$ such that

$$\Omega_0 = 2 \sum_{j=1}^{p} dx_j \wedge dy_j - 2 \sum_{j=p+1}^{n} dx_j \wedge dy_j \tag{6.9}$$

is the standard (indefinite) Kähler form of $\mathbb{C}^n$ and $g_0 = \Omega_0(J_{\mathbb{C}}\cdot,\cdot)$ is the pseudo-complex euclidean metric of signature (p,q): $ds^2 = |dz_1|^2+\cdots+|dz_p|^2-|dz_{p+1}|^2-\cdots-|dz_n|^2$ $(p+q=n)$. In other words, the CR structure J on $\operatorname{Null}\omega_{\mathcal{N}}$ is obtained from the standard complex structure $J_{\mathbb{C}}$ on $\mathbb{C}^n$ by the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Null}\omega_{\mathcal{N}} & \xrightarrow{P_*} & T(\mathbb{C}^n) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J_{\mathbb{C}}} \\ \operatorname{Null}\omega_{\mathcal{N}} & \xrightarrow{P_*} & T(\mathbb{C}^n). \end{array} \tag{6.10}$$

Let $\rho\colon \mathbb{R}\to\mathbb{R}\times T^n$ be the representation defined by

$$\rho(t) = ((t,0),(e^{\boldsymbol{i}t\cdot a_1},\dots,e^{\boldsymbol{i}t\cdot a_p},e^{\boldsymbol{i}t\cdot b_{p+1}},\dots,e^{\boldsymbol{i}t\cdot b_n}))$$

such that

$$b_{p+1}\le\cdots\le b_n\le 0\le a_1\le\cdots\le a_p. \tag{6.11}$$

Note that if all $a_i=0, b_j=0$, then $\rho(\mathbb{R})$ is the center of $\mathcal{N}$. In general, $\rho(\mathbb{R})$ is a closed subgroup of $\mathbb{R}\times T^n\subset \operatorname{Psh}(\mathcal{N})$ isomorphic to $\mathbb{R}$. As $\mathbb{R}\times T^n$ acts properly on $\mathcal{N}$, $\rho(\mathbb{R})$ acts properly and freely on $\mathcal{N}$. Let

$$\rho(\mathbb{R})\to\mathcal{N}\xrightarrow{\nu}\mathcal{N}/\rho(\mathbb{R}) \tag{6.12}$$

be the principal bundle. As ν_* maps $\operatorname{Null}\omega_{\mathcal{N}}$ isomorphically onto $T(\mathcal{N}/\rho(\mathbb{R}))$ at each point, J induces a complex structure $\hat{J}$ on $\mathcal{N}/\rho(\mathbb{R})$. We note the following proposition whose proof is the same as that of [20].

Proposition 6.1. *The orbit space $(\mathcal{N}/\rho(\mathbb{R}),\hat{J})$ is biholomorphic to $(\mathbb{C}^n,J_{\mathbb{C}})$.*

Put $A_t=(e^{\boldsymbol{i}t\cdot a_1},\dots,e^{\boldsymbol{i}t\cdot a_p},e^{\boldsymbol{i}t\cdot b_{p+1}},\dots,e^{\boldsymbol{i}t\cdot b_n})$. Recall that $\rho(\mathbb{R})$ acts on $\mathcal{N}$ by

$$\rho(t)(s,z) = (s+t,A_tz)\ \ ((s,z)\in\mathcal{N}).$$

Let ξ be the vector field on $\mathcal{N}$ induced by $\rho(\mathbb{R})$. Then,

$$\xi = \frac{d}{dt}+\sum_{j=1}^{n}a_j\left(x_j\frac{d}{dy_j}-y_j\frac{d}{dx_j}\right)-\sum_{j=p+1}^{n}b_j\left(x_j\frac{d}{dy_j}-y_j\frac{d}{dx_j}\right).$$

Using (6.7), $\omega_{\mathcal{N}}(\xi) = 1+(\sum_{j=1}^{p}a_j|z_j|^2-\sum_{j=p+1}^{n}b_j|z_j|^2)$. By the hypothesis (6.11), $\omega_{\mathcal{N}}(\xi)>0$ everywhere on $\mathcal{N}$. We have the contact form

$$\eta(Z) = \frac{1}{\omega_{\mathcal{N}}(\xi)}\cdot\omega_{\mathcal{N}}(Z)\quad\text{for all } Z\in T\mathcal{N}. \tag{6.13}$$

As in (6.2), it follows that

$$\eta(\xi)=1,\quad d\eta(\xi,X)=0 \text{ for all } X\in T\mathcal{N}. \tag{6.14}$$

From (6.3), we have an indefinite Hermitian metric on $(\mathcal{N}/\rho(\mathbb{R}), \hat{J}) = (\mathbb{C}^n, J_{\mathbb{C}})$:

$$\hat{g}(\hat{X}, \hat{Y}) = d\eta(JX, Y),$$

where $X, Y \in \text{Null } \eta$ such that $\nu_*(X) = \hat{X}$, $\nu_*(Y) = \hat{Y}$. Let $\hat{\Omega}(\hat{X}, \hat{Y}) = \hat{g}(\hat{X}, \hat{J}\hat{Y})$ be the fundamental two-form on $\mathcal{N}/\rho(\mathbb{R}) = \mathbb{C}^n$. Using (6.14), it follows that $\nu^*\hat{\Omega} = d\eta$, i.e., $d\hat{\Omega} = 0$. Therefore, $\hat{\Omega}$ is a pseudo-Kähler form on $\mathbb{C}^n$. Thus we obtain a Bochner flat pseudo-Kähler structure $(\hat{g}, \hat{\Omega}, J_{\mathbb{C}})$ on $\mathbb{C}^n$. For $(a, b) = (a_1, \ldots, a_p, b_{p+1}, \ldots, b_n)$, we put $\hat{g} = g_{a,b}$. Since $\nu^*\hat{\Omega} = d\eta$ and $(\text{Null } \eta, J)$ is a nondegenerate spherical CR structure (i.e., $S(\eta, J) = 0$), we note that $B(g_{a,b}, J_{\mathbb{C}}) = 0$. Hence, $(\mathbb{C}^n, g_{a,b})$ is a Bochner flat pseudo-Kähler manifold.

Problem 6.2. We do not know whether the pseudo-Kähler metric $g_{a,b}$ is geodesically complete whenever $b_{p+1} \leq \cdots \leq b_n \leq 0 \leq a_1 \leq \cdots \leq a_p$.

Remark 6.3. Put $\omega = \omega_{\mathcal{N}}$. The new pseudo-Kähler metric $\hat{g} = g_{a,b}$ on $\mathbb{C}^n$ has been obtained by deforming ω to $\eta = u \cdot \omega$ where $u = 1/\omega(\xi)$. The effect under this change is to produce a globally conformal pseudo-Kähler metric $\hat{g}_\omega$ starting from the standard pseudo-complex euclidean metric g_0 of (6.9). Because $d\eta = u \cdot d\omega$ on $\text{Null } \omega$, the pushforward by ν gives $\hat{\Omega} = \hat{u} \cdot \Theta$ on $\mathbb{C}^n$ where $\Theta(\nu_* X, \nu_* Y) = d\omega(X, Y)$. Let $\hat{g}_\omega(A, B) = \Theta(J_{\mathbb{C}} A, B)$. Then $\hat{u} \cdot \hat{g}_\omega(A, B) = \hat{\Omega}(J_{\mathbb{C}} A, B) = \hat{g}(A, B)$. Hence the nondegenerate Hermitian metric $\hat{g}_\omega$ is globally conformal to the pseudo-Kähler metric $\hat{g}$. This deformation is carried within the locally conformal (indefinite) Kähler geometry.

6.2 Bochner flat structures on weighted pseudo-Kähler projective orbifolds

There is an equivariant principal bundle from (3.2):

$$\mathrm{U}(1) \to (\mathrm{U}(p+1, q), \Sigma_{\mathbb{C}}^{2p+1,2q}) \xrightarrow{\pi} (\mathrm{PU}(p+1, q), \mathbb{CP}^{p,q}).$$

The maximal compact Lie group of $\mathrm{U}(p+1, q)$ is conjugate to $\mathrm{U}(p+1) \times \mathrm{U}(q)$ which contains the maximal torus T^{n+1}. Choose a representation of $\mathbb{R}$ such that

$$\rho(t) = (e^{\mathrm{i}t \cdot a_1}, \ldots, e^{\mathrm{i}t \cdot a_{p+1}}, e^{\mathrm{i}t \cdot b_{p+2}}, \ldots, e^{\mathrm{i}t \cdot b_{n+1}}), \tag{6.15}$$

where the a_i, b_j's are rational numbers $(p + q = n)$. So the image $\rho(\mathbb{R}) = S^1$. In particular, $\rho(\mathbb{R})$ acts properly on $\Sigma_{\mathbb{C}}^{2p+1,2q}$. It acts freely if and only if $a_1 = \cdots = a_{p+1} = b_{p+2} = \cdots = b_{n+1}$. It is easy to see that $\rho(\mathbb{R})$ induces the vector field

$$\xi = \sum_{j=1}^{p+1} a_j \left(x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j} \right) + \sum_{j=p+2}^{n+1} b_j \left(x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j} \right) \tag{6.16}$$

on $\Sigma_{\mathbb{C}}^{2p+1,2q}$. For the 1-form ω_0 of (5.1), let $\eta_1 = \boldsymbol{i}\,\omega_0 = \mathrm{Im}\langle z, dz\rangle$ which is a real 1-form. Calculate that

$$\eta_1(\xi) = \sum_{j=1}^{p+1} a_j|z_j|^2 - \sum_{j=p+2}^{n+1} b_j|z_j|^2. \tag{6.17}$$

Note that $\eta_1(\xi) > 0$ whenever

$$b_{p+2} \le \cdots \le b_{n+1} < 0 < a_1 \le a_2 \le \cdots \le a_{p+1}.$$

As remarked, $\rho(\mathbb{R}) = S^1$ acts properly but not necessarily freely. When all a_i, b_j are mutually distinct, the quotient is an orbifold. The quotient orbifold

$$\begin{aligned}&\mathbb{CP}^{p,q}(a_1,\dots,a_{p+1};b_{p+2},\dots,b_{n+1})\\ &\quad = \Sigma_{\mathbb{C}}^{2p+1,2q}/\langle e^{\mathrm{i}t\cdot a_1},\dots,e^{\mathrm{i}t\cdot a_{p+1}},e^{\mathrm{i}t\cdot b_{p+2}},\dots,e^{\mathrm{i}t\cdot b_{n+1}}\rangle\end{aligned} \tag{6.18}$$

is called the *weighted pseudo-Kähler projective space*. It is not necessarily compact (cf. Remark 6.5). In the same manner we obtain the following result.

Corollary 6.4. *Suppose that all a_i, b_j are mutually distinct positive rational numbers. Then an orbifold $\mathbb{CP}^{p,q}(a_1,\dots,a_{p+1};b_{p+2},\dots,b_{n+1})$ admits a Bochner flat pseudo-Kähler (singular) metric.*

Remark 6.5. If $p = n$, $q = 0$, then the above *compact* quotient is the weighted complex projective space $\mathbb{CP}^n(a_1,\dots,a_{n+1})$. If $p = 0$, $q = n$, the above *noncompact* quotient $V_{-1}^{2n+1}/\langle e^{\mathrm{i}t\cdot a_1}, e^{\mathrm{i}t\cdot b_2},\dots,e^{\mathrm{i}t\cdot b_{n+1}}\rangle$ is the complex hyperbolic orbifold $\mathbb{H}^n_{\mathbb{C}}(a_1;b_2,\dots,b_{n+1})$. Here $V_{-1}^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid \mathcal{B}(z,z) = -1\}$.

7 Pseudo-conformal quaternionic CR geometry

Definition 7.1. A $4n$-dimensional orientable subbundle $\mathcal{D}$ equipped with a quaternionic structure Q is called an *almost pseudo-conformal quaternionic CR* structure (almost p-c qCR structure) on M if the following properties are satisfied.

(i) $\mathcal{D} \cup [\mathcal{D},\mathcal{D}] = TM$.

(ii) At any point $p \in M$, the 3-dimensional quotient bundle $TM/\mathcal{D}$ is isomorphic to the Lie algebra $\mathrm{Im}\,\mathbb{H} = \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$.

(iii) There exists an $\mathrm{Im}\,\mathbb{H}$-valued 1-form $\omega = \omega_1\boldsymbol{i} + \omega_2\boldsymbol{j} + \omega_3\boldsymbol{k}$ locally defined on a neighborhood of M such that $\mathcal{D} = \mathrm{Null}\,\omega = \bigcap_{\alpha=1}^3 \mathrm{Null}\,\omega_\alpha$ and $d\omega_\alpha|\mathcal{D}$ is nondegenerate. Here each ω_α is a real valued 1-form ($\alpha = 1,2,3$).

(iv) The endomorphism $J_\gamma = (d\omega_\beta|\mathcal{D})^{-1} \circ (d\omega_\alpha|\mathcal{D})\colon \mathcal{D} \to \mathcal{D}$ constitutes the quaternionic structure Q on $\mathcal{D}$: ${J_\gamma}^2 = -1$, $J_\alpha J_\beta = J_\gamma = -J_\beta J_\alpha$ ($\gamma = 1,2,3$) etc.

The form ω is called an admissible 1-form and $\{J_\alpha\}_{\alpha=1,2,3}$ are hypercomplex structures on $\mathcal{D}$ associated to ω. We have the following (cf. [3]):

Lemma 7.2. *If we put* $\sigma_\alpha = (d\omega_\alpha|\mathcal{D})$ *on* $\mathcal{D}$, *then the equality* $\sigma_1(J_1X, Y) = \sigma_2(J_2X, Y) = \sigma_3(J_3X, Y)$ *holds for all* $X, Y \in \mathcal{D}$.

Moreover, the form

$$g^{\mathcal{D}} = \sigma_\alpha \circ J_\alpha \tag{7.1}$$

is a nondegenerate Q*-invariant symmetric bilinear form on* $\mathcal{D}$.

In general, there is no canonical choice of ω which annihilates $\mathcal{D}$. The fiber of the quotient bundle $TM/\mathcal{D}$ is orientable and isomorphic to Im $\mathbb{H}$ by ω on a neighborhood U by (ii). If ω' is another admissible 1-form such that Null $\omega' = \mathcal{D}$ on U', then ω' is uniquely determined on $U \cap U'$ by

$$\begin{aligned} &\omega' = u \cdot a \cdot \omega \cdot \bar{a} \quad \text{for some functions } a \in \mathrm{Sp}(1),\ u > 0, \\ &J'_\alpha = a_{\beta\alpha} \cdot J_\alpha. \end{aligned} \tag{7.2}$$

Here $\{J'_\alpha\}$ are the associated hypercomplex structures to ω' for which $A = (a_{\alpha\beta}) \in \mathrm{SO}(3)$ is obtained from the conjugate by $a \in \mathrm{Sp}(1)$. Thus Definition 7.1 does not depend on the choice of ω' satisfying (7.2). By Lemma 7.2, we may assume that $g^{\mathcal{D}}$ on $\mathcal{D}|U$ has signature $(4p, 4q)$ $(p+q = n)$. As above put $g'^{\mathcal{D}}(X, Y) = d\omega'_\alpha(J'_\alpha X, Y)$ $(X, Y \in \mathcal{D})$.

Corollary 7.3. *If* $\omega' = u \cdot a \cdot \omega \cdot \bar{a}$ *on* $U \cap U'$, *then* $g'^{\mathcal{D}} = u \cdot g^{\mathcal{D}}$. *As a consequence, the signature* $(4p, 4q)$ *is constant on* $U \cap U'$ *(and hence everywhere on* M*) under the change* $\omega' = u \cdot a \cdot \omega \cdot \bar{a}$.

We shall consider an integrability condition on the almost p-c qCR structure $\mathcal{D}$.

Definition 7.4. Put the skew symmetric 2-forms ρ_α locally on a neighborhood:

$$\rho_\alpha = d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma, \tag{7.3}$$

where $(\alpha, \beta, \gamma) \sim (1, 2, 3)$ up to cyclic permutation. If ρ_α satisfies that

$$\mathrm{Null}\,\rho_1 = \mathrm{Null}\,\rho_2 = \mathrm{Null}\,\rho_3, \tag{7.4}$$

then the pair (ω, Q) is a (*local*) *quaternionic CR structure* (qCR structure) on M.

(See [7], [2].) When ω is entirely defined on M, it is called a quaternionic CR structure. Using the two Definitions 7.1 and 7.4, we arrive at the following notion due to Libermann [28].

Definition 7.5 (p-c qCR structure). The pair $(\mathcal{D}, Q)$ on M is said to be a *pseudo-conformal quaternionic CR* structure if there exists locally an admissible 1-form ω with Null $\omega = \mathcal{D}$ on a neighborhood U at each point of M such that ω is a qCR structure on U.

Note that a quaternionic CR structure $(\omega, \{I, J, K\})$ gives an almost pseudo-conformal quaternionic CR structure $(\mathrm{Null}\,\omega, \{I, J, K\})$ on M. We can choose three (characteristic CR) vector fields $\{\xi_1, \xi_1, \xi_3\}$ which constitute the distribution of (7.4) such that $\omega_\alpha(\xi_\beta) = \delta_{\alpha\beta}$. Then the almost complex structure J_α naturally extends to an almost complex structure $\bar{J}_\alpha$ on $\mathcal{D} \oplus \{\xi_\beta, \xi_\gamma\} = \mathrm{Null}\,\omega_\alpha$ such that

$$\bar{J}_\alpha|\mathcal{D} = J_\alpha, \quad \bar{J}_\alpha\xi_\beta = \xi_\gamma, \quad \bar{J}_\alpha\xi_\gamma = -\xi_\beta.$$

Contrary to the nondegenerate CR structure, the almost complex structure on $\mathcal{D}$ is not assumed to be *integrable*. However, by the requirement of structure equations defining the qCR structure, we can prove the integrability of the quaternionic structure. (Compare [3], [2] for the proof.)

Theorem 7.6. *Each almost complex structure $\bar{J}_\alpha$ of the quaternionic CR structure is a complex structure on the codimension-*1 *contact subbundle* $\mathrm{Null}\,\omega_\alpha$ $(\alpha = 1, 2, 3)$.

This implies that *the integrability of three almost complex structures $\{\bar{J}_\alpha\}_{\alpha=1,2,3}$ is equivalent with the condition that (M, g) is a pseudo-Sasakian 3-structure*, i.e., the notion of a nondegenerate quaternionic CR structure is equivalent to the notion of a pseudo-Sasakian 3-structure (cf. [2]). The 3-Sasakian metric on M is defined by

$$g(X, Y) = \sum_{\alpha=1}^{3} \omega_\alpha(X) \cdot \omega_\alpha(Y) + d\omega_1(\bar{J}_1 X, Y) \quad \text{for all } X, Y \in TM. \tag{7.5}$$

Note that $d\omega_1(\bar{J}_1, \ ) = d\omega_2(\bar{J}_2, \ ) = d\omega_3(\bar{J}_3, \ )$.

Remark 7.7. Consider the structure equation of the nondegenerate CR case. Let ω be a 1-form which represents a CR structure $(\mathrm{Null}\omega, J)$. The CR structure has no such equality (7.4). There corresponds only one structure equation (5.3). Here J is assumed to be integrable. However, by the coincidence (7.4) in the p-c qCR case, Theorem 7.6 shows that each almost complex structure $\bar{J}_\alpha$ $(\alpha = 1, 2, 3)$ is integrable. Moreover, each characteristic vector field ξ_α of ω_α is a CR-vector field which never occurs only from (5.3) for the nondegenerate CR structure in general.

7.1 Construction of the qCR structure on $\Sigma_{\mathbb{H}}^{3+4p,4q}$

We now consider the quaternionic pseudo-Riemannian space form $\Sigma_{\mathbb{H}}^{4p+3,4q}$ of signature $(4p + 3, 4q)$ (cf. Section 3). As in (5.1), we put

$$\omega_0 = -(\bar{z}_1 dz_1 + \cdots + \bar{z}_{p+1} dz_{p+1} - \bar{w}_1 dw_1 - \cdots - \bar{w}_q dw_q). \tag{7.6}$$

Then it is easy to check that ω_0 is an $\mathfrak{sp}(1)$-valued 1-form on $\Sigma_{\mathbb{H}}^{3+4p,4q}$. Let ξ_1, ξ_2, ξ_3 be the vector fields on $\Sigma_{\mathbb{H}}^{3+4p,4q}$ induced by the one-parameter subgroups $\{e^{\boldsymbol{i}\theta}\}_{\theta\in\mathbb{R}}$, $\{e^{\boldsymbol{j}\theta}\}_{\theta\in\mathbb{R}}$, $\{e^{\boldsymbol{k}\theta}\}_{\theta\in\mathbb{R}}$, respectively, which is equivalent to $\xi_1 = IN$, $\xi_2 = JN$, $\xi_3 =$

KN. Note that $\{\xi_1, \xi_2, \xi_3\}$ generates the fields of Lie algebra of $\mathrm{Sp}(1)$. A calculation shows that

$$\omega_0(\xi_1) = \boldsymbol{i}\,, \quad \omega_0(\xi_2) = \boldsymbol{j}\,, \quad \omega_0(\xi_3) = \boldsymbol{k}\,. \tag{7.7}$$

By the formula for ω_0, if $a \in \mathrm{Sp}(1)$, then the right translation R_a on $\Sigma^{3+4p,4q}_{\mathbb{H}}$ satisfies

$$\mathrm{R}_a^*\omega_0 = a \cdot \omega_0 \cdot \bar{a}\,. \tag{7.8}$$

Therefore, ω_0 is a connection form of the principal bundle given in (3.2). Note that $\mathrm{Sp}(p+1, q)$ leaves ω_0 invariant. From (3.6) we have the $4n$-dimensional bundle $\mathcal{D}_{\mathbb{H}} = \{IN, JN, KN\}^{\perp}$. For this, we shall check that $\Sigma^{4p+3,4q}_{\mathbb{H}}$ satisfies the conditions (i), (ii), (iii) and (iv) of Definition 7.1 and (7.4). In order to prove (iv) and (7.4), we quote the following lemma from [4].

Lemma 7.8. *Put $J_1 = I$, $J_2 = J$, $J_3 = K$. Then*

$$d\omega_\alpha(X, Y) = g^{\mathbb{H}}(X, J_\alpha Y) \quad (X, Y \in \mathcal{D}_{\mathbb{H}},\ \alpha = 1, 2, 3).$$

Theorem 7.9. *$(\Sigma^{3+4p,4q}_{\mathbb{H}}, \omega_0, \{I, J, K\}, g^{\mathbb{H}})$ is a $(4n+3)$-dimensional homogeneous quaternionic CR-manifold of signature $(3+4p, 4q)$ such that $\mathrm{Null}\,\omega_0 = \mathcal{D}_{\mathbb{H}}$ $(p + q = n \geq 1)$. Moreover, there exists the equivariant principal bundle of the pseudo-Riemannian submersion over (homogeneous) pseudo-quaternionic Kähler projective space $\mathbb{HP}^{p,q}$ of signature $(4p, 4q)$:*

$$\mathrm{Sp}(1) \to (\mathrm{Sp}(p+1, q) \cdot \mathrm{Sp}(1), \Sigma^{3+4p,4q}_{\mathbb{H}}, g^{\mathbb{H}}) \xrightarrow{\pi} (\mathrm{PSp}(p+1, q), \mathbb{HP}^{p,q}, \hat{g}^{\mathbb{H}}).$$

Proof. First of all, let $\omega_0 = \omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}$. Then it follows that

$$\begin{aligned} {\omega_0}^3 \wedge {d\omega_0}^{2n} &= \omega_0 \wedge \omega_0 \wedge \omega_0 \wedge \underbrace{(d\omega_0 \wedge d\omega_0) \wedge \cdots \wedge (d\omega_0 \wedge d\omega_0)}_{n\text{-times}} \\ &= 6\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge ({d\omega_1}^2 + {d\omega_2}^2 + {d\omega_3}^2)^n \neq 0 \end{aligned}$$

at any point of $\Sigma^{3+4p,4q}_{\mathbb{H}}$. (Compare, for example, [19], [29].) This calculation shows (i) and (ii). In particular, each ω_a is a nondegenerate contact form on $\Sigma^{3+4p,4q}_{\mathbb{H}}$. Using (7.8) and the fact that ξ_1 generates $\{e^{\boldsymbol{i}\theta}\}_{\theta \in \mathbb{R}} \subset \mathrm{Sp}(1)$, this implies that

$$\mathcal{L}_{\xi_1}\omega_1 = 0. \tag{7.9}$$

(Similarly $\mathcal{L}_{\xi_2}\omega_2 = \mathcal{L}_{\xi_3}\omega_3 = 0$.) Since $\omega_a(\xi_a) = 1$ from (7.7), it follows that

$$\mathcal{L}_{\xi_a}\omega_a = \iota_{\xi_a} d\omega_a = 0. \tag{7.10}$$

In particular, each ξ_a is the characteristic vector field for ω_a. This implies that $\mathcal{D}_{\mathbb{H}} = \bigcap_{a=1}^{3} \mathrm{Null}\omega_a$ for which there is the decomposition $T\Sigma^{3+4p,4q}_{\mathbb{H}} = \{\xi_1, \xi_2, \xi_3\} \oplus \mathcal{D}_{\mathbb{H}}$. This shows (iii). From Lemma 7.8, $d\omega_\alpha(X, Y) = g^{\mathbb{H}}(X, J_\alpha Y) = g^{\mathbb{H}}(X, J_\beta J_\gamma Y) = d\omega_\beta(X, J_\gamma Y)$, this shows (iv), i.e., $J_\gamma = (d\omega_\beta|\mathcal{D}_{\mathbb{H}})^{-1} \circ (d\omega_\alpha|\mathcal{D}_{\mathbb{H}})$. If $\{e_i\}_{i=1,\ldots,4n}$ is the orthonormal basis of $\mathcal{D}_{\mathbb{H}}$, then the dual frame θ^i is obtained as $\theta^i(e_j) = \delta^i_j$ and $\theta^i(\xi_1) = \theta^i(\xi_2) = \theta^i(\xi_3) = 0$. Then $d\omega_a(e_i, e_j) = g(e_i, J_a e_j) = -\boldsymbol{J}^a_{ij}$.

Since $\{\xi_1, \xi_2, \xi_3\}$ generates $\mathrm{Sp}(1)$ of the bundle (3.2), we obtain $d\omega_a + 2\omega_b \wedge \omega_c = -\boldsymbol{J}^a_{ij}\theta^i \wedge \theta^j$. Applying to J, K similarly, we obtain the following structure equation of the bundle (3.2):

$$d\omega_0 + \omega_0 \wedge \omega_0 = -(\boldsymbol{I}_{ij}\boldsymbol{i} + \boldsymbol{J}_{ij}\boldsymbol{j} + \boldsymbol{K}_{ij}\boldsymbol{k})\theta^i \wedge \theta^j. \tag{7.11}$$

From this equation, condition (7.4) is easily checked and hence $\mathrm{Null}\,\rho_\alpha = \{\xi_1, \xi_2, \xi_3\}$ $(\alpha = 1, 2, 3)$. □

Remark 7.10. It is shown in [3] that $(\Sigma^{3+4p,4q}_{\mathbb{H}}, \{I, J, K\}, g^{\mathbb{H}})$ is a pseudo 3-Sasakian space form of constant positive curvature with type $(4p+3, 4q)$. Using O'Neill's formula, the pseudo-quaternionic Kähler projective space $\mathbb{HP}^{4p,4q}$ is of constant positive quaternionic curvature.

7.2 Construction of the flat p-c qCR structure on $S^{4p+3,4q}$

Put $\mathcal{D} = \mathcal{D}_{\mathbb{H}}$. If N_x is the normal vector at $x \in \Sigma^{3+4p,4q}_{\mathbb{H}} \subset \mathbb{H}^{n+1}_{\mathbb{H}}$, recall that $(\mathrm{Null}\,\omega_0)_x = \mathcal{D}_x = \{IN_x, JN_x, KN_x\}^\perp \subset T_x\Sigma^{3+4p,4q}_{\mathbb{H}} = N_x^\perp$ with respect to $g^{\mathbb{H}} = \mathrm{Re}\langle\,,\,\rangle$. If we note that $\{N_x, IN_x, JN_x, KN_x\} = x\mathbb{H}$, then $\mathcal{D}_x = x\mathbb{H}^\perp$ with respect to $\langle\,,\,\rangle$. Let $\iota\colon \mathbb{H}^{n+1} \to \mathbb{H}^{n+2}$ be the embedding defined by $\iota(z_1, \ldots, z_{n+1}) = (z_1, \ldots, z_{n+1}, 1)$ so that

$$\begin{aligned} \iota(\Sigma^{3+4p,4q}_{\mathbb{H}}) &\subset V_0^{4n+7}, \\ P \circ \iota(\Sigma^{3+4p,4q}_{\mathbb{H}}) &\subset P(V_0) = S^{4p+3,4q} \quad \text{on the quotient.} \end{aligned} \tag{7.12}$$

As $\Sigma^{3+4p,4q}_{\mathbb{H}}$ has the transitive isometry group $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ (cf. Definition 3.1), this embedding implies that $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ is identified with the subgroup $P(\mathrm{Sp}(p+1,q) \times \mathrm{Sp}(1))$ of $\mathrm{PSp}(p+1, q+1)$ leaving the last component z_{n+2} invariant in $V_0^{4n+7} \subset \mathbb{H}^{n+2}$. As usual, we define a 1-form η_0 on $P \circ \iota(\Sigma^{3+4p,4q}_{\mathbb{H}})$ by $(P \circ \iota)^*\eta_0 = \omega_0$. As $\langle \iota_*((\mathrm{Null}\,\omega_0)_x), \iota(x)\rangle = \langle((\mathrm{Null}\,\omega_0)_x, 0), (x, 1))\rangle = 0$, it follows that $\iota_*((\mathrm{Null}\,\omega_0)_x) = \iota(x)^\perp = (\iota(x)\mathbb{H})^\perp$ and hence

$$(\mathrm{Null}\,\eta_0)_{[\iota(x)]} = P_*\iota_*((\mathrm{Null}\,\omega_0)_x) = P_*((\iota(x)\mathbb{H})^\perp). \tag{7.13}$$

In particular, $(\mathrm{Null}\,\eta_0)_{[\iota(x)]}$ is invariant under $\{I, J, K\}$. If $f \in \mathrm{PSp}(p+1, q+1)$, then it follows that

$$f_*((\mathrm{Null}\,\eta_0)_{[\iota(x)]}) = (\mathrm{Null}\,{f^{-1}}^*\eta_0)_{[f(\iota(x))]} = P_*((f(\iota(x))\mathbb{H})^\perp)$$

on $f(P \circ \iota(\Sigma^{3+4p,4q}_{\mathbb{H}}))$. Suppose that $h([\iota(x)]) = [\iota(x)]$. Then $h(\iota(x)) = \iota(x)\cdot\lambda$ for $\lambda \in \mathbb{H}^*$. Then we have

$$h_*((\mathrm{Null}\,\eta_0)_{[\iota(x)]}) = P_*((\iota(x)\lambda\mathbb{H})^\perp) = (\mathrm{Null}\,\eta_0)_{[\iota(x)]}. \tag{7.14}$$

For any point $b \in S^{4p+3,4q}$, we have $f([\iota(x)]) = b$ for some element $f \in \mathrm{PSp}(p+1,q+1)$. From (7.14), $\mathcal{H}_b = \{(\mathrm{Null}\, f^{-1*}\eta_0)_{[f(\iota(x))]}\}_{f\in \mathrm{PSp}(p+1,q+1)}$ is the well-defined distribution at b.

Put $\mathcal{H} = \bigcup_{b\in S^{4p+3,4q}} \mathcal{H}_b$. As above, it is invariant under $\{I, J, K\}$. Obviously $\{f(P \circ \iota(\Sigma^{3+4p,4q}_{\mathbb{H}}))\}_{f\in \mathrm{PSp}(p+1,q+1)}$ covers $S^{4p+3,4q}$ and each $f^{-1*}\eta_0$ is a qCR structure on $U = f(P \circ \iota(\Sigma^{3+4p,4q}_{\mathbb{H}}))$. By Definition 7.5, we get:

Theorem 7.11. *$S^{4p+3,4q}$ supports a pseudo-conformal qCR structure, i.e., there exists a qCR structure ω locally on each neighborhood U such that*

$$\mathcal{H}|U = \mathrm{Null}\,\omega.$$

Moreover, the automorphism group of pseudo-conformal qCR transformations $\mathrm{Aut}_{\mathrm{pc\,qCR}}(S^{4p+3,4q})$ with respect to this p-c qCR structure on $S^{4p+3,4q}$ is $\mathrm{PSp}(p+1,q+1)$.

Proof. We have shown in [4] that there is a canonical pseudo-conformal qCR structure $\mathcal{H}^{\mathrm{can}}$ on $S^{3+4p,4q}$ invariant under $\mathrm{PSp}(p+1,q+1)$. In fact, we have $\mathcal{H} = \mathcal{H}^{\mathrm{can}}$. As for a proof of the second part of the theorem, we will explain the pseudo-conformal quaternionic Heisenberg geometry in the next section. This geometry is used to prove $\mathrm{Aut}(S^{4p+3,4q}) = \mathrm{Aut}_{\mathrm{pc\,qCR}}(S^{4p+3,4q}) = \mathrm{PSp}(p+1,q+1)$ (cf. [4]). □

7.3 Pseudo-conformal quaternionic Heisenberg geometry

Let $\mathrm{PSp}(p+1,q+1)$ be the group of all automorphisms preserving the flat p-c qCR structure of $S^{4p+3,4q} = \mathrm{PSp}(p+1,q+1)/P^+(\mathbb{H})$ (cf. [3]). We consider the stabilizer of the point at infinity $\{\infty\} = [1,0,\ldots,0,1] \in S^{4p+3,4q}$. Recall the (indefinite) Heisenberg nilpotent Lie group $\mathcal{M} = \mathcal{M}(p,q)$ from [19]. It is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law

$$(a,y)\cdot(b,z) = (a+b-\mathrm{Im}\langle y,z\rangle, y+z).$$

Here $\langle\,,\rangle$ is the Hermitian inner product of signature (p,q) on $\mathbb{H}^n$ as in (2.1) and $\mathrm{Im}\langle\,,\rangle$ is the imaginary part ($p+q=n$). It is nilpotent because the commutator subgroup $[\mathcal{M},\mathcal{M}] = \mathbb{R}^3$ which is the center of $\mathcal{M}$ consisting of elements of the form $(a,0)$. In particular, there is the central extension

$$1 \to \mathbb{R}^3 \to \mathcal{M} \longrightarrow \mathbb{H}^n \to 1. \tag{7.15}$$

Denote by $\mathrm{Sim}(\mathcal{M})$ the semidirect product $\mathcal{M} \rtimes (\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+)$ where the action $(A\cdot g,t) \in \mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+$ on $(a,y)\in\mathcal{M}$ is given by

$$(A\cdot g,t)\circ(a,y) = (t^2\cdot gag^{-1},\ t\cdot Ayg^{-1}). \tag{7.16}$$

Denote the origin by $O = [1,0,\ldots,0,-1] \in S^{4p+3,4q} - \{\infty\}$. The stabilizer $\mathrm{Aut}(S^{4p+3,4q})_\infty$ is isomorphic to $\mathrm{Sim}(\mathcal{M})$ (cf. [21]). The orbit $\mathcal{M}\cdot O$ is a dense

open subset of $S^{4p+3,4q}$. The embedding ι is defined by

$$((a,b,c),(z_+,z_-)) \in \mathcal{M} \xrightarrow{\iota} \begin{bmatrix} \frac{\|z_+\|^2-\|z_-\|^2}{2} - 1 + \boldsymbol{i}\,a + \boldsymbol{j}\,b + \boldsymbol{k}\,c \\ \sqrt{2}z_+ \\ \sqrt{2}z_- \\ \frac{\|z_+\|^2-\|z_-\|^2}{2} + 1 + \boldsymbol{i}\,a + \boldsymbol{j}\,b + \boldsymbol{k}\,c \end{bmatrix}. \tag{7.17}$$

The pair $(\mathrm{Sim}(\mathcal{M}), \mathcal{M})$ is said to be a *pseudo-conformal quaternionic Heisenberg geometry* which is a subgeometry of flat p-c qCR geometry $(\mathrm{Aut}(S^{4p+3,4q}), S^{4p+3,4q})$.

Proposition 7.12. *There exists a global* $\mathrm{Im}\,\mathbb{H}$*-valued* 1*-form* $\omega_{\mathcal{M}}$ *on the* (*indefinite*) *Heisenberg nilpotent group* $\mathcal{M}$ *which represents the pseudo-conformal qCR structure* $\mathcal{H}|_{\mathcal{M}}$.

As in (6.7), $\omega_{\mathcal{M}}$ is obtained explicitly. If $(t_1,t_2,t_3,(z_1,\dots,z_n))$ is the coordinate of $\mathcal{M} = \mathbb{R}^3 \times \mathbb{H}^n$, then

$$\omega_{\mathcal{M}} = dt_1\boldsymbol{i} + dt_2\boldsymbol{j} + dt_3\boldsymbol{k} + \mathrm{Im}\langle z, dz\rangle.$$

Note that $\omega_{\mathcal{M}}$ is not an admissible 1-form by the definition (because $\{\xi_\alpha\}_{\alpha=1,2,3}$ generates $\mathbb{R}^3$) but it is *locally conformal* to ω_0, i.e., $\omega_{\mathcal{M}} = \lambda\cdot\omega_0\cdot\bar{\lambda}$ for some $\mathbb{H}^*$-valued function λ on $\mathcal{M}\cap\Sigma^{3+4p,4q}_{\mathbb{H}} \subset S^{4p+3,4q}$. It follows that $\mathrm{Null}\,\omega_{\mathcal{M}} = \mathrm{Null}\,\omega_0 = \mathcal{H}|_{\mathcal{M}}$ under the identification $\Sigma^{3+4p,4q}_{\mathbb{H}} = P\circ\iota(\Sigma^{3+4p,4q}_{\mathbb{H}})$. So $\mathcal{M}$ is a pseudo-conformal quaternionic CR-manifold but not a qCR-manifold.

Taking a discrete uniform subgroup $\Delta \subset \mathcal{M}\rtimes(\mathrm{Sp}(p)\times\mathrm{Sp}(q))\cdot\mathrm{Sp}(1)$ and an infinite cyclic subgroup $\mathbb{Z}$ from $(\mathrm{Sp}(p)\times\mathrm{Sp}(q))\times\mathbb{R}^+$, we obtain that

Corollary 7.13. *There exist compact flat pseudo-conformal quaternionic CR-manifolds* $\mathcal{M}/\Delta$ *and* $S^{4n+2}\times S^1$.

Here the universal covering of the latter manifold is $\mathcal{M} - \{O\} \approx S^{4n+2}\times\mathbb{R}^+$ $(\subset S^{4n+3,4q} - \{O,\infty\})$.

Remark 7.14. Pseudo-conformal qCR-manifolds contain the class of 3-pseudo-Sasakian manifolds. For example, $\Sigma^{3+4p,4q}_{\mathbb{H}}/\Gamma$, where Γ is a subgroup of $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ that acts properly and freely. But it is very difficult whether there exist compact 3-pseudo-Sasakian manifolds in general. As a special case, when $q=0$ or $p=0$, we can find discrete cocompact subgroups from $\mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$ or $\mathrm{Sp}(1,n)\cdot\mathrm{Sp}(1)$ that act properly and freely on $\Sigma^{3+4n,0}_{\mathbb{H}} = S^{4n+3}$ or $\Sigma^{3,4n}_{\mathbb{H}} = V^{4n+3}_{-1} = \mathrm{Sp}(1,n)/\mathrm{Sp}(n)$, respectively. Thus, we obtain compact 3-pseudo-Sasakian manifolds (nondegenerate qCR-manifolds); (i) The spherical space form S^{4n+3}/F which is $\mathrm{Sp}(1)$ or $\mathrm{SO}(3)$-bundle over the quaternionic Kähler projective orbifold $\mathbb{HP}^n/F^*$ of positive scalar curvature. ($F\subset\mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$ is a finite group.) (ii) The pseudo-Riemannian standard space form V^{4n+3}_{-1}/Γ of signature $(4n,3)$ with constant sectional curvature -1 which is an $\mathrm{Sp}(1)$-bundle over the quaternionic Kähler hyperbolic orbifold $\mathbb{H}^n_{\mathbb{H}}/\Gamma^*$ of

negative scalar curvature. ($\Gamma^* \subset \mathrm{PSp}(1,n)$ is a discrete uniform subgroup.) It is unknown whether there exist compact 3-pseudo-Sasakian manifolds (or qCR-manifolds) whose pseudo-quaternionic Kähler quotients (orbifolds) have zero Ricci curvature. However, (iii) the indefinite quaternionic Heisenberg nilmanifold $\mathcal{M}/\Delta$ is a compact *p-c qCR* manifold whose quotient has the zero Ricci curvature. Some finite cover of $\mathcal{M}/\Delta$ is a Heisenberg nilmanifold which is a principal 3-torus bundle over the flat quaternionic n-torus $T^{p,q}_{\mathbb{H}}$ of signature $(4p, 4q)$ $(p+q=n)$. Obviously those compact manifolds are not positive-definite 3-Sasakian manifolds. (See [19], [21] more generally.) Refer to [8], [10], [9], [11], [31], [32] for (positive definite) Sasakian 3-structures.

Remark that there is a noncompact pseudo-quaternionic Kähler manifold with zero Ricci curvature (non-isometric to $\mathbb{H}^n$).

7.4 Curvature forms in $\mathcal{R}(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1))$

Concerning geometric invariants, there is the curvature tensor T of a p-c qCR structure in [3]. This is thought of as a quaternionic analogue of Chern–Moser's CR curvature tensor. On a $(4n+3)$-dimensional almost p-c qCR-manifold $(M, \mathcal{D}, Q)$, there is an $\mathfrak{sp}(p+1,q+1)$-valued Cartan form κ whose associated curvature form Π has zero curvature if and only if $(M,\mathcal{D})$ is locally isomorphic to $(\mathrm{PSp}(p+1,q+1)/P^+(\mathbb{H}),\mathcal{H})$. We do not know whether the curvature tensor T on M could be derived only from the Cartan form Π and its curvature form Ω of the almost p-c qCR structure $\mathcal{D}$ because $\mathcal{D}$ lacks the structure equations representing the integrability conditions unlike the p-c qCR structure, see (7.11). However, with the aid of the pseudo-Riemannian connection of the pseudo-Sasakian 3-structure, which is locally equivalent to a p-c qCR structure, we can define a *quaternionic CR curvature tensor* $\mathcal{R}$. Based on the construction of this tensor $\mathcal{R}$, we shall obtain a curvature tensor

$$T = \mathcal{R} - R_{\mathbb{HP}^{p,q}}. \tag{7.18}$$

Here $R_{\mathbb{HP}^{p,q}}$ is the curvature tensor of the pseudo-quaternionic Kähler projective space $\mathbb{HP}^{p,q}$ in dimensional $4n$ $(p+q=n)$. Remark that if T vanishes for a p-c qCR structure, then Π also vanishes with respect to the underlying almost p-c qCR structure. Let $\mathcal{R}(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1))$ be the space of all curvature tensors. (See, for example, [1].) It decomposes into the direct sum $\mathcal{R}(\mathrm{Sp}(p,q))\oplus\mathcal{R}_{\mathbb{HP}}(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1))$ $(n\geq 2)$. Here $\mathcal{R}(\mathrm{Sp}(p,q))$ is the space of those curvatures with zero Ricci forms and $\mathcal{R}_{\mathbb{HP}} \approx \mathbb{R}$ is the space of curvature tensors of $\mathbb{HP}^{p,q}$. Since T is locally defined by our Definition 7.5, we have to show that it becomes a global invariant on M, i.e., T is invariant under the equivalence of p-c qCR structures. In [3] we have shown the following result.

Theorem 7.15. *The fourth-order curvature tensor* $T=(T^i_{jk\ell}) \in \mathcal{R}(\mathrm{Sp}(p,q))$ *exists globally on any pseudo-conformal quaternionic CR-manifold of dimension* $4n+3$ $(n\geq 1)$.

When $n = 1$, then $T = W^- \in \mathcal{R}(\mathrm{Sp}(1))$ where W^- is anti-self-dual ($W^+ = 0$). We now state the uniformization theorem from [3].

Theorem 7.16. (1) *Let M be a $(4n+3)$-dimensional p-c qCR manifold ($n \geq 1$). If the curvature tensor T vanishes, then M is locally modeled on $S^{4p+3,4q}$ whose coordinate changes belong to the group* $\mathrm{PSp}(p+1, q+1)$.

(2) *If M is a 3-dimensional p-c qCR manifold, then there is a global Riemannian metric g so that (M, g) is locally isometric to the standard 3-sphere S^3.*

8 Remarks and problems

8.1 Conformally flat Lorentz metrics

In Section 4, we have obtained the standard pseudo-Riemannian metric g_1 on $S^{p,q}$ in Proposition 4.1. It is locally conformal to the euclidean metric g_0 of signature (p, q) on $\mathbb{R}^n$ ($p+q = n \geq 2$). When $p = 2n+1, q = 1$, the metric g_1 is a conformally flat Lorentz metric on the $(2n+2)$-manifold $S^{2n+1,1} = S^{2n+1} \times S^1/\mathbb{Z}_2$ with the group of conformal transformations $\mathrm{O}(2n+2, 2)$. Note that $\mathrm{U}(n+1, 1) \subset \mathrm{O}(2n+2, 2)$ is the natural embedding.

Problem 8.1. How does g_1 on $S^{2n+1,1}$ relate to the Fefferman Lorentz metric obtained from the standard CR-manifold S^{2n+1}?

8.2 Exotic quaternionic invariant closed 4-forms on $\mathbb{H}^n$

Let $\mathcal{M}$ be the (positive definite) quaternionic Heisenberg Lie group as before (cf. Section 7.3). The abelian Lie subgroup $\mathbb{R}^3 \times T^n$ consists of the direct product of the center $\mathbb{R}^3$ of $\mathcal{M}$ and the maximal torus T^n of $\mathrm{Sp}(n)$. Note that $\mathbb{R}^3 \times T^n$ is the subgroup of the similarity group $\mathrm{Sim}(\mathcal{M}) = \mathcal{M} \times (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$. Let $s = (s_1, s_2, s_3) \in \mathbb{R}^3$ and $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$. We introduce a representation $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3 \times T^n$ such that

$$\begin{aligned}
\rho(t_1)((s_1, s_2, s_3), (z_1, \ldots, z_n)) &= ((s_1 + t_1, s_2, s_3), (e^{\mathrm{i}t \cdot a_1} \cdot z_1, \ldots, e^{\mathrm{i}t \cdot a_n} \cdot z_n)), \\
\rho(t_2)((s_1, s_2, s_3), z) &= ((s_1, s_2 + t_2, s_3), z), \\
\rho(t_3)((s_1, s_2, s_3), z) &= ((s_1, s_2, s_3 + t_3), z).
\end{aligned} \tag{8.1}$$

See (7.16) for the action of $\mathbb{R}^3 \times T^n$ on $\mathcal{M}$. As the $\rho(\mathbb{R}^3)$-action on $\mathcal{M}$ is proper, we have the principal bundle

$$\mathbb{R}^3 \to \mathcal{M} \xrightarrow{\pi} \mathcal{M}/\rho(\mathbb{R}^3). \tag{8.2}$$

Identifying $\mathbb{H}^n$ with $\mathbb{R}^{4n}$, the group $\rho(\mathbb{R}^3)$ induces the three vector fields

$$\begin{aligned}\xi_1 = {} & \frac{d}{dt_1} + a_1\left(x_1\frac{d}{dx_2} - x_2\frac{d}{dx_1}\right) + a_1\left(x_3\frac{d}{dx_4} - x_4\frac{d}{dx_3}\right) \\ & + \cdots + a_n\left(x_{4n-3}\frac{d}{x_{4n-2}} - x_{4n-2}\frac{d}{dx_{4n-3}}\right) \\ & + a_n\left(x_{4n-1}\frac{d}{dx_{4n}} - x_{4n}\frac{d}{dx_{4n-1}}\right), \\ \xi_2 = {} & \frac{d}{dt_2} \quad \text{and} \quad \xi_3 = \frac{d}{dt_3}.\end{aligned} \tag{8.3}$$

We define real 1-forms on $\mathcal{M}$ as follows:

$$\begin{aligned}\omega_1 = {} & dt_1 + (x_1dx_2 - x_2dx_1) + (x_3dx_4 - x_4dx_3) + \cdots \\ & \cdots + (x_{4n-3}dx_{4n-2} - x_{4n-2}dx_{4n-3}) + (x_{4n-1}dx_{4n} - x_{4n}dx_{4n-1}), \\ \omega_2 = {} & dt_2 + (x_1dx_3 - x_3dx_1) - (x_2dx_4 - x_4dx_2) + \cdots \\ & \cdots + (x_{4n-3}dx_{4n-1} - x_{4n-1}dx_{4n-3}) - (x_{4n-2}dx_{4n} - x_{4n}dx_{4n-2}), \\ \omega_3 = {} & dt_3 + (x_1dx_4 - x_4dx_1) + (x_2dx_3 - x_3dx_2) + \cdots \\ & \cdots + (x_{4n-3}dx_{4n} - x_{4n}dx_{4n-3}) + (x_{4n-2}dx_{4n-1} - x_{4n-1}dx_{4n-2}).\end{aligned} \tag{8.4}$$

Setting

$$\omega = \omega_1\boldsymbol{i} + \omega_2\boldsymbol{j} + \omega_3\boldsymbol{k}, \tag{8.5}$$

ω is an $\mathrm{Im}\,\mathbb{H}$-valued 1-form on $\mathcal{M}$. Then it follows that $\omega_1(\xi_1) = 1 + a_1|z_1|^2 + \cdots + a_n|z_n|^2$. Suppose that

$$0 \le a_1 \le \cdots \le a_n. \tag{8.6}$$

Then $\omega_1(\xi_1) > 0$, so we introduce new 1-forms

$$\eta_1 = \frac{1}{\omega_1(\xi_1)} \cdot \omega_1, \quad \eta_2 = \omega_2 \quad \text{and} \quad \eta_3 = \omega_3. \tag{8.7}$$

We obtain that

$$\begin{aligned}\eta_i(\xi_j) &= \delta_{ij}, \\ d\eta_i(\xi_i, X) &= 0 \quad \text{for all } X \in T\mathcal{M}.\end{aligned} \tag{8.8}$$

Then the 4-form on $\mathcal{M}$,

$$\Omega = d\eta_1 \wedge d\eta_1 + d\eta_2 \wedge d\eta_2 + d\eta_3 \wedge d\eta_3 \tag{8.9}$$

is $\{I, J, K\}$-invariant on $\mathcal{D} = \bigcap_{i=1}^3 \mathrm{Null}\,\omega_i$. As the projection π induces an isomorphism of $\mathcal{D}$ onto $T(\mathcal{M}/\rho(\mathbb{R}^3))$ at each point from (8.2), we can show that

Lemma 8.2. *The quaternion structure $Q = \mathrm{Span}\{I, J, K\}$ on $\mathcal{D}$ induces the quaternion structure $\widehat{Q} = \mathrm{Span}\{\hat{I}, \hat{J}, \hat{K}\}$ on the quotient $\mathcal{M}/\rho(\mathbb{R}^3)$. In particular, $\mathcal{M}/\rho(\mathbb{R}^3)$ endowed with $\widehat{Q}$ is isomorphic to the standard quaternionic number space $\mathbb{H}^n$.*

Now using (8.8), the 4-form Ω induces an $\{\hat{I}, \hat{J}, \hat{K}\}$-invariant 4-form $\widehat{\Omega}$ on $(\mathcal{M}/\rho(\mathbb{R}^3), \hat{Q}) = (\mathbb{H}^n, Q)$ such that

$$\pi^*\widehat{\Omega} = \Omega. \tag{8.10}$$

Since Ω is closed, $d\widehat{\Omega} = 0$.

Proposition 8.3. *There exists a closed* 4*-form* $\widehat{\Omega}$ *of signature* $(4p, 4q)$ *on* $\mathbb{H}^n$ *which is not isomorphic to the standard closed* 4*-form of signature* $(4p, 4q)$ *on* $\mathbb{H}^n$ $(p+q = n)$.

Proof. Let Ω_0 be the canonical 4-form on $\mathcal{M}$ which induces the standard quaternionic Kähler 4-form $\widehat{\Omega}_0$ on $\mathbb{H}^n$. If $\widehat{\Omega}$ is isometric to $\widehat{\Omega}_0$, then it is easy to see that the isometry extends to an isometry between Ω and Ω_0 on $\mathcal{M}$. Then this isometry conjugates the group $\rho(\mathbb{R}^3)$ to the standard central group $\mathbb{R}^3$ inside $\mathrm{PSp}(p+1, q+1)$. However, by construction of the representation ρ, this is impossible. □

On the other hand, setting

$$\hat{g}_\eta(\nu_* X, \nu_* Y) = d\eta_1(IX, Y) \quad \text{for all } X, Y \in \mathcal{D}, \tag{8.11}$$

$\hat{g}_\eta$ is a well-defined $\{I, J, K\}$-invariant *Riemannian* metric on $\mathbb{H}^n$. However the equality that $d\eta_1(IX, Y) = d\eta_2(JX, Y) = d\eta_3(KX, Y)$ $(X, Y \in \mathcal{D})$ does not hold although $d\omega_1(IX, Y) = d\omega_2(JX, Y) = d\omega_3(KX, Y)$ by (8.4). So $\hat{g}_\eta$ is not a quaternionic Kähler metric. Obviously when we set $\hat{g}_\omega = d\eta_2(J\cdot, \cdot) = d\eta_3(K\cdot, \cdot)$, it is the standard quaternionic Kähler metric on $\mathbb{H}^n$ by the above equality.

Problem 8.4. Does there exist a nonstandard quaternionic Kähler metric on $\mathbb{H}^n$?

8.3 Global rigidity of compact nondegenerate geometric manifolds with noncompact geometric flow

We pose the following vague conjecture whose supporting results are the Obata and Lelong-Ferrand's theorem, and the analogue of compact strictly pseudoconvex CR-manifolds with noncompact CR flow, and also compact positive definite p-c qCR manifolds with noncompact p-c qCR flow. See [18]. (We refer to [16] for the related works and references therein, especially [27], [30].)

Conjecture 8.5. If a compact nondegenerate, conformal (respectively CR, pseudo-conformal quaternionic CR) manifold M supports a noncompact closed $\mathbb{R}$-flow consisting of conformal (respectively CR, pseudo-conformal quaternionic CR) transformations, then M is flat, moreover M is isomorphic to a geometric manifold covered by a domain of $S^{c(p+1)-1,cq}$ $(c = 1, 2, 4)$.

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Part H

Other topics of recent interest

Chapter 25

Linear wave equations on Lorentzian manifolds

Christian Bär

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1 Introduction

In General Relativity spacetime is modelled by a Lorentzian manifold, see e.g. [8], [15]. Many physical phenomena, such as electro-magnetic radiation, are described by solutions to certain linear wave equations defined on this spacetime manifold. Thus a good understanding of the theory of wave equations is crucial. This includes initial value problems (the Cauchy problem), fundamental solutions, and inverse operators (Green's operators). The classical textbooks on partial differential equations contain the relevant results for small domains in Lorentzian manifolds or for very special manifolds such as Minkowski space.

In this chapter we summarize the global analytic results obtained in [4], see also Leray's unpublished lecture notes [13] and Choquet-Bruhat's exposition [7]. In order to obtain a good solution theory one has to impose certain geometric conditions on the underlying manifold. The situation is similar to the study of elliptic operators on Riemannian manifolds. In order to ensure that the Laplace–Beltrami operator on a Riemannian manifold M is essentially self-adjoint one may make the natural assumption that M be complete. Unfortunately, there is no good notion of completeness for Lorentzian manifolds. It will turn out that the analysis of wave operators works out nicely if one assumes that the underlying Lorentzian manifold be globally hyperbolic.

Completeness of Riemannian manifolds and global hyperbolicity of Lorentzian manifolds are indeed related. If (S, g_0) is a Riemannian manifold, then the Lorentzian cylinder $M = \mathbb{R} \times S$ with product metric $g = -dt^2 + g_0$ is globally hyperbolic if and only if (S, g_0) is complete.

We will start by collecting some material on distributional sections in vector bundles. Then we will summarize the theory of globally hyperbolic Lorentzian manifolds. Then we will define wave operators, also called normally hyperbolic operators, and give some examples. After that we consider the basic initial value problem, the Cauchy problem. It turns out that on a globally hyperbolic manifold solutions exist and are unique. They depend continuously on the initial data. The support of the solutions can be controlled which is physically nothing than the statement that a wave can never propagate faster than with the speed of light. In the subsequent section we use the results on the Cauchy problem to show existence and uniqueness of fundamental solutions. This is closely related to the existence and uniqueness of Green's operators.

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2 Distributional sections in vector bundles

Let us start by giving some definitions and by fixing the terminology for distributions on manifolds. We will confine ourselves to those facts that we will actually need later on. A systematic and much more complete introduction may be found e.g. in [9].

2.1 Preliminaries on distributional sections

Let M be a manifold equipped with a smooth volume density dV. Later on we will use the volume density induced by a Lorentzian metric but this is irrelevant for now. We consider a real or complex vector bundle $E \to M$. We will always write $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex. The space of compactly supported smooth sections in E will be denoted by $\mathcal{D}(M, E)$. We equip E and the cotangent bundle T^*M with connections, both denoted by ∇. They induce connections on the tensor bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$, again denoted by ∇. For a continuously differentiable section $\varphi \in C^1(M, E)$ the covariant derivative is a continuous section in $T^*M \otimes E$, $\nabla\varphi \in C^0(M, T^*M \otimes E)$. More generally, for $\varphi \in C^k(M, E)$ we get $\nabla^k\varphi \in C^0(M, \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ factors}} \otimes E)$.

We choose an auxiliary Riemannian metric on T^*M and an auxiliary Riemannian or Hermitian metric on E depending on whether E is real or complex. This induces metrics on all bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$. Hence the norm of $\nabla^k\varphi$ is defined at all points of M.

For a subset $A \subset M$ and $\varphi \in C^k(M, E)$ we define the C^k-norm by

$$\|\varphi\|_{C^k(A)} := \max_{j=0,\dots,k} \sup_{x \in A} |\nabla^j \varphi(x)|. \tag{1}$$

If A is compact, then different choices of the metrics and the connections yield equivalent norms $\|\cdot\|_{C^k(A)}$. For this reason there will be no need to explicitly specify the metrics and the connections.

The elements of $\mathcal{D}(M, E)$ are referred to as test sections in E. We define a notion of convergence of test sections.

Definition 2.1. Let $\varphi, \varphi_n \in \mathcal{D}(M, E)$. We say that the sequence $(\varphi_n)_n$ *converges to* φ *in* $\mathcal{D}(M, E)$ if the following two conditions hold:

(1) There is a compact set $K \subset M$ such that the supports of φ and of all φ_n are contained in K, i.e., $\operatorname{supp}(\varphi), \operatorname{supp}(\varphi_n) \subset K$ for all n.

(2) The sequence $(\varphi_n)_n$ converges to φ in all C^k-norms over K, i.e., for each $k \in \mathbb{N}$

$$\|\varphi - \varphi_n\|_{C^k(K)} \xrightarrow[n\to\infty]{} 0.$$

We fix a finite-dimensional $\mathbb{K}$-vector space W. Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex. Denote by E^* the vector bundle over M dual to E.

Definition 2.2. A $\mathbb{K}$-linear map $F : \mathcal{D}(M, E^*) \to W$ is called a *distribution in E with values in W* or a *distributional section in E with values in W* if it is continuous in the sense that for all convergent sequences $\varphi_n \to \varphi$ in $\mathcal{D}(M, E^*)$ one has $F[\varphi_n] \to F[\varphi]$. We write $\mathcal{D}'(M, E, W)$ for the space of all W-valued distributions in E.

Note that since W is finite-dimensional all norms $|\cdot|$ on W yield the same topology on W. Hence there is no need to specify a norm on W for Definition 2.2 to make sense. Note moreover, that distributional sections in E act on test sections in E^*.

Example 2.3. Pick a bundle $E \to M$ and a point $x \in M$. The *delta-distribution* δ_x is a distributional section in E with values in E_x^*. For $\varphi \in \mathcal{D}(M, E^*)$ it is defined by

$$\delta_x[\varphi] = \varphi(x).$$

Example 2.4. Every locally integrable section $f \in L^1_{\mathrm{loc}}(M, E)$ can be regarded as a $\mathbb{K}$-valued distribution in E by setting for any $\varphi \in \mathcal{D}(M, E^*)$

$$f[\varphi] := \int_M \varphi(f)\ \mathrm{dV}.$$

Here $\varphi(f)$ denotes the $\mathbb{K}$-valued L^1-function with compact support on M obtained by pointwise application of $\varphi(x) \in E_x^*$ to $f(x) \in E_x$.

2.2 Differential operators acting on distributions

Let E and F be two $\mathbb{K}$-vector bundles over the manifold M, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Consider a linear differential operator $P\colon C^\infty(M,E) \to C^\infty(M,F)$. There is a unique linear differential operator $P^*\colon C^\infty(M,F^*) \to C^\infty(M,E^*)$ called the *formal adjoint of* P such that for any $\varphi \in \mathcal{D}(M,E)$ and $\psi \in \mathcal{D}(M,F^*)$

$$\int_M \psi(P\varphi)\ \mathrm{dV} = \int_M (P^*\psi)(\varphi)\ \mathrm{dV}. \tag{2}$$

If P is of order k, then so is P^* and (2) holds for all $\varphi \in C^k(M,E)$ and $\psi \in C^k(M,F^*)$ such that $\operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi)$ is compact. With respect to the canonical identification $E = (E^*)^*$ we have $(P^*)^* = P$.

Any linear differential operator $P\colon C^\infty(M,E) \to C^\infty(M,F)$ extends canonically to a linear operator $P\colon \mathcal{D}'(M,E,W) \to \mathcal{D}'(M,F,W)$ by

$$(PT)[\varphi] := T[P^*\varphi]$$

where $\varphi \in \mathcal{D}(M,F^*)$. If a sequence $(\varphi_n)_n$ converges in $\mathcal{D}(M,F^*)$ to 0, then the sequence $(P^*\varphi_n)_n$ converges to 0 as well because P^* is a differential operator. Hence $(PT)[\varphi_n] = T[P^*\varphi_n] \to 0$. Therefore PT is indeed again a distribution.

The map $P\colon \mathcal{D}'(M,E,W) \to \mathcal{D}'(M,F,W)$ is $\mathbb{K}$-linear. If P is of order k and φ is a C^k-section in E, seen as a $\mathbb{K}$-valued distribution in E, then the distribution $P\varphi$ coincides with the continuous section obtained by applying P to φ classically.

An important special case occurs when P is of order 0, that is, if P lies in $C^\infty(M, \operatorname{Hom}(E,F))$. Then $P^* \in C^\infty(M, \operatorname{Hom}(F^*,E^*))$ is the pointwise adjoint. In particular, for a function $f \in C^\infty(M,\mathbb{K})$ we have

$$(fT)[\varphi] = T[f\varphi].$$

2.3 Supports

Definition 2.5. The *support* of a distribution $T \in \mathcal{D}'(M,E,W)$ is defined as the set

$$\operatorname{supp}(T) := \{x \in M \mid \text{for all neighborhoods } U \text{ of } x \text{ there exists } \varphi \in \mathcal{D}(M,E) \text{ with } \operatorname{supp}(\varphi) \subset U \text{ and } T[\varphi] \neq 0\}.$$

It follows from the definition that the support of T is a closed subset of M. In case T is a L^1_{loc}-section this notion of support coincides with the usual one for sections.

If for $\varphi \in \mathcal{D}(M,E^*)$ the supports of φ and T are disjoint, then $T[\varphi] = 0$. Namely, for each $x \in \operatorname{supp}(\varphi)$ there is a neighborhood U of x such that $T[\psi] = 0$ whenever $\operatorname{supp}(\psi) \subset U$. Cover the compact set $\operatorname{supp}(\varphi)$ by finitely many such open sets $U_1, \ldots, U_k$. Using a partition of unity one can write $\varphi = \psi_1 + \cdots + \psi_k$ with $\psi_j \in \mathcal{D}(M,E^*)$ and $\operatorname{supp}(\psi_j) \subset U_j$. Hence

$$T[\varphi] = T[\psi_1 + \cdots + \psi_k] = T[\psi_1] + \cdots + T[\psi_k] = 0.$$

Be aware that it is not sufficient to assume that φ vanishes on $\operatorname{supp}(T)$ in order to ensure $T[\varphi] = 0$. For example, if $M = \mathbb{R}$ and E is the trivial $\mathbb{K}$-line bundle let $T \in \mathcal{D}'(\mathbb{R}, \mathbb{K})$ be given by $T[\varphi] = \varphi'(0)$. Then $\operatorname{supp}(T) = \{0\}$ but $T[\varphi] = \varphi'(0)$ may well be nonzero while $\varphi(0) = 0$.

If $T \in \mathcal{D}'(M, E, W)$ and $\varphi \in C^\infty(M, E^*)$, then the evaluation $T[\varphi]$ can be defined if $\operatorname{supp}(T) \cap \operatorname{supp}(\varphi)$ is compact even if the support of φ itself is noncompact. To do this pick a function $\sigma \in \mathcal{D}(M, \mathbb{R})$ that is constant 1 on a neighborhood of $\operatorname{supp}(T) \cap \operatorname{supp}(\varphi)$ and put

$$T[\varphi] := T[\sigma\varphi].$$

This definition is independent of the choice of σ since for another choice σ' we have

$$T[\sigma\varphi] - T[\sigma'\varphi] = T[(\sigma - \sigma')\varphi] = 0$$

because $\operatorname{supp}((\sigma - \sigma')\varphi)$ and $\operatorname{supp}(T)$ are disjoint.

Let $T \in \mathcal{D}'(M, E, W)$ and let $\Omega \subset M$ be an open subset. Each test section $\varphi \in \mathcal{D}(\Omega, E^*)$ can be extended by 0 and yields a test section $\varphi \in \mathcal{D}(M, E^*)$. This defines an embedding $\mathcal{D}(\Omega, E^*) \subset \mathcal{D}(M, E^*)$. By the restriction of T to Ω we mean its restriction from $\mathcal{D}(M, E^*)$ to $\mathcal{D}(\Omega, E^*)$.

Definition 2.6. The *singular support* $\operatorname{sing\,supp}(T)$ of a distribution $T \in \mathcal{D}'(M, E, W)$ is the set of points which do not have a neighborhood restricted to which T coincides with a smooth section.

The singular support is also closed and we always have $\operatorname{sing\,supp}(T) \subset \operatorname{supp}(T)$.

Example 2.7. For the delta-distribution δ_x we have $\operatorname{supp}(\delta_x) = \operatorname{sing\,supp}(\delta_x) = \{x\}$.

2.4 Convergence of distributions

The space $\mathcal{D}'(M, E)$ of distributions in E will always be given the *weak topology*. This means that $T_n \to T$ in $\mathcal{D}'(M, E, W)$ if and only if $T_n[\varphi] \to T[\varphi]$ for all $\varphi \in \mathcal{D}(M, E^*)$. Linear differential operators P are always continuous with respect to the weak topology. Namely, if $T_n \to T$, then we have for every $\varphi \in \mathcal{D}(M, E^*)$

$$PT_n[\varphi] = T_n[P^*\varphi] \to T[P^*\varphi] = PT[\varphi].$$

Hence

$$PT_n \to PT.$$

Remark 2.8. Let $T_n, T \in C^0(M, E)$ and suppose $\|T_n - T\|_{C^0(M)} \to 0$. Consider T_n and T as distributions. Then $T_n \to T$ in $\mathcal{D}'(M, E)$. In particular, for every linear differential operator P we have $PT_n \to PT$.

3 Globally hyperbolic Lorentzian manifolds

Next we summarize some notions and facts from Lorentzian geometry. More comprehensive introductions can be found in [2] and in [14].

By a *Lorentzian manifold* we mean a semi-Riemannian manifold whose metric has signature $(-,+,\dots,+)$. We denote the Lorentzian metric by g or by $\langle\cdot,\cdot\rangle$. A tangent vector $X \in TM$ is called *timelike* if $\langle X, X\rangle < 0$, *lightlike* if $\langle X, X\rangle = 0$ and $X \neq 0$, *causal* if it is timelike or lightlike, and *spacelike* otherwise. At each point $p \in M$ the set of timelike vectors in T_pM decomposes into two connected components. A *timeorientation* on M is a choice of one of the two connected components of timelike vectors in T_pM which depends continuously on p. This means that we can find a continuous timelike vector field on M taking values in the chosen connected components. Tangent vectors in the chosen connected component are called *future directed*, those in the other component are called *past directed*.

Let M be a timeoriented Lorentzian manifold. A piecewise C^1-curve in M is called *timelike, lightlike, causal, spacelike, future directed*, or *past directed* if its tangent vectors are timelike, lightlike, causal, spacelike, future directed, or past directed respectively.

The *chronological future* $I_+^M(x)$ of a point $x \in M$ is the set of points that can be reached from x by future directed timelike curves. Similarly, the *causal future* $J_+^M(x)$ of a point $x \in M$ consists of those points that can be reached from x by causal curves and of x itself. The *chronological future* of a subset $A \subset M$ is defined to be $I_+^M(A) := \bigcup_{x\in A} I_+^M(x)$. Similarly, the *causal future* of A is $J_+^M(A) := \bigcup_{x\in A} J_+^M(x)$. The *chronological past* $I_-^M(A)$ and the *causal past* $J_-^M(A)$ are defined by replacing future directed curves by past directed curves. One has in general that $I_\pm^M(A)$ is the interior of $J_\pm^M(A)$ and that $J_\pm^M(A)$ is contained in the closure of $I_\pm^M(A)$. The chronological future and past are open subsets but the causal future and past are not always closed even if A is closed.

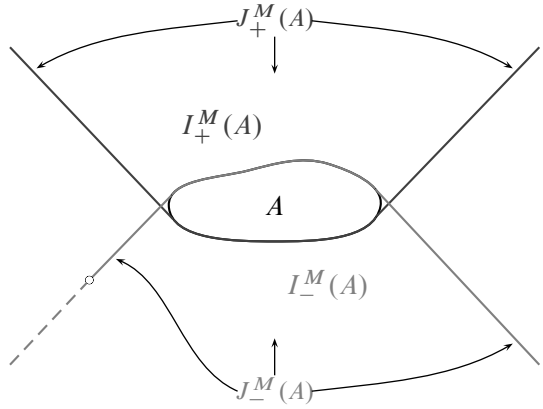

Figure 1. Causal and chronological future resp. past of A.

We will also use the notation $J^M(A) := J^M_-(A) \cup J^M_+(A)$. A subset $A \subset M$ is called *past compact* if $A \cap J^M_-(p)$ is compact for all $p \in M$. Similarly, one defines *future compact* subsets.

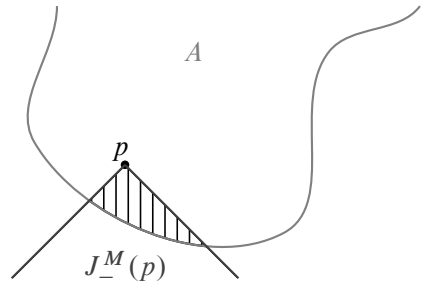

Figure 2. Past compact subset.

Definition 3.1. A subset S of a connected timeoriented Lorentzian manifold is called *achronal* if each timelike curve meets S in at most one point. A subset S of a connected timeoriented Lorentzian manifold is called *acausal* if each causal curve meets S in at most one point. A subset S of a connected timeoriented Lorentzian manifold is a *Cauchy hypersurface* if each inextendible timelike curve in M meets S at exactly one point.

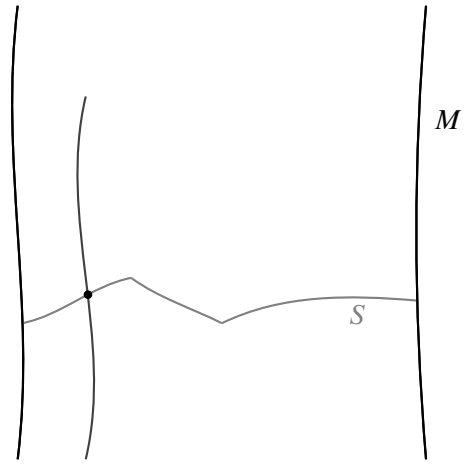

Figure 3. Cauchy hypersurface.

Obviously every acausal subset is achronal, but the reverse is wrong. Any Cauchy hypersurface is achronal. Moreover, it is a closed topological hypersurface and it is

hit by each inextendible causal curve in at least one point. Any two Cauchy hypersurfaces in M are homeomorphic. Furthermore, the causal future and past of a Cauchy hypersurface is past and future compact respectively.

Definition 3.2. A Lorentzian manifold is said to satisfy the *causality condition* if it does not contain any closed causal curve.

A Lorentzian manifold is said to satisfy the *strong causality condition* if there are no almost closed causal curves. More precisely, for each point $p \in M$ and for each open neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that each causal curve in M starting and ending in V is entirely contained in U.

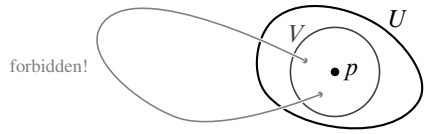

Figure 4. Strong causality condition.

Obviously, the strong causality condition implies the causality condition.

In order to get a good analytical theory for wave operators we must impose certain geometric conditions on the Lorentzian manifold. Here are several equivalent formulations.

Theorem 3.3. *Let M be a connected timeoriented Lorentzian manifold. Then the following are equivalent:*

(1) *M satisfies the strong causality condition and for all $p, q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$ is compact.*

(2) *There exists a Cauchy hypersurface in M.*

(3) *There exists a smooth spacelike Cauchy hypersurface in M.*

(4) *M is foliated by smooth spacelike Cauchy hypersurfaces. More precisely, M is isometric to $\mathbb{R} \times S$ with metric $-\beta dt^2 + g_t$ where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.*

That (1) implies (4) has been shown by Bernal and Sánchez in [5, Theorem 1.1] using work of Geroch [11, Theorem 11]. See also [8, Proposition 6.6.8] and [15, p. 209] for earlier mentioning of this fact. The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are trivial. That (2) implies (1) is well known, see e.g. [14, Corollary 39, p. 422].

Definition 3.4. A connected timeoriented Lorentzian manifold satisfying one and hence all conditions in Theorem 3.3 is called *globally hyperbolic*.

Remark 3.5. If M is a globally hyperbolic Lorentzian manifold, then a nonempty open subset $\Omega \subset M$ is itself globally hyperbolic if and only if for any $p, q \in \Omega$ the intersection $J_+^\Omega(p) \cap J_-^\Omega(q) \subset \Omega$ is compact. Indeed non-existence of almost closed causal curves in M directly implies non-existence of such curves in Ω.

Remark 3.6. It should be noted that global hyperbolicity is a conformal notion. The definition of a Cauchy hypersurface requires only causal concepts. Hence if (M, g) is globally hyperbolic and we replace the metric g by a conformally related metric $\hat{g} = f \cdot g$, f a smooth positive function on M, then $(M, \hat{g})$ is again globally hyperbolic.

Examples 3.7. Minkowski space is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. One can write Minkowski space as $\mathbb{R} \times \mathbb{R}^{n-1}$ with the metric $-dt^2 + g_t$ where g_t is the Euclidean metric on $\mathbb{R}^{n-1}$ and does not depend on t.

Let (S, g_0) be a connected Riemannian manifold and $I \subset \mathbb{R}$ an interval. The manifold $M = I \times S$ with the metric $g = -dt^2 + g_0$ is globally hyperbolic if and only if (S, g_0) is complete. This applies in particular if S is compact.

More generally, if $f : I \to \mathbb{R}$ is a smooth positive function we may equip $M = I \times S$ with the metric $g = -dt^2 + f(t)^2 \cdot g_0$. Again, (M, g) is globally hyperbolic if and only if (S, g_0) is complete. *Robertson–Walker spacetimes* and, in particular, *Friedmann cosmological models*, are of this type. They are used to discuss big bang, expansion of the universe, and cosmological redshift, compare [14, Chapter 12]. Another example of this type is *deSitter spacetime*, where $I = \mathbb{R}$, $S = S^{n-1}$, g_0 is the canonical metric of S^{n-1} of constant sectional curvature 1, and $f(t) = \cosh(t)$. But *Anti-deSitter spacetime* is not globally hyperbolic.

The interior and exterior *Schwarzschild spacetimes* are globally hyperbolic. They model the universe in the neighborhood of a massive static rotationally symmetric body such as a black hole. They are used to investigate perihelion advance of Mercury, the bending of light near the sun and other astronomical phenomena, see [14, Chapter 13].

Lemma 3.8. *Let S be a Cauchy hypersurface in a globally hyperbolic Lorentzian manifold M and let $K, K' \subset M$ be compact.*

Then $J_\pm^M(K) \cap S$, $J_\pm^M(K) \cap J_\mp^M(S)$, and $J_+^M(K) \cap J_-^M(K')$ are compact.

4 Wave operators

Let M be a Lorentzian manifold and let $E \to M$ be a real or complex vector bundle. A linear differential operator $P : C^\infty(M, E) \to C^\infty(M, E)$ of second order will be called a *wave operator* or a *normally hyperbolic operator* if its principal symbol is given by the metric

$$\sigma_P(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id}_{E_x}$$

for all $x \in M$ and all $\xi \in T_x^*M$. In other words, if we choose local coordinates $x^1, \dots, x^n$ on M and a local trivialization of E, then

$$P = -\sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x)\frac{\partial}{\partial x^j} + B(x)$$

where A_j and B are matrix-valued coefficients depending smoothly on x and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$ with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Example 4.1. Let E be the trivial line bundle so that sections in E are just functions. The d'Alembert operator $P = \Box = -\operatorname{div} \circ \operatorname{grad}$ is a wave operator.

Example 4.2. Let E be a vector bundle and let ∇ be a connection on E. This connection together with the Levi-Civita connection on T^*M induces a connection on $T^*M \otimes E$, again denoted ∇. We define the *connection-d'Alembert operator* $\Box^\nabla$ to be minus the composition of the following three maps

$$\begin{aligned} C^\infty(M,E) &\xrightarrow{\nabla} C^\infty(M, T^*M \otimes E) \\ &\xrightarrow{\nabla} C^\infty(M, T^*M \otimes T^*M \otimes E) \xrightarrow{\operatorname{tr}\otimes \mathrm{id}_E} C^\infty(M,E), \end{aligned}$$

where $\operatorname{tr}\colon T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\operatorname{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$. We compute the principal symbol,

$$\sigma_{\Box^\nabla}(\xi)\varphi = -(\operatorname{tr}\otimes \mathrm{id}_E) \circ \sigma_\nabla(\xi) \circ \sigma_\nabla(\xi)(\varphi) = -(\operatorname{tr}\otimes \mathrm{id}_E)(\xi \otimes \xi \otimes \varphi) = -\langle \xi, \xi \rangle \varphi.$$

Hence $\Box^\nabla$ is a wave operator.

Example 4.3. Let $E = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ increases the degree by one while the codifferential $\delta\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ decreases the degree by one. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = d\delta + \delta d$ is a wave operator.

Example 4.4. If M carries a Lorentzian metric and a spin structure, then one can define the spinor bundle ΣM and the Dirac operator

$$D\colon C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M),$$

see [1] or [3] for the definitions. The principal symbol of D is given by Clifford multiplication,

$$\sigma_D(\xi)\psi = \xi^\sharp \cdot \psi.$$

Hence

$$\sigma_{D^2}(\xi)\psi = \sigma_D(\xi)\sigma_D(\xi)\psi = \xi^\sharp \cdot \xi^\sharp \cdot \psi = -\langle \xi, \xi \rangle \psi.$$

Thus $P = D^2$ is a wave operator.

5 The Cauchy problem

We now come to the basic initial value problem for wave operators, the *Cauchy problem*. The local theory of linear hyperbolic operators can be found in basically any textbook on partial differential equations. In [10] and [12] the local theory for wave operators on Lorentzian manifolds is developed. The results of this section are of global nature. They make statements about solutions to the Cauchy problem which are defined globally on a manifold. Proofs of the results of this section can be found in [4, Section 3.2].

Theorem 5.1 (Existence and uniqueness of solutions). *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S. Let E be a vector bundle over M and let P be a wave operator acting on sections in E.*

Then for each $u_0, u_1 \in \mathcal{D}(S, E)$ and for each $f \in \mathcal{D}(M, E)$ there exists a unique $u \in C^\infty(M, E)$ satisfying $Pu = f$, $u|_S = u_0$, and $\nabla_\nu u|_S = u_1$.

It is unclear how to even formulate the Cauchy problem on a Lorentzian manifold which is not globally hyperbolic. One would have to replace the concept of a Cauchy hypersurface by something different to impose the initial conditions upon. Here are two examples which illustrate what can typically go wrong.

Example 5.2. Let $M = S^1 \times \mathbb{R}^{n-1}$ with the metric $g = -d\theta^2 + g_0$ where $d\theta^2$ is the standard metric on S^1 of length 1 and g_0 is the Euclidean metric on $\mathbb{R}^{n-1}$. The universal covering of M is Minkowski space.

Let us try to impose a Cauchy problem on $\{\theta_0\} \times \mathbb{R}^{n-1}$ which is the image of a Cauchy hypersurface in Minkowski space. Such a solution would lift to Minkowski space where it indeed exists uniquely due to Theorem 5.1. But such a solution on Minkowski space is in general not time periodic, hence does not descend to a solution on M.

Therefore existence of solutions fails. The problem is here that M violates the causality condition, i.e., there are closed causal curves.

Remark 5.3. Compact Lorentzian manifolds always possess closed timelike curves and are therefore never well suited for the analysis of wave operators.

Example 5.4. Let M be a timelike strip in 2-dimensional Minkowski space, i.e., $M = \mathbb{R} \times (0, 1)$ with metric $g = -dt^2 + dx^2$. Let $S := \{0\} \times (0, 1)$. Given any $u_0, u_1 \in \mathcal{D}(S, E)$ and any $f \in \mathcal{D}(M, E)$, there exists a solution u to the Cauchy problem. One can simply take the solution in Minkowski space and restrict it to M. But this solution is not unique in M. Choose x in Minkowski space, $x \notin M$, such that $J_+^{\text{Mink}}(x)$ intersects M in the future of S and of $\operatorname{supp}(f)$. The advanced fundamental solution $w = F_+(x)$ (see next section) has support contained in $J_+^{\text{Mink}}(x)$ and satisfies $Pw = 0$ away from x. Hence $u + w$ restricted to M is again a solution to the Cauchy problem on M with the same initial data.

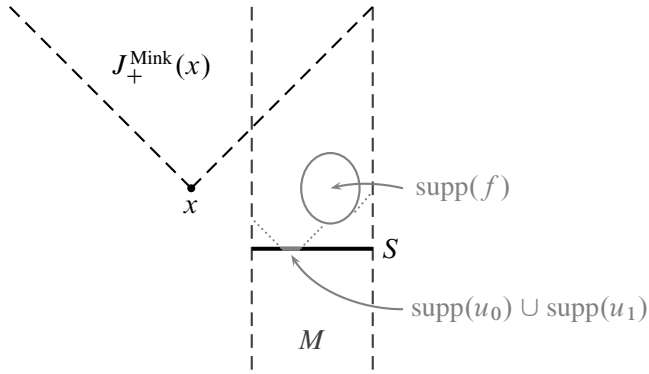

Figure 5. Nonunique solution to Cauchy problem.

The problem is here that S is acausal but not a Cauchy hypersurface. Physically, a wave "from outside the manifold" enters into M.

The physical statement that a wave can never propagate faster than with the speed of light is contained in the following.

Theorem 5.5 (Finite propagation speed). *The solution u from Theorem 5.1 satisfies* $\operatorname{supp}(u) \subset J^M(K)$ *where* $K = \operatorname{supp}(u_0) \cup \operatorname{supp}(u_1) \cup \operatorname{supp}(f)$.

The solution to the Cauchy problem depends continuously on the data.

Theorem 5.6 (Stability). *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S. Let E be a vector bundle over M and let P be a wave operator acting on sections in E.*

Then the map $\mathcal{D}(M,E) \oplus \mathcal{D}(S,E) \oplus \mathcal{D}(S,E) \to C^\infty(M,E)$ sending (f, u_0, u_1) to the unique solution u of the Cauchy problem $Pu = f$, $u|_S = u|_0$, $\nabla_\nu u = u_1$ is linear continuous.

This is essentially an application of the open mapping theorem for Fréchet spaces.

6 Fundamental solutions

Definition 6.1. Let M be a timeoriented Lorentzian manifold, let $E \to M$ be a vector bundle and let $P\colon C^\infty(M,E) \to C^\infty(M,E)$ be a wave operator. Let $x \in M$. A *fundamental solution* of P at x is a distribution $F \in \mathcal{D}'(M,E,E_x^*)$ such that

$$PF = \delta_x.$$

In other words, for all $\varphi \in \mathcal{D}(M, E^*)$ we have

$$F[P^*\varphi] = \varphi(x).$$

If $\operatorname{supp}(F(x)) \subset J_+^M(x)$, then we call F an *advanced fundamental solution*, if $\operatorname{supp}(F(x)) \subset J_-^M(x)$, then we call F a *retarded fundamental solution*.

Using the knowledge about the Cauchy problem from the previous section it is now not hard to find global fundamental solutions on a globally hyperbolic manifold.

Theorem 6.2. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M.*

Then for every $x \in M$ there is exactly one fundamental solution $F_+(x)$ for P at x with past compact support and exactly one fundamental solution $F_-(x)$ for P at x with future compact support. They satisfy

(1) $\operatorname{supp}(F_\pm(x)) \subset J_\pm^M(x)$,

(2) *for each $\varphi \in \mathcal{D}(M, E^*)$ the maps $x \mapsto F_\pm(x)[\varphi]$ are smooth sections in E^* satisfying the differential equation $P^*(F_\pm(\cdot)[\varphi]) = \varphi$.*

Sketch of proof. We do not do the uniqueness part. To show existence fix a foliation of M by spacelike Cauchy hypersurfaces $S_t, t \in \mathbb{R}$ as in Theorem 3.3. Let ν be the future directed unit normal field along the leaves S_t. Let $\varphi \in \mathcal{D}(M, E^*)$. Choose t so large that $\operatorname{supp}(\varphi) \subset I_-^M(S_t)$. By Theorem 5.1 there exists a unique $\chi_\varphi \in C^\infty(M, E^*)$ such that $P^*\chi_\varphi = \varphi$ and $\chi_\varphi|_{S_t} = (\nabla_\nu \chi_\varphi)|_{S_t} = 0$. One can check that χ_φ does not depend on the choice of t.

Fix $x \in M$. By Theorem 5.6 χ_φ depends continuously on φ. Since the evaluation map $C^\infty(M, E) \to E_x$ is continuous, the map $\mathcal{D}(M, E^*) \to E_x^*$, $\varphi \mapsto \chi_\varphi(x)$, is also continuous. Thus $F_+(x)[\varphi] := \chi_\varphi(x)$ defines a distribution. By definition $P^*(F_+(\cdot)[\varphi]) = P^*\chi_\varphi = \varphi$.

Now $P^*\chi_{P^*\varphi} = P^*\varphi$, hence $P^*(\chi_{P^*\varphi} - \varphi) = 0$. Since both $\chi_{P^*\varphi}$ and φ vanish along S_t the uniqueness part which we have omitted shows $\chi_{P^*\varphi} = \varphi$. Thus

$$(PF_+(x))[\varphi] = F_+(x)[P^*\varphi] = \chi_{P^*\varphi}(x) = \varphi(x) = \delta_x[\varphi].$$

Hence $F_+(x)$ is a fundamental solution of P at x.

It remains to show $\operatorname{supp}(F_+(x)) \subset J_+^M(x)$. Let $y \in M \setminus J_+^M(x)$. We have to construct a neighborhood of y such that for each test section $\varphi \in \mathcal{D}(M, E^*)$ whose support is contained in this neighborhood we have $F_+(x)[\varphi] = \chi_\varphi(x) = 0$. Since M is globally hyperbolic $J_+^M(x)$ is closed and therefore $J_+^M(x) \cap J_-^M(y') = \emptyset$ for all y' sufficiently close to y. We choose $y' \in I_+^M(y)$ and $y'' \in I_-^M(y)$ so close that $J_+^M(x) \cap J_-^M(y') = \emptyset$ and $\big(J_+^M(y'') \cap \bigcup_{t \le t'} S_t\big) \cap J_+^M(x) = \emptyset$ where $t' \in \mathbb{R}$ is such that $y' \in S_{t'}$.

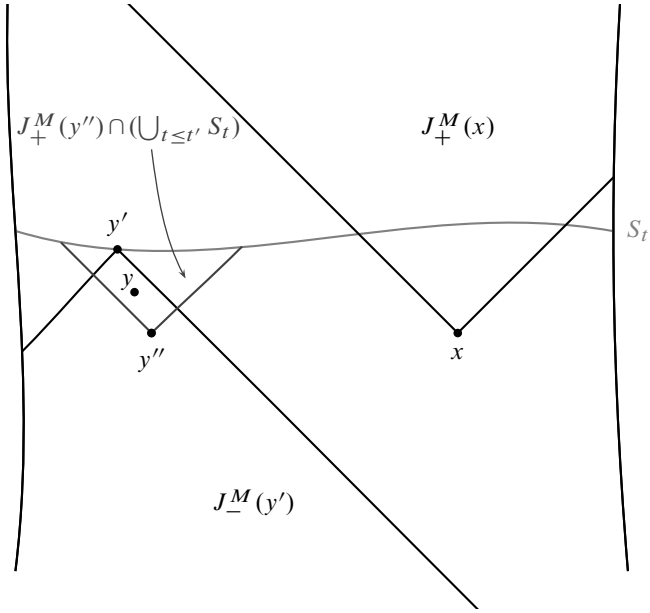

Figure 6. Construction of y, y' and y''.

Now $K := J_-^M(y') \cap J_+^M(y'')$ is a compact neighborhood of y. Let $\varphi \in \mathcal{D}(M, E^*)$ be such that $\operatorname{supp}(\varphi) \subset K$. By Theorem 5.1 $\operatorname{supp}(\chi_\varphi) \subset J_+^M(K) \cup J_-^M(K) \subset J_+^M(y'') \cup J_-^M(y')$. By the independence of χ_φ of the choice of $t > t'$ we have that χ_φ vanishes on $\bigcup_{t>t'} S_t$. Hence $\operatorname{supp}(\chi_\varphi) \subset \big(J_+^M(y'') \cap \bigcup_{t \le t'} S_t\big) \cup J_-^M(y')$ and is therefore disjoint from $J_+^M(x)$. Thus $F_+(x)[\varphi] = \chi_\varphi(x) = 0$ as required. □

For a complete proof see [4, Section 3.3].

7 Green's operators

Now we want to find "solution operators" for a given wave operator P. More precisely, we want to find operators which are inverses of P when restricted to suitable spaces of sections. We will see that existence of such operators is basically equivalent to the existence of fundamental solutions.

Definition 7.1. Let M be a timeoriented connected Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M. A linear map $G_+ \colon \mathcal{D}(M, E) \to C^\infty(M, E)$ satisfying

(i) $P \circ G_+ = \mathrm{id}_{\mathcal{D}(M,E)}$,

(ii) $G_+ \circ P|_{\mathcal{D}(M,E)} = \mathrm{id}_{\mathcal{D}(M,E)}$,

(iii) $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathcal{D}(M,E)$,

is called an *advanced Green's operator for* P.

Similarly, a linear map $G_-\colon \mathcal{D}(M,E) \to C^\infty(M,E)$ satisfying (i), (ii), and

(iii′) $\mathrm{supp}(G_-\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathcal{D}(M,E)$

instead of (iii) is called a *retarded Green's operator for* P.

Fundamental solutions and Green's operators are closely related.

Theorem 7.2. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M.*

Then there exist unique advanced and retarded Green's operators $G_\pm\colon \mathcal{D}(M,E) \to C^\infty(M,E)$ for P.

Proof. By Theorem 6.2 there exist families $F_\pm(x)$ of advanced and retarded fundamental solutions for the adjoint operator P^* respectively. We know that $F_\pm(x)$ depend smoothly on x and the differential equation $P(F_\pm(\cdot)[\varphi]) = \varphi$ holds. By definition we have

$$P(G_\pm\varphi) = P(F_\mp(\cdot)[\varphi]) = \varphi$$

thus showing (i). Assertion (ii) follows from the fact that the $F_\pm(x)$ are fundamental solutions,

$$G_\pm(P\varphi)(x) = F_\mp(x)[P\varphi] = P^*F_\mp(x)[\varphi] = \delta_x[\varphi] = \varphi(x).$$

To show (iii) let $x \in M$ such that $(G_+\varphi)(x) \neq 0$. Since $\mathrm{supp}(F_-(x)) \subset J_-^M(x)$ the support of φ must hit $J_-^M(x)$. Hence $x \in J_+^M(\mathrm{supp}(\varphi))$ and therefore $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$. The argument for G_- is analogous. □

We have seen that existence of fundamental solutions for P^* depending nicely on x implies existence of Green's operators for P. This construction can be reversed. Then uniqueness of fundamental solutions in Theorem 6.2 implies uniqueness of Green's operators.

Lemma 7.3. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M. Let $G_\pm$ be the Green's operators for P and $G_\pm^*$ the Green's operators for the adjoint operator P^*. Then*

$$\int_M (G_\pm^*\varphi)\cdot\psi \ \mathrm{dV} = \int_M \varphi\cdot(G_\mp\psi) \ \mathrm{dV} \tag{1}$$

holds for all $\varphi \in \mathcal{D}(M,E^*)$ and $\psi \in \mathcal{D}(M,E)$.

Proof. For the Green's operators we have $PG_\pm = \mathrm{id}_{\mathcal{D}(M,E)}$ and $P^*G_\pm^* = \mathrm{id}_{\mathcal{D}(M,E^*)}$ and hence

$$\begin{aligned}\int_M (G_\pm^*\varphi)\cdot\psi \ \mathrm{dV} &= \int_M (G_\pm^*\varphi)\cdot(PG_\mp\psi)\ \mathrm{dV}\\ &= \int_M (P^*G_\pm^*\varphi)\cdot(G_\mp\psi)\ \mathrm{dV}\\ &= \int_M \varphi\cdot(G_\mp\psi)\ \mathrm{dV}.\end{aligned}$$

Notice that $\mathrm{supp}(G_\pm\varphi)\cap\mathrm{supp}(G_\mp\psi) \subset J_\pm^M(\mathrm{supp}(\varphi))\cap J_\mp^M(\mathrm{supp}(\psi))$ is compact in a globally hyperbolic manifold so that the partial integration in the second equation is justified. □

Notation 7.4. We write $C^\infty_{\mathrm{sc}}(M,E)$ for the set of all $\varphi\in C^\infty(M,E)$ for which there exists a compact subset $K\subset M$ such that $\mathrm{supp}(\varphi)\subset J^M(K)$. Obviously, $C^\infty_{\mathrm{sc}}(M,E)$ is a vector subspace of $C^\infty(M,E)$.

The subscript "sc" should remind the reader of "space-like compact". Namely, if M is globally hyperbolic and $\varphi\in C^\infty_{\mathrm{sc}}(M,E)$, then for every Cauchy hypersurface $S\subset M$ the support of $\varphi|_S$ is contained in $S\cap J^M(K)$ hence compact by Lemma 3.8. In this sense sections in $C^\infty_{\mathrm{sc}}(M,E)$ have space-like compact support.

Definition 7.5. We say that a sequence of elements $\varphi_j\in C^\infty_{\mathrm{sc}}(M,E)$ *converges in* $C^\infty_{\mathrm{sc}}(M,E)$ *to* $\varphi\in C^\infty_{\mathrm{sc}}(M,E)$ if there exists a compact subset $K\subset M$ such that

$$\mathrm{supp}(\varphi)\subset J^M(K)\quad\text{and}\quad\mathrm{supp}(\varphi_j)\subset J^M(K)$$

for all j, and

$$\|\varphi_j-\varphi\|_{C^k(K',E)}\to 0$$

for all $k\in\mathbb{N}$ and all compact subsets $K'\subset M$.

If G_+ and G_- are advanced and retarded Green's operators for P respectively, then we get a linear map

$$G := G_+ - G_- \colon \mathcal{D}(M,E)\to C^\infty_{\mathrm{sc}}(M,E).$$

Much of the solution theory of wave operators on globally hyperbolic Lorentzian manifolds is collected in the following theorem.

Theorem 7.6. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M. Let G_+ and G_- be advanced and retarded Green's operators for P respectively.*

Then

$$0\to\mathcal{D}(M,E)\xrightarrow{P}\mathcal{D}(M,E)\xrightarrow{G}C^\infty_{\mathrm{sc}}(M,E)\xrightarrow{P}C^\infty_{\mathrm{sc}}(M,E) \tag{2}$$

is an exact sequence of linear maps.

Proof. Properties (i) and (ii) in Definition 7.1 of Green's operators directly yield $G \circ P = 0$ and $P \circ G = 0$, both on $\mathcal{D}(M, E)$. Properties (iii) and (iii') ensure that G maps $\mathcal{D}(M, E)$ to $C^\infty_{\mathrm{sc}}(M, E)$. Hence the sequence of linear maps forms a complex.

Exactness at the first $\mathcal{D}(M, E)$ means that

$$P : \mathcal{D}(M, E) \to \mathcal{D}(M, E)$$

is injective. To see injectivity let $\varphi \in \mathcal{D}(M, E)$ with $P\varphi = 0$. Then $\varphi = G_+ P\varphi = G_+ 0 = 0$.

Next let $\varphi \in \mathcal{D}(M, E)$ with $G\varphi = 0$, i.e., $G_+\varphi = G_-\varphi$. We put $\psi := G_+\varphi = G_-\varphi \in C^\infty(M, E)$ and we see $\mathrm{supp}(\psi) = \mathrm{supp}(G_+\varphi) \cap \mathrm{supp}(G_-\varphi) \subset J^M_+(\mathrm{supp}(\varphi)) \cap J^M_-(\mathrm{supp}(\varphi))$. Since (M, g) is globally hyperbolic $J^M_+(\mathrm{supp}(\varphi)) \cap J^M_-(\mathrm{supp}(\varphi))$ is compact, hence $\psi \in \mathcal{D}(M, E)$. From $P(\psi) = P(G_+(\varphi)) = \varphi$ we see that $\varphi \in P(\mathcal{D}(M, E))$. This shows exactness at the second $\mathcal{D}(M, E)$.

Finally, let $\varphi \in C^\infty_{\mathrm{sc}}(M, E)$ such that $P\varphi = 0$. Without loss of generality we may assume that $\mathrm{supp}(\varphi) \subset I^M_+(K) \cup I^M_-(K)$ for a compact subset K of M. Using a partition of unity subordinated to the open covering $\{I^M_+(K), I^M_-(K)\}$ write φ as $\varphi = \varphi_1 + \varphi_2$ where $\mathrm{supp}(\varphi_1) \subset I^M_-(K) \subset J^M_-(K)$ and $\mathrm{supp}(\varphi_2) \subset I^M_+(K) \subset J^M_+(K)$. For $\psi := -P\varphi_1 = P\varphi_2$ we see that $\mathrm{supp}(\psi) \subset J^M_-(K) \cap J^M_+(K)$, hence $\psi \in \mathcal{D}(M, E)$.

We check that $G_+\psi = \varphi_2$. For all $\chi \in \mathcal{D}(M, E^*)$ we have

$$\int_M \chi \cdot (G_+ P\varphi_2)\ \mathrm{dV} = \int_M (G^*_-\chi) \cdot (P\varphi_2)\ \mathrm{dV} = \int_M (P^* G^*_- \chi) \cdot \varphi_2\ \mathrm{dV} = \int_M \chi \cdot \varphi_2\ \mathrm{dV}$$

where G^*_- is the Green's operator for the adjoint operator P^* according to Lemma 7.3. Notice that for the second equation we use the fact that $\mathrm{supp}(\varphi_2) \cap \mathrm{supp}(G^*_-\chi) \subset J^M_+(K) \cap J^M_-(\mathrm{supp}(\chi))$ is compact. Similarly, one shows $G_-\psi = -\varphi_1$.

Now $G\psi = G_+\psi - G_-\psi = \varphi_2 + \varphi_1 = \varphi$, hence φ is in the image of G. □

Proposition 7.7. *Let M be a globally hyperbolic Lorentzian manifold, let P be a wave operator acting on sections in a vector bundle E over M. Let G_+ and G_- be the advanced and retarded Green's operators for P respectively.*

Then $G_\pm : \mathcal{D}(M, E) \to C^\infty_{\mathrm{sc}}(M, E)$ and all maps in the complex (2) are continuous.

Proof. The maps $P : \mathcal{D}(M, E) \to \mathcal{D}(M, E)$ and $P : C^\infty_{\mathrm{sc}}(M, E) \to C^\infty_{\mathrm{sc}}(M, E)$ are continuous simply because P is a differential operator. It remains to show that $G : \mathcal{D}(M, E) \to C^\infty_{\mathrm{sc}}(M, E)$ is continuous.

Let $\varphi_j, \varphi \in \mathcal{D}(M, E)$ and $\varphi_j \to \varphi$ in $\mathcal{D}(M, E)$ for all j. Then there exists a compact subset $K \subset M$ such that $\mathrm{supp}(\varphi_j) \subset K$ for all j and $\mathrm{supp}(\varphi) \subset K$. Hence $\mathrm{supp}(G\varphi_j) \subset J^M(K)$ for all j and $\mathrm{supp}(G\varphi) \subset J^M(K)$. From the proof of Theorem 6.2 we know that $G_+\varphi$ coincides with the solution u to the Cauchy problem $Pu = \varphi$ with initial conditions $u|_{S_-} = (\nabla_\nu u)|_{S_-} = 0$ where $S_- \subset M$ is a spacelike Cauchy hypersurface such that $K \subset I^M_+(S_-)$. Theorem 5.6 tells us that if $\varphi_j \to \varphi$

in $\mathcal{D}(M,E)$, then the solutions $G_+\varphi_j \to G_+\varphi$ in $C^\infty(M,E)$. The proof for G_- is analogous and the statement for G follows. □

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Chapter 26

Survey of D-branes and K-theory

Daniel S. Freed

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Introduction

K-theory was introduced into mathematics by Grothendieck in the mid-1950s as part of his generalization of the Riemann–Roch–Hirzebruch theorem in algebraic geometry [BS58]. A few years later Atiyah and Hirzebruch [AH61] adopted these ideas in a purely topological context, where they proved a powerful tool in longstanding geometric problems. The most far-reaching application was to linear analysis, the index of elliptic operators [AS63], and this drove subsequent development of the subject. One source of delight for topologists over the past decade has been the reemergence of K-theory in a new context: superstring theory. Namely, the charge of a D-brane in a superstring background – the Cartesian product of a spatial 9-manifold Y and the time line $\mathbb{R}$ – is measured in the topological K-theory of Y. There are many variations, depending on the details of the string theory, and these have led to developments on the mathematical side as well. We will not review the purely mathematical developments – see [Ati01] for one account – but will instead recount the justification for positing K-theory as the home for D-brane charges.

We mention straightaway the seminal work of Polchinski [Pol95] which demonstrates that D-branes are a source of electric and magnetic current for Ramond-Ramond gauge fields. In a classical gauge theory the charges, which are the de Rham cohomology classes of the currents, vary continuously. *Dirac charge quantization* asserts that these charges are quantized in the quantum theory: the possible values form a lattice. Even better, charges lie in an abelian group which is part of a generalized

cohomology theory. The physical question is: which cohomology theory? The traditional answer, integral cohomology, applies to abelian gauge fields in many theories. (In more concrete terms a current quantized by integral cohomology has integer periods when integrated over cycles in space.) The novelty with Ramond-Ramond fields is that the fluxes and charges are quantized by K-theory. This chapter is a survey of justifications for this assertion. There are many answers to "Why K-theory?": couplings and anomalies in the low energy 10-dimensional theory, anomalies in the 2-dimensional worldsheet theory, tachyon condensation, torsion charges, and the relation to M-theory. We review each in turn. More evidence can be marshaled in support of K-theory from the worldsheet point of view; see [Moo04] for an extensive account, which includes the example of D-branes in the WZW model.

We have written expository texts elsewhere [Fre00], [Fre02, Part 3] which elaborate on some aspects in detail. In addition to the survey [Moo04], we also recommend the survey [Wit01]. The list of references here is by no means exhaustive. We have endeavored to open a path into the literature, which may be further pursued via the bibliographies in the references.

Couplings and anomalies in the low energy theory

Low energy Type II superstring theory, in its Euclidean version, is formulated on a 10-dimensional spin manifold X. The bosonic fields are a Riemannian metric; a gauge field, the B-field, whose field strength is a global 3-form; a scalar field, the dilaton field; and the Ramond-Ramond field. There are two theories, Types IIA and IIB, and the Ramond-Ramond field strength is an inhomogeneous closed differential form of even degree in IIA and odd degree in IIB. There are also fermionic fields – spinor fields and Rarita–Schwinger fields. These are the fields of a supergravity theory, and the starting point for a supersymmetric action functional is that determined by supergravity. In both cases the supergravity theory is free of anomalies [AGW83].

A D-brane, in the worldsheet string theory, is a Dirichlet boundary condition for an open string. It is modeled by a submanifold $W \subset Y$ on which the ends of the open string are constrained. One should think of W as analogous to the worldline of a particle – a 1-dimensional oriented submanifold – in the ordinary Maxwell theory of electromagnetism. In the classical Maxwell theory the gauge field is locally a 1-form A, and the classical action has an *electric coupling*

$$-\int_W qA = -\int_X q\tau_W \wedge A, \tag{1}$$

where $q \in \mathbb{R}$ is the electric charge of the particle represented by the worldline and τ_W is a differential form Poincaré dual form to W. (It is a distributional form, or current in the sense of de Rham.) The form $j_E = q\tau_W$ is the electric current in the Maxwell

equation

$$d * F = j_E, \tag{2}$$

where the 2-form $F = dA$ is the field strength. The other Maxwell equation asserts that F is closed:

$$dF = 0. \tag{3}$$

Now from the canonical quantization of the open string one computes the low energy theory on the D-brane. The fields on W are a scalar field (which controls deformations of the position of the submanifold W) and a gauge field whose field strength is a 2-form F; there are also fermions. The fermions lead to anomalies in the quantum theory. The *anomaly inflow mechanism*, first introduced in another context by Callan and Harvey [CH85], cancels these anomalies by simultaneously introducing a term in the action of the form (1) and modifying the Maxwell equation (3), termed the "Bianchi identity" in this context, by postulating a nonzero right hand side.[1] This scheme was carried out for Ramond-Ramond fields and D-branes in [GHM97], [MM98], and [CY98]. The result is that the Hirzebruch $\hat{A}$ genus, introduced in the 1950s as part of the developments surrounding the Riemann–Roch theorem, appears in the electric coupling. Minasian and Moore [MM98] use the Atiyah–Hirzebruch topological version of Grothendieck–Riemann–Roch, and input from Graeme Segal and Maxim Kontsevich, to interpret this coupling in K-theory. They then postulate that Ramond-Ramond charge induced by the D-brane is an element of topological K-theory.

The inflow mechanism is closely related to what is known as the *Green–Schwarz mechanism* [IT94]. From the latter point of view, the right hand side in the amended Maxwell equation (3) is a magnetic current j_B and the anomaly

$$\int_X j_B \wedge j_E \tag{4}$$

is due to the simultaneous electric and magnetic currents. This anomaly formula (4) is for a family $\mathcal{X} \to S$ of spacetimes parametrized by a base space S (of other fields in the theory on which the currents depend), and the integral in (4) is over the fibers in that family. The resulting 2-form on S is the *local* anomaly, or in physics language, the local anomaly is the descent mechanism applied to the form $j_B \wedge j_E$. In more geometric terms, the anomaly is measured by a hermitian line bundle with connection over a space of parameters, and (4) is the formula for the curvature of that connection.

The next step was the author's realization that if charges are quantized by K-theory then so are the fluxes of the Ramond-Ramond fields. Thus Ramond-Ramond fields are geometric objects which on the one hand have a field strength which is a global closed differential form and on the other hand represent a class in topological K-theory. Analogous geometric objects are well-known: a connection on a circle

[1] We remark that physicists often write formulas which don't quite make sense: gauge potentials are treated as global, yet with charge quantization they cannot be; the geometric ideas which enable sensible semiclassical formulas are mentioned below.

bundle has a curvature which is a global closed 2-form and the circle bundle itself represents a class in degree two *integral* cohomology. Higher degree generalizations of connections on circle bundles, or more precisely equivalence classes of such, were introduced into mathematics by Cheeger and Simons [CS85], and their use in physics was pioneered by Gawędski [edz88]. A geometric generalization for degree three integral cohomology, called a *gerbe with connection*, appears in [Bry93], [Hit01] and many other places. At that time, in the late 1990s, a very general theory of such geometric objects was being developed by Hopkins and Singer [HS05]. This gave the language to propose [FH00] that a Ramond-Ramond field is a "cocycle" or "object" in a category representing *differential K-theory*. The electric coupling formula of Minasian–Moore and others has a natural, and simple, expression in this theory. Furthermore, the anomaly cancellation now works on the nose – not just at the level of differential forms. For example, in (4) the currents are refined to objects in differential K-theory and the result of the integral is mapped to degree two differential cohomology. The anomaly cancellation is for Type IIA is more delicate than for Type IIB as the anomalies are torsion of order two. In differential K-theory the Euler class of an odd rank real vector bundle shows up as the anomaly (whereas the image of this Euler class in topological K-theory vanishes). The quantization of Ramond-Ramond flux by K-theory and anomaly cancellations are also discussed in [MW00].

An important property of the Ramond-Ramond field is its self-duality. Witten explained special features of self-dual fields in [Wit97a], [Wit00] and the partition function of Ramond-Ramond fields is further described in [DMW03]. Mathematical aspects of self-dual fields are developed in [HS05] from the Lagrangian point of view and in [FMS] from a Hamiltonian point of view. It is an open problem to integrate these points of view and so construct a complete theory of a self-dual field. For a self-dual field the magnetic current determines the electric current and the bilinear pairing (4) between magnetic and electric currents is refined to a quadratic form on the single current, all this on the level of objects in the differential theory so that charge and flux quantization is taken into account. For the Ramond-Ramond field the self-dual current – think of it as magnetic – may be represented by a vector bundle E with connection. Then the corresponding electric current is represented by $\overline{E}$, and the quadratic form lifts the tensor product $E \otimes \overline{E}$ to a real bundle, then integrates it over the 10-manifold X.[2]

Type I supergravity is inconsistent due to an anomaly [AGW83], and it was a great advance in 1984 when Green and Schwarz realized how low energy effects of the Type I superstring modify supergravity to an anomaly-free theory [GS84]. This is the original example of the Green–Schwarz mechanism. They worked at the level of differential forms – the local anomaly – and so the question arises to incorporate charge quantization. Already Moore and Minasian [MM98] recognized that charges induced by D-branes in Type I should be quantized by real K-theory, that is, by KO-theory. Then, following the logic above, the Ramond-Ramond field in Type I should be an object in differential KO-theory and must be treated as a self-dual field. The necessary

[2] By the index theorem that integral is the pfaffian line bundle of a Dirac operator coupled to $E \otimes \overline{E}$.

quadratic form was supplied in [Fre00] and it was used to refine the Green–Schwarz anomaly cancellation to a statement about both local and global anomalies.

We summarize as follows. The anomaly due to a fermion is measured by a line bundle with connection over the space of bosonic fields. This bundle is defined as the pfaffian line bundle of a family of Dirac operators. The Atiyah–Singer index theorem computes the topology of this bundle in terms of topological K-theory, and there is a presumed geometric refinement to differential K-theory which computes the pfaffian line bundle with its geometry. (See [Fre02, Part 2] for details.) There is a general anomaly (4) due to simultaneous electric and magnetic current, and if Ramond-Ramond fluxes and charges are quantized by topological K-theory, then this formula too lies in differential K-theory. This makes possible the cancellation of fermion anomalies against the electric coupling anomaly, termed the Green–Schwarz mechanism. We remark that there are examples in lower dimensions, such as 2-dimensional worldsheet heterotic string theory, where there is a Green–Schwarz mechanism and yet the gauge field is quantized in ordinary integer cohomology. This is possible since in low dimensions the index formula can sometimes be expressed in integer cohomology rather than K-theory; see [Fre87, §5], for example.

Worldsheet considerations

As mentioned in the introduction, the article [Moo04] by Greg Moore motivates K-theory as the home for Ramond-Ramond charge using more worldsheet considerations than we recount here.

We simply mention in passing one simple harbinger of K-theory which dates from the early days of string theory, when strings were thought to be a theory of hadrons. Namely, in the elementary theory of an open string one may attach "Chan–Paton factors" on the ends of an open string [GSW87, §1.5.3]. In Type II this is a $U(n)$ gauge field, so it is natural as the boundary of the string moves over a D-brane W in a spacetime X to postulate that there is a complex vector bundle with connection on W and the Chan–Paton factor is the pullback to the boundary of the open string. A complex vector bundle $E \to W$ determines a class $[E] \in \mathrm{K}(W)$. Suppose the normal bundle ν to $W \subset X$ is oriented for K-theory, that is, carries a spin^c structure. Then the inclusion $i : W \hookrightarrow X$ induces a pushforward[3] $i_* \colon \mathrm{K}(W) \to \mathrm{K}(X)$, and the associated charge of the D-brane is $i_*[E]$. Of course, the charge depends on the orientation of the normal bundle, and one should find that orientation choice in the physics. Where?

Anomalies in the worldsheet theory, analyzed in [FW99], provide the answer. Let Σ be a compact oriented 2-manifold with boundary, the worldsheet of an open string. The worldsheet theory includes fermions with local boundary conditions; the partition function of the fermions is then a section of a line bundle with connection[4] over the space of bosonic fields, which includes the space $\mathrm{Map}\big((\Sigma, \partial\Sigma); (X, W)\big)$ of maps of Σ

[3] with a degree shift we omit from the notation.

to X which take the boundary $\partial\Sigma$ into the D-brane W. The main result is that this line bundle is flat and holonomies are measured by $w_2(\nu)$, the second Stiefel–Whitney class of the normal bundle ν to W in X. Here W is assumed oriented. This anomaly is canceled by a Green–Schwarz mechanism for the gauge field on W, assumed here to be abelian; the generalization to the nonabelian case was taken up by Kapustin [Kap00]. Assume the B-field on X vanishes. The gauge field on W is locally a 1-form A, and there is an "electric" coupling

$$\int_{\partial\Sigma} A \tag{1}$$

to the boundary of the worldsheet. This becomes anomalous if we postulate a magnetic current for this gauge field. Precisely, we take the Dirac quantization law for the gauge field A to be given by integer cohomology, so that a magnetic current is a cocycle for a degree three differential cohomology class on W. There is a natural choice, the gerbe which obstructs the existence of a spin^c structure on the normal bundle ν. It is a flat gerbe whose holonomies are given by $w_2(\nu)$. The gauge field A is postulated to be a (non-flat) trivialization of this gerbe. Topologically, then, it defines a spin^c structure on ν and so the pushforward map i_* necessary to define the charge in K-theory. Geometrically, the gauge field A is usually termed a *spin^c connection*.

This argument supposed that the B-field on X vanishes, and there is a natural generalization to include nonzero B. In fact, B is also quantized by integer cohomology, so is a cocycle for a degree three differential cohomology class on X. This is exactly the geometric object which twists complex K-theory, and this led Witten [Wit98] to propose that the proper home for Ramond-Ramond charges is twisted K-theory in case $B \neq 0$, at least if the flux of B is torsion. (It was later proposed for all B fields; see [BM00], for example.) The worldsheet story with $B \neq 0$ is consistent with that proposal.

Tachyon condensation

Suppose now that W is a D-brane and W' an *anti*-D-brane. So open strings have both endpoints on W, both on W', or one on each. The analysis of low energy states in the quantized open string with both endpoints on the same brane gives a gauge field, as analyzed above. If one endpoint is on each, then there is a sole low energy mode which is tachyonic, that is, not physical. A physical mechanism of brane–antibrane annihilation via tachyon condensation was proposed by Sen [Sen98] and others. Witten [Wit98] interprets those arguments in terms of K-theory and so gives more evidence that Ramond-Ramond charge lies in K-theory. Suppose that $E \to W$ and $F \to W'$ are the Chan–Paton vector bundles. The low energy modes come into

[4] In [FW99] doubling is used to replace the Dirac operator on Σ with an operator on the double. The recent Ph.D. thesis of Matthew Scholl [Sch06] analyzes determinants directly for the operator on Σ with local boundary conditions.

play when the length of the open string goes to zero, and for that to happen we must take $W = W'$. Then Witten interprets the tachyon field as a linear map $T : E \to F$. Brane–antibrane annihilation occurs if T is an isomorphism; then the configuration is equivalent to zero. This annihilation harks back to the *difference construction* in topological K-theory, introduced in [AH62, §3] and [Bot69] and elaborated in [ABS64, Part II]. Witten interprets the annihilation as an equivalence relation on configurations of branes and antibranes, and so identifies the equivalence classes with K-theory.

Torsion charges

The charges associated to an abelian gauge field take values in an abelian group, which may include torsion. The quotient by the torsion, a free abelian group, naturally sits in a real vector space and periods of the current around cycles in space define points in this free group. There may also be configurations which carry torsion charge; these cannot be measured by periods. The possible torsion charges depend heavily on the choice of cohomology theory with which to quantize charge. In [Wit98], following a suggestion of [MM98], this is used as further evidence of the correctness of K-theory for Ramond-Ramond charge. He works with the Type I theory, in which the charges are quantized by KO-theory. Taking space to be flat $\mathbb{R}^9$ the charge group in Type I is $\mathrm{KO}^0_c(\mathbb{R}^9)$, where the subscript denotes classes of compact support. (Charges are always assumed compactly supported in spatial directions – no charge "leaks out to infinity".) By Bott periodicity this group is isomorphic to $\mathrm{KO}^{-9}(\text{point}) \cong \mathbb{Z}/2\mathbb{Z}$. Witten identifies the Type I 0-brane as a D-brane whose associated charge is the nontrivial element of this group.

For more on torsion charges, see [BD02] and [BDM02].

Relation to M-theory

The Type IIA superstring in 10 dimensions is identified with a certain limit of 11-dimensional M-theory. More precisely, Type IIA on a 10-dimensional spacetime X is M-theory on $X \times S^1$ (or a circle bundle over X) in the limit where S^1 is very small. Thus one can ask to recover facts about Type IIA, in particular the quantization of Ramond-Ramond charge by K-theory, from facts about M-theory. Diaconescu, Moore, and Witten [DMW03] analyze the partition function of M-theory on $X \times S^1$ to test if the partition functions agree. Their analysis reveals explicit indications that the Ramond-Ramond field is quantized by K-theory. We review a few of the main ideas.

The subtleties in the analysis already start with the gauge field, sometimes called the C-field, in M-theory. Note that its dimensional reduction to 10 dimensions is (part of) the Ramond-Ramond field. The C-field has field strength a *closed* form of

degree 4, so the magnetic current as a differential form vanishes. However, there is a torsion magnetic current – its quantization law is shifted [Wit97b] – and this affects the computations. Explicit geometric models can be given for the C-field which make this shift manifest. One particularly nice one, following Witten, is in terms of E_8-connections [DFM]. Now in the limit of large metric on X which Diaconescu-Moore-Witten consider, the contribution to the Type IIA partition function of the Ramond-Ramond field strength, which is a 0-form + 2-form + 4-form, is dominated by the 4-form component. This sets up the possibility of a comparison with the contribution of the M-theory 4-form field strength to its partition function.

The first bit of analysis concerns the topological Chern–Simons term in the M-theory action. On an 11-manifold of the form $X \times S^1$ it reduces to a topological invariant that depends only on the topological class of the C-field in $H^4(X)$. (This class lives on $X \times S^1$, but is presumed pulled back from X since the analysis assumes that the Neveu–Schwarz B-field in the Type IIA theory vanishes.) This invariant is a $\mathbb{Z}/2\mathbb{Z}$-valued quadratic function on $H^4(X)$ whose underlying bilinear form was studied by Landweber–Stong [LS87]. The key step is to compute the integral over C-fields in M-theory by first averaging over shifts by flat C-fields. The Steenrod square Sq^3 appears in this computation. This same operation is the first nontrivial differential in the spectral sequence relating cohomology and K-theory. Careful analysis leads to the conclusion that the topological class of Ramond-Ramond fields which contribute to the partition function are best regarded in K-theory.

There are refinements to the argument in [DFM].

Concluding remarks

There are variations on the arguments recounted here which further match features of Ramond-Ramond fields in superstring theory with topological K-theory. For example, in case the B-field in Type II is nonzero Witten [Wit98, §5.3] suggests that a twisted form of K-theory quantizes Ramond-Ramond charges and fluxes. This fits well with the worldsheet anomaly, as mentioned above. There is also a precise geometric home for the Ramond-Ramond field in this case: it is an object in twisted differential K-theory, the twisting being defined by the B-field, which itself is an object in differential cohomology. For more on twistings by B-fields, see [BM00], [MMS03] and many other references. Other variations include orbifolds and orientifolds, which involve both equivariant K-theory and further twistings of K-theory (including KR-theory [Guk00], [BS]). There is also a vast literature on D-branes in topological string theory, much of which postulates that a D-brane is a complex of sheaves on a Calabi–Yau manifold. In this context we are back to Grothendieck's original picture of K-theory.

The reemergence of K-theory in the context of D-branes has been the catalyst for mathematical developments in several directions. The theory of twistings, for

K-theory and also other generalized cohomology theories, has been worked out on a variety of spaces – manifolds, orbifolds, smooth stacks, algebraic stacks, etc. – in various contexts – topological K-theory, K-theory of operator algebra, K-theory of sheaves, etc. Old ideas are reinterpreted in terms of twistings, and there are new occurrences as well, for example the identification of the Verlinde ring as a twisted equivariant K-theory ring.

There remain mysteries in this area, perhaps the most pressing being the seeming incompatibility of the K-theory home for Ramond-Ramond charges with S-duality in the Type IIB theory [DMW03, §11].

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